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Examining Impacts of Mass-Diameter (m-D) and Area-Diameter (A-D) Relationships of Ice Particles on Retrievals of Effective Radius and Ice Water Content from Radar and Lidar Measurements

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Abstract

Mass-diameter (m-D) and projected area-diameter (A-D) relations are often used to describe the shape of nonspherical ice particles. This study analytically investigates how retrieved effective radius (r_{eff}) and ice water content (IWC) from radar and lidar measurements depend on the assumption of m-D [$m(D) = a D^b$] and A-D [$A(D) = \gamma D^\delta$] relationships. We assume that unattenuated reflectivity factor (Z) and visible extinction coefficient (k_{ext}) by cloud particles are available from the radar and lidar measurements, respectively. A sensitivity test shows that r_{eff} increases with increasing a , decreasing b , decreasing γ , and increasing δ . It also shows that a 10% variation of a , b , γ , and δ induces more than a 100% change of r_{eff} . In addition, we consider both gamma and lognormal particle size distributions (PSDs), and examine the sensitivity of r_{eff} to the assumption of PSD. It is shown that r_{eff} increases by up to 10% with increasing dispersion (μ) of the gamma PSD by 2, when large ice particles are predominant. Moreover, r_{eff} decreases by up to 20% with increasing the width parameter (ω) of the lognormal PSD by 0.1. We also derive an analytic conversion equation between two effective radii when different particle shapes and PSD assumptions are used. When applying the conversion equation to nine types of m-D and A-D relationships, r_{eff} easily changes up to 30%. The proposed r_{eff} conversion method can be used to eliminate the inconsistency of assumptions that made in a cloud retrieval algorithm and a forward radiative transfer model.

Keywords

Ice particle shape; mass-Diameter (m-D); Area-Diameter (A-D); effective radius; ice water content (IWC); radar; lidar; reflectivity; visible extinction coefficient; particle size distribution (PSD)

1. Introduction

Nonspherical particles have a smaller mass and a projected area than spherical particles for a given maximum diameter (or maximum dimension), D . Numerous field campaigns using improved instruments and techniques have measured individual ice particle shapes [e.g., Field et al., 2006, Lawson et al., 2006; McFarquhar et al., 2007; Lawson, 2011; Um et al., 2015], and provided relationships between mass and D (m-D), and projected area and D (A-

D). Ice particle shapes of liquid-topped clouds in temperature between -20°C and -3°C are relatively well-known [Myagkov et al., 2016]. However, for colder temperatures, mass and area of ice particles significantly vary with region, temperature, and cloud type, implying that large uncertainties exist in describing the m-D and A-D relationships.

Space-borne radar and lidar sensors such as Cloud-Aerosol Lidar with Orthogonal Polarization (CALIOP) aboard Cloud-Aerosol Lidar and Infrared Pathfinder Satellite Observations (CALIPSO) [Winker et al., 2003, 2009] and Cloud Profiling Radar (CPR) aboard CloudSat [Stephens et al., 2002, 2008] provide an opportunity of cloud retrievals from combined radar and lidar sensors at a global scale, as shown in Okamoto et al. [2003], Tinel et al. [2005], Delanoë and Hogan [2008, 2010], Stein et al. [2011], and Deng et al. [2010, 2013]. Since the radar and lidar have different sensitivities to cloud optical properties, combining these two active instruments, in principle, brings more detailed and accurate vertical structures of cloud layers than a single active sensor or a passive sensor. However, the radar and lidar retrieval algorithms require an assumption of m-D and A-D relationships, because the radar reflectivity factor is proportional to the mass-squared, and the lidar extinction coefficient is proportional to the projected area of ice particles. Since the m-D and A-D relationships depend on particle shape, retrieved cloud properties differ depending on the assumption of particle shape used for the radar and lidar retrievals.

Several studies have pointed out the importance of the knowledge of particle shape in radar and/or lidar cloud retrievals. Donovan and Van Lammeren [2001] suggested a factor of 3 of differences in retrieved effective radius (r_{eff}) due to a particle shape assumption. Hogan et al. [2006a] applied two different particle shapes from Francis et al. [1998] and Mitchell et al. [1996], and found 30% of differences in retrieved r_{eff} and ice water content (IWC). Fontaine et al. [2014] examined impacts of m-D and A-D relationships in determining a reflectivity-IWC (Z-IWC) relationship. Stein et al. [2011] examined a sensitivity of radar-lidar and passive retrieval algorithms to particle shape. Mace and Benson [2017] found 30–200% of differences in retrieving precipitation rate from a Doppler radar depending on ice bulk density, which is predominantly a function of ice particle shape. Other studies also point out importance of particle shape in radar reflectivity forward model. For example, Sato and Okamoto [2006] examined how the radar reflectivity changes with particle shape, and they found 5dB of radar reflectivity differences for $r_{eff} < 100\ \mu\text{m}$, and 13 dB for $100\ \mu\text{m} < r_{eff} < 600\ \mu\text{m}$. Hammonds et al. [2014] also suggested 4 dB of uncertainties in radar reflectivity simulation depending on mass-dimensional relationship.

When one computes irradiance profiles at a global scale, one might need to use cloud properties such as r_{eff} and optical depth derived from different cloud algorithms because no single retrieval algorithm can provide the properties everywhere all the time. Because the ice r_{eff} particularly depends on the assumption of ice particle shape, one needs to use r_{eff} with a consistent particle shape assumption in the forward radiative transfer model and cloud retrieval. Another option is to develop a relationship to convert the ice r_{eff} derived with a specific particle shape into r_{eff} with a different particle shape assumption for the consistency.

In this study, we analytically derive the relationship between two r_{eff} retrieved from different particle shape assumptions. This differs from earlier studies [e.g., Hogan et al., 2006a;

Fontaine et al., 2014; Stein et al., 2011] that examined impacts of particle shape on r_{eff} numerically. We start with an assumption that lidar extinction and radar reflectivity factor are known (or fixed) from lidar and radar observations, respectively. Then r_{eff} and IWC are expressed by coefficients of m-D and A-D relationships. This approach is similar to the one by Donovan and Van Lammeren [2001]. They examined how particle shape assumptions change the relationship between r_{eff} and r_{eff}' , where r_{eff}' is defined as the ratio of radar reflectivity to lidar-derived extinction coefficient, hereafter referred as radar-lidar-ratio. In this study, we directly relate r_{eff} to the measured radar-lidar-ratio, instead of using r_{eff}' for various particle shapes. We also use the first derivative of the analytical expression to quantify the sensitivity of r_{eff} to particle shape.

In addition, we examine how well radar and lidar observations can constrain the effective radius, which is a function of particle size distribution (PSD). Generally, the number of unknowns in the PSD is greater than the number of equations that can be set up from observations. Assumptions of one or two parameters of a PSD are often made to reduce the number of unknowns but they introduce an error. We examine the sensitivity of retrieved effective radius to frequently-assumed parameters in the PSD.

Section 2 compares pre-existing m-D and A-D relationships, and Section 3 derives integrated optical properties such as effective radius (r_{eff}) and IWC with a gamma PSD. Then uncertainties in retrievals of r_{eff} and IWC are further examined with the derivative of equations of r_{eff} with respect to parameters of m-D and A-D relations. Section 4 uses a lognormal PSD, and compares the results with those from the gamma PSD. Section 5 demonstrates simple applications of this study, a conversion of r_{eff} when different m-D and A-D relationships and/or PSD are used between two radar-lidar algorithms.

2. Methodology

2.1. Mass-Diameter (m-D) and Area-Diameter (A-D) relationships

Often power laws are used to describe the mass or area distribution of nonspherical ice particles [e.g., Brown and Francis, 1995; Mitchell, 1996; Mitchell et al., 1996; Francis et al., 1998; Heymsfield et al., 2013]:

$$m(D) = aD^b, \quad (1)$$

$$A(D) = \gamma D^\delta, \quad (2)$$

where m is the mass of cloud particles, A is the projected area of cloud particles, and D is the maximum diameter (or the maximum linear dimension of the particle). Unless noted, all variables have centimeter-gram-second (CGS) units throughout this study. Therefore, D is in the unit of cm, a is in the unit of g cm^{-b} , $m(D)$ is in gram, γ is in the unit of $\text{cm}^{2-\delta}$ and $A(D)$ is in cm^2 .

Table 1 summarizes coefficients a , b , γ , and δ of power laws used in several studies. Brown and Francis [1995] provided a m-D relation for $D = 97 \times 10^{-4} \text{ cm}$ ($= 97 \mu\text{m}$), while spherical

assumption can be used for $D < 97 \times 10^{-4}$ cm. Francis et al. [1998] further defined a A-D relation from the same field experiments, which holds for $D \geq 128 \times 10^{-4}$ cm, while a spherical assumption can be used for smaller particles. For the analytical integration of mass and area over PSD, we compute a single set of a , b , γ , and δ valid for all sizes of D (case (3) of Table 1). In doing so, we compute $m(D)$ for 1×10^{-4} cm $\leq D \leq 200 \times 10^{-4}$ cm, using Eq. (1) with coefficients a and b (cases (1) and (2) of Table 1). Then linear regression is performed between $\ln(D)$ and $\ln[m(D)]$ to get coefficients γ and δ (case (3) of Table 1). Similarly, coefficients γ and δ (case (3) of Table 1) are obtained from linear regression between $\ln(D)$ and $\ln[A(D)]$. Obtained correlation coefficients are > 0.99 , and root mean square (RMS) errors for mass and area are 2.33×10^{-7} g and 8.49×10^{-6} cm², respectively. Hereafter, the single coefficient set of a , b , γ , and δ for all size D (case (3) in Table 1) is referred to as *Brown and Francis*.

