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The periodic traveling waves in a diffusive periodic SIR epidemic model with nonlinear incidence

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ARTICLE INFO

Article history:

Received 14 August 2020

Revised 24 November 2020

Accepted 5 January 2021

Available online 21 January 2021

Keywords:

Reaction diffusion

SIR epidemic model

Periodic traveling waves

T-map

Fixed point theorem

Nonlinear incidence

ABSTRACT

In this paper, a reaction-diffusion SIR epidemic model is proposed. It takes into account the individuals mobility, the time periodicity of the infection rate and recovery rate, and the general nonlinear incidence function, which contains a number of classical incidence functions. In our model, due to the introduction of the general nonlinear incidence function, the boundedness of the infected individuals can not be obtained, so we study the existence and nonexistence of periodic traveling wave solutions of original system with the aid of auxiliary system. The basic reproduction number $\mathcal{R}_0$ and the critical wave speed c^* are given. We obtained the existence and uniqueness of periodic traveling waves for each wave speed $c > c^*$ using the Schauder's fixed points theorem when $\mathcal{R}_0 > 1$. The nonexistence of periodic traveling waves for two cases (i) $\mathcal{R}_0 > 1$ and $0 < c < c^*$, (ii) $\mathcal{R}_0 \leq 1$ and $c \geq 0$ are also obtained. These results generalize and improve the existing conclusions. Finally, the numerical experiments support the theoretical results. The differences of traveling wave solution between periodic system and general aperiodic coefficient system are analyzed by numerical simulations.

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1. Introduction

Consider the several outbreaks of infectious disease in human history, such as the Black Death in the late Middle Ages, from the 16th to the 18th century, smallpox, measles, typhoid fever, plague, in the middle of the 20th century, Ebola, SARS and bird flu, these infectious diseases have brought huge impacts on human society. Most recently, in 2020, the COVID-19 outbreak has spread globally in just a few months and has infected more than 50 million people, killed more than one million two hundred as of mid-November 2020, and had an incalculable impact on human lives and the global economy. In the history of mankind, the researches on this kind of terrible infectious disease have never stopped, and the prediction, prevention and control of the disease spread are the research focus. It is a straightforward and effective way to study the spread and control of disease applying appropriate mathematical models.

Besides, to make the model better reflects or describes the real environment, traveling wave solutions, as a special form solution, have been studied more and more widely, the traveling wave solutions cover a wide range of fields, such as in chemical reactions, biological population, genetic engineering, fluid mechanics, mathe-

matical ecology, etc. (See [1–6]). Particularly, in the epidemic models, traveling wave solutions can be understood as a state of disease propagation in space at a specific speed from a certain area to the surroundings. The existence of traveling waves and the minimum wave speed have important significance for the prediction and control of diseases (See [7–15]). At present, the traveling waves for autonomous reaction-diffusion and nonlocal diffusion epidemic models have been studied extensively. Some important results can be found in the literatures [7–10,13,16–23] and their references.

On the other hand, we know that the disease transmission coefficient and recovery rate are not always constant, due to the influence of factors such as weather, temperature, season and so on (See [24–30] and references therein). Consequently, the nonautonomous (especially, time periodic) systems have received widespread attention from scholars [31–33,35]. In [31], Zhang et al. proposed the following reaction-diffusion SIR model with time-dependent

$$\begin{cases} \frac{\partial S}{\partial t} = d_1 \Delta S(t, x) - \beta(t)S(t, x)I(t, x), \\ \frac{\partial I}{\partial t} = d_2 \Delta I(t, x) + \beta(t)S(t, x)I(t, x) - \gamma(t)I(t, x), \\ \frac{\partial R}{\partial t} = d_3 \Delta R(t, x) + \gamma(t)I(t, x), \end{cases} \quad (1)$$

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It is worth noting that, different from the previous model with constant transmission rate and recovery rate in [34], this model presents the traveling waves of periodic parabolic systems, which render the previous methods in [34] unusable. In order to effectively break through these problems, the authors turn the problem into a truncated problem, and then investigate the periodic traveling waves by applying some limiting arguments. They established the existence of periodic traveling waves when wave speed c is large than critical wave speed $c^* = 2\sqrt{\frac{d_2}{T} \int_0^T [\beta(t) - \gamma(t)]dt}$ and the basic reproduction number $\mathcal{R}_0 = \frac{\int_0^T \beta(t)dt}{\int_0^T \gamma(t)dt} > 1$. They also obtained the nonexistence for either $\mathcal{R}_0 \leq 1$ or $c < c^*$ and $\mathcal{R}_0 > 1$.

Recently, Wang et al. in [35] investigated the following reaction-diffusion time periodic SIR model

$$\begin{cases} \frac{\partial S}{\partial t} = d_1 \Delta S(t, x) - \frac{\beta(t)S(t, x)I(t, x)}{S(t, x) + I(t, x)}, \\ \frac{\partial I}{\partial t} = d_2 \Delta I(t, x) + \frac{\beta(t)S(t, x)I(t, x)}{S(t, x) + I(t, x)} - \gamma(t)I(t, x), \\ \frac{\partial R}{\partial t} = d_3 \Delta R(t, x) + \gamma(t)I(t, x). \end{cases} \quad (2)$$

The authors have used fixed-point theories to investigate the periodic traveling waves. Theoretical results show that the system has periodic traveling waves for each wave speed c large than the critical value $c^* = 2\sqrt{\frac{d_2}{T} \int_0^T [\beta(t) - \gamma(t)]dt}$ when the basic reproduction number $\mathcal{R}_0 = \frac{\int_0^T \beta(t)dt}{\int_0^T \gamma(t)dt} > 1$, and there is no periodic traveling wave when $\mathcal{R}_0 \leq 1$ or $\mathcal{R}_0 > 1$ and $c < c^*$.

It is well known that the incidence function in many infectious disease models are bilinear or standard incidence. From the perspective of the development mechanism of infectious diseases, the bilinear incidence is generally used for small-scale susceptible group and exposure. As the population size increases, the bilinear incidence will become infinite and lose its practical significance. At this time, the standard incidence is adopted, which is applicable to a large number of people. However, more and more nonlinear incidence have been mentioned many times (See [36–38]). Korobeinikov and Maini in [39] investigated the the nonlinear incidence $f(S)g(I)$ in a variety of epidemic models. So here, we wonder if the system has a variety of incidence functions, can we come to the same conclusions? Incorporating these above factors, it is then very interesting and natural to investigate the periodic traveling wave solution of periodic epidemic model with general nonlinear incidence.

Motivated by the articles in [31,35], we investigate the following reaction-diffusion time periodic SIR epidemic model with general nonlinear incidence:

$$\begin{cases} \frac{\partial S}{\partial t} = d_1 \Delta S(t, x) - \beta(t)f(S(t, x), I(t, x)), \\ \frac{\partial I}{\partial t} = d_2 \Delta I(t, x) + \beta(t)f(S(t, x), I(t, x)) - \gamma(t)I(t, x), \\ \frac{\partial R}{\partial t} = d_3 \Delta R(t, x) + \gamma(t)I(t, x), \end{cases} \quad (3)$$

where S , I and R denote the sizes of susceptible, infective and removed individuals in location x and at time t , respectively; $d_i \geq 0$ ($i = 1, 2, 3$) stand for the diffusion coefficients for the three individuals, respectively; $\beta(t)$ and $\gamma(t)$ are considered as strictly positive and T -periodic functions; the term $f(S, I)$ is the nonlinear incidence.

It is worth mentioning that our model makes the boundedness of infected individuals difficult to obtain due to the introduction of general response function, which will lead to the Schauder's fixed point theorem cannot be used like in [31] and [35]. To overcome these difficulties, we have to establish an auxiliary system to further construct a bounded closed convex set, and use the fixed point theorem to obtain the traveling wave solution of the auxiliary system, and use the Arzelà-Ascoli theorem to extract diagonal elements to obtain the existence of the traveling wave solution of the system (3).

This paper is organized as follows. Next Section, some important lemmas are introduced and the critical wave speed c^* and the basic reproduction number $\mathcal{R}_0$ are given. Furthermore, the auxiliary system is introduced. In Section 3, a pair of sub- and super-solutions are constructed for the auxiliary system. In Section 4, the convex and closed set is constructed. The fixed point problems derivation from model (3) are discussed. In Section 5, the existence of periodic travelling wave solutions will be firstly proved for the auxiliary system. Then, the corresponding conclusions are further proved for model (3). In Section 6, we study the nonexistence of periodic traveling wave solutions. Section 7, the numerical examples are given to verify our conclusions and a simple comparison is made with the general aperiodic model. Lastly, in Section 8, a brief conclusion is given.

2. Preliminary

We first introduce some notations, important lemmas and definitions to be used in the next sections.

Denote $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{N}$ and $\mathbb{N}_+$ be the sets of natural numbers and positive natural numbers, respectively. Denote $\Gamma := BUC(\mathbb{R}, \mathbb{R})$ be a Banach space of bounded and uniformly continuous functions with the supremum norm. Let $\Gamma^+ := \{v \in \Gamma : v(\xi) \geq 0, \xi \in \mathbb{R}\}$. From Theorem 1.5 in [40] we know that the Γ -realization $d\Delta_\Gamma$ of $d\Delta$ generates a strongly continuous analytic semigroup $T(t)$ on Γ and $T(t)\Gamma^+ \subset \Gamma^+$ for any $t \geq 0$. Additionally, from (Section 2, [31]) or (Chapter I, [40]) we obtain

$$(T(t)v)(\xi) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(\xi-z)^2}{4t}} v(z) dz, \quad t > 0, \quad \xi \in \mathbb{R}, \quad v(\cdot) \in \Gamma. \quad (4)$$

For any given constants $\mu > 0$ and $T > 0$, let $BUC([0, T] \times \mathbb{R}, \mathbb{R})$ be a space of uniformly bounded and uniformly continuous functions $[0, T] \times \mathbb{R} \mapsto \mathbb{R}$. Define

$$B_\mu([0, T] \times \mathbb{R}, \mathbb{R}^2) := \left\{ v = (v_1, v_2) : \begin{array}{l} v_i \in BUC([0, T] \times \mathbb{R}, \mathbb{R}), \sup_{t \in [0, T], \xi \in \mathbb{R}} \{e^{-\mu|\xi|} |v_i(t, \xi)|\} < \infty, \\ v_i(0, \xi) = v_i(T, \xi), \xi \in \mathbb{R}, i = 1, 2. \end{array} \right\}$$

with the norm

$$\|v\|_\mu := \max \left\{ \sup_{t \in [0, T], \xi \in \mathbb{R}} e^{-\mu|\xi|} |v_1(t, \xi)|, \sup_{t \in [0, T], \xi \in \mathbb{R}} e^{-\mu|\xi|} |v_2(t, \xi)| \right\}.$$

Next, we need to introduce the following important lemmas on the parabolic initial value problems and Harnack inequalities (See [41,42]), which will be used in the proofs of Lemma 10 and Theorem 3, respectively.

Lemma 1. (See [42]) Let $\rho(t, x) \in C([0, T] \times \mathbb{R}, \mathbb{R})$, initial value $\mu_0 \in C(\mathbb{R})$. Set

$$\mu(t, x) = (T(t)\mu_0)(x) + \int_0^t T(t-s)\rho(s, \cdot)ds(x), \quad t \in [0, T], x \in \mathbb{R}^n. \quad (5)$$

Then, $\mu(t, x) \in C([0, T] \times \mathbb{R}) \cap C^{\theta, 2\theta}([0, T] \times \mathbb{R})$ for each $\epsilon \in (0, T)$ and $\theta \in (0, 1)$. Furthermore, the following inequalities hold for some constants $M > 0$ and $M(\epsilon, \theta) > 0$

$$\|\mu\|_\infty \leq M(\|\mu_0\|_\infty + \|\rho\|_\infty), \quad (6)$$

and

$$\|\mu\|_{C^{\theta, 2\theta}([0, T] \times \mathbb{R})} \leq M(\epsilon, \theta)(\epsilon^{-\theta} \|\mu_0\|_\infty + \|\rho\|_\infty), \quad (7)$$

where

$$\|\mu\|_\infty = \sup_{(t, x) \in [0, T] \times \mathbb{R}} \|\mu(t, x)\|, \quad \|\mu\|_{C^{\theta, 2\theta}([0, T] \times \mathbb{R})} = \sup_{x \in \mathbb{R}} \|\mu(\cdot, x)\|_{C^\theta([0, T])} + \sup_{0 \leq t \leq T} \|\mu(t, \cdot)\|_{C^{2\theta}(\mathbb{R})}.$$

Lemma 2. (See [43]) (Harnack inequalities) Let the operators

$$\mathbf{L}_i := \sum_{j,k=1}^n a_{j,k}^i(t, \xi) \frac{\partial^2}{\partial \xi_j \partial \xi_k} + \sum_{j=1}^n b_j^i(t, \xi) \frac{\partial}{\partial \xi_j} - \frac{\partial}{\partial t}, \quad i = 1, 2, \dots, l,$$

be uniformly parabolic in an open domain $(\tau, M) \times \Omega$ of $\mathbb{R} \times \mathbb{R}^n$, that is, there exists $\alpha_0 > 0$ such that the inequality $\sum_{j,k=1}^n a_{j,k}^i(t, \xi) \xi_j \xi_k \geq \alpha_0 \sum_{j=1}^n \xi_j^2$ always holds for every $(\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$, where $-\infty < \tau < M \leq +\infty$ and Ω is a bounded open set. Assume that $a_{j,k}^i(t, \xi), b_j^i(t, \xi) \in C((\tau, M) \times \Omega, \mathbb{R})$ and satisfy

$$\max_{(t, \xi) \in (\tau, M) \times \Omega} |a_{j,k}^i(t, \xi)| + |b_j^i(t, \xi)| \leq \beta_0$$

for some $\beta_0 > 0$. Suppose that

$$\mathbf{w} = (\omega_1, \omega_2, \dots, \omega_l) \in C((\tau, M) \times \overline{\Omega}, \mathbb{R}^l) C^{1,2}((\tau, M) \times \Omega, \mathbb{R}^l)$$

satisfies

$$\sum_{s=1}^l c^{i,s}(t, \xi) \omega_s + \mathbf{L}_i \omega_i \leq 0, \quad (t, \xi) \in (\tau, M) \times \Omega, i = 1, 2, \dots, l, \quad (8)$$

where $c^{i,s}(t, \xi) \in C((\tau, M) \times \Omega, \mathbb{R})$ and $c^{i,s} \geq 0$ if $i \neq s$, and there exists a constant $\gamma_0 > 0$ such that

$$\max_{(t, \xi) \in (\tau, M) \times \Omega} |c^{i,s}(t, \xi)| \leq \gamma_0, \quad i, s = 1, 2, \dots, l.$$

Assume that $D \subset \subset U \subset \Omega$ and there exist some positive constants ϱ and ϵ such that $\text{dist}(\overline{D}, \partial U) > \varrho$ and $|D| > \epsilon$. Let $\theta > 0$ and $\tau + 4\theta > M$, then there exist positive constants p, ω_1 and ω_2 dependent on $\alpha_0, \beta_0, \gamma_0, \varrho, \epsilon, n$, $\text{diam } \Omega$ and θ satisfying

$$\inf_{(\tau+3\theta, \tau+4\theta) \times D} \omega_k \geq \omega_1 [(\omega_k)^+]_{p, (\tau+\theta, \tau+2\theta) \times D} - \omega_2 \max_{j=1,2,\dots,k} \sup_{\tau \times U[\tau, \tau+4\theta] \times \partial U} (\omega_j)^-.$$

Where $(\omega_k)^+ = \max\{\omega_k, 0\}$, $(\omega_k)^- = \max\{-\omega_k, 0\}$, $[(\omega_k)^+]_{p, \Omega} = (\frac{1}{|\Omega|} \int_\Omega |(\omega_k)^+|^p dx dt)^{\frac{1}{p}}$, particularly, $[(\omega_k)^+]_{\infty, \Omega} = \sup_\Omega |(\omega_k)^+|$. Furthermore, if all inequalities in (8) become equalities, then the lemma is also valid for $p = \infty$ and ω_1, ω_2 independent of ϵ .