While Brown and Francis [1995] and Francis et al. [1998] provide fixed m-D and A-D relations regardless of temperature, Heymsfield et al. [2013] provide temperature-dependent m-D and A-D relations based on a wide geographical range of field experiments from Tropics through Arctic as

$$a = 0.0081 \exp(0.013T), \quad (3)$$

$$b = 2.31 + 0.0054 T, \quad (4)$$

$$\gamma = \frac{\pi}{4} (0.2833 + 0.006913T + 8.09 \times 10^{-5} T^2), \text{ and} \quad (5)$$

$$\delta = -0.2026 + 0.009681T + 1.19 \times 10^{-4} T^2 + 2, \quad (6)$$

where T is the temperature in Celsius, and $-86^\circ\text{C} \leq T \leq 0^\circ\text{C}$. We consider three different temperatures as -30°C , -45°C , and -60°C to get a , b , γ , and δ in Table 1 (cases (4)-(6)).

Yang et al. [2000] computed the mass and area of ice particles for plates, hexagonal columns, and bullets. Table 2 of Yang et al. [2000] provides coefficients of fourth order polynomials of $\ln(D)$ to compute the mass and area. Using these fourth order polynomials, we compute $m(D)$ and $A(D)$ over the size range 1×10^{-4} cm $\leq D \leq 200 \times 10^{-4}$ cm, and derive coefficients a , b , γ , and δ by linear regression (cases (7)-(9) of Table 1). For plates, hexagonal columns, and bullets, the correlation coefficients between original values and obtained values are > 0.99 , and RMS errors for mass and area are $< 9.96 \times 10^{-8}$ g and $< 2.79 \times 10^{-6}$ cm², respectively.

In addition, using single particle shape properties of Yang et al. [2000], the mass and area of habit mixtures are also derived in this study, while similar work had been performed in Deng et al. [2010, 2013]. We use habit fractions defined in Baum et al. [2005a, b]; For $D < 60 \times 10^{-4}$ cm, 100% droxtals, and for 60×10^{-4} cm $\leq D < 1000 \times 10^{-4}$ cm, 15% of 6-branch bullets, 50% of solid hexagonal columns, and 35% of plates are assumed. The coefficients a , b , γ , and δ for mixtures are given in case (10) of Table 1, while RMS errors for mass and

area are 9.88×10^{-8} g and 8.86×10^{-6} cm², respectively. Case (11) of Table 1 provides coefficients of power laws for spherical particles, with an assumption of solid ice density (ρ_i) as 0.917 g cm⁻³. Therefore, $a = \rho_i \pi / 6$, $b = 3$, $\gamma = \pi / 4$, and $\delta = 2$.

Figure 1 shows the mass and projected area of ice particles as a function of D from a , b , γ , and δ listed in Table 1. As expected a spherical particle has a larger mass and a projected area than nonspherical particles for a given D . Among nonspherical particles used in this study, the mass and projected area by *Brown and Francis* are closest to those for spherical particles. Bullet with 6 branches by Yang et al. [2000] has the smallest mass and projected area for a given D . Temperature-dependent particle shapes described by Heymsfield et al. [2013] show that the mass decreases, and projected area slightly increases with increasing temperature (-60°C to -30°C).

Figure 2 shows how the different m-D and A-D relationships, which are determined by particle shape, affect effective radius (r_{eff}) retrievals. As discussed in Section 2.2, radar reflectivity factor of a particle is proportional to $m(D)^2$. Therefore, total reflectivity of N_t number of particles with a size D is proportional to $m(D)^2 \times N_t$. In addition, the extinction coefficient of N_t particles at visible wavelengths is given by $Q_{\text{ext}} A(D) \times N_t$, where Q_{ext} is extinction efficiency at visible wavelengths. If we take the ratio of reflectivity to the extinction coefficient, N_t is canceled out, and the ratio is proportional to $m(D)^2 / A(D)$. Moreover, the effective radius is proportional to $m(D) / A(D)$ (Section 2.2). Therefore, Fig. 2 shows a relationship between radar reflectivity to lidar-radar ratio [$\sim m(D)^2 / A(D)$] and effective radius [$\sim m(D) / A(D)$]. In this figure, plates and bullets by Yang et al. [2000] produce the smallest effective radius for a given lidar-radar ratio. In contrast, Heymsfield et al. [2013] at $T = -60^\circ\text{C}$ gives the largest cloud effective radius for $m(D)^2 / A(D) < 0.03 \times 10^{-7}$ g² cm⁻², while spherical assumption gives the largest effective radius for $m(D)^2 / A(D) > 0.03 \times 10^{-7}$ g² cm⁻². In Sections 3 and 4, we consider more realistic particle size distributions (PSDs) with gamma and lognormal distributions. However, similar conclusions are found to those obtained from the single particle size assumption shown in Fig. 2.

Note that several m-D and A-D relationships considered in this study were obtained from in-situ measurements [Brown and Francis, 1995; Francis et al., 1998; Heymsfield et al., 2013]. Recent studies [Field et al., 2006; Lawson, 2011; Korolev and Field, 2015] have reported that shattered ice fragments by instruments artificially increase the number of small particles. In this study, we only use particle shape parameters (a , b , γ and δ) instead of number concentrations $N(D)$ from the in-situ measurements. Therefore, the impacts of shattering artifacts would be relatively small, once the particle shapes of large ice particles are properly measured. Examining impacts of shattering effects on the m-D and A-D relationships remains a topic of future work.

2.2. Size-integrated optical parameters

In a Rayleigh-scattering regime, the equivalent radar reflectivity factor of ice particles can be computed [Brown et al., 1995; Schneider and Stephens, 1995; McFarlane and Evans, 2004; Hogan et al., 2006a, 2006b] as

$$Z_{e, Ray} = \frac{|K_i|^2}{|K_w|^2} \frac{36}{\pi^2 \rho_i^2} \int [m(D)]^2 N(D) dD \quad (7)$$

where $Z_{e, Ray}$ is the equivalent radar reflectivity factor with Rayleigh scattering theory, $|K_i|^2$ is the dielectric factor of solid ice, $|K_w|^2$ is the dielectric factor of water, $N(D)$ is the number of particles with the particle size D in a unit volume ($\text{cm}^{-3} \text{cm}^{-1}$), m is the mass in gram, and ρ_i is the density of solid ice (g cm^{-3}). However, for ice particles $> 100 \mu\text{m}$, Mie scattering is not negligible and the effect should be considered in 94-GHz (3.2 mm) radar measurements. In this study, we use a Mie correction factor by following Benedetti et al. [2003] and Austin et al. [2009] as

$$Z_e = f_{Mie} Z_{e, Ray} = f_{Mie} \frac{|K_i|^2}{|K_w|^2} \frac{36}{\pi^2 \rho_i^2} \int [m(D)]^2 N(D) dD \quad (8)$$

where Z_e includes both Mie and Rayleigh scattering effects, and f_{Mie} is the Mie correction factor. f_{Mie} is 1 for small ice particles ($< 100 \mu\text{m}$), and it decreases with an increasing ice particle size [Austin et al., 2009]. In addition, we define the radar reflectivity factor of ice particles (Z), which can be inferred from Z_e using the dielectric factors [Smith, 1984; Atlas, 1995]:

$$Z = \frac{|K_w|^2}{|K_i|^2} Z_e = f_{Mie} \frac{36}{\pi^2 \rho_i^2} \int [m(D)]^2 N(D) dD. \quad (9)$$

Combining Eqs. (1) and (9), we obtain

$$Z = \frac{36 f_{Mie}}{\pi^2 \rho_i^2} a^2 \int D^{2b} N(D) dD. \quad (10)$$

The cloud extinction coefficient (k_{ext} in the unit of cm^{-1}) at a visible wavelength is an integration of the extinction cross section over the PSD,

$$k_{ext} = \int A(D) N(D) Q_{ext} dD = 2\gamma \int D^\delta N(D) dD, \quad (11)$$

where Q_{ext} is the visible extinction efficiency, and approximated as 2 in this study. IWC (g cm^{-3}) is the total ice mass in a unit volume, which is an integration of $m(D)$ over PSD [e.g., Bouldala et al., 2002; McFarlane and Evans, 2004],

$$IWC = \int m(D) N(D) dD = a \int D^b N(D) dD. \quad (12)$$

In this study, effective radius (r_{eff} , in the unit of cm) is defined as [Foot, 1988; Brown et al., 1995; Hogan et al., 2006a, 2006b; Donovan and Van Lammeren, 2001; Delanoë and Hogan, 2008]:

$$r_{eff} = \frac{IWC}{k_{ext}} \frac{3}{2\rho_i}. \quad (13)$$

2.3. Assumptions made in this study

Most importantly, we assume that the unattenuated radar reflectivity factor (Z) and visible extinction (k_{ext}) by cloud particles are available from radar and lidar measurements, respectively. In obtaining the unattenuated reflectivity factor from the radar measurements, attenuation by gas and hydrometeors should be corrected [Marchand et al., 2008]. The gas attenuation can be estimated directly from temperature and humidity profiles based on satellite infrared/microwave sounding observations or reanalysis [e.g. Aumann et al., 2003; Tobin et al., 2006; Rienecker et al., 2011]. The attenuation by ice-phase hydrometeors is negligible since imaginary part of the refractive index of ice is in the order of 10^{-3} at 94 GHz (3.2 mm). Multiple scattering of the radar signal by cloud particles is generally negligible for non-precipitating clouds [Battaglia et al., 2005, 2007; Lebsock, 2011]. Therefore, we target non-precipitating clouds in this study.