In this paper, for the convenience, let $f(\xi)$ with $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ be a continuously differentiable function, we denote $f_{\xi_i}(\xi) = \frac{\partial f(\xi)}{\partial \xi_i}$.

For nonlinear incidence $f(S, I)$ in model (3) in this paper we always assume that the following assumptions hold.

(A1) $f(S, I)$ is twice continuously differentiable for $S, I \geq 0$ and $f(0, I) = f(S, 0) = 0$ for all $S, I > 0$.

(A2) $f_i(S, I) > 0$ and $f_{II}(S, I) \leq 0$ for all $S > 0$ and $I \geq 0$. For any fixed $I \geq 0$, $f_S(S, I)$ is positive and bounded for all $S \geq 0$.

There are some concrete incidence functions satisfying assumptions **(A1)** and **(A2)**, such as $f(S, I) = SI$ (See [31]), $f(S, I) = \frac{SI}{S+I}$ (See [35]), $f(S, I) = \frac{SI}{1+kl}$ with constant $k \geq 0$ (See [43]) and $f(S, I) = \frac{SI}{1+aS+bl}$ with constants $a, b \geq 0$ (See [44]).

Because the first two equations of (3) are relatively independent, it suffices to consider the reduced system

$$\begin{cases} \frac{\partial S}{\partial t} = d_1 \Delta S(t, x) - \beta(t)f(S(t, x), I(t, x)), \\ \frac{\partial I}{\partial t} = d_2 \Delta I(t, x) + \beta(t)f(S(t, x), I(t, x)) - \gamma(t)I(t, x). \end{cases} \quad (9)$$

Definition 1. The T-periodic traveling wave of system (9) is defined as a special solution $(S(t, x), I(t, x)) = (\phi(t, z), \varphi(t, z))$ with $z = x + ct$ satisfying

$$(\phi(t + T, z), \varphi(t + T, z)) \equiv (\phi(t, z), \varphi(t, z)). \quad (10)$$

Obviously, such solutions $(\phi(t, z), \varphi(t, z))$ with $z = x + ct$ must satisfy

$$\begin{cases} \phi_t(t, z) = d_1 \phi_{zz}(t, z) - c\phi_z(t, z) - \beta(t)f(\phi(t, z), \varphi(t, z)), \\ \varphi_t(t, z) = d_2 \varphi_{zz}(t, z) - c\varphi_z(t, z) + \beta(t)f(\phi(t, z), \varphi(t, z)) - \gamma(t)\varphi(t, z). \end{cases} \quad (11)$$

In this paper, we investigate the traveling wave solutions $(\phi(t, x + ct), \varphi(t, x + ct))$ of system (9) satisfying the following asymptotic boundary conditions

$$\begin{pmatrix} \phi(t, -\infty) \\ \varphi(t, -\infty) \end{pmatrix} = \begin{pmatrix} S_0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \phi(t, \infty) \\ \varphi(t, \infty) \end{pmatrix} = \begin{pmatrix} S^\infty \\ 0 \end{pmatrix} \text{ uniformly for } t \in \mathbb{R}. \quad (12)$$

Here $S_0 > 0$ is thought to be the sizes of susceptible individuals in the initial state. The constant $S^\infty \geq 0$ stands for the sizes of susceptible individuals after the epidemic. Define the basic reproduction number of periodic system (9) as below

$$\mathcal{R}_0 := \frac{\int_0^T \beta(t) f_I(S_0, 0) dt}{\int_0^T \gamma(t) dt}.$$

In fact, $\mathcal{R}_0$ reflects the average number of newly infected individuals generated by a single infected individual at the beginning of the disease infection process.

Linearizing the second equation of system (11) at $(S_0, 0)$, yields

$$M_t = d_2 M_{zz}(t, z) - cM_z(t, z) + (\beta(t)f_I(S_0, 0) - \gamma(t))M(t, z). \quad (13)$$

Define

$$\Theta_c(\lambda) = d_2 \lambda^2 - c\lambda + \omega, \quad c \in \mathbb{R}, \quad \lambda \in \mathbb{R},$$

where $\omega := \frac{1}{T} \int_0^T (\beta(t)f_I(S_0, 0) - \gamma(t)) dt$. Clearly, $\omega > 0$ if and only if $\mathcal{R}_0 = \frac{\int_0^T \beta(t)f_I(S_0, 0) dt}{\int_0^T \gamma(t) dt} > 1$. Let

$$\lambda_1 = \frac{c - \sqrt{c^2 - 4d_2\omega}}{2d_2} \text{ and } \lambda_2 = \frac{c + \sqrt{c^2 - 4d_2\omega}}{2d_2}.$$

If $c > c^* := 2\sqrt{d_2\omega}$, then we have $\Theta_c(\lambda_1) = \Theta_c(\lambda_2) = 0$ and $\Theta_c(\lambda) < 0$ for all $\lambda \in (\lambda_1, \lambda_2)$.

In this paper, we will consider the periodic travelling wave solutions for model (3) by using the auxiliary system method. We introduce the auxiliary system

$$\begin{cases} \phi_t(t, z) = d_1 \phi_{zz}(t, z) - c\phi_z(t, z) - \beta(t)f(\phi(t, z), \varphi(t, z)), \\ \varphi_t(t, z) = d_2 \varphi_{zz}(t, z) - c\varphi_z(t, z) + \beta(t)f(\phi(t, z), \varphi(t, z)) - \gamma(t)\varphi(t, z) - \varepsilon\varphi^2(t, z), \end{cases} \quad (14)$$

where $\varepsilon > 0$ is a constant. Our idea is that firstly we will discuss the existence of travelling wave solution $(\phi_\varepsilon(t, z), \varphi_\varepsilon(t, z))$ which is dependent of parameter ε for auxiliary system (14) with the asymptotic boundary conditions (12) when $\mathcal{R}_0 > 1$ and $c > c^*$ by using the sub- and super-solutions method and the fixed point theorems. Then taking $\varepsilon \rightarrow 0$, we will further prove that there exists the limit function $(\phi^*(t, z), \varphi^*(t, z))$ of $(\phi_\varepsilon(t, z), \varphi_\varepsilon(t, z))$, which is the travelling wave solution of model (3) with the asymptotic boundary conditions (12) when $\mathcal{R}_0 > 1$ and $c > c^*$.

3. Sub- and super-solutions

In what follows, we always assume that $\mathcal{R}_0 > 1$ and $c > c^*$. Define function $K(t) = \exp(\int_0^t [d_2 \lambda_1^2 - c\lambda_1 + (\beta(s)f_I(S_0, 0) - \gamma(s))] ds)$. We easily verify $K(t + T) = K(t)$ for all $t \geq 0$. Let

$$\phi^+(t, z) := S_0, \quad \phi^-(t, z) := \max\{S_0(1 - M_1 e^{\varepsilon_1 z}), 0\},$$

$$\varphi^+(t, z) := K(t)e^{\lambda_1 z}, \quad \varphi^-(t, z) := \max\{K(t)e^{\lambda_1 z}(1 - M_2 e^{\varepsilon_2 z}), 0\},$$

where $t > 0, z \in \mathbb{R}$. Parameters M_i and ε_i ($i = 1, 2$) are positive constants and will be given later. Now, we will prove that (ϕ^-, φ^-) and (ϕ^+, φ^+) are the upper-lower solutions of system (14).

Lemma 3. Function $\phi^+(t, z) = S_0$ satisfies the inequality $\phi_t^+ \geq d_1 \phi_{zz}^+ - c\phi_z^+ - \beta(t)f(\phi^+, \varphi^-)$.

The proof of Lemma 3 is simple since $\beta(t)f(\phi^+, \varphi^-) \geq 0$.

Lemma 4. Function $\varphi^+(t, z) = K(t)e^{\lambda_1 z}$ satisfies the inequality

$$\varphi_t^+ \geq d_2 \varphi_{zz}^+ - c \varphi_z^+ + \beta(t)f(\phi^+, \varphi^+) - \gamma(t)\varphi^+ - \varepsilon \varphi^{+2}.$$

Proof. By a simple calculation, we get

$$\begin{aligned} \varphi_t^+(t, z) &= K'(t)e^{\lambda_1 z} = [d_2 \lambda_1^2 - c \lambda_1 + (\beta(t)f_1(S_0, 0) - \gamma(t))]K(t)e^{\lambda_1 z} \\ &= d_2 \varphi_{zz}^+ - c \varphi_z^+ + (\beta(t)f_1(S_0, 0) - \gamma(t))\varphi^+. \end{aligned} \quad (15)$$

Assumption (A2) implies that $f(\phi^+, \varphi^+) \leq f_1(S_0, 0)\varphi^+$ for $\varphi^+ \geq 0$. Thus

$$\begin{aligned} d_2 \varphi_{zz}^+ - c \varphi_z^+ + \beta(t)f(\phi^+, \varphi^+) - \gamma(t)\varphi^+ - \varepsilon \varphi^{+2} \\ \leq d_2 \varphi_{zz}^+ - c \varphi_z^+ + \beta(t)f_1(S_0, 0)\varphi^+ - \gamma(t)\varphi^+ = \varphi_t^+(t, z). \end{aligned} \quad (16)$$

This is end of the proof. $\square$

Lemma 5. Suppose $0 < \varepsilon_1 < \min\{\lambda_1, \frac{c}{d_1}\}$ and $M_1 > 1$ large enough, then function $\phi^-(t, z)$ satisfies the inequality

$$\phi_t^- - d_1 \phi_{zz}^- + c \phi_z^- \leq -\beta(t)f(\phi^-, \varphi^+), \quad (17)$$

for any $z \neq z_1 := -\varepsilon_1^{-1} \ln M_1$.

Proof. If $z > -\varepsilon_1^{-1} \ln M_1$, then $\phi^-(t, z) = 0$ and (17) holds. If $z < -\varepsilon_1^{-1} \ln M_1$, then $\phi^-(t, z) = S_0(1 - M_1 e^{\varepsilon_1 z})$ and $\varphi^+(t, z) = K(t)e^{\lambda_1 z}$. The inequality (17) is equivalent to

$$d_1 S_0 M_1 \varepsilon_1^2 e^{\varepsilon_1 z} - c S_0 M_1 \varepsilon_1 e^{\varepsilon_1 z} \leq -\beta(t)f(S_0(1 - M_1 e^{\varepsilon_1 z}), K(t)e^{\lambda_1 z}),$$

that is,

$$S_0 M_1 \varepsilon_1 (c - d_1 \varepsilon_1) \geq \frac{\beta(t)f(S_0(1 - M_1 e^{\varepsilon_1 z}), K(t)e^{\lambda_1 z})}{e^{\varepsilon_1 z}}. \quad (18)$$

From assumptions (A1) and (A2) one can see that $f(S_0(1 - M_1 e^{\varepsilon_1 z}), K(t)e^{\lambda_1 z}) < f(S_0, K(t)e^{\lambda_1 z})$ and

$$\frac{f(S_0, K(t)e^{\lambda_1 z})}{e^{\varepsilon_1 z}} = K(t) \frac{f(S_0, K(t)e^{\lambda_1 z})}{K(t)e^{\varepsilon_1 z}} \leq f_1(S_0, 0)K(t).$$

To obtain inequality (18), it is sufficient to verify

$$S_0 M_1 \varepsilon_1 (c - d_1 \varepsilon_1) \geq \beta(t)f_1(S_0, 0)K(t). \quad (19)$$

Noting the periodicity and positivity of $\beta(t)$ and $K(t)$, keeping $0 < \varepsilon_1 < \min\{\lambda_1, \frac{c}{d_1}\}$ and taking $M_1 > 1$ sufficiently large, then inequality (19) holds. This proof is completed. $\square$

Lemma 6. Assume that $\varepsilon_2 \in (0, \min\{\varepsilon_1, \lambda_2 - \lambda_1\})$ small enough and $M_2 > 1$ large enough such that $-\varepsilon_2^{-1} \ln M_2 < -\varepsilon_1^{-1} \ln M_1$. Then function $\varphi^-(t, z)$ satisfies the inequality

$$\varphi_t^- - d_2 \varphi_{zz}^- + c \varphi_z^- \leq \beta(t)f(\phi^-, \varphi^-) - \gamma(t)\varphi^- - \varepsilon \varphi^{-2}, \quad (20)$$

for any $z \neq z_2 := -\varepsilon_2^{-1} \ln M_2$.

Proof. It follows from $-\varepsilon_2^{-1} \ln M_2 < -\varepsilon_1^{-1} \ln M_1$ that $z_2 < z_1 < 0$. If $z > -\varepsilon_2^{-1} \ln M_2$, then $\varphi^-(t, z) = 0$, which implies that (20) holds. If $z < -\varepsilon_2^{-1} \ln M_2$, then $\varphi^-(t, z) = K(t)e^{\lambda_1 z}(1 - M_2 e^{\varepsilon_2 z})$ and $\phi^-(t, z) = S_0(1 - M_1 e^{\varepsilon_1 z})$. Rewriting inequality (20), we have

$$\begin{aligned} \varphi_t^- - d_2 \varphi_{zz}^- + c \varphi_z^- - (\beta(t)f_1(S_0, 0) - \gamma(t))\varphi^- \\ \leq \beta(t)[f(\phi^-, \varphi^-) - f_1(S_0, 0)\varphi^-] - \varepsilon \varphi^{-2}. \end{aligned} \quad (21)$$

By direct calculations, we can obtain

$$\begin{aligned} \varphi_t^- - d_2 \varphi_{zz}^- + c \varphi_z^- - (\beta(t)f_1(S_0, 0) - \gamma(t))\varphi^- \\ = e^{\varepsilon_1 z} [K'(t) - d_2 \lambda_1^2 K(t) + c \lambda_1 K(t) - (\beta(t)f_1(S_0, 0) - \gamma(t))K(t)] \\ - M_2 e^{(\lambda_1 + \varepsilon_2)z} [K'(t) - d_2 (\lambda_1 + \varepsilon_2)^2 K(t) + c (\lambda_1 + \varepsilon_2) K(t) - (\beta(t)f_1(S_0, 0) - \gamma(t))K(t)] \\ = -M_2 e^{(\lambda_1 + \varepsilon_2)z} K(t) [(d_2 \lambda_1^2 - c \lambda_1) - (d_2 (\lambda_1 + \varepsilon_2)^2 - c (\lambda_1 + \varepsilon_2))] \\ = M_2 e^{(\lambda_1 + \varepsilon_2)z} K(t) \Theta_c(\lambda_1 + \varepsilon_2). \end{aligned}$$

Then inequality (21) is equivalent to

$$-M_2 e^{(\lambda_1 + \varepsilon_2)z} K(t) \Theta_c(\lambda_1 + \varepsilon_2) \geq \beta(t)[f_1(S_0, 0)\varphi^- - f(\phi^-, \varphi^-)] + \varepsilon \varphi^{-2}. \quad (22)$$