If an ice particle is larger than 100 μm , the particle is not a Rayleigh scatterer anymore. In this study, we use a Mie correction factor (f_{Mie}) to take into account Mie scattering, following approaches of Benedetti et al. [2003] and Austin et al. [2009]. In their studies, f_{Mie} is parameterized with the width parameter (ω) and geometrical diameter (D_g) of a lognormal PSD. When D_g is 100 μm , Eqs. (14)-(17) of Austin et al. [2009] give $f_{Mie} \sim 0.9$. This approach can be applied for other PSDs, such as a gamma PSD for which f_{Mie} is parameterized with dispersion (μ) and slope parameters (Λ). Therefore, we assume that f_{Mie} is not a function of D . A more sophisticated formula that takes into account Mie scattering in a radar wavelength can be developed for future applications.

While direct measurements of the extinction coefficient are available from High Spectral Resolution Lidar (HSRL) or Raman lidar [Burton et al., 2012; Whiteman et al., 2004; Haarig et al., 2016], lidar ratio and multiple scattering factors are required to compute the extinction coefficient from elastic backscatter lidars such as CALIOP [Platt, 1979; Platt et al., 1998; Young and Vaughan, 2009]. The lidar ratio and multiple scattering factor can be estimated and evaluated from two-transmission method [Young and Vaughan, 2009], from comparisons with other independent observations [Garnier et al., 2015; Holz et al. 2016], or by an iteration method [Hogan et al., 2006a; Seifert et al., 2007; Kienast-Sjögren et al. 2016]. Once reasonable lidar ratio and multiple scattering factors are determined, attenuation by hydrometeors can be estimated, provided that Rayleigh scattering by gas molecules is already corrected using the atmospheric profiles. Young and Vaughan [2009] and Hogan et al. [2006a] provide detailed discussions of how the visible extinction coefficient is estimated from lidar backscatter measurements.

The density of solid ice changes up to 1% with temperature. Cloud ice particles, however, can have a much smaller density than the solid ice particle due to porosities (or bubbles). Sato and Okamoto [2006] defined the ice bulk density (ρ_b) as a ratio of ice mass to exterior volume of ice particle including air bubbles. If there is no bubble in the ice particle, ρ_b becomes a density of solid ice around 0.917 g cm^{-3} , but measured ρ_b is actually around 0.81 g cm^{-3} [Sato and Okamoto, 2006]. Heymsfield et al. [2004] defined an effective density (ρ_e) as a ratio of ice mass to volume of the circumscribed sphere of a nonspherical particle. They found that ρ_e can be related to the slope (Λ) of a gamma PSD. The range of ρ_e shown in Heymsfield et al. [2004] is quite large; 0.15 g cm^{-3} to 0.91 g cm^{-3} . Note that ρ_e is smaller than ρ_b , since ρ_e uses the enclosed sphere volume of nonspherical ice particle, while ρ_b uses the exterior volume of a nonspherical ice particle. In this study, we assume the ice bulk density (ρ_b) to equal the density of solid ice ($= 0.917 \text{ g cm}^{-3}$, ρ_i) because a change of ρ_b from 0.60 to 0.92 g cm^{-3} only causes $<1 \text{ dB}$ differences in the radar reflectivity [Sato and Okamoto, 2006]. However, we take into account variations of the effective density (ρ_e) by considering different ice particle shapes (or m-D and A-D relations).

The phase identification is important in estimating radar reflectivity (Z) from equivalent radar reflectivity factor (Z_e) (Eq. (9)). We assume that cloud particles are all in ice phase and no mixed phase is involved. In addition, the expression we derive here requires that both radar and lidar signals are available, i.e. a cloud layer needs to be optically thin so that it does not fully attenuate the lidar signal. Further studies are required to extend our expressions to lidar- or radar-only observations.

In the following sections, we examine how coefficients in m-D and A-D relations affect the retrieved effective radius in the radar and lidar observations. The retrieval algorithm is generally based on an inversion method that starts with an initial guess. The algorithm goes through iterations to minimize a cost function till the cost function becomes smaller than a threshold value. Optimal Estimation allows quantification of the retrieval errors, once uncertainties of input empirical data are known. Even though estimating the uncertainties of input data is also challenging [Mace and Benson, 2017], we assume that the inversion method converges to a solution with a reasonable accuracy. Then the analytic relationship derived here can be used for converting the effective radius derived with different particle shape assumptions to the effective radius with a common particle shape assumption for consistent radiative transfer computations.

Lastly, this study uses power laws to express distributions of mass and projected area as in Eqs. (1) and (2). Erfani and Mitchell [2016] noted that the power laws can overestimate particle mass and area for small particle sizes. They found that the second-order polynomials as functions of $\ln(D)$ are more feasible to describe mass and projected area of ice particles over the diverse range of D . However, because the power laws can be handled easily in analytic integrations of mass and projected area over PSD, we use the power laws throughout this study.

3. Analytic derivation using a gamma particle size distribution (PSD)

In this section, we consider a gamma size distribution in deriving radar reflectivity factor (Z), ice water content (IWC), visible extinction coefficient (k_{ext}), and effective radius (r_{eff}). Then the sensitivity of r_{eff} to coefficients of m-D and A-D relationships is analytically examined. We also show similar derivations with a lognormal size distribution in Section 4.

3.1. Sensitivity of r_{eff} and IWC to coefficients of m-D and A-D relationships

The gamma particle size distribution (PSD) [e.g., Kosarev and Mazin, 1991; Mitchell, 1991] is defined as

$$N(D) = N_0 D^\mu \exp(-\Lambda D), \quad (14)$$

where Λ is the slope (cm^{-1}), μ is the dispersion (unitless), and N_0 ($\text{cm}^{-\mu-4}$) is the intercept.

In this equation, $N(D)$ decreases more rapidly toward large D with increasing Λ , and the inflection point of $N(D)$ moves toward zero with decreasing μ . This means that the particle effective radius decreases with increasing Λ or decreasing μ . The j th Moment Generating Function (MGF) of gamma distribution is

$$M_j = \int N(D) D^j dD = N_0 \frac{\Gamma(j + \mu + 1)}{\Lambda^{j + \mu + 1}}. \quad (15)$$

The total number (N_T) of the gamma distribution in the unit of cm^{-3} is obtained from the zeroth moment of MGF:

$$N_T = N_0 \frac{\Gamma(\mu + 1)}{\Lambda^{\mu + 1}}. \quad (16)$$

Combining Eqs. (10) and (15), the radar reflectivity factor in the unit of $\text{cm}^6 \text{cm}^{-3}$ ($= 10^{12} \text{mm}^6 \text{m}^{-3}$) is

$$Z = \frac{36 f_{Mie}^2 a^2 N_0 \Gamma(2b + \mu + 1)}{\pi^2 \rho_i^2 \Lambda^{2b + \mu + 1}}. \quad (17)$$

Similarly, k_{ext} (cm^{-1}), IWC (g cm^{-3}), and r_{eff} (cm) are expressed as

$$k_{ext} = 2 N_0 \gamma \frac{\Gamma(\delta + \mu + 1)}{\Lambda^{\delta + \mu + 1}}, \quad (18)$$

$$IWC = a N_0 \frac{\Gamma(b + \mu + 1)}{\Lambda^{b + \mu + 1}}, \text{ and} \quad (19)$$

$$r_{eff} = 3aN_0 \frac{\Gamma(b+\mu+1)}{\Lambda^{b+\mu+1}} \frac{1}{4\rho_i N_0 \gamma} \frac{\Lambda^{\delta+\mu+1}}{\Gamma(\delta+\mu+1)} = \frac{3a}{4\rho_{si}\gamma} \frac{\Gamma(b+\mu+1)}{\Gamma(\delta+\mu+1)} \Lambda^{\delta-b}. \quad (20)$$

Once we take the ratio of Z to k_{ext} (radar reflectivity-to-lidar-extinction ratio), N_0 cancels out and results in

$$\frac{Z}{k_{ext}} = \frac{18f_{Mie}}{\pi^2\rho_i^2} \frac{a^2}{\gamma} \frac{\Gamma(2b+\mu+1)}{\Gamma(\delta+\mu+1)} \Lambda^{\delta-2b}. \quad (21)$$

where Z/k_{ext} is in the unit of cm^4 . Figure 3 shows a typical range of Z/k_{ext} using CloudSat and CALIPSO measurements. CloudSat provides equivalent radar reflectivity factor in dB (Z_{dB}) (Fig. 3a), where $Z_{dB} = 10 \log Z_e$. Then Eq. (9) can be used to obtain radar reflectivity (Z) from equivalent radar reflectivity factor (Z_e). Combining CloudSat Z with CALIPSO cloud extinction coefficient (k_{ext}) results in Z/k_{ext} in Fig. 3c for ice clouds. The ice clouds are selected when $k_{ext} > 0.01 \text{ km}^{-1}$ and air temperature $< 253 \text{ K}$. Z/k_{ext} generally increases with Z (Fig. 3d), and Z/k_{ext} is between 10^{-10} and 10^{-6} cm^4 (Fig. 3e).