Assumption (A2) implies that $\frac{f(S, I)}{I}$ is a decreasing function on $(0, +\infty)$. Thus, for any given $\epsilon \in (0, f_1(S_0, 0))$, we can find a number $\sigma_0 > 0$ such that $\frac{f(S, I)}{I} \geq f_1(S_0, 0) - \epsilon$ as $S_0 - \sigma_0 < S < S_0$ and $0 < I < \sigma_0$. Since $\varphi^- \rightarrow 0$ and $\phi^- \rightarrow S_0$ as $z \rightarrow -\infty$, for given above $\epsilon \in (0, f_1(S_0, 0))$, we can choose a $M'_2 > 0$ such that $0 < \varphi^- < \sigma_0$ and $S_0 - \sigma_0 \leq \phi^- \leq S_0$ when $z < -M'_2$. Let $M_2 > \max\{M_1^{\varepsilon_2/\varepsilon_1}, e^{\varepsilon_2 M'_2}\}$, then we get

$$\begin{aligned} f_1(S_0, 0)\varphi^- - f(\phi^-, \varphi^-) &= (f_1(S_0, 0) - \frac{f(\phi^-, \varphi^-)}{\varphi^-})\varphi^- \\ &\leq \left(\frac{f_1(S_0, 0) - \frac{f(\phi^-, \varphi^-)}{\varphi^-} + \varphi^-}{2} \right)^2 \leq (f_1(S_0, 0) - (f_1(S_0, 0) - \epsilon) + \varphi^-)^2. \end{aligned} \quad (23)$$

Due to inequality (23) is valid for any $\epsilon \in (0, f_1(S_0, 0))$, which implies that $f_1(S_0, 0)\varphi^- - f(\phi^-, \varphi^-) \leq \varphi^{-2}$. That is,

$$\beta(t)[f_1(S_0, 0)\varphi^- - f(\phi^-, \varphi^-)] + \varepsilon\varphi^{-2} \leq (\beta(t) + \varepsilon)\varphi^{-2}. \quad (24)$$

Noting (24), to obtain (22), it suffices to verify

$$-M_2 e^{(\lambda_1 + \varepsilon_2)z} K(t) \Theta_c(\lambda_1 + \varepsilon_2) \geq (\beta(t) + \varepsilon) K^2(t) e^{2\lambda_1 z} (1 - M_2 e^{\varepsilon_2 z})^2. \quad (25)$$

In view of $0 < (1 - M_2 e^{\varepsilon_2 z})^2 < 1$, to obtain (25) it suffices to verify

$$-M_2 e^{(\lambda_1 + \varepsilon_2)z} K(t) \Theta_c(\lambda_1 + \varepsilon_2) \geq (\beta(t) + \varepsilon) K^2(t) e^{2\lambda_1 z}. \quad (26)$$

That is,

$$-M_2 \Theta_c(\lambda_1 + \varepsilon_2) \geq (\beta(t) + \varepsilon) K(t) e^{(\lambda_1 - \varepsilon_2)z}. \quad (27)$$

Since $\varepsilon_2 < \min\{\varepsilon_1, \lambda_2 - \lambda_1\}$ and $0 < \varepsilon_1 < \lambda_1$, we have $\lambda_1 - \varepsilon_2 > 0$. Hence, for $z < -\varepsilon_2^{-1} \ln M_2$, it is sufficient to verify

$$-M_2 \Theta_c(\lambda_1 + \varepsilon_2) \geq (\beta(t) + \varepsilon) K(t) M_2^{\frac{\varepsilon_2 - \lambda_1}{\varepsilon_2}}.$$

In fact, it is easy to see that $\lambda_1 + \varepsilon_2 \in (\lambda_1, \lambda_2)$ and $\Theta_c(\lambda_1 + \varepsilon_2) < 0$. The above inequality is true for $M_2 > \max\{M_1^{\varepsilon_2/\varepsilon_1}, e^{\varepsilon_2 M_2'}\}$ large enough. The proof of this lemma is shown. $\square$

4. Fixed-point problems

Now, we construct a set $\mathcal{D}$ as follows

$$\mathcal{D} := \{(\tilde{\phi}, \tilde{\varphi}) \in B_\mu([0, T] \times \mathbb{R}, \mathbb{R}^2) : \phi^- \leq \tilde{\phi} \leq \phi^+, \varphi^- \leq \tilde{\varphi} \leq \min\{\varphi^+, \Lambda_\varepsilon\}\},$$

where $\Lambda_\varepsilon > 0$ is sufficiently large constant satisfying $\beta(t)f(S_0, \Lambda_\varepsilon) - \gamma(t)\Lambda_\varepsilon - \varepsilon\Lambda_\varepsilon^2 < 0$ for all $t \in [0, T]$. In fact, we can choose $\Lambda_\varepsilon = \max_{t \in [0, T]} \{\frac{\beta(t)f_1(S_0, 0) - \gamma(t)}{\varepsilon}\}$. For any $(\tilde{\phi}, \tilde{\varphi}) \in \mathcal{D}$, let

$$f_1[\tilde{\phi}, \tilde{\varphi}] = \alpha_1 \tilde{\phi} - \beta(t)f(\tilde{\phi}, \tilde{\varphi}), \quad f_2[\tilde{\phi}, \tilde{\varphi}] = \alpha_2 \tilde{\varphi} + \beta(t)f(\tilde{\phi}, \tilde{\varphi}) - \gamma(t)\tilde{\varphi} - \varepsilon\tilde{\varphi}^2,$$

where α_1 and α_2 are positive constants satisfying $\alpha_1 > \max\{m\hat{\beta}, \frac{\ln 4}{T}\}$ with $\hat{\beta} := \max_{t \in [0, T]} \beta(t)$, $m := \sup_{0 \leq S \leq S_0} \{f_S(S, \Lambda_\varepsilon)\}$ and $\alpha_2 > \max_{t \in [0, T]} \gamma(t) + 2\varepsilon\Lambda_\varepsilon$. For a given $(\tilde{\phi}, \tilde{\varphi}) \in \mathcal{D}$, we investigate the following parabolic initial value problem:

$$\begin{cases} \phi_t - d_1 \phi_{zz} + c\phi_z + \alpha_1 \phi = f_1[\tilde{\phi}, \tilde{\varphi}], & 0 < t \leq T, \quad z \in \mathbb{R}, \\ \varphi_t - d_2 \varphi_{zz} + c\varphi_z + \alpha_2 \varphi = f_2[\tilde{\phi}, \tilde{\varphi}], & 0 < t \leq T, \quad z \in \mathbb{R}, \\ \phi(0, z) = \phi_0(z), \quad \varphi(0, z) = \varphi_0(z), & z \in \mathbb{R}. \end{cases} \quad (28)$$

Rewrite (28) as an integral system:

$$\begin{cases} \phi(t) = T_1(t)\phi_0 + \int_0^t T_1(t-s)f_1[\tilde{\phi}, \tilde{\varphi}](s)ds, \\ \varphi(t) = T_2(t)\varphi_0 + \int_0^t T_2(t-s)f_2[\tilde{\phi}, \tilde{\varphi}](s)ds, \end{cases} \quad (29)$$

where $(\phi(t), \varphi(t)) = (\phi(t, z), \varphi(t, z))$ and $T_i(t)$ is the analytic semigroup (See [41]) generated by operator $\mathcal{A}_i : D(\mathcal{A}_i) \rightarrow C(\mathbb{R})$ defined by $\mathcal{A}_i v = d_i v_{zz} - c v_z - \alpha_i v$, $i = 1, 2$, and

$$D(\mathcal{A}_i) := \left\{ v \in \bigcap_{1 \leq p < \infty} W_{loc}^{2,p}(\mathbb{R}) : \mathcal{A}_i v \in C(\mathbb{R}) \right\}, \quad i = 1, 2,$$

where the space $W_{loc}^{2,p}(\mathbb{R})$ can be found in [40]. Furthermore, from (4) we have

$$(T_i(t)v)(x) = e^{-\alpha_i t} \frac{1}{\sqrt{4\pi d_i t}} \int_{\mathbb{R}} e^{-\frac{(x-\xi-\varepsilon)^2}{4d_i t}} v(\xi) d\xi, \quad t > 0, \quad x \in \mathbb{R}, \quad v(\cdot) \in \Gamma. \quad (30)$$

The solution of integral system (29) is the mild solution of problem (28). For a given $\eta > 0$, define

$$\tilde{B}_\eta(\mathbb{R}, \mathbb{R}^2) := \left\{ v = (v_1, v_2) : v_i \in \Gamma, \sup_{\xi \in \mathbb{R}} e^{-\eta|\xi|} |v_i(\xi)| < \infty, \xi \in \mathbb{R}, i = 1, 2 \right\},$$

equipped with the norm

$$\|v\|_\mu := \max \left\{ \sup_{\xi \in \mathbb{R}} e^{-\eta|\xi|} |v_1(\xi)|, \sup_{\xi \in \mathbb{R}} e^{-\eta|\xi|} |v_2(\xi)| \right\}.$$

Let

$$\tilde{\mathcal{D}} := \left\{ (\phi_0(\cdot), \varphi_0(\cdot)) \in \tilde{B}_\eta(\mathbb{R}, \mathbb{R}^2) : \begin{aligned} &\phi^-(0, z) \leq \phi_0(z) \leq \phi^+(0, z) \\ &\varphi^-(0, z) \leq \varphi_0(z) \leq \min\{\varphi^+(0, z), \Lambda_\varepsilon\} \end{aligned} \right\}.$$

Obviously, $\tilde{\mathcal{D}}$ is convex and closed set. Next, we show that the system (29) has a unique solution $(\phi^*, \varphi^*) \in \mathcal{D}$ for any $(\tilde{\phi}, \tilde{\varphi}) \in \mathcal{D}$.

Now, we show that the invariance of integral system (29). To obtain this, we introduce the following Lemma.

Lemma 7. The functions ϕ^+ , φ^+ , ϕ^- and φ^- satisfy the following inequalities:

$$\phi^+(t) \geq T_1(t)\phi^+(0) + \int_0^t T_1(t-s)f_1[\phi^+, \varphi^-](s)ds, \quad (31)$$

$$\varphi^+(t) \geq T_2(t)\varphi^+(0) + \int_0^t T_2(t-s)f_2[\phi^+, \varphi^+](s)ds, \quad (32)$$

$$\phi^-(t) \leq T_1(t)\phi^-(0) + \int_0^t T_1(t-s)f_1[\phi^-, \varphi^+](s)ds \quad (33)$$

and

$$\varphi^-(t) \leq T_2(t)\varphi^-(0) + \int_0^t T_2(t-s)f_2[\phi^-, \varphi^-](s)ds. \quad (34)$$

Proof. Since $\phi^+(t, z) \equiv S_0$ and $\mathcal{A}_1\phi = d_1\phi_{zz} - c\phi_z - \alpha_1\phi$, we have

$$\phi_t^+(t) = d_1\phi_{zz}^+(t) - c\phi_z^+(t) - \alpha_1\phi^+(t) + \alpha_1\phi^+(t) = \mathcal{A}_1\phi^+(t) + \alpha_1\phi^+(t).$$

Since the positivity of semigroup $T_1(t)$, we know that

$$\begin{aligned} \phi_t^+ &= T_1(t)\phi^+(0) + \int_0^t T_1(t-s)\alpha_1\phi^+(s)ds \\ &\geq T_1(t)\phi^+(0) + \int_0^t T_1(t-s)[\alpha_1\phi^+(s) - \beta(s)f(\phi^+(s), \varphi^-(s))]ds \\ &= T_1(t)\phi^+(0) + \int_0^t T_1(t-s)f_1[\phi^+, \varphi^-](s)ds, \end{aligned}$$

which implies that inequality (31) holds.

From the Eq. (15) in Lemma 4 and $\mathcal{A}_2\varphi = d_2\varphi_{zz} - c\varphi_z - \alpha_2\varphi$, we know that

$$\begin{aligned} \varphi_t^+ &= d_2\varphi_{zz}^+ - c\varphi_z^+ + (\beta(t)f_1(S_0, 0) - \gamma(t))\varphi^+ \\ &= \mathcal{A}_2\varphi^+ + \alpha_2\varphi^+(t) + (\beta(t)f_1(S_0, 0) - \gamma(t))\varphi^+(t). \end{aligned}$$

Since the positivity of semigroup $T_2(t)$, one has

$$\begin{aligned} \varphi^+(t) &= T_2(t)\varphi^+(0) + \int_0^t T_2(t-s)[\alpha_2\varphi^+(s) + (\beta(s)f_1(S_0, 0) - \gamma(s))\varphi^+(s)]ds \\ &\geq T_2(t)\varphi^+(0) + \int_0^t T_2(t-s)[\alpha_2\varphi^+(s) + \beta(s)f(\phi^+(s), \varphi^+(s)) - \gamma(s)\varphi^+(s) - \varepsilon\varphi^{+2}(s)]ds \\ &= T_2(t)\varphi^+(0) + \int_0^t T_2(t-s)f_2[\phi^+, \varphi^+](s)ds. \end{aligned}$$

Hence, inequality (32) holds.

Next, we consider $\phi^-(t)$ and $\varphi^-(t)$. Wherein we let $\hat{\phi}^-(t, z) = \phi^-(t, z + ct)$ and $\hat{\varphi}^-(t, z) = \varphi^-(t, z + ct)$, $(t, z) \in [0, T] \times \mathbb{R}$. It follows from Lemma 5 that

$$\hat{\phi}_t^-(t, z) - d_1\hat{\phi}_{zz}^-(t, z) + \alpha_1\hat{\phi}^-(t, z) - f_1[\hat{\phi}^-, \hat{\varphi}^+](t, z) \leq 0,$$

for any $z \neq \hat{z}_1(t) := -\varepsilon_1^{-1} \ln M_1 - ct$. By a direct computation, yields

$$\frac{\partial \hat{\phi}^-(t, \hat{z}_1(t) - 0)}{\partial z} = \lim_{z \rightarrow \hat{z}_1(t) - 0} \{-S_0 M_1 \varepsilon_1 e^{\varepsilon_1(z+ct)}\} = -\varepsilon_1 S_0 < 0.$$

Let

$$\Gamma(t, z) := -\hat{\phi}_t^-(t, z) + d_1\hat{\phi}_{zz}^-(t, z) - \alpha_1\hat{\phi}^-(t, z) + f_1[\hat{\phi}^-, \hat{\varphi}^+](t, z) \geq 0$$

and

$$H(\hat{\phi}^-)(t, z, s) := \frac{e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}} \int_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1(t-s)}} \hat{\phi}^-(s, y) dy.$$

Calculating the partial derivative of $H(\hat{\phi}^-)(t, z, s)$, one has

$$\begin{aligned} \frac{\partial}{\partial s} H(\hat{\phi}^-)(t, z, s) &= \frac{\alpha_1 e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}} \int_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1(t-s)}} \hat{\phi}^-(s, y) dy + \frac{e^{-\alpha_1(t-s)}}{2(t-s)\sqrt{4\pi d_1(t-s)}} \int_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1(t-s)}} \hat{\phi}^-(s, y) dy \\ &\quad - \frac{e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}} \int_{\mathbb{R}} \frac{(z-y)^2}{4d_1(t-s)^2} e^{-\frac{(z-y)^2}{4d_1(t-s)}} \hat{\phi}^-(s, y) dy \\ &\quad + \frac{d_1 e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}} \int_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1(t-s)}} \frac{\partial^2 \hat{\phi}^-(s, y)}{\partial y^2} dy - \frac{e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}} \int_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1(t-s)}} \alpha_1 \hat{\phi}^-(s, y) dy \\ &\quad + \frac{e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}} \int_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1(t-s)}} [f_1[\hat{\phi}^-, \hat{\varphi}^+](s, y) - \Gamma(s, y)] dy. \end{aligned}$$

Noting $\hat{\phi}^-(t, z) = 0$ for $z > \hat{z}_1(t)$, and using the integration by parts formula, one also has

$$\begin{aligned} \frac{d_1 e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}} \int_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1(t-s)}} \frac{\partial^2 \hat{\phi}^-(s, y)}{\partial y^2} dy &= \frac{d_1 e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}} \int_{-\infty}^{\hat{z}_1(s)} e^{-\frac{(z-y)^2}{4d_1(t-s)}} \frac{\partial^2 \hat{\phi}^-(s, y)}{\partial y^2} dy \\ &= \frac{d_1 e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}} e^{-\frac{(z-\hat{z}_1(s))^2}{4d_1(t-s)}} \frac{\partial \hat{\phi}^-(s, \hat{z}_1(s) - 0)}{\partial z} - \frac{e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}} \int_{-\infty}^{\hat{z}_1(s)} \frac{e^{-\frac{(z-y)^2}{4d_1(t-s)}}}{2(t-s)} \hat{\phi}^-(s, y) dy \\ &\quad + \frac{e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}} \int_{-\infty}^{\hat{z}_1(s)} \frac{(z-y)^2}{4d_1(t-s)^2} e^{-\frac{(z-y)^2}{4d_1(t-s)}} \hat{\phi}^-(s, y) dy. \end{aligned}$$

Recalling $\frac{\partial \hat{\phi}^-(t, \hat{z}_1(t)-0)}{\partial z} = -\varepsilon_1 S_0$, we obtain

$$\begin{aligned} \frac{\partial}{\partial s} H(\hat{\phi}^-)(t, z, s) &= -\varepsilon_1 S_0 \frac{d_1 e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}} e^{-\frac{(z-\hat{z}_1(s))^2}{4d_1(t-s)}} \\ &\quad + \frac{d_1 e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}} \int_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1(t-s)}} [f_1[\hat{\phi}^-, \hat{\phi}^+](s, y) - \Gamma(s, y)] dy. \end{aligned}$$

Since

$$\frac{d_1 e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}} e^{-\frac{(z-\hat{z}_1(s))^2}{4d_1(t-s)}} \frac{\partial \hat{\phi}^-(s, \hat{z}_1(s) - 0)}{\partial z}$$

is integrable on $s \in [0, t]$, noting the continuity of $\frac{\partial}{\partial s} H(\hat{\phi}^-)(t, z, s)$ in $s \in [0, t]$, from (Chapter I, [40]) we know

$$\lim_{s \rightarrow t-0} T(t-s) \hat{\phi}^-(t, z) = \lim_{s \rightarrow t-0} \frac{e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}} \int_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1(t-s)}} \hat{\phi}^-(s, y) dy = \hat{\phi}^-(t, z).$$

Therefore,

$$\begin{aligned} \hat{\phi}^-(t, z) &= \lim_{\eta \rightarrow 0^+} H(\hat{\phi}^-)(t, z, t-\eta) = H(\hat{\phi}^-)(t, z, 0) + \lim_{\eta \rightarrow 0^+} \int_0^{t-\eta} \frac{\partial}{\partial s} H(\hat{\phi}^-)(t, z, s) ds \\ &= \frac{e^{-\alpha_1 t}}{\sqrt{4\pi d_1 t}} \int_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1 t}} \hat{\phi}^-(0, y) dy - \varepsilon_1 S_0 \int_0^t \frac{d_1 e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}} e^{-\frac{(z-\hat{z}_1(s))^2}{4d_1(t-s)}} ds \\ &\quad + \int_0^t \frac{e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}} \int_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1(t-s)}} [f_1[\hat{\phi}^-, \hat{\phi}^+](s, y) - \Gamma(s, y)] dy ds. \end{aligned}$$

Due to $\Gamma(s, y) \geq 0$, it follows from above equality that

$$\begin{aligned} \hat{\phi}^-(t, z) &\leq \frac{e^{-\alpha_1 t}}{\sqrt{4\pi d_1 t}} \int_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1 t}} \hat{\phi}^-(0, y) dy \\ &\quad + \int_0^t \frac{e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}} \int_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1(t-s)}} f_1[\hat{\phi}^-, \hat{\phi}^+](s, y) dy ds. \end{aligned}$$

By the definition of $\hat{\phi}^-(t, z)$ and $\hat{\phi}^+(t, z)$ and applying an integral transformation, we obtain

$$\begin{aligned} \phi^-(t, z+ct) &\leq \frac{e^{-\alpha_1 t}}{\sqrt{4\pi d_1 t}} \int_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1 t}} \phi^-(0, y) dy + \int_0^t \frac{e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}} \int_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1(t-s)}} f_1[\phi^-, \phi^+](s, y+cs) dy ds \\ &= \frac{e^{-\alpha_1 t}}{\sqrt{4\pi d_1 t}} \int_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1 t}} \phi^-(0, y) dy + \int_0^t \frac{e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}} \int_{\mathbb{R}} e^{-\frac{(z-y+cs)^2}{4d_1(t-s)}} f_1[\phi^-, \phi^+](s, y) dy ds. \end{aligned}$$

In the above inequality, let $\hat{z} = z + ct$, we obtain

$$\begin{aligned} \phi^-(t, \hat{z}) &\leq \frac{e^{-\alpha_1 t}}{\sqrt{4\pi d_1 t}} \int_{\mathbb{R}} e^{-\frac{(\hat{z}-y)^2}{4d_1 t}} \phi^-(0, y) dy \\ &\quad + \int_0^t \frac{e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}} \int_{\mathbb{R}} e^{-\frac{(\hat{z}-y-c(t-s))^2}{4d_1(t-s)}} f_1[\phi^-, \phi^+](s, y) dy ds \\ &= T_1(t) \phi^-(0) + \int_0^t T_1(t-s) f_1[\phi^-, \phi^+](s) ds, \end{aligned}$$

which implies that inequality (33) holds. By using the same scheme as above, we can also prove that inequality (34) holds. The proof is completed. $\square$

With the aids of the discussions of above lemma, we obtain the following invariance of integral system (29).

Lemma 8. Let $(\phi(t, z; \phi_0, \varphi_0), \varphi(t, z; \phi_0, \varphi_0))$ be the solution of integral system (29) with initial value $(\phi_0, \varphi_0) \in \tilde{\mathcal{D}}$. Then

$$\phi^-(t, z) \leq \phi(t, z; \phi_0, \varphi_0) \leq \phi^+(t, z), \quad \varphi^-(t, z) \leq \varphi(t, z; \phi_0, \varphi_0) \leq \min\{\varphi^+(t, z), \Lambda_\varepsilon\},$$

for $(t, z) \in [0, T] \times \mathbb{R}$.

Proof. Since $(\tilde{\phi}, \tilde{\varphi}) \in \mathcal{D}$ and $(\phi_0, \varphi_0) \in \tilde{\mathcal{D}}$, then $\phi^- \leq \tilde{\phi} \leq \phi^+$, $\varphi^- \leq \tilde{\varphi} \leq \min\{\varphi^+, \Lambda_\varepsilon\}$ and

$$\phi^-(0, z) \leq \phi_0(z) \leq \phi^+(0, z), \quad \varphi^-(0, z) \leq \varphi_0(z) \leq \min\{\varphi^+(0, z), \Lambda_\varepsilon\}.$$

Recalling $\phi^+(t, z) = S_0$, we have $\phi_t^+ = d_1 \phi_{zz}^+ - c \phi_z^+ - \alpha_1 \phi^+ + \alpha_1 \phi^+ = \mathcal{A}_1 \phi^+ + \alpha_1 \phi^+$. Since the analytic semigroup $T_1(t)$ generated by operator $\mathcal{A}_1$, we have

$$\phi^+(t) = T_1(t)\phi^+(0) + \alpha_1 \int_0^t T_1(t-s)\phi^+(s)ds. \quad (35)$$

Combining the positivity of semigroup $T_1(\cdot)$ and (35), one has

$$\begin{aligned} \int_0^t T_1(t-s)f_1[\tilde{\phi}, \tilde{\varphi}](s)ds &\leq \alpha_1 \int_0^t T_1(t-s)\phi^+(s)ds \\ &= \phi^+(t) - T_1(t)\phi^+(0) \leq \phi^+(t) - T_1(t)\phi_0, \quad t \in (0, T]. \end{aligned}$$

Consequently $T_1(t)\phi_0 + \int_0^t T_1(t-s)f_1[\tilde{\phi}, \tilde{\varphi}](s)ds \leq \phi^+(t)$, thus $\phi(t, z; \phi_0, \varphi_0) \leq \phi^+(t, z)$ for any $(t, z) \in [0, T] \times \mathbb{R}$.

Now we prove $\phi(t, z; \phi_0, \varphi_0) \geq \phi^-(t, z)$. Let $\Phi(t, z) = \phi(t, z; \phi_0, \varphi_0) - \phi^-(t, z)$, from inequality (33), we obtain

$$\begin{aligned} \Phi(t) &= T_1(t)[\phi_0 - \phi^-(0)] + \int_0^t T_1(t-s)(\alpha_1[\tilde{\phi}(s) - \phi^-(s)] - \beta(s)[f(\tilde{\phi}, \tilde{\varphi}) - f(\phi^-, \varphi^+)](s))ds \\ &\geq T_1(t)[\phi_0 - \phi^-(0)] + \int_0^t T_1(t-s)(\alpha_1[\tilde{\phi} - \phi^-](s) - m\beta(s)[\tilde{\phi} - \phi^-](s))ds. \end{aligned}$$

Since $\alpha_1 > m\hat{\beta}$ and $\tilde{\phi} - \phi^- \geq 0$, we have $\Phi(t) \geq 0$, therefore, $\phi(t, z; \phi_0, \varphi_0) \geq \phi^-(t, z)$, $(t, z) \in [0, T] \times \mathbb{R}$. Next, we consider $\varphi(t, z; \phi_0, \varphi_0)$. Recalling $\alpha_2 > \max_{t \in [0, T]} \gamma(t) + 2\varepsilon\Lambda_\varepsilon$, we know that

$$\int_0^t T_2(t-s)f_2[\tilde{\phi}, \tilde{\varphi}](s)ds = \int_0^t T_2(t-s)[\alpha_2\tilde{\varphi} + \beta f(\tilde{\phi}, \tilde{\varphi}) - \gamma\tilde{\varphi} - \varepsilon\tilde{\varphi}^2](s)ds \leq \int_0^t T_2(t-s)[\alpha_2\varphi^+ + (\beta f_1(S_0, 0) - \gamma)\varphi^+ - \varepsilon\varphi^{+2}](s)ds,$$

for $t \in (0, T]$. Recalling (32), we have

$$\int_0^t T_2(t-s)f_2[\tilde{\phi}, \tilde{\varphi}](s)ds \leq \varphi^+(t) - T_2(t)\varphi^+(0) \leq \varphi^+(t) - T_2(t)\varphi_0, \quad t \in (0, T].$$

According to $\varphi(0, z) = \varphi_0(z) \leq \varphi^+(t, z)$ for all $z \in \mathbb{R}$, yields

$$\varphi(t, z; \phi_0, \varphi_0) \leq \varphi^+(t, z), \quad \text{for all } t \in [0, T], z \in \mathbb{R}.$$

Since $\varphi_{\Lambda_\varepsilon}^+(t, z) \equiv \Lambda_\varepsilon$ satisfies $\varphi_{\Lambda_\varepsilon}^+ = \mathcal{A}_1\varphi_{\Lambda_\varepsilon}^+ + \alpha_2\varphi_{\Lambda_\varepsilon}^+$, we have

$$\varphi_{\Lambda_\varepsilon}^+(t, z) = T_2(t)\varphi_{\Lambda_\varepsilon}^+(0, z) + \alpha_2 \int_0^t T_2(t-s)\varphi_{\Lambda_\varepsilon}^+(s, z)ds.$$

Combining $\beta(t)f(S_0, \Lambda_\varepsilon) - \gamma(t)\Lambda_\varepsilon - \varepsilon\Lambda_\varepsilon^2 < 0$, and a similar arguments as above, one has

$$\varphi(t, z; \phi_0, \varphi_0) \leq \varphi_{\Lambda_\varepsilon}^+(t, z) \equiv \Lambda_\varepsilon \quad \text{for all } t \in [0, T], z \in \mathbb{R}.$$

Additionally, from the positiveness of $T_2(\cdot)$, $\alpha_2 > \max_{t \in [0, T]} \gamma(t) + 2\varepsilon\Lambda_\varepsilon$ and (34), we obtain

$$\int_0^t T_2(t-s)f_2[\tilde{\phi}, \tilde{\varphi}](s)ds \geq \int_0^t T_2(t-s)f_2[\phi^-, \varphi^-](s)ds \geq \varphi^-(t) - T_2(t)\varphi^-(0) \geq \varphi^-(t) - T_2(t)\varphi_0,$$

therefore, $\varphi(t) = T_2(t)\varphi_0 + \int_0^t T_2(t-s)f_2[\tilde{\phi}, \tilde{\varphi}](s)ds \geq \varphi^-(t)$ for $t \in (0, T]$. Since $\varphi(0, z) = \varphi_0(z) \geq \varphi^-(0, z)$ for any $z \in \mathbb{R}$, we get $\varphi(t, z; \phi_0, \varphi_0) \geq \varphi^-(t, z)$, $(t, z) \in [0, T] \times \mathbb{R}$. This completes the proof. $\square$

Define time-T map of system (29) $(\phi_0(z), \varphi_0(z)) \mapsto (\phi(T, z; \phi_0, \varphi_0), \varphi(T, z; \phi_0, \varphi_0))$ as follows.

$$\Gamma_{(\tilde{\phi}, \tilde{\varphi})}(\phi_0(\cdot), \varphi_0(\cdot)) = (\phi(T, \cdot; \phi_0, \varphi_0), \varphi(T, \cdot; \phi_0, \varphi_0)).$$

Theorem 1. For any given $(\tilde{\phi}, \tilde{\varphi}) \in \mathcal{D}$, map $\Gamma_{(\tilde{\phi}, \tilde{\varphi})}$ has a unique fixed point $(\phi^*, \varphi^*) \in \mathcal{D}$.