In Eqs. (20) and (21), r_{eff} and Z/k_{ext} are expressed with a , b , γ , δ , μ , and Λ . Note that impacts of μ and Λ largely offset in $N(D)$ for a given D , since $N(D)$ increases with increasing μ and with decreasing Λ . Either μ or Λ in above equations can be eliminated using Eq. (21). To eliminate Λ , we solve Eq. (21) for Λ

$$\Lambda = \left\{ \frac{Z}{k_{ext}} \frac{\pi^2\rho_i^2}{18f_{Mie}} \frac{\gamma}{a^2} \frac{\Gamma(\delta+\mu+1)}{\Gamma(2b+\mu+1)} \right\}^{\frac{1}{\delta-2b}}, \quad (22)$$

and substitute Eq. (22) into Eq. (20) to obtain

$$r_{eff} = \frac{3a}{4\rho_i\gamma} \frac{\Gamma(b+\mu+1)}{\Gamma(\delta+\mu+1)} \left\{ \frac{Z}{k_{ext}} \frac{\pi^2\rho_i^2}{18f_{Mie}} \frac{\gamma}{a^2} \frac{\Gamma(\delta+\mu+1)}{\Gamma(2b+\mu+1)} \right\}^{\frac{b-\delta}{2b-\delta}}. \quad (23)$$

Resulting Eq. (23) is a function of a , b , γ , δ , and μ . Using asymptotic theory, we get

$$\left(\frac{\Gamma(x+p)}{\Gamma(x+q)} \right) \sim \left(x + \frac{p+q-1}{2} + o(x^{-1}) \right)^{p-q} \quad \text{as } x \rightarrow \infty, \quad (24)$$

where $x = \mu + \delta + 1$, $p = b - \delta$, and $q = 0$ (see Appendix A for more detailed expressions). When using the first two terms in the right side of Eq. (24) and ignoring higher terms, errors are $<15\%$, $<4\%$, and $<2\%$ for $\mu = -2$, $\mu = 0$, and $\mu = 2$, respectively (Appendix A, Fig. A1). Using Eq. (24), we can approximate Eq. (23) as

$$\begin{aligned}
 r_{eff} &= \frac{3a}{4\rho_i\gamma} \left\{ \mu + \frac{b+\delta+1}{2} \right\}^{b-\delta} \left\{ \frac{Z}{k_{ext}} \frac{\pi^2 \rho_i^2}{18f_{Mie} a^2} \gamma \right\}^{\frac{b-\delta}{2b-\delta}} \left\{ \mu + \frac{2b+\delta+1}{2} \right\}^{-b+\delta} \\
 &= \frac{3}{4\rho_i} a^{\frac{\delta}{2b-\delta}\gamma - \frac{b}{2b-\delta}} \left\{ \frac{Z}{k_{ext}} \frac{\pi^2 \rho_i^2}{18f_{Mie}} \right\}^{\frac{b-\delta}{2b-\delta}} \left\{ \frac{2\mu+b+\delta+1}{2\mu+2b+\delta+1} \right\}^{b-\delta}.
 \end{aligned} \tag{25}$$

Note that this approximated Eq. (25) is only used for analytic expressions of the first derivatives in Eqs. (27)-(30), and (32). Full equation Eq. (23) is used for all other derivations. We take the natural logarithm of Eq. (25),

$$\begin{aligned}
 \ln r_{eff} &= \ln \frac{3}{4\rho_i} + \frac{\delta}{2b-\delta} \ln a - \frac{b}{2b-\delta} \ln \gamma + \left(\frac{b-\delta}{2b-\delta} \right) \ln \left(\frac{Z}{k_{ext}} \frac{\pi^2 \rho_i^2}{18f_{Mie}} \right) + (b-\delta) \\
 &\quad \ln \left(\frac{2\mu+b+\delta+1}{2\mu+2b+\delta+1} \right),
 \end{aligned} \tag{26}$$

where Z and k_{ext} are known values since they are assumed to be available from the radar and lidar measurements (Section 2.3). We assume that μ is not a function of a , b , γ , and δ (i.e. the size distribution does not depend on particle shape) and take derivatives of r_{eff} with respect to a , b , γ , and δ . These derivatives can be interpreted as a sensitivity of r_{eff} to assumption of particle shape factor, in terms of a , b , γ , and δ . The first derivatives of Eq. (26) with respect to a , b , γ , and δ are

$$\frac{\partial(\ln r_{eff})}{\partial a} = \frac{\delta}{2b-\delta} \frac{1}{a}, \tag{27}$$

$$\begin{aligned}
 \frac{\partial(\ln r_{eff})}{\partial b} &= \frac{\delta}{(2b-\delta)^2} \ln \left(\frac{Z}{k_{ext}} \frac{\pi^2 \rho_i^2}{18f_{Mie} a^2} \gamma \right) + \ln \left(\frac{2\mu+b+\delta+1}{2\mu+2b+\delta+1} \right) \\
 &\quad - \frac{(b-\delta)(2\mu+\delta+1)}{(2\mu+b+\delta+1)(2\mu+2b+\delta+1)},
 \end{aligned} \tag{28}$$

$$\frac{\partial(\ln r_{eff})}{\partial \gamma} = -\frac{1}{\gamma} \frac{b}{2b-\delta}, \text{ and} \tag{29}$$

$$\begin{aligned}
 \frac{\partial(\ln r_{eff})}{\partial \delta} &= -\frac{b}{(2b-\delta)^2} \ln \left(\frac{Z}{k_{ext}} \frac{\pi^2 \rho_i^2}{18f_{Mie} a^2} \gamma \right) - \ln \left(\frac{2\mu+b+\delta+1}{2\mu+2b+\delta+1} \right) \\
 &\quad + \frac{b(b-\delta)}{(2\mu+b+\delta+1)(2\mu+2b+\delta+1)}.
 \end{aligned} \tag{30}$$

Equation (27) > 0 , Eq. (28) < 0 , Eq. (29) < 0 , and Eq. (30) > 0 , because $a > 0$, $b > \delta > 0$, $\gamma > 0$, and $0 < \frac{Z}{k_{ext}} \frac{\pi^2 \rho_i^2}{18f_{Mie} a^2} \gamma < 1$. Therefore, r_{eff} increases with increasing a , decreasing b , decreasing γ , or increasing δ .

Figure 4 shows the sensitivity of r_{eff} for changing a , b , γ , and δ by $\pm 10\%$ using Eq. (23). We set reference values of a , b , γ , and δ using *Brown and Francis* (case (3) of Table 1). Then two of four parameters a , b , γ , and δ are perturbed by 10% from the reference values in each panel of Fig. 4. We consider two values of Z/k_{ext} in Eq. (23), 10^{-10} and 10^{-6} cm⁴, which are, respectively, the lower and upper limit of a typical range (Fig. 3e). Also, μ is fixed as -1 in Fig. 4. The sensitivity of r_{eff} to μ is separately examined in Section 3.2. In addition, f_{Mie} is fixed as 1 in Fig. 4. For $D_g = 100$ μ m in the lognormal PSD, f_{Mie} is around 0.9 (Section 2.3). If we use f_{Mie} of 0.9 instead of 1, r_{eff} shows almost the same sensitivity to a , b , γ , and δ (not shown).

Figure 4 shows that retrieved r_{eff} increases with increasing a , with decreasing b , with decreasing γ , or with increasing δ , which are consistent with signs of Eqs. (27)–(30). Both $Z/k_{ext} = 10^{-10}$ and 10^{-6} cm⁴ show almost the same sensitivity of r_{eff} to a , b , γ , and δ . The 10% changes of a , b , γ , and δ can change r_{eff} by more than 100% (each panel of Fig. 4). In particular, r_{eff} is more sensitive to b and δ , in comparison to a and γ , simply because b and δ are exponents for mass and projected area distributions, while a and γ are scaling factors.