Proof. Lemma 8 implies $\Gamma_{(\tilde{\phi}, \tilde{\varphi})}(\tilde{\mathcal{D}}) \subset \tilde{\mathcal{D}}$. Let $(\tilde{\phi}, \tilde{\varphi}) \in \mathcal{D}$, then $f_i[\tilde{\phi}, \tilde{\varphi}](t, \cdot) \in C([0, T], C(\mathbb{R}))$, and $f_i[\tilde{\phi}, \tilde{\varphi}]$ are uniformly bounded with respect to $(\tilde{\phi}, \tilde{\varphi}) \in \mathcal{D}$, $i = 1, 2$. Using Lemma 1 we know that (ϕ, φ) defined by system (29) belongs to $C([0, T] \times \mathbb{R}, \mathbb{R}) \cap C^{\theta, 2\theta}([0, T] \times \mathbb{R}, \mathbb{R})$ for arbitrary $\epsilon \in (0, T)$ and $0 < \theta < 1$, moreover, there exist $C_1, C_2 > 0$ dependent on ϵ, θ satisfying

$$\|\phi(T, \cdot)\|_{C^{2\theta}(\mathbb{R})} \leq C_1(\epsilon^{-\theta}\|\phi_0\|_\infty + \|f_1[\tilde{\phi}, \tilde{\varphi}]\|_\infty) \quad (36)$$

and

$$\|\varphi(T, \cdot)\|_{C^{2\theta}(\mathbb{R})} \leq C_2(\epsilon^{-\theta}\|\varphi_0\|_\infty + \|f_2[\tilde{\phi}, \tilde{\varphi}]\|_\infty). \quad (37)$$

Combining the estimates (36) and (37), we know that $\{(\phi(T, \cdot; \phi_0, \varphi_0), \varphi(T, \cdot; \phi_0, \varphi_0)) : (\phi_0, \varphi_0) \in \tilde{\mathcal{D}}\}$ is compact on $C(\Omega, \mathbb{R}^2)$. Next, we need to prove the compactness and continuity of $\Gamma_{(\tilde{\phi}, \tilde{\varphi})} : \tilde{\mathcal{D}} \rightarrow \tilde{\mathcal{D}}$. For any given sequences $\{(\phi_{0n}, \varphi_{0n})\} \subset \tilde{\mathcal{D}}$, let $\Gamma(\phi_{0n}, \varphi_{0n}) = (\phi_n(T, \cdot; \phi_{0n}, \varphi_{0n}), \varphi_n(T, \cdot; \phi_{0n}, \varphi_{0n}))$, $n = 1, 2, \dots$. Due to $\phi_n(T, \cdot; \phi_{0n}, \varphi_{0n})$ and $\varphi_n(T, \cdot; \phi_{0n}, \varphi_{0n})$ satisfy the (36) and (37), therefore, we can choose a subsequence of $\{(\phi_n(T, \cdot; \phi_{0n}, \varphi_{0n}), \varphi_n(T, \cdot; \phi_{0n}, \varphi_{0n}))\}$, for presentation convenience, still labeled by $\{(\phi_n(T, \cdot; \phi_{0n}, \varphi_{0n}), \varphi_n(T, \cdot; \phi_{0n}, \varphi_{0n}))\}$, satisfying $(\phi_n(T, \cdot; \phi_{0n}, \varphi_{0n}), \varphi_n(T, \cdot; \phi_{0n}, \varphi_{0n})) \rightarrow (\phi_*(T, \cdot; \phi_{0*}, \varphi_{0*}), \varphi_*(T, \cdot; \phi_{0*}, \varphi_{0*}))$ as $n \rightarrow \infty$ in $C_{loc}([0, T] \times \mathbb{R}, \mathbb{R}^2)$, in other words, for any $Z \in \mathbb{R}^+$,

$$\lim_{n \rightarrow \infty} \|(\phi_n(T, \cdot; \phi_{0n}, \varphi_{0n}), \varphi_n(T, \cdot; \phi_{0n}, \varphi_{0n})) - (\phi_*(T, \cdot; \phi_{0*}, \varphi_{0*}), \varphi_*(T, \cdot; \phi_{0*}, \varphi_{0*}))\|_{C([0, T] \times [-Z, Z], \mathbb{R}^2)} = 0. \quad (38)$$

Next, we will prove that

$$\lim_{n \rightarrow \infty} \|(\phi_n(T, \cdot; \phi_{0n}, \varphi_{0n}), \varphi_n(T, \cdot; \phi_{0n}, \varphi_{0n})) - (\phi_*(T, \cdot; \phi_{0*}, \varphi_{0*}), \varphi_*(T, \cdot; \phi_{0*}, \varphi_{0*}))\|_{\mu} = 0.$$

The boundedness of $\tilde{\mathcal{D}}$ means

$$\|(\phi_n(T, \cdot; \phi_{0n}, \varphi_{0n}), \varphi_n(T, \cdot; \phi_{0n}, \varphi_{0n})) - (\phi_*(T, \cdot; \phi_{0*}, \varphi_{0*}), \varphi_*(T, \cdot; \phi_{0*}, \varphi_{0*}))\|_{\mu}$$

also uniformly bounded for $n \in \mathbb{N}_+$. For any given $\delta > 0$, we can choose constant $Z^* > 0$ satisfying

$$e^{-\mu|z|} |(\phi_n(T, \cdot; \phi_{0n}, \varphi_{0n}), \varphi_n(T, \cdot; \phi_{0n}, \varphi_{0n})) - (\phi_*(T, \cdot; \phi_{0*}, \varphi_{0*}), \varphi_*(T, \cdot; \phi_{0*}, \varphi_{0*}))| < \delta \quad (39)$$

for $|z| > Z^*$ and $n \in \mathbb{N}_+$. Additionally, it follows from (38) that there exists $N^* \in \mathbb{N}_+$ such that

$$e^{-\mu|z|} |(\phi_n(T, \cdot; \phi_{0n}, \varphi_{0n}), \varphi_n(T, \cdot; \phi_{0n}, \varphi_{0n})) - (\phi_*(T, \cdot; \phi_{0*}, \varphi_{0*}), \varphi_*(T, \cdot; \phi_{0*}, \varphi_{0*}))| < \delta \quad (40)$$

for $z \in [-Z^*, Z^*]$ and $n > N^*$. Thus, it follows from the inequalities (39) and (40) that $(\phi_n(T, \cdot; \phi_{0n}, \varphi_{0n}), \varphi_n(T, \cdot; \phi_{0n}, \varphi_{0n})) \rightarrow (\phi_*(T, \cdot; \phi_{0*}, \varphi_{0*}), \varphi_*(T, \cdot; \phi_{0*}, \varphi_{0*}))$ as $n \rightarrow \infty$ with respect to the norm $\|\cdot\|_{\mu}$. Therefore, the Schauder's fixed point theorem ensures that $\Gamma_{(\tilde{\phi}, \tilde{\varphi})}$ admits a fixed point $(\phi_0^*, \varphi_0^*) \in \tilde{\mathcal{D}}$ satisfying $\phi(T, z; \phi_0^*, \varphi_0^*) = \phi_0^*(z)$ and $\varphi(T, z; \phi_0^*, \varphi_0^*) = \varphi_0^*(z)$ for any $z \in \mathbb{R}$. Finally, we show that the uniqueness of (ϕ_0^*, φ_0^*) . Suppose that there exists $(\phi_0^{**}, \varphi_0^{**})$ satisfying $\phi(T, z; \phi_0^{**}, \varphi_0^{**}) = \phi_0^{**}(z)$, $\varphi(T, z; \phi_0^{**}, \varphi_0^{**}) = \varphi_0^{**}(z)$ for any $z \in \mathbb{R}$. We have

$$\begin{aligned} |\phi(T, z; \phi_0^*, \varphi_0^*) - \phi(T, z; \phi_0^{**}, \varphi_0^{**})| &\leq e^{-\alpha_1 T} \int_{\mathbb{R}} \frac{e^{-\frac{(z-x-cT)^2}{4d_1 T}}}{\sqrt{4\pi d_1 T}} |\phi_0^*(x) - \phi_0^{**}(x)| dx \\ &\leq \|\phi_0^*(\cdot) - \phi_0^{**}(\cdot)\|_{L^\infty} e^{-\alpha_1 T} \int_{\mathbb{R}} \frac{e^{-\frac{(z-x-cT)^2}{4d_1 T}}}{\sqrt{4\pi d_1 T}} dx = e^{-\alpha_1 T} \|\phi_0^*(\cdot) - \phi_0^{**}(\cdot)\|_{L^\infty}. \end{aligned}$$

However, due to $\phi(T, \cdot; \phi_0^*, \varphi_0^*) = \phi_0^*(\cdot)$, $\phi(T, \cdot; \phi_0^{**}, \varphi_0^{**}) = \phi_0^{**}(\cdot)$, thus, one has

$$\|\phi_0^*(\cdot) - \phi_0^{**}(\cdot)\|_{L^\infty} \leq e^{-\alpha_1 T} \|\phi_0^*(\cdot) - \phi_0^{**}(\cdot)\|_{L^\infty},$$

which contradicts with $e^{-\alpha_1 T} < 1$. Consequently, $\phi_0^*(\cdot) = \phi_0^{**}(\cdot)$, $\varphi_0^*(\cdot) = \varphi_0^{**}(\cdot)$ can be obtained similarly. $\square$

Remark 1. It is clear that the fixed point (ϕ^*, φ^*) of map $\Gamma_{(\tilde{\phi}, \tilde{\varphi})}$ is the solution of system (29) defined on $[0, T]$.

From Theorem 1 we define an operator $\mathcal{F} : \mathcal{D} \rightarrow B_\mu$ as follows

$$\mathcal{F}(\tilde{\phi}, \tilde{\varphi}) = (\phi^*, \varphi^*), \quad (\tilde{\phi}, \tilde{\varphi}) \in \mathcal{D}.$$

To obtain the fixed point of operator $\mathcal{F}$, we need to prove the following conclusions.

Lemma 9. The operator $\mathcal{F} : \mathcal{D} \rightarrow \mathcal{D}$ is continuous in $B_\mu([0, T] \times \mathbb{R}, \mathbb{R}^2)$.

Proof. For any given $(\tilde{\phi}_i, \tilde{\varphi}_i) \in \mathcal{D}$, let $(\phi_i^*(t, z; \tilde{\phi}_i, \tilde{\varphi}_i), \varphi_i^*(t, z; \tilde{\phi}_i, \tilde{\varphi}_i)) = \mathcal{F}(\tilde{\phi}_i, \tilde{\varphi}_i)$, $i = 1, 2$. Using (30) and the first equation of system (29), we deduce that

$$\begin{aligned} \phi_i^*(T, z; \tilde{\phi}_i, \tilde{\varphi}_i) &= e^{-\alpha_1 T} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1 T}} e^{-\frac{(z-y-cT)^2}{4d_1 T}} \phi_i^*(0, y) dy + \int_0^T e^{-\alpha_1(T-s)} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1(T-s)}} e^{-\frac{(z-y-c(T-s))^2}{4d_1(T-s)}} f_1[\tilde{\phi}_i, \tilde{\varphi}_i](s, y) dy ds \\ &= e^{-\alpha_1 T} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1 T}} e^{-\frac{(z-y-cT)^2}{4d_1 T}} \phi_i^*(0, y) dy + \int_0^T e^{-\alpha_1 s} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1 s}} e^{-\frac{(z-y-cs)^2}{4d_1 s}} f_1[\tilde{\phi}_i, \tilde{\varphi}_i](T-s, y) dy ds. \end{aligned}$$

Let $b := \max\{\beta f(S_0)g'(0), \beta mg(\Lambda)\}$. Taking μ small enough such that $e^{d_1 T \mu^2 + cT \mu - \alpha_1 T} \leq \frac{1}{4}$, yields

$$\begin{aligned} |\phi_1^*(T, z; \tilde{\phi}_1, \tilde{\varphi}_1) - \phi_2^*(T, z; \tilde{\phi}_2, \tilde{\varphi}_2)| e^{-\mu|z|} &\leq e^{-\alpha_1 T} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1 T}} e^{-\frac{(z-y-cT)^2}{4d_1 T}} |\phi_1^*(0, y) - \phi_2^*(0, y)| dy e^{-\mu|z|} \\ &\quad + \int_0^T e^{-\alpha_1 s} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1 s}} e^{-\frac{(z-y-cs)^2}{4d_1 s}} (\alpha_1 + b)(|\tilde{\phi}_1 - \tilde{\phi}_2| + |\tilde{\varphi}_1 - \tilde{\varphi}_2|)(T-s, y) dy ds e^{-\mu|z|} \\ &\leq e^{-\alpha_1 T} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1 T}} e^{-\frac{(z-y-cT)^2}{4d_1 T}} |\phi_1^*(0, y) - \phi_2^*(0, y)| e^{-\mu|y|} e^{\mu|y-z|} dy \\ &\quad + \int_0^T e^{-\alpha_1 s} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1 s}} e^{-\frac{(z-y-cs)^2}{4d_1 s}} (\alpha_1 + b)(|\tilde{\phi}_1 - \tilde{\phi}_2| + |\tilde{\varphi}_1 - \tilde{\varphi}_2|)(T-s, y) e^{-\mu|y|} \times e^{\mu|y-z|} dy ds \end{aligned}$$

$$\begin{aligned}
&\leq e^{-\alpha_1 T + \mu c T} |\phi_1^*(0) - \phi_2^*(0)|_\mu \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1 T}} e^{-\frac{(z-y-cT)^2}{4d_1 T}} e^{\mu|z-y-cT|} dy \\
&\quad + (\alpha_1 + b) (|\tilde{\phi}_1 - \tilde{\phi}_2|_\mu + |\tilde{\varphi}_1 - \tilde{\varphi}_2|_\mu) \int_0^T e^{(\mu c - \alpha_1)s} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1 s}} e^{-\frac{(z-y-cs)^2}{4d_1 s}} e^{\mu|z-y-cs|} dy ds \\
&\leq 2e^{(d_1 \mu^2 + c\mu - \alpha_1)T} |\phi_1^*(0) - \phi_2^*(0)|_\mu + (\alpha_1 + b) (|\tilde{\phi}_1 - \tilde{\phi}_2|_\mu + |\tilde{\varphi}_1 - \tilde{\varphi}_2|_\mu) \int_0^T 2e^{(d_1 \mu^2 + c\mu - \alpha_1)s} ds \\
&\leq \frac{1}{2} |\phi_1^*(0) - \phi_2^*(0)|_\mu + \frac{2(\alpha_1 + b)(e^{(d_1 \mu^2 + c\mu - \alpha_1)T} - 1)}{d_1 \mu^2 + c\mu - \alpha_1} (|\tilde{\phi}_1 - \tilde{\phi}_2|_\mu + |\tilde{\varphi}_1 - \tilde{\varphi}_2|_\mu).
\end{aligned}$$

Let $L := \frac{4(\alpha_1 + b)(e^{(d_1 \mu^2 + c\mu - \alpha_1)T} - 1)}{d_1 \mu^2 + c\mu - \alpha_1}$. Noting $\phi_i^*(T, z, \tilde{\phi}_i, \tilde{\varphi}_i) = \phi_i^*(0, z)$, $i = 1, 2$, we obtain that

$$|\phi_1^*(0) - \phi_2^*(0)|_\mu \leq L (|\tilde{\phi}_1 - \tilde{\phi}_2|_\mu + |\tilde{\varphi}_1 - \tilde{\varphi}_2|_\mu).$$

Since $\phi_i^*(t, z, \tilde{\phi}_i, \tilde{\varphi}_i)$ satisfies

$$\phi_i^*(t, z, \tilde{\phi}_i, \tilde{\varphi}_i) = e^{-\alpha_i t} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1 t}} e^{-\frac{(z-y-cs)^2}{4d_1 t}} \phi_i^*(0, y) dy + \int_0^t e^{-\alpha_i s} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1 s}} e^{-\frac{(z-y-cs)^2}{4d_1 s}} f_1[\tilde{\phi}_i, \tilde{\varphi}_i](t-s, y) dy ds.$$

Therefore, the continuity of $\phi^*(t, z, \tilde{\phi}, \tilde{\varphi})$ and $\varphi^*(t, z, \tilde{\phi}, \tilde{\varphi})$ in $B_\mu([0, T] \times \mathbb{R}, \mathbb{R}^2)$ can be obtained similarly. $\square$

Lemma 10. The operator $\mathcal{F} : \mathcal{D} \rightarrow \mathcal{D}$ is compact in $B_\mu([0, T] \times \mathbb{R}, \mathbb{R}^2)$.