Figure 5 represents computed r_{eff} using Eq. (23) for different sets of a , b , γ , and δ listed in Table 1. Similar to Fig. 4, μ is fixed as -1 , but Z/k_{ext} changes from 10^{-10} to 10^{-6} cm⁴. In addition, f_{Mie} is assumed to be 1 in Fig. 5. When f_{Mie} is assumed to be 0.9 (Section 2.3), the retrieved r_{eff} is 1.5–3% larger than r_{eff} with $f_{Mie} = 1$ (not shown). This is simply because r_{eff} is proportional to $f_{Mie}^{-(b-\delta)/(2b-\delta)}$ in Eq. (23), while $-(b-\delta)/(2b-\delta)$ changes between -0.15 and -0.25 depending on m-D and A-D relationships.

The vertical spread of curves in Fig. 5 is basically the uncertainty in the retrieved r_{eff} due to ice particle shape (a , b , γ and δ) assumptions. When $Z/k_{ext} < 10^{-7}$ cm⁴, the particle shape of Heymsfield et al. [2013] at $T = -60^\circ$ C gives the largest r_{eff} , while plates and bullets of Yang et al. [2000] give the smallest r_{eff} . For $Z/k_{ext} < 10^{-8}$ cm⁴, r_{eff} derived with Heymsfield et al. [2013] at the temperature of -60° C is almost twice of r_{eff} derived with plates or bullets of Yang et al. [2000]. These results are consistent with Fig. 2, in which size distribution is not considered (i.e. mono-disperse). This suggests that relative changes of r_{eff} due to different a , b , γ , and δ might not be limited to a specific PSD assumption.

A similar type of comparisons to those in this section was performed by Donovan and Van Lammeren [2001]. Figure 10 of Donovan and Van Lammeren [2001] shows that an assumption of spherical particles leads to the largest r_{eff} , while compact polycrystal leads to the smallest r_{eff} for the given r_{eff}' , where r_{eff}' is defined from the radar-to-lidar ratio.

The sensitivity of IWC to ice particle shape can be computed by multiplying Eqs. (27)–(30) by $\partial(\ln IWC)/\partial(\ln r_{eff})$. Note that IWC , k_{ext} , and r_{eff} are related by Eq. (13), and k_{ext} is fixed because it is known from the lidar measurements. Therefore, $\partial(\ln k_{ext}) = 0$, and $\partial(\ln IWC) = \partial(\ln r_{eff})$ or $\partial(\ln IWC)/\partial(\ln r_{eff}) = 1$. This suggests that IWC has the same sensitivity to a , b , γ , and δ as r_{eff} .

3.2. Sensitivity of r_{eff} to assumption of μ

When the sensitivity of r_{eff} to m-D and A-D relationships (in terms of a , b , γ , and δ) is analyzed in Section 3.1, μ is fixed as -1 . Data from field campaigns suggest that μ varies between -2 to 10 [Heymsfield et al., 2002, 2013; Patade et al., 2015; Hou et al., 2014]. In this section, we examine how μ in the gamma PSD influences the solution of r_{eff} .

Instead of fixing μ as -1 , we can simultaneously retrieve μ along with other parameters. The problem with this approach is that we have three unknowns, N_0 , μ , and Λ , to fully describe the gamma PSD, but we only have two measured values of radar reflectivity factor (Z) and visible extinction (k_{ext}). This means that N_0 , μ , and Λ are not uniquely determined. As a result, the radar and lidar algorithm requires additional information to constrain the solution of N_0 , μ , and Λ . Since our derivation of r_{eff} in Eq. (23) includes μ , we can use a relationship between μ and temperature based on in-situ measurements [Heymsfield et al., 2013]:

$$\mu = -0.84 - 0.0915 T - 2.936 \times 10^{-3} T^2 - 3.653 \times 10^{-5} T^3 - 2.157 \times 10^{-8} T^4, \quad (31)$$

where T is the temperature in Celsius between -86°C and 0°C . Note that in Fig. 9 of Heymsfield et al. [2013], actual μ deviates up to ± 2 from the temperature-based value in Eq. (31). This suggests that constraining μ with Eq. (31) brings uncertainties of μ by ± 2 .

The sensitivity of r_{eff} to μ can be obtained from the first derivative of r_{eff} with respect to μ :

$$\frac{\partial \ln r_{eff}}{\partial \mu} = \frac{(b - \delta)}{(2\mu + b + \delta + 1)} \frac{2b}{(2\mu + 2b + \delta + 1)}. \quad (32)$$

Eq. (32) is positive, and only a function of μ , b and δ , but not a and γ . The sensitivity increases with increasing b , decreasing δ , or decreasing μ . If $b = \delta$ or $b = 0$, Eq. (32) is zero, and the solution of r_{eff} is not affected by the choice of μ . These conditions are, however, unrealistic (Appendix B).

Figure 6 shows how much r_{eff} changes when μ is increased by 2, considering the actual μ can deviate from temperature-based μ (Eq. (31)) by up to ± 2 . In addition, f_{Mie} is assumed to be 1 because f_{Mie} does not change $r_{eff}(\mu + 2)/r_{eff}(\mu)$. In Fig. 6, the sensitivity of r_{eff} to μ [$= \partial(\ln r_{eff})/\partial \mu$] is larger for a smaller μ , which is consistent with Eq. (32).

Among m-D and A-D relationships in Fig. 6, the ice mixture by Yang et al. [2000] shows the largest sensitivity, while the particle shape of Heymsfield et al. [2013] at -60°C shows the smallest sensitivity. This is because the mixture by Yang et al. [2000] has the largest coefficient b , while the particle shape of Heymsfield et al. [2013] at -60°C has the smallest b , which essentially determines the magnitude of Eq. (32).

In Fig. 6, a large uncertainty of r_{eff} occurs for $\mu_0 < 0$, resulting ratios of $r_{eff}(\mu = \mu_0 + 2)$ to $r_{eff}(\mu = \mu_0) > 1.2$. This means that $>20\%$ errors in r_{eff} are expected when increasing μ by 2. However, when μ_0 is positive in Fig. 6, most of the shapes show the ratio less than 1.1 ($<10\%$ errors in r_{eff}). The negative disperse (μ) means a sub-exponential particle size distribution, which is often associated with small ice particles or smaller Λ [Patade et al.,

2015]. In other words, when ice clouds are predominantly composed of larger ice particles, $\mu > 0$ and r_{eff} is relatively insensitive to the assumption of μ . In addition, Fig. 9b of Patade et al. [2015] shows a strong relationship between μ and Λ for subdivided temperature ranges. This suggests that the uncertainty of r_{eff} due to the assumption of μ can be significantly reduced if the relationship between μ and Λ is used in the retrievals.

4. Analytic derivation using a lognormal PSD

In this section, we derive size-integrated optical parameters using a lognormal PSD, and the results are compared with those from the gamma PSD (Section 3). We consider the lognormal PSD as follows:

$$N(D) = N_T \frac{1}{\sqrt{2\pi\omega D}} \exp\left[-\frac{(\ln D - \ln D_g)^2}{2\omega^2}\right]. \quad (33)$$

where N_T is a total number of particles in a unit volume (cm^{-3}), D_g is a geometrical diameter (cm), and ω is a width parameter (unitless). The j th Moment Generating Function (MGF) of the lognormal distribution is given by

$$M_j = \int N(D) D^j dD = N_T D_g^j \exp\left(\frac{1}{2} j^2 \omega^2\right). \quad (34)$$

If we apply Eq. (34) to Eqs. (10)-(13), we get

$$Z = \frac{36 f_{Mie}}{\pi^2 \rho_i^2} a^2 N_T D_g^{2b} \exp(2b^2 \omega^2), \quad (35)$$

$$k_{ext} = 2\gamma N_T D_g^\delta \exp\left(\frac{1}{2} \delta^2 \omega^2\right), \quad (36)$$

$$IWC = a N_T D_g^b \exp\left(\frac{1}{2} b^2 \omega^2\right), \text{ and} \quad (37)$$

$$r_{eff} = \frac{3}{4\rho_i} \frac{a}{\gamma} D_g^{b-\delta} \exp\left(\frac{1}{2} (b^2 - \delta^2) \omega^2\right). \quad (38)$$

The ratio Z/k_{ext} can be expressed as a function of a , b , γ , δ , D_g , and ω :

$$\frac{Z}{k_{ext}} = \frac{18 f_{Mie} a^2}{\pi^2 \rho_i^2 \gamma} D_g^{2b-\delta} \exp\left(\frac{4b^2 - \delta^2}{2} \omega^2\right). \quad (39)$$

Note that for a given D , impacts of D_g and ω on $N(D)$ largely offset since $N(D)$ increases with increasing D_g and with decreasing ω . We can eliminate one of D_g and ω using Eq. (39). Rearranging Eq. (39), we get

$$D_g = \left\{ \frac{z}{k_{ext}} \frac{\pi^2 \rho_i^2 \gamma}{18 f_{Mie} a^2} \right\}^{\frac{1}{2b-\delta}} \exp\left(-\frac{2b+\delta}{2} \omega^2\right). \quad (40)$$