Proof. For any given $(\tilde{\phi}, \tilde{\varphi}) \in \mathcal{D}$, let $\mathcal{F}(\tilde{\phi}, \tilde{\varphi}) = (\phi^*, \varphi^*)$, where (ϕ^*, φ^*) is the solution of system (29). From the estimates (36) and (37), and the uniform boundedness of $f_1[\tilde{\phi}, \tilde{\varphi}]$ and $f_2[\tilde{\phi}, \tilde{\varphi}]$ for any $(\tilde{\phi}, \tilde{\varphi}) \in \mathcal{D}$, we know that $\|\phi^*(0)\|_{C^{2\theta}(\mathbb{R})} = \|\phi^*(T)\|_{C^{2\theta}(\mathbb{R})} \leq C$ and $\|\varphi^*(0)\|_{C^{2\theta}(\mathbb{R})} = \|\varphi^*(T)\|_{C^{2\theta}(\mathbb{R})} \leq C$ for some constant $C > 0$. Thanks to Lemma 1, we know that $(\phi^*, \varphi^*) \in C^{\theta, 2\theta}([0, T] \times \mathbb{R}, \mathbb{R}^2)$ and there exist some C_1, C_2 and $\tilde{K} > 0$ satisfying

$$\|\phi^*\|_{C^{\theta, 2\theta}([0, T] \times \mathbb{R})} \leq C_1 (\|\phi^*(0)\|_{C^{2\theta}(\mathbb{R})} + \|f_1[\tilde{\phi}, \tilde{\varphi}]\|_\infty) \leq \tilde{K} \quad (41)$$

and

$$\|\varphi^*\|_{C^{\theta, 2\theta}([0, T] \times \mathbb{R})} \leq C_2 (\|\varphi^*(0)\|_{C^{2\theta}(\mathbb{R})} + \|f_2[\tilde{\phi}, \tilde{\varphi}]\|_\infty) \leq \tilde{K}. \quad (42)$$

For any given sequences $\{(\tilde{\phi}_n, \tilde{\varphi}_n)\} \subset \mathcal{D}$, let $\mathcal{F}(\tilde{\phi}_n, \tilde{\varphi}_n) = (\phi_n^*, \varphi_n^*)$, $n = 1, 2, \dots$. Due to ϕ_n^* and φ_n^* satisfying the (41) and (42) respectively, we can choose a subsequence of $\{(\phi_n^*, \varphi_n^*)\}$, without loss of generality, still labeled by $\{(\phi_n^*, \varphi_n^*)\}$, satisfying

$$(\phi_n^*, \varphi_n^*) \rightarrow (\phi^{**}, \varphi^{**}) \text{ in } C_{loc}([0, T] \times \mathbb{R}, \mathbb{R}^2),$$

By a similar method as in Theorem 1, we can obtain $\|(\phi_n^*(t, z), \varphi_n^*(t, z)) - (\phi^{**}(t, z), \varphi^{**}(t, z))\|_\mu \rightarrow 0$ as $n \rightarrow \infty$, therefore, the operator $\mathcal{F}$ is compact in $B_\mu([0, T] \times \mathbb{R}, \mathbb{R}^2)$. $\square$

Applying Lemma 9, Lemma 10 and the Schauder's fixed point theorem, we further obtain the following theorem.

Theorem 2. The operator $\mathcal{F} : \mathcal{D} \rightarrow \mathcal{D}$ has a fixed point $(\tilde{\phi}^*, \tilde{\varphi}^*) \in \mathcal{D}$.

Remark 2. It is clear that the fixed point $(\tilde{\phi}^*, \tilde{\varphi}^*) \in \mathcal{D}$ of operator $\mathcal{F}$ is the solution of system (14) defined on $[0, T]$.

5. Existence of periodic traveling waves

Now, on the basis of the above discussion we show that the existence of periodic solution for auxiliary system (14).

Theorem 3. Suppose $\mathcal{R}_0 > 1$ and $c > c^*$. Then auxiliary system (14) admits a nontrivial time periodic solution (ϕ^*, φ^*) satisfying asymptotic boundary conditions (12). Moreover,

$$0 < \frac{1}{T} \int_0^T \varphi^*(t, z) dt \leq S_0 - S^\infty, \quad z \in \mathbb{R}$$

and

$$\frac{1}{T} \int_{\mathbb{R}} \int_0^T \gamma(t) \varphi^*(t, z) + \varepsilon \varphi^{*2}(t, z) dt dz = \frac{1}{T} \int_{\mathbb{R}} \int_0^T \beta(t) f(\phi^*(t, z), \varphi^*(t, z)) dt dz = c[S_0 - S^\infty]. \quad (43)$$

Proof. By the above discussion, for every $c > c^*$, operator $\mathcal{F}$ admits a unique fixed point $(\phi^*, \varphi^*) \in \mathcal{D}$ satisfying $(\phi^*(T, \cdot), \varphi^*(T, \cdot)) = (\phi^*(0, \cdot), \varphi^*(0, \cdot))$, moreover

$$\begin{cases} \phi^*(t) = T_1(t) \phi^*(0) + \int_0^t T_1(t-s) f_1[\phi^*, \varphi^*](s) ds, \\ \varphi^*(t) = T_2(t) \varphi^*(0) + \int_0^t T_2(t-s) f_2[\phi^*, \varphi^*](s) ds, \end{cases} \quad (44)$$

for $0 \leq t \leq T$. Denote $(\hat{\phi}^*(t, z), \hat{\varphi}^*(t, z)) = (\phi^*(t - nT, z), \varphi^*(t - nT, z))$, where integers n can be chosen to satisfy $nT \leq t < (n+1)T$. We can verify that $\hat{\phi}^*(t, z)$ and $\hat{\varphi}^*(t, z)$ are T-periodic function for $(t, z) \in \mathbb{R} \times \mathbb{R}$. From $(\phi^*, \varphi^*) \in C^{\theta, 2\theta}([0, T] \times \mathbb{R}, \mathbb{R}^2)$, we know that $(\hat{\phi}^*, \hat{\varphi}^*) \in C^{\theta, 2\theta}(\mathbb{R} \times \mathbb{R}, \mathbb{R}^2)$. It is clear that $(\hat{\phi}^*, \hat{\varphi}^*)$ satisfies following system

$$\begin{cases} \hat{\phi}^*(t) = T_1(t) \hat{\phi}^*(0) + \int_0^t T_1(t-s) f_1[\hat{\phi}^*, \hat{\varphi}^*](s) ds, \\ \hat{\varphi}^*(t) = T_2(t) \hat{\varphi}^*(0) + \int_0^t T_2(t-s) f_2[\hat{\phi}^*, \hat{\varphi}^*](s) ds, \end{cases} \quad (45)$$

for $t \in \mathbb{R}$. Without loss of generality, $(\hat{\phi}^*, \hat{\varphi}^*)$ still labeled by (ϕ^*, φ^*) . Thanks to ([41], Theorem 5.1.2, 5.1.3, 5.1.4), it follows that $(\phi^*, \varphi^*) \in C^{1,2+2\theta}(\mathbb{R} \times \mathbb{R}, \mathbb{R}^2)$ satisfies

$$\begin{cases} \phi_t^* = d_1 \phi_{zz}^* - c \phi_z^* - \beta(t) f(\phi^*, \varphi^*), \\ \varphi_t^* = d_2 \varphi_{zz}^* - c \varphi_z^* + \beta(t) f(\phi^*, \varphi^*) - \gamma(t) \varphi^* - \varepsilon \varphi^{*2} \end{cases} \quad (46)$$

for $t \in \mathbb{R}, z \in \mathbb{R}$. Moreover,

$$\|\phi^*\|_{C^{1,2+2\theta}(\mathbb{R} \times \mathbb{R}, \mathbb{R})} + \|\varphi^*\|_{C^{1,2+2\theta}(\mathbb{R} \times \mathbb{R}, \mathbb{R})} < \infty \quad \text{for some } \theta \in (0, 1). \quad (47)$$

It is clear that $\lim_{z \rightarrow -\infty} \phi^*(t, z) = S_0$ and $\lim_{z \rightarrow -\infty} \varphi^*(t, z) = 0$ uniformly for $t \in \mathbb{R}$. Now, we intend to show that the second equality of (43). To obtain this, we need to investigate the asymptotic behavior of (ϕ_z^*, φ_z^*) and $(\phi_{zz}^*, \varphi_{zz}^*)$ as $z \rightarrow -\infty$. Using the estimate (47) and Landau type inequalities (See [45]), for any given $M < \infty$, we have

$$|\phi_z^*|_{L^\infty([0, T] \times (-\infty, M])} \leq 2|\phi^* - S_0|_{L^\infty([0, T] \times (-\infty, M])}^{\frac{1}{2}} |\phi_{zz}^*|_{L^\infty([0, T] \times (-\infty, M])}^{\frac{1}{2}},$$

and

$$|\varphi_z^*|_{L^\infty([0, T] \times (-\infty, M])} \leq 2|\varphi^* - S_0|_{L^\infty([0, T] \times (-\infty, M])}^{\frac{1}{2}} |\varphi_{zz}^*|_{L^\infty([0, T] \times (-\infty, M])}^{\frac{1}{2}}.$$

Consequently,

$$\lim_{z \rightarrow -\infty} (\phi_z^*(t, z), \varphi_z^*(t, z)) = (0, 0) \quad \text{uniformly for } t \in \mathbb{R}. \quad (48)$$

Taking the partial derivatives with respect to z on both sides of the first equation of (46), yields

$$(\phi_z^*)_t = d_1 (\phi_z^*)_{zz} - c (\phi_z^*)_z - \beta(t) (f_5(\phi^*, \varphi^*) \phi_z^* + f_1(\phi^*, \varphi^*) \varphi_z^*). \quad (49)$$

Applying ([41], Theorem 5.1.3, 5.1.4), we know that $\phi_z^* \in C^{1,2+2\theta}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\|\phi_z^*\|_{C^{1,2+2\theta}(\mathbb{R} \times \mathbb{R}, \mathbb{R})} < \infty$. Repeating the above procedure and Landau type inequality, one can see that

$$\lim_{z \rightarrow -\infty} (\phi_{zz}^*(t, z), \varphi_{zz}^*(t, z)) = (0, 0) \quad \text{uniformly for } t \in \mathbb{R}. \quad (50)$$

Consider the system (46), combining the (48) and (50), we have

$$\lim_{z \rightarrow -\infty} (\phi_t^*(t, z), \varphi_t^*(t, z)) = (0, 0) \quad \text{uniformly for } t \in \mathbb{R}.$$

Setting $\Phi(z) := \frac{1}{T} \int_0^T \phi^*(t, z) dt$. It follows from (48) that $\Phi_z(z) \rightarrow 0$ as $z \rightarrow -\infty$. From the first equation of (46), we have

$$c \Phi_z = d_1 \Phi_{zz} - \frac{1}{T} \int_0^T \beta(t) f(\phi^*, \varphi^*) dt. \quad (51)$$

By integrating (51) from $-\infty$ to z , we obtain

$$d_1 \Phi_z(z) = c[\Phi(z) - S_0] + \frac{1}{T} \int_{-\infty}^z \int_0^T \beta(t) f(\phi^*(t, y), \varphi^*(t, y)) dt dy. \quad (52)$$

It follows from uniform boundedness of $\phi^*(t, z)$ and $\phi_z^*(t, z)$ that $\Phi(z) = \frac{1}{T} \int_0^T \phi^*(t, z) dt$ and $\Phi_z(z) = \frac{1}{T} \int_0^T \phi_z^*(t, z) dt$ are bounded. Therefore, from (52) we know

$$\frac{1}{T} \int_0^T \beta(t) f(\phi^*(t, z), \varphi^*(t, z)) dt < \infty.$$

Combining (51), it follows that

$$(e^{-cz/d_1} \Phi_z)_z = e^{-cz/d_1} (\Phi_{zz} - c \Phi_z/d_1) = \frac{e^{-cz/d_1}}{d_1 T} \int_0^T \beta(t) f(\phi^*(t, z), \varphi^*(t, z)) dt.$$

Integrating above equality from z to ∞ , we find

$$e^{-cz/d_1} \Phi_z(z) = -\frac{1}{d_1 T} \int_z^\infty e^{-cy/d_1} \int_0^T \beta(t) f(\phi^*(t, y), \varphi^*(t, y)) dt dy.$$

Hence, $\Phi_z(z) < 0$ for all $z \in \mathbb{R}$, which indicates that $\Phi(+\infty)$ exists and let $\Phi(+\infty) := S^\infty$. Clearly, $S^\infty = \Phi(+\infty) < \Phi(-\infty) = S_0$. On the other hand, applying the Barblat's lemma (See [46]), it follows that $\Phi_z(z) \rightarrow 0$ as $z \rightarrow +\infty$. Taking $z \rightarrow \infty$ in (52), we find that

$$\frac{1}{T} \int_{-\infty}^\infty \int_0^T \beta(t) f(\phi^*, \varphi^*) dt dz = c[S_0 - \Phi(\infty)] = c[S_0 - S^\infty]. \quad (53)$$

Next, we consider the limitation of $\varphi^*(t, z)$ as $z \rightarrow \infty$. Let $\underline{\gamma} := \min_{t \in [0, T]} \gamma(t)$ and $\Psi(z) := \frac{1}{T} \int_0^T \varphi^*(t, z) dt$. Combining the second equation of (46), one has

$$-d_2 \Psi_{zz} + c \Psi_z + \underline{\gamma} \Psi = \frac{1}{T} \int_0^T [\beta(t) f(\phi^*, \varphi^*) - (\gamma(t) - \underline{\gamma}) \varphi^* - \varepsilon \varphi^{*2}] dt. \quad (54)$$

The two roots of the characteristic equation $-d_2 \lambda^2 + c \lambda + \underline{\gamma} = 0$ are denoted by

$$\lambda^- = \frac{c - \sqrt{c^2 + 4d_2 \underline{\gamma}}}{2d_2} < 0, \quad \lambda^+ = \frac{c + \sqrt{c^2 + 4d_2 \underline{\gamma}}}{2d_2} > 0.$$

Let $\tilde{\rho} := d_2(\lambda^+ - \lambda^-) = \sqrt{c^2 + 4d_2\gamma}$. Then, it follows from (54) that

$$\Psi(z) \leq \frac{1}{\tilde{\rho}T} \int_0^\infty e^{\lambda^-y} \int_0^T \beta(t) f(\phi^*(t, z-y), \varphi^*(t, z-y)) dt dy + \frac{1}{\tilde{\rho}T} \int_{-\infty}^0 e^{\lambda^+y} \int_0^T \beta(t) f(\phi^*(t, z-y), \varphi^*(t, z-y)) dt dy.$$

Integrating from η to ξ on both sides of above equation, we have

$$\int_\eta^\xi \Psi(z) dz \leq \frac{1}{\tilde{\rho}T} \int_0^\infty e^{\lambda^-y} \int_\eta^\xi \int_0^T \beta(t) f(\phi^*(t, z-y), \varphi^*(t, z-y)) dt dz dy + \frac{1}{\tilde{\rho}T} \int_{-\infty}^0 e^{\lambda^+y} \int_\eta^\xi \int_0^T \beta(t) f(\phi^*(t, z-y), \varphi^*(t, z-y)) dt dz dy.$$

From (53) we know that $\int_0^T \beta(t) f(\phi^*, \varphi^*) dt$ is integrable on $\mathbb{R}$. The Fubini's theorem (See [47]) ensures that $\Psi(z)$ is integrable on $\mathbb{R}$, and

$$\int_{-\infty}^\infty \Psi(z) dz \leq \frac{1}{\gamma T} \int_{-\infty}^\infty \int_0^T \beta(t) f(\phi^*(t, z), \varphi^*(t, z)) dt dz.$$

By a similar argument as the boundedness of $\Phi_z(z)$, we can obtain that $\Psi_z(z)$ is bounded on $\mathbb{R}$. Therefore, the Barbărlat's lemma guarantees $\Psi(z) \rightarrow 0$ as $z \rightarrow \infty$. We consider the second equation of system (46), taking $\tau = -T, \theta = T$ and $D = (\xi - \frac{1}{4}, \xi + \frac{1}{4}), U = (\xi - \frac{1}{2}, \xi + \frac{1}{2}), \Omega = (\xi - 1, \xi + 1)$ with $\xi \in \mathbb{R}$. Then, the Lemma 2 means that there is a constant $C_0 > 0$ independent of D such that

$$\sup_{(0,T) \times D} \varphi^*(t, z) \leq C_0 \inf_{(2T, 3T) \times D} \varphi^*(t, z) = C_0 \min_{[2T, 3T] \times \bar{D}} \varphi^*(t, z) \leq C_0 \min_{\bar{D}} \varphi^*(0, z).$$

Now, we show that $\lim_{z \rightarrow \infty} \varphi^*(0, z) = 0$. Suppose that this conclusion is false, then there exists a constant $\delta > 0$ and sequence $\{z_n\}$ satisfying $z_n \rightarrow \infty$ as $n \rightarrow \infty$ such that $\lim_{n \rightarrow \infty} \varphi^*(0, z_n) \geq \delta$. The continuity of $\varphi^*(t, z)$ with respect to t implies that there exists constant $\sigma > 0$ such that $\lim_{n \rightarrow \infty} \varphi^*(t, z_n) \geq \delta$ for all $0 \leq t \leq \sigma$. We deduce that

$$\lim_{n \rightarrow \infty} \Psi(z_n) = \lim_{n \rightarrow \infty} \frac{1}{T} \int_0^T \varphi^*(t, z_n) dt \geq \lim_{n \rightarrow \infty} \frac{1}{T} \int_0^\sigma \varphi^*(t, z_n) dt \geq \frac{\delta \sigma}{T},$$

which is a contradiction with $\lim_{z \rightarrow \infty} \Psi(z) = 0$. Thus $\lim_{z \rightarrow \infty} \varphi^*(0, z) = 0$, combining the periodicity of φ^* , we can obtain $\varphi^*(t, z) \rightarrow 0$ uniformly for $t \in \mathbb{R}$ as $z \rightarrow \infty$.

Next, let

$$\limsup_{z \rightarrow \infty} \max_{t \in [0, T]} \phi^*(t, z) := S_+^\infty, \quad \liminf_{z \rightarrow \infty} \min_{t \in [0, T]} \phi^*(t, z) := S_-^\infty. \quad (55)$$

Thanks to the T-periodicity of ϕ^* , we need to show $S_+^\infty = S_-^\infty = S^\infty$. From (55), we know that there exist sequences $\{t_n\}$ and $\{z_n\}$ with $\{t_n\} \subset [0, T]$ and $z_n \rightarrow \infty$ as $n \rightarrow \infty$, satisfying $\lim_{n \rightarrow \infty} \phi^*(t_n, z_n) = S_+^\infty$. Letting $\phi_n(t, z) := \phi^*(t + t_n, z + z_n)$ and $\varphi_n(t, z) := \varphi^*(t + t_n, z + z_n)$, $n \in \mathbb{N}_+$, $t \in \mathbb{R}, z \in \mathbb{R}$. The estimation (47) implies that we can choose a subsequence of $\{(\phi_n(t, z), \varphi_n(t, z))\}$, without loss of generality, still denoted by $\{(\phi_n(t, z), \varphi_n(t, z))\}$, satisfying $(\phi_n(t, z), \varphi_n(t, z)) \rightarrow (\phi_*(t, z), 0) \in C_{loc}^{\theta, 2\theta}(\mathbb{R} \times \mathbb{R})$ as $n \rightarrow \infty$. Consequently, $\phi_*(0, 0) = S_+^\infty$, $\phi_*(t, z)$ is a T-periodic function, and $\phi_*(t, z) \leq S_+^\infty$ for all $(t, z) \in \mathbb{R} \times \mathbb{R}$.

Due to $\{t_n\} \subset [0, T]$, there exists a $t^* \in [0, T]$ such that $t_n \rightarrow t^*$ as $n \rightarrow \infty$. Let $\phi_*^+(t, z) := \phi_*(t - t^*, z)$, then $\phi_*^+(t, z)$ satisfies

$$\phi_*^+(t) = T_1(t) \phi_*^+(0) + \int_0^t T_1(t-s) f_1[\phi_*^+, 0](s) ds = T_1(t) \phi_*^+(0) + \int_0^t T_1(t-s) \alpha_1 \phi_*^+(s) ds.$$

Further, we know $\phi_*^+ = d_1 \phi_{*zz}^+ - c \phi_{*z}^+$. Due to $\phi_*^+(t^*, 0) = S_+^\infty$ and $\phi_*^+(t, z) \leq S_+^\infty$, then the maximum principle guarantees $\phi_*^+(t, z) \equiv S_+^\infty$, that is $\frac{1}{T} \int_0^T \phi_*^+(t, z) dt \equiv S_+^\infty$ for all $t \in \mathbb{R}$. Next, let $\Phi_*^+(z) := \frac{1}{T} \int_0^T \phi_*^+(t, z) dt$, then $\Phi_*^+(z) \equiv S_+^\infty$. Furthermore, one can see

$$\Phi_*^+(z) = \frac{1}{T} \int_0^T \phi_*^+(t, z) dt = \lim_{n \rightarrow \infty} \frac{1}{T} \int_0^T \phi^*(t - t^* + t_n, z + z_n) dt = S^\infty,$$

which indicates $S_+^\infty = S^\infty$. Thus $\lim_{z \rightarrow \infty} \sup \max_{t \in [0, T]} \phi^*(t, z) = S^\infty$. Similarly, we can show that $\lim_{z \rightarrow \infty} \inf \min_{t \in [0, T]} \phi^*(t, z) = S^\infty$. Therefore, $\phi_*^+(t, z) \rightarrow S^\infty$ uniformly for $t \in \mathbb{R}$ as $z \rightarrow \infty$.

Finally, by using Landau type inequalities to $\phi_z^*(t, z)$ and $\phi_{zz}^*(t, z)$, we can deduce

$$\lim_{z \rightarrow \pm\infty} \varphi_z^*(t, z) = \lim_{z \rightarrow \pm\infty} \varphi_{zz}^*(t, z) = 0, \quad \text{uniformly for } t \in \mathbb{R}. \quad (56)$$

From the (54), we have

$$-d_2 \Psi_{zz} + c \Psi_z = \frac{1}{T} \int_0^T [\beta(t) f(\phi^*, \varphi^*) - \gamma(t) \varphi^* - \varepsilon \varphi^{*2}] dt. \quad (57)$$

Making an integration to Eq. (57) on $\mathbb{R}$, combining the (53) and (56), we obtain

$$\frac{1}{T} \int_{\mathbb{R}} \int_0^T [\gamma(t) \varphi^* + \varepsilon \varphi^{*2}] dt dz = \frac{1}{T} \int_{\mathbb{R}} \int_0^T \beta(t) f(\phi^*, \varphi^*) dt dz = c[S_0 - S^\infty]. \quad (58)$$

Define

$$\Psi^*(z) := \frac{1}{cT} \int_{-\infty}^z \int_0^T [\gamma(t) \varphi^*(t, y) + \varepsilon \varphi^{*2}(t, y)] dt dy + \frac{1}{cT} \int_z^\infty e^{\frac{c(z-y)}{d_2}} \int_0^T [\gamma(t) \varphi^*(t, y) + \varepsilon \varphi^{*2}(t, y)] dt dy. \quad (59)$$

It is clear that function $\Psi^*(z)$ satisfies

$$c \Psi_z^*(z) = d_2 \Psi_{zz}^*(z) + \frac{1}{T} \int_0^T [\gamma(t) \varphi^*(t, z) + \varepsilon \varphi^{*2}(t, z)] dt, \quad \forall z \in \mathbb{R}. \quad (60)$$

Combining (56), (58) and L'Hôpital's rule, we have

$$\lim_{z \rightarrow -\infty} \Psi^*(z) = 0, \quad \lim_{z \rightarrow \infty} \Psi^*(z) = \frac{1}{cT} \int_{\mathbb{R}} \int_0^T [\gamma(t) \varphi^*(t, y) + \varepsilon \varphi^{*2}(t, y)] dt dy = S_0 - S^\infty \quad (61)$$

and $\lim_{z \rightarrow \pm\infty} \Psi_z^*(z) = 0$. Note that $\Psi(z) = \frac{1}{T} \int_0^T \varphi^*(t, z) dt$. Let $\hat{\Psi}(z) := \Psi(z) + \Psi^*(z)$, $z \in \mathbb{R}$. Combining (57) and (60), we find

$$c\hat{\Psi}_z(z) = d_2\hat{\Psi}_{zz}(z) + \frac{1}{T} \int_0^T \beta(t)f(\phi^*, \varphi^*)dt, \quad z \in \mathbb{R}. \quad (62)$$

Multiplying by e^{-cz/d_2} to two sides of the (62) and integrating from z to ∞ , yields

$$\hat{\Psi}_z(z) = \frac{1}{d_2 T} \int_z^\infty e^{-\frac{c(y-z)}{d_2}} \int_0^T \beta(t)f(\phi^*, \varphi^*)dt \geq 0,$$

which implies that $\hat{\Psi}(z)$ is nondecreasing in $\mathbb{R}$. On the other hand, it follows from (62) and (63) that $\lim_{z \rightarrow \infty} \hat{\Psi}(z) = S_0 - S^\infty$. Therefore, $\Psi(z) \leq \hat{\Psi}(z) \leq S_0 - S^\infty$, that is $0 < \frac{1}{T} \int_0^T \varphi^*(t, z) dt \leq S_0 - S^\infty$ for all $z \in \mathbb{R}$. $\square$

Now, to return system (11), we obtain the following conclusion.

Theorem 4. Suppose $\mathcal{R}_0 > 1$ and $c > c^*$. Then system (11) admits a time periodic solution (ϕ^*, φ^*) satisfying asymptotic boundary conditions (12). Moreover,

$$0 < \frac{1}{T} \int_0^T \varphi^*(t, z) dt \leq S_0 - S^\infty, \quad z \in \mathbb{R}$$

and

$$\frac{1}{T} \int_{\mathbb{R}} \int_0^T \gamma(t)\varphi^*(t, z) dt dz = \frac{1}{T} \int_{\mathbb{R}} \int_0^T \beta(t)f(\phi^*(t, z), \varphi^*(t, z)) dt dz = c[S_0 - S^\infty]. \quad (63)$$

Proof. Let $\{\varepsilon_n\} \subset (0, 1)$ is a decreasing sequence and $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Theorem 3 guarantees that system (14) admits a solution sequence $\{(\phi_n^*(t, z), \varphi_n^*(t, z))\}$ with $\varepsilon = \varepsilon_n$, $(n = 1, 2, \dots)$, satisfying

$$\begin{cases} (\phi_n^*)_t = d_1(\phi_n^*)_{zz} - c(\phi_n^*)_z - \beta(t)f(\phi_n^*, \varphi_n^*), \\ (\varphi_n^*)_t = d_2(\varphi_n^*)_{zz} - c(\varphi_n^*)_z + \beta(t)f(\phi_n^*, \varphi_n^*) - \gamma(t)\varphi_n^* - \varepsilon_n\varphi_n^{*2} \end{cases} \quad (64)$$

and

$$0 < \frac{1}{T} \int_0^T \varphi_n^*(t, z) dt \leq S_0 - S^\infty, \quad z \in \mathbb{R}.$$

Moreover,

$$\frac{1}{T} \int_{\mathbb{R}} \int_0^T \gamma(t)\varphi_n^*(t, z) + \varepsilon_n\varphi_n^{*2}(t, z) dt dz = \frac{1}{T} \int_{\mathbb{R}} \int_0^T \beta(t)f(\phi_n^*(t, z), \varphi_n^*(t, z)) dt dz = c[S_0 - S^\infty]. \quad (65)$$

Furthermore, we know that

$$\varphi^-(t, z) \leq \varphi_n^*(t, z) \leq S_0 \text{ and } \varphi^-(t, z) \leq \varphi_n^*(t, z) \leq \min\{\varphi^+(t, z), \Lambda_{\varepsilon_n}\}, \quad n \in \mathbb{N}_+.$$

It follows from $\Lambda_\varepsilon = \max_{t \in [0, T]} \left\{ \frac{\beta(t)f(S_0, 0) - \gamma(t)}{\varepsilon} \right\}$ that $\Lambda_{\varepsilon_n} \rightarrow \infty$ as $n \rightarrow \infty$.

For $z \in [-k, k]$, $k = 1, 2, \dots$, there exists $n_k \in \mathbb{N}_+$ such that $\varphi^+(t, z) = K(t)e^{\lambda_1 z} < \Lambda_{\varepsilon_n}$ for all $n > n_k$, $z \in [-k, k]$ and $t \in [0, T]$. Therefore, solution sequences $\{(\phi_n^*(t, z), \varphi_n^*(t, z))\}$ are equi-continuous and uniformly bounded on $z \in [-k, k]$. The Arzelà-Ascoli theorem means that there exists a uniformly convergent subsequences $\{(\phi_{k,m}^*(t, z), \varphi_{k,m}^*(t, z))\}$ of $\{(\phi_n^*(t, z), \varphi_n^*(t, z))\}$ as $m \rightarrow \infty$.

It's not hard to see that there is a subsequence $\{(\phi_{m,m}^*(t, z), \varphi_{m,m}^*(t, z))\}$, which uniformly converges on every interval $[-k, k]$ ($k = 1, 2, 3, \dots$). Let $\lim_{m \rightarrow \infty} (\phi_{m,m}^*(t, z), \varphi_{m,m}^*(t, z)) = (\phi^*(t, z), \varphi^*(t, z))$. For every $m \in \mathbb{N}_+$, we know

$$\begin{cases} (\phi_{m,m}^*)_t = d_1(\phi_{m,m}^*)_{zz} - c(\phi_{m,m}^*)_z - \beta(t)f(\phi_{m,m}^*, \varphi_{m,m}^*), \\ (\varphi_{m,m}^*)_t = d_2(\varphi_{m,m}^*)_{zz} - c(\varphi_{m,m}^*)_z + \beta(t)f(\phi_{m,m}^*, \varphi_{m,m}^*) - \gamma(t)\varphi_{m,m}^* - \varepsilon_{m,m}(\varphi_{m,m}^*)^2. \end{cases} \quad (66)$$

Let $m \rightarrow \infty$, we know that $\lim_{m \rightarrow \infty} \varepsilon_{m,m} = 0$. Thus, $(\phi^*(t, z), \varphi^*(t, z))$ satisfies system (11). Combining Theorem 3 and (65), we know that $0 \leq \frac{1}{T} \int_0^T \varphi^*(t, z) dt \leq S_0 - S^\infty$ and (63) holds. Finally, we need to prove $\frac{1}{T} \int_0^T \varphi^*(t, z) dt > 0$. It follows from $\varphi^-(t, z) \leq \varphi^*(t, z) \leq \varphi^+(t, z)$ and $\varphi^-(t, z) = \max\{K(t)e^{\lambda_1 z}(1 - M_2 e^{\varepsilon_2 z}), 0\}$ that

$$\varphi^*(t, z) \geq 0 \text{ for all } (t, z) \in \mathbb{R} \times \mathbb{R}$$

and

$$\varphi^*(t, z) > 0 \text{ for any } t \in [0, T], z < z_2 = -\varepsilon_2^{-1} \ln M_2. \quad (67)$$

Therefore, we have

$$\begin{cases} \varphi_t^*(t, z) = d_2\varphi_{zz}^* - c\varphi_z^* + \beta(t)f(\phi^*, \varphi^*) - \gamma(t)\varphi^* \\ \quad \geq d_2\varphi_{zz}^* - c\varphi_z^* - \gamma(t)\varphi^*, \quad (t, z) \in (0, T] \times \mathbb{R}, \\ \varphi^*(t, z) \geq 0, \quad (t, z) \in (0, T] \times \mathbb{R}, \\ \varphi^*(0, z) \geq 0, \quad z \in \mathbb{R}. \end{cases}$$

From (67), we have $\varphi^*(0, z) \not\equiv 0$ for $z \in \mathbb{R}$. By the maximum principle (see, [48]), we know that $\varphi^*(t, z) > 0$ on $\mathbb{R} \times \mathbb{R}$ and $\frac{1}{T} \int_0^T \varphi^*(t, z) dt > 0$. This completes the proof. $\square$

Remark 3. It is clear that when incidence functions $f(S, I) = SI$ and $f(S, I) = \frac{SI}{S+I}$, then assumptions (A1) and (A2) are satisfied. Therefore, the main results on the existence of periodic travelling wave solutions for models (1) and (2) obtained in [31,35] are the direct corollaries of Theorem 4.