Combining Eqs. (38) and (40) results in

$$r_{eff} = \frac{3}{4\rho_i} \frac{a}{\gamma} \left\{ \frac{Z}{k_{ext}} \frac{\pi^2 \rho_i^2 \gamma}{18 f_{Mie} a^2} \right\}^{\frac{b-\delta}{2b-\delta}} \exp\left(-\frac{b}{2}(b-\delta)\omega^2\right). \quad (41)$$

Equation (41) becomes a function of a , b , γ , δ , and ω , while D_g is eliminated in the equation. By taking the natural logarithm of Eq. (41),

$$\ln r_{eff} = \ln\left(\frac{3}{4\rho_i}\right) + \frac{\delta}{2b-\delta} \ln a - \frac{b}{2b-\delta} \ln \gamma + \frac{b-\delta}{2b-\delta} \ln\left\{ \frac{Z}{k_{ext}} \frac{\pi^2 \rho_i^2}{18 f_{Mie}} \right\} - \frac{1}{2}(b^2 - b\delta)\omega^2. \quad (42)$$

As in Section 3, we get the first derivatives of r_{eff} with respect to a , b , γ , and δ .

$$\frac{\partial(\ln r_{eff})}{\partial a} = \frac{\delta}{2b-\delta} \frac{1}{a} > 0, \quad (43)$$

$$\frac{\partial(\ln r_{eff})}{\partial b} = \frac{\delta}{(2b-\delta)^2} \ln\left\{ \frac{Z}{k_{ext}} \frac{\pi^2 \rho_i^2 \gamma}{18 f_{Mie} a^2} \right\} - \frac{\omega^2}{2}(2b-\delta) < 0, \quad (44)$$

$$\frac{\partial(\ln r_{eff})}{\partial \gamma} = -\frac{b}{2b-\delta} \frac{1}{\gamma} < 0, \text{ and} \quad (45)$$

$$\frac{\partial(\ln r_{eff})}{\partial \delta} = -\frac{b}{(2b-\delta)^2} \ln\left\{ \frac{Z}{k_{ext}} \frac{\pi^2 \rho_i^2 \gamma}{18 f_{Mie} a^2} \right\} + \frac{\omega^2}{2} > 0. \quad (46)$$

Equations (43)-(46) show consistent signs to those found in Eqs. (27)-(30). In addition, Eqs. (43) and (45) are equal to Eqs. (27) and (29), respectively. This means that sensitivity of r_{eff} to a and γ are the same when either gamma or lognormal PSD is used. In contrast, the sensitivity of r_{eff} to b and δ depends on μ in the gamma PSD and ω in the lognormal PSD.

As in the gamma PSD, the lognormal PSD has three unknown parameters N_T , D_g , and ω , while we only have two measured parameters as k_{ext} and Z . Therefore, a unique solution of

N_b , D_g , and ω does not exist, and the retrieval algorithm requires additional information about N_b , D_g , or ω . Since our expression of r_{eff} in Eq. (41) is a function of ω , we can use in-situ measurements of ω to constrain the solution [e.g., Tian et al., 2010, Austin et al., 2009]. For example, Austin et al. [2009] set up a *a priori* value of ω depending on air temperature in CloudSat 2B-CWC algorithm:

$$\omega = 0.694582 + 0.00650884T, \quad (47)$$

where T is the temperature in Celsius. Figure 7 shows that a *a priori* value of ω and retrieved ω by the 2B-CWC algorithm. For each temperature level, retrieved ω deviates from a *a priori* value (red line in Fig. 7) by about 0.1. Therefore, when we use the temperature-based ω in Eq. (47), the uncertainty of ω is about 0.1, and it also causes the uncertainty in r_{eff} . The sensitivity of r_{eff} to the assumption of ω is quantified by

$$\frac{\partial(\ln r_{eff})}{\partial \omega} = -b(b - \delta)\omega. \quad (48)$$

Equation (48) is negative, and the magnitude increases with increasing ω , increasing b , or decreasing δ . If $b = \delta$ or $b = 0$, Eq. (48) becomes zero, and the solution of r_{eff} is not affected by choice of ω , but these conditions are unrealistic (Appendix B).

Figure 8 shows changes of r_{eff} when ω is increased by 0.1. As in Fig. 6, f_{Mie} is fixed as 1 because f_{Mie} does not change $r_{eff}(\omega+0.1)/r_{eff}(\omega)$. When ω is larger, the sensitivity of r_{eff} to ω is larger, which is consistent with Eq. (48). In addition, among m-D and A-D relationships used in Fig. 8, the mixture of Yang et al. [2000] shows the largest sensitivity (the largest deviation of ratio from 1), and the particle shape of Heymsfield et al. [2013] at -60°C shows the smallest sensitivity. This is consistent with those found in Section 3.2 with the gamma PSD. Generally, uncertainties of r_{eff} related to the assumption of ω are smaller than 20% for all particle shapes.

5. Conversion of r_{eff}

In this section, we use analytical relationships derived in Sections 3 and 4 to demonstrate the conversion of r_{eff} derived with different particle shapes (Section 5.1) or PSD (Section 5.2) assumptions. In Section 5.3, we discuss a more general case that both particle shape and PSD are different between two radar-lidar algorithms.

5.1. Conversions of r_{eff} when different particle shapes are used in the gamma PSD

If two retrieval algorithms use the same gamma PSD, but assume different particle shapes (a , b , γ and δ), retrieved effective radii would differ as shown in Fig. 5. Let us assume that $r_{eff,1}$ is retrieved from a coefficient set of a_1 , b_1 , γ_1 , and δ_1 , and $r_{eff,2}$ is retrieved from a coefficient set of a_2 , b_2 , γ_2 , and δ_2 . We also assume that both algorithms use the same value μ . If we want to convert $r_{eff,1}$ into $r_{eff,2}$, we can use analytic expressions discussed in Section 3. First, we can express Z/k_{ext} with $r_{eff,1}$, a_1 , b_1 , γ_1 , and δ_1 using Eq. (23) as follows:

$$\frac{Z}{k_{\text{ext}}} = \left\{ r_{\text{eff},1} \frac{4\rho_i\gamma_1}{3a_1} \frac{\Gamma(\delta_1 + \mu + 1)}{\Gamma(b_1 + \mu + 1)} \right\}^{\frac{2b_1 - \delta_1}{b_1 - \delta_1}} \frac{18f_{\text{Mie},1}}{\pi^2\rho_i^2} \frac{a_1^2}{\gamma_1} \frac{\Gamma(2b_1 + \mu + 1)}{\Gamma(\delta_1 + \mu + 1)}. \quad (49)$$

Combining Eqs. (23) and (49), $r_{\text{eff},2}$ can be further expressed with $r_{\text{eff},1}$, a_1 , b_1 , γ_1 , and δ_1 , as follows:

$$\begin{aligned} r_{\text{eff},2} &= \frac{3a_2}{4\rho_i\gamma_2} \frac{\Gamma(b_2 + \mu + 1)}{\Gamma(\delta_2 + \mu + 1)} \left\{ \frac{\pi^2\rho_i^2}{18f_{\text{Mie},2}} \frac{\gamma_2}{a_2^2} \frac{\Gamma(\delta_2 + \mu + 1)}{\Gamma(2b_2 + \mu + 1)} \right\}^{\frac{b_2 - \delta_2}{2b_2 - \delta_2}} \left\{ \frac{Z}{k_{\text{ext}}} \right\}^{\frac{b_2 - \delta_2}{2b_2 - \delta_2}} \\ &= r_{\text{eff},1} \frac{2b_1 - \delta_1 b_2 - \delta_2}{b_1 - \delta_1 2b_2 - \delta_2} \frac{3a_2}{4\rho_i\gamma_2} \frac{\Gamma(b_2 + \mu + 1)}{\Gamma(\delta_2 + \mu + 1)} \left\{ \frac{f_{\text{Mie},1}}{f_{\text{Mie},2}} \frac{a_1^2}{\gamma_1} \frac{\gamma_2}{a_2^2} \right\}^{\frac{b_2 - \delta_2}{2b_2 - \delta_2}} \\ &\times \left\{ \frac{\Gamma(\delta_2 + \mu + 1)}{\Gamma(2b_2 + \mu + 1)} \frac{\Gamma(2b_1 + \mu + 1)}{\Gamma(\delta_1 + \mu + 1)} \right\}^{\frac{b_2 - \delta_2}{2b_2 - \delta_2}} \left\{ \frac{4\rho_i\gamma_1}{3a_1} \frac{\Gamma(\delta_1 + \mu + 1)}{\Gamma(b_1 + \mu + 1)} \right\}^{\frac{2b_1 - \delta_1 b_2 - \delta_2}{b_1 - \delta_1 2b_2 - \delta_2}}. \end{aligned} \quad (50)$$

Eq. (50) gives a conversion formula from $r_{\text{eff},1}$ to $r_{\text{eff},2}$, or vice versa. Note that Eq. (50) becomes $r_{\text{eff},2} = r_{\text{eff},1}$, if two algorithms use the same set of a , b , γ , and δ (i.e. $a_1 = a_2$, $b_1 = b_2$, $\gamma_1 = \gamma_2$, $\delta_1 = \delta_2$) and Mie correction factor ($f_{\text{Mie},1} = f_{\text{Mie},2}$).