6. Nonexistence of periodic traveling waves

In what follows, we shall establish the non-existence of periodic traveling wave solutions for $\mathcal{R}_0 \leq 1$ or $\mathcal{R}_0 > 1$ and $0 < c < c^*$.

Theorem 5. Assume that $\mathcal{R}_0 \leq 1$ and $c \geq 0$, then there does not exist nonnegative and periodic traveling wave solution (ϕ^*, φ^*) satisfying asymptotic boundary conditions (12).

Proof. Suppose that there exists a periodic traveling wave solution (ϕ^*, φ^*) for system (11) satisfying (12). Then, we have

$$\varphi_t^* = d_2 \varphi_{zz}^* - c \varphi_z^* + \beta(t) f_I(S_0, 0) \varphi^* - \gamma(t) \varphi^* + \beta(t) f(\phi^*, \varphi^*) - \beta(t) f_I(S_0, 0) \varphi^*, \quad (68)$$

for $t > 0, z \in \mathbb{R}$. Let $\tilde{\Psi}(t) := \int_{\mathbb{R}} \varphi^*(t, z) dz$. Integrating (68) with respect to z on $\mathbb{R}$, and combining asymptotic boundary conditions (12) and (15), yields

$$\tilde{\Psi}_t(t) = (\beta(t) f_I(S_0, 0) - \gamma(t)) \tilde{\Psi}(t) + m(t),$$

where $m(t) = \beta(t) \int_{\mathbb{R}} [f(\phi^*, \varphi^*) - f_I(S_0, 0) \varphi^*] dz < 0$. Due to $\tilde{\Psi}(t)$ and $m(t)$ are T -periodic, we have

$$\frac{\tilde{\Psi}_t(t)}{\tilde{\Psi}(t)} = (\beta(t) f_I(S_0, 0) - \gamma(t)) + \frac{m(t)}{\tilde{\Psi}(t)}, \quad t \in \mathbb{R}.$$

Integrating above equality from 0 to T , combining the periodicity of $\tilde{\Psi}(t)$, yields

$$0 = \int_0^T (\beta(t) f_I(S_0, 0) - \gamma(t)) dt + \int_0^T \frac{m(t)}{\tilde{\Psi}(t)} dt.$$

Note that $\mathcal{R}_0 \leq 1$, which implies $\int_0^T [\beta(t) f_I(S_0, 0) - \gamma(t)] dt \leq 0$. Combining the $m(t) < 0$, yields

$$0 = \int_{\mathbb{R}} (\beta(t) f_I(S_0, 0) - \gamma(t)) dt + \int_{\mathbb{R}} \frac{m(t)}{\tilde{\Psi}(t)} < 0,$$

which is a contradiction. This is the end of the proof. $\square$

Theorem 6. Assume $\mathcal{R}_0 > 1$, then system (11) does not have a periodic traveling wave solution (ϕ^*, φ^*) satisfying asymptotic boundary conditions (12) for any $c \in (0, c^*)$.

Proof. Assume that the solution $(\phi(t, x + \tilde{c}t), \varphi(t, x + \tilde{c}t))$ is a traveling wave solution satisfying conditions (12) for some $0 < \tilde{c} < c^* = 2\sqrt{\frac{d_2 \int_0^T [\beta(t) f_I(S_0, 0) - \gamma(t)] dt}{T}}$. Since $\mathcal{R}_0 = \frac{\int_0^T \beta(t) f_I(S_0, 0) dt}{\int_0^T \gamma(t) dt} > 1$ means $\int_0^T [\beta(t) f_I(S_0, 0) - \gamma(t)] dt > 0$, which implies that there exists a sufficiently small $\delta_0 > 0$ such that $\int_0^T [\beta(t) (f_I(S_0, 0) - \delta_0) - \gamma(t)] dt > 0$. Let

$$\omega^\delta := \frac{1}{T} \int_0^T [\beta(t) (f_I(S_0, 0) - \delta) - \gamma(t)] dt, \quad \delta \in (0, \delta_0).$$

We can choose a $\delta \in (0, \delta_0)$ satisfying $\tilde{c} < 2\sqrt{d_2 \omega^\delta}$.

Fixed a constant $c_0 \in (\tilde{c}, 2\sqrt{d_2 \omega^\delta})$ and let $N_{c_0} = \frac{\sqrt{4d_2 \omega^\delta - c_0^2}}{2d_2}$. Define

$$G^\delta(t) = \exp\left(\int_0^t [\beta(s) (f_I(S_0, 0) - \delta) - \gamma(s)] ds - \omega^\delta t\right).$$

Obviously, $G_t^\delta(t) = [\beta(t) (f_I(S_0, 0) - \delta) - \gamma(t)] G^\delta(t) - \omega^\delta G^\delta(t)$. Define a function $\psi^{c_0}(t, z) := e^{\frac{c_0 z}{2d_2}} \sin(N_{c_0} z) G^\delta(t)$. We can deduce $\psi^{c_0}(t + T, z) = \psi^{c_0}(t, z)$ for any $z \in \mathbb{R}$. Then, by a direct calculation, yields

$$\psi_t^{c_0}(t, z) = d_2 \psi_{zz}^{c_0}(t, z) - c_0 \psi_z^{c_0}(t, z) + [\beta(t) (f_I(S_0, 0) - \delta) - \gamma(t)] \psi^{c_0}(t, z). \quad (69)$$

We choose $n_0 \in \mathbb{N}_+$ such that $\frac{(2n_0-1)\pi}{N_{c_0}} > N_\delta$. Further, we can choose $y_1 = -\frac{2n_0\pi}{N_{c_0}}, y_2 = -\frac{(2n_0-1)\pi}{N_{c_0}}$. Then, we have $\sin(N_{c_0} y_1) = \sin(N_{c_0} y_2) = 0$ and $\sin(N_{c_0} z) > 0$ for all $z \in (y_1, y_2)$. Since $\varphi(0, z)$ is strictly positive on interval $[y_1, y_2]$, thus we can find a constant $\epsilon > 0$ satisfying $\epsilon \psi^{c_0}(0, z) \leq \varphi(0, z)$ for all $z \in [y_1, y_2]$. Define $\hat{\varphi}(t, x) := \varphi(t, x + (c - c_0)t)$. Since $(\phi(t, z), \varphi(t, z))$ is a solution of system (11), we have

$$\hat{\varphi}_t(t, x) = d_2 \hat{\varphi}_{xx}(t, x) - c_0 \hat{\varphi}_x(t, x) + \beta(t) f(\phi(t, x + (c - c_0)t), \hat{\varphi}(t, x)) - \gamma(t) \hat{\varphi}(t, x). \quad (70)$$

Due to $\lim_{z \rightarrow -\infty} \phi(t, z) = S_0$ and $\lim_{z \rightarrow -\infty} \varphi(t, z) = 0$ uniformly for $t \in \mathbb{R}$, combining (A2), we can deduce that there exists a $N_\delta > 0$ satisfying $f(\phi, \varphi) \geq (f_I(S_0, 0) - \delta)\varphi$ for all $z < -N_\delta$ uniformly for $t \in \mathbb{R}$. Since $c - c_0 < 0$ and $y_1 < y_2 < -N_\delta$, we have $x + (c - c_0)t \leq -N_\delta$ for all $t \geq 0, x \in [y_1, y_2]$. Therefore, from (70) we know

$$\hat{\varphi}_t(t, x) \geq d_2 \hat{\varphi}_{xx}(t, x) - c_0 \hat{\varphi}_x(t, x) + [\beta(t) (f_I(S_0, 0) - \delta) - \gamma(t)] \hat{\varphi}(t, x), \quad (71)$$

for $x \in [y_1, y_2]$. Let $\check{\varphi}(t, x) := \varphi(t, x + (c - c_0)t) - \epsilon \psi^{c_0}(t, z) = \hat{\varphi}(t, x) - \epsilon \psi^{c_0}(t, z)$ for all $t \geq 0, x \in [y_1, y_2]$. It follows from (69) and (71) that

$$\begin{cases} \check{\varphi}_t(t, x) \geq d_2 \check{\varphi}_{xx}(t, x) - c_0 \check{\varphi}_x(t, x) \\ \quad + [\beta(t) (f_I(S_0, 0) - \delta) - \gamma(t)] \check{\varphi}(t, x), \quad t \geq 0, x \in [y_1, y_2], \\ \check{\varphi}(0, x) \geq 0, x \in [y_1, y_2], \quad \check{\varphi}(t, y_i) \geq 0, i = 1, 2. \end{cases}$$

Fig. 1. Numerical simulation show that there exists traveling wave solution connecting the initial disease-free steady state $(S_0, 0)$ and the steady state $(S^\infty, 0)$ after the epidemic. Moreover, from the numerical stimulation we can obtain $S^\infty \approx 1.601$.

Fig. 2. Numerical simulation show that there exists traveling wave solution connecting the initial disease-free steady state $(S_0, 0)$ and the steady state $(S^\infty, 0)$ when $c > c^* = 0.7212$, and there does not exist such traveling wave solutions if $c < c^*$.

Applying the maximum principle, we can derive that $\tilde{\varphi} \geq 0$ for all $t > 0$ and $x \in [y_1, y_2]$, that is $\varphi(t, x + (c - c_0)t) \geq \epsilon \psi^{c_0}(t, z)$ for all $t > 0$ and $x \in [y_1, y_2]$, which contradict with $\varphi(t, x + (c - c_0)t) \rightarrow 0$ when $t \rightarrow \infty$ since $c - c_0 < 0$. This is the end of the proof. $\square$

Remark 4. We easily see that the main results on the non-existence of periodic travelling wave solutions for models (1) and (2) established in [31,35] can be directly obtained from Theorems 5 and 6.

7. Numerical examples

In this section, we present some numerical simulations to illustrate the theoretic results. Assume that the incidence function $f(S, I) = SI$. We choose the following initial conditions

$$S(0, x) = \begin{cases} 1 + 0.5 \sin x, & \text{if } -20 \leq x < 0, \\ 5.5, & \text{if } 0 \leq x \leq 20, \end{cases} \quad I(0, x) = \begin{cases} 0.1 + 0.1 \sin x, & \text{if } -20 \leq x < 0, \\ 0, & \text{if } 0 \leq x \leq 20, \end{cases}$$

All values of the parameters are given: $\beta(t) = 0.15 * (0.1 * \sin(t) + 1)$, $\gamma(t) = 0.5 * (0.1 * \cos(t) + 1)$, $S_0 = 5.5$, $d_1 = 0.09648$, $d_2 = 0.4$ and $d_3 = 0.17$. By simple calculation, we obtain $\mathcal{R}_0 = 1.65 > 1$ and $c^* = 0.7212$. Theorem 4 means that there exists traveling wave solutions satisfying the asymptotical boundary conditions (12). That is, $\phi(t, -\infty) = S^\infty$, $\varphi(t, -\infty) = 0$, $\phi(t, \infty) = S_0$ and $\varphi(t, \infty) = 0$. The numerical simulations are given in Fig. 1.

Remark 5. It is clear that there exists a constant S^∞ , which stands for the steady state of susceptible individuals after the epidemic, satisfying $S^\infty < S_0 = 5.5$. However, we still don't obtain the exact value of S^∞ through theoretical analysis.

Now, we observe the effect of the critical wave speed c by using the equipotential diagram. We choose all conditions and parameters are same as previous, that is $\mathcal{R}_0 = 1.65 > 1$ and $c^* = 0.7212$. It follows from Theorem 3 and Theorem 6 that there exists traveling waves connecting $(S_0, 0)$ and $(S^\infty, 0)$ for every speed $c > c^*$, and there does not exist such traveling waves for any $c < c^*$. The numerical simulations are given in Fig. 2.

In order to find out the difference of traveling wave solution when the infection coefficient and recovery rate are related to time t or independent of time t . We choose $\beta = 0.15$, $\gamma = 0.5$ and assume that the rest of the parameters are same as above. We choose the following initial conditions

$$S(0, x) = \begin{cases} 1, & \text{if } -20 \leq x < 0, \\ 5.5, & \text{if } 0 \leq x \leq 20, \end{cases} \quad I(0, x) = \begin{cases} 0.1, & \text{if } -30 \leq x < 0, \\ 0, & \text{if } 0 \leq x \leq 30, \end{cases}$$

The numerical simulations are given in Figs. 3 and 4.

Fig. 3. Numerical simulation show that the traveling wave solution about susceptible individuals is more oscillatory or has a periodicity when the infection coefficient and recovery rate are related to time t than usual.

Fig. 4. Numerical simulation show that the traveling wave solution about infected individuals is more oscillatory or has a periodicity when the infection coefficient and recovery rate are related to time t than usual.

8. Conclusion

In this paper, we proposed a nonautonomous and diffusive SIR epidemic model with general nonlinear incidence given by system (3) assuming that the transmission rates and recovery rates are positive and T -periodic functions. We further investigated the existence and nonexistence of periodic traveling waves for system (3). It is important to mention that the classical methods such as shooting method [34], two-sided Laplace transform [18] and monotone iteration methods [49] can not be used directly, since the introduction of nonautonomous and nonlinear incidence. Consequently, we further refined and developed the previous methods in [31,35]. More precisely, we constructed an auxiliary system (14) to obtain the boundedness of the infected individual, which guarantees the suitable bounded closed convex set between the upper and lower solutions. Finally, the existence of traveling wave solutions are obtained by the fixed-point theorem and some limiting arguments. Many of the methods and arguments in this investigation can be widely applied to the existence and nonexistence of periodic traveling wave solutions for other periodic reaction-diffusion systems.

The conclusions of this article show that there exists a critical wave speed $c^* > 0$ such that the system (3) admits a T -periodic traveling wave solutions $(\phi^*(t, z), \varphi^*(t, z))$ satisfying the asymptotic boundary conditions (12) when the reproductive number $\mathcal{R}_0 > 1$, and there is no such solution for either $\mathcal{R}_0 \leq 1$ or $\mathcal{R}_0 > 1$ and $0 < c < c^*$. In fact, it is not difficult to find that the literatures [35] and [31] are the direct corollaries of our conclusions. Furthermore, we provided some numerical simulations to verify our theoretic results. It is not difficult to find that the traveling wave solution $\phi^*(t, z)$ tends to a constant S^∞ less than S_0 as $z \rightarrow \infty$ when $\mathcal{R}_0 > 1$ and $c > c^*$, and the exact value of S^∞ can not be obtained by theoretical analysis, which will be our further research direction.

Finally, there are some very interesting problems that deserve further study. For example, on the one hand, we can consider the traveling wave solutions of the nonlocal diffusion time-periodic model. On the other hand, we can investigate the periodic model with susceptible input and natural death, because there are disease-free periodic solution and endemic periodic solution when the basic reproduction number is more than unit, so we can study the existence of traveling wave solutions connecting these two periodic solutions.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Weixin Wu: Writing - review & editing, Visualization. **Zhidong Teng:** Funding acquisition, Supervision.

Acknowledgement

This work was supported by the [Natural Science Foundation of China](#) (Grant No. 11771373, 11861065).

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