Figure 9 shows relationships between $r_{\text{eff},1}$ and $r_{\text{eff},2}$, when $r_{\text{eff},1}$ is retrieved from the m-D and A-D relations of *Brown and Francis* (case (3) of Table 1), while $r_{\text{eff},2}$ is retrieved from other m-D and A-D relationships shown in Table 1 (cases (4)-(11)). We also assume in Fig. 9 that the same Mie correction factor is used in two algorithms ($f_{\text{Mie},1} = f_{\text{Mie},2}$). In Fig. 9, we use two values of μ as 4.16 and -0.45 , corresponding temperature -75°C and -5°C based on Eq. (31). However, if other values of μ are used in the retrieval algorithms, the corresponding values should be used in Eq. (50) for the effective radius conversion.

Figure 9 shows that the impact of μ on the relationship between $r_{\text{eff},1}$ and $r_{\text{eff},2}$ is almost negligible, as long as the same μ is applied to $r_{\text{eff},1}$ and $r_{\text{eff},2}$, while $r_{\text{eff},1}$ significantly differs from $r_{\text{eff},2}$. For the given $r_{\text{eff},1}$, the spherical assumption or the ice particle shape by Heymsfield et al. [2013] at $T = -60^\circ\text{C}$ gives the largest $r_{\text{eff},2}$, while plates or bullets from Yang et al. [2000] give the smallest $r_{\text{eff},2}$. These results are consistent with those shown in Figs. 2 and 5.

Deng et al. [2013] showed that r_{eff} from DARDAR products [Delanoë and Hogan, 2008, 2010] is greater than r_{eff} from CloudSat 2C-ICE products [Deng et al., 2010, 2013, 2015]. The CloudSat 2C-ICE algorithm uses particle shape from the habit mixtures from Yang et al. [2000], while the DARDAR algorithm uses the particle shape from *Brown and Francis* [Brown and Francis, 1995, Francis et al., 1998]. Figure 9 shows that $r_{\text{eff},2}$ derived with the habit mixtures from Yang et al. [2000] is smaller than $r_{\text{eff},1}$ derived with the particle shape from *Brown and Francis*, for $r_{\text{eff},1} < 120 \mu\text{m}$. Considering the effective radius is typically smaller than $100 \mu\text{m}$, e.g., Fig. 10 of Deng et al. [2013], Fig. 9 is consistent with the result of Deng et al. [2013].

5.2. Conversions of r_{eff} when different PSDs are used but with the same particle shape

In this section, we assume that two algorithms use different PSDs (gamma versus lognormal) but use the same coefficients of a , b , γ , and δ . If $r_{eff,Gam}$ is retrieved with a gamma PSD, while $r_{eff,LN}$ is retrieved with a lognormal PSD, the conversion from $r_{eff,Gam}$ to $r_{eff,LN}$ can also be made using equations derived in Sections 3 and 4. Similar to the relationship derived in Section 5.1, Z/k_{ext} can be expressed with $r_{eff,Gam}$, a , b , γ , and δ as in Eq. (49). This can be used to express Z/k_{ext} in Eq. (41) as

$$\begin{aligned}
 r_{eff,LN} &= \frac{3}{4\rho_i} \frac{a}{\gamma} \left\{ \frac{\pi^2 \rho_i^2 \gamma}{18 f_{Mie,LN} a^2} \right\}^{\frac{b-\delta}{2b-\delta}} \exp \left[-\frac{b}{2}(b-\delta)\omega^2 \right] \left\{ \frac{Z}{k_{ext}} \right\}^{\frac{b-\delta}{2b-\delta}} \\
 &= \frac{3}{4\rho_i} \frac{a}{\gamma} \left\{ \frac{\pi^2 \rho_i^2 \gamma}{18 f_{Mie,LN} a^2} \right\}^{\frac{b-\delta}{2b-\delta}} \exp \left[-\frac{b}{2}(b-\delta)\omega^2 \right] \\
 &\quad \times \left\{ r_{eff,Gam} \frac{4\rho_i \gamma \Gamma(\delta + \mu + 1)}{3a \Gamma(b + \mu + 1)} \right\} \left\{ \frac{18 f_{Mie,Gam} a^2 \Gamma(2b + \mu + 1)}{\pi^2 \rho_i^2 \gamma \Gamma(\delta + \mu + 1)} \right\}^{\frac{b-\delta}{2b-\delta}} \\
 &= r_{eff,Gam} \exp \left[-\frac{b}{2}(b-\delta)\omega^2 \right] \frac{\Gamma(\delta + \mu + 1)}{\Gamma(b + \mu + 1)} \left\{ \frac{f_{Mie,Gam} \Gamma(2b + \mu + 1)}{f_{Mie,LN} \Gamma(\delta + \mu + 1)} \right\}^{\frac{b-\delta}{2b-\delta}}.
 \end{aligned} \tag{51}$$

Therefore, $r_{eff,LN}$ is directly proportional to $r_{eff,Gam}$ and the ratio is determined by both μ and ω . Figure 10 shows the ratio of $r_{eff,Gam}$ to $r_{eff,LN}$ for various combinations of μ and ω , while a , b , γ , and δ are from *Brown and Francis* (case (3) of Table 1). It is also assumed in Fig. 10 that the same Mie correction factor is used between two algorithms ($f_{Mie,LN} = f_{Mie,Gam}$). The ratio of $r_{eff,Gam}$ to $r_{eff,LN}$ is less than 1 for a smaller ω and μ , indicating $r_{eff,LN}$ is larger than $r_{eff,Gam}$. In contrast, the ratio is larger than 1 for a larger ω and μ , i.e. $r_{eff,Gam}$ is larger than $r_{eff,LN}$. In Eq. (51), $r_{eff,Gam}$ equals to $r_{eff,LN}$ when

$$\begin{aligned}
 \omega &= \sqrt{\frac{2}{b(b-\delta)} \ln \left\{ \frac{\Gamma(\delta + \mu + 1)}{\Gamma(b + \mu + 1)} \left[\frac{\Gamma(2b + \mu + 1)}{\Gamma(\delta + \mu + 1)} \right]^{\frac{b-\delta}{2b-\delta}} \right\}} \\
 &\approx \sqrt{\frac{2}{b} \ln \left\{ \frac{2\mu + 2b + \delta + 1}{2\mu + b + \delta + 1} \right\}}.
 \end{aligned} \tag{52}$$

The constant line of the ratio = 1 in Fig. 10 satisfies the condition of Eq. (52). Therefore, the retrieved r_{eff} from two algorithms are the same once 1) two algorithms use the same particle shape in terms of a , b , γ , and δ , and 2) μ of the gamma PSD and ω of the lognormal PSD satisfy the Eq. (52). Otherwise, Eq. (51) should be applied for converting r_{eff} derived with a gamma PSD into r_{eff} derived with a lognormal PSD, or vice versa.

5.3. Conversion of r_{eff} when different PSDs and particle shapes are used

In this section, we consider two algorithms that use different particle shapes and PSDs. Let us assume that $r_{eff,1}$ is retrieved from a set of coefficients a_1 , b_1 , γ_1 , and δ_1 and a gamma PSD, and $r_{eff,2}$ is retrieved from a set of coefficients a_2 , b_2 , γ_2 , and δ_2 and a lognormal PSD. Similar to Eq. (49), Z/k_{ext} can be expressed with $r_{eff,1}$, a_1 , b_1 , γ_1 , and δ_1 . Then Z/k_{ext} in Eq. (41) is substituted with Eq. (49), and we obtain

$$\begin{aligned}
r_{eff,2} &= \frac{3}{4\rho_i} \frac{a_2}{\gamma_2} \left\{ \frac{\pi^2 \rho_i^2 \gamma_2}{18 f_{Mie,2} a_2^2} \right\}^{\frac{b_2 - \delta_2}{2b_2 - \delta_2}} \exp\left(-\frac{b_2}{2}(b_2 - \delta_2)\omega^2\right) \left\{ \frac{Z}{k_{ext}} \right\}^{\frac{b_2 - \delta_2}{2b_2 - \delta_2}} \\
&= \frac{3}{4\rho_i} \frac{a_2}{\gamma_2} \left\{ \frac{a_1^2 \gamma_2 f_{Mie,1}}{\gamma_1 a_2^2 f_{Mie,2}} \right\}^{\frac{b_2 - \delta_2}{2b_2 - \delta_2}} \exp\left(-\frac{b_2}{2}(b_2 - \delta_2)\omega^2\right) \\
&\quad \times \left\{ r_{eff,1} \frac{4\rho_i \gamma_1 \Gamma(\delta_1 + \mu + 1)}{3a_1 \Gamma(b_1 + \mu + 1)} \right\}^{\frac{2b_1 - \delta_1 b_2 - \delta_2}{b_1 - \delta_1 2b_2 - \delta_2}} \left\{ \frac{\Gamma(2b_1 + \mu + 1)}{\Gamma(\delta_1 + \mu + 1)} \right\}^{\frac{b_2 - \delta_2}{2b_2 - \delta_2}}.
\end{aligned} \tag{53}$$

Equation (53) is a function of two sets of a , b , γ , and δ , as well as μ and ω . To simplify the relation in Eq. (53), we can use *priori* values of μ and ω that are used for r_{eff} retrievals such as Eq. (31) or (47).

In Fig. 11, we consider two temperatures, -75°C , and -5°C , and compute μ using Eq. (31) and ω using (47). This corresponds to $\mu = 4.16$ and -0.45 , and $\omega = 0.21$ and 0.66 , respectively. We also assume in Fig. 11 that the same Mie correction factor is used in two algorithms $f_{Mie,1} = f_{Mie,2}$. In Fig. 11, $r_{eff,1}$ is from m-D and A-D relationships of *Brown and Francis* (case (3) of Table 1) and a gamma PSD, while $r_{eff,2}$ is from other m-D and A-D relationships from Table 1 and a lognormal PSD. Compared to Fig. 9, two different temperatures produce significantly different relationships between $r_{eff,1}$ and $r_{eff,2}$. This is because of different dependencies of μ and ω on the temperature, i.e. Eq. (31) versus Eq. (47).

Deng et al. [2013] compared DARDAR with CloudSat 2B-CWC products, and they found that r_{eff} from 2B-CWC is larger by 0–30% (see Fig. 7 of the reference). In Fig. 11, 2B-CWC corresponds to $r_{eff,2}$ using a spherical assumption (red line), and DARDAR corresponds to $r_{eff,1}$. When the temperature is -75°C , $r_{eff,2}$ with a spherical assumption is 30% larger than $r_{eff,1}$ (solid red line). In contrast, when the temperature is -5°C , $r_{eff,2}$ with a spherical assumption is 10% smaller than $r_{eff,1}$ (dashed red line). Therefore, a diverse range (0–30%) of differences between DARDAR and 2B-CWC found in Deng et al. [2013] can be explained by the range of temperature. Note that other factors influence the differences between DARDAR and 2B-CWC r_{eff} because 2B-CWC uses radar only, while DARDAR uses radar and lidar. This study addresses the differences only caused by the assumption of particle shape and PSD. Equation (53) provides a possible conversion formula to overcome differences caused by particle shape and PSD assumptions.

6. Summary

This study analytically examines the impact of assumptions of ice particle shape on the effective radius derived from radar-lidar observations. We define the particle shape by four parameters, a , b , γ , and δ , expressing the relationships between mass and maximum diameter (m-D), and projected area and maximum diameter (A-D). The m-D and A-D relationships are expressed using power laws for analytic integration of mass and projected area over the particle size distribution (PSD). We use gamma and lognormal PSDs in computing size-integrated optical properties such as radar reflectivity factor (Z), visible

extinction coefficient (k_{ext}), effective radius (r_{eff}), and ice water content (IWC). Throughout the analysis, we assume that radar reflectivity factor and visible extinction are available, respectively, from radar and lidar measurements. We then express r_{eff} and IWC as functions of four parameters used in the m-D and A-D relationships.

Different particle shape assumptions used in earlier studies lead to different m-D and A-D relationships (Fig. 1 and Table 1). This also results in a significant difference of mass-to-area ratio, which is directly related to the effective radius (r_{eff}) (Fig. 2). Among relationships examined in this study, the particle shape from Heymsfield et al. [2013] for $T = -60^\circ\text{C}$ gives the largest effective radius, while plates and bullets defined by Yang et al. [2000] give the smallest effective radius for a given Z/k_{ext} . These are obtained either we assume mono-disperse particles (Fig. 2) or a gamma PSD (Fig. 5).

Effects of a , b , γ , and δ on cloud retrievals are also quantified using the first-order derivatives. The signs of the derivatives for gamma (Eqs. (27)–(30)) and lognormal (Eqs. (43)–(46)) PSDs are consistent. The results indicate that the effective radius increases with increasing a , decreasing b , decreasing γ , and increasing δ . Altering a , b , γ , and δ by 10% changes r_{eff} by more than 100% (Fig. 4). When we apply different m-D and A-D relationships shown in Table 1 (and thus different a , b , γ , and δ), the largest r_{eff} is almost twice as large as the smallest r_{eff} (Fig. 5). The sensitivity of IWC to a , b , γ , and δ is the same to r_{eff} .

Because most radar-lidar inversion methods retrieve a larger number of unknown parameters than the number of equations that can be set up from measurements, they quite depend on *a priori* assumption of parameters in PSD. Therefore, we also examine how r_{eff} is affected by the assumption of μ in gamma PSD. As μ increases, r_{eff} also increases ($\partial(\ln r_{eff})/\partial\mu > 0$ in Eq. (32)). In addition, the sensitivity of r_{eff} to μ increases with increasing b , decreasing δ , or decreasing μ (magnitude of Eq. (32)). In contrast, a and γ do not change the sensitivity of r_{eff} to μ . When μ is increased by a factor of 2, r_{eff} increases by 20–50% for $p < 0$, while r_{eff} increases by $< 10\%$ for $\mu > 0$ (Fig. 6). We also examine effects of ω on r_{eff} when a lognormal PSD is used. As ω increases, a smaller r_{eff} is obtained $\partial(\ln r_{eff})/\partial\omega < 0$ in Eq. (48). The sensitivity of r_{eff} to μ increases with increasing ω , increasing b , or decreasing δ (magnitude of Eq. (48)). Among m-D and A-D relationships considered in this study, the particle shape of Heymsfield et al. [2013] at the temperature of -60°C shows the smallest dependence of r_{eff} on μ and ω , while the ice mixture by Yang et al. [2000] shows the largest dependence (Figs. 6 and 8).

We demonstrate the conversion method of r_{eff} when different assumptions of particle shape and size distribution are used. First, we consider two retrieval algorithms that use the same gamma PSD, but assume different particle shapes, in terms of a , b , γ , and δ . The effective radii derived from these two algorithms are related by Eq. (50). The relationship is a function of two sets of a , b , γ , and δ and dispersion parameter (μ) of the gamma PSD. Different m-D and A-D relationships produce significant differences up to 100% in the retrieved r_{eff} (Fig. 9).

Second, we consider two retrieval algorithms that use different PSDs, i.e. gamma and lognormal PSDs, but use the same particle shape (a , b , γ , and δ). In this case, two values of r_{eff} from the gamma and lognormal PSDs are related to each other by Eq. (51). The ratio of r_{eff} depends on the dispersion parameter (μ) of the gamma PSD and the width parameter (ω) of the lognormal PSD (Fig. 10). When ω and μ are smaller (larger), a lognormal PSD leads to a larger (smaller) r_{eff} than r_{eff} derived with a gamma PSD (Fig. 10). The condition in which both PSDs derive the same r_{eff} is given by Eq. (52).

Third, we consider two algorithms that use different PSDs and particle shapes. The relation of r_{eff} is expressed with two sets of a , b , γ , δ , μ of the gamma PSD, and ω of the lognormal PSD (Eq. (53)). We can simplify this relation using *a priori* μ and ω used for r_{eff} retrievals. The relationship between two r_{eff} from two algorithms depends on temperature because μ and ω have different dependencies on the temperature change.

Throughout this study, we assume that the Mie correction factor is independent of maximum dimension, and it is treated as a constant scaling factor when integrating the radar backscatter cross section over the particle size distribution (Eq. (8)). In addition, ice bulk density (ρ_b) is assumed to be the density of solid ice (ρ_i , 0.917 g cm^{-3}) (Eqs. (7), (8)), following Sato and Okamoto [2006]. Future studies are needed related to assumptions of the Mie scattering correction and ice bulk density.

Results of this study can be used to convert r_{eff} derived with different particle shape and size distribution assumptions. Equations derived in this work provide an efficient way to avoid inconsistency between assumptions used in r_{eff} retrievals and forward radiative transfer computations. Particle shape and PSD assumptions used in retrievals are not necessarily correct. Making the same assumptions in radiative transfer computations, however, eliminates the error caused by inconsistent assumptions.

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Appendix A:: Error analysis of the asymptotic theory for a ratio of two gamma functions

According to Buri and Elezovic [2011] and Olver et al. [2010],

$$\left(\frac{\Gamma(x+p)}{\Gamma(x+q)}\right)^{\frac{1}{p-q}} \sim x + f_0 + \frac{1}{x}f_1 + \frac{1}{x^2}f_2 + \frac{1}{x^3}f_3 + \frac{1}{x^4}f_4 + o(x^{-5}), \quad (\text{A1})$$

where

$$f_0(p, q) = \frac{p + q - 1}{2}, \quad (\text{A2})$$

$$f_1(p, q) = \frac{1}{24} [1 - (p - q)^2], \quad (\text{A3})$$

$$f_2(p, q) = -f_0 f_1, \quad (\text{A4})$$

$$f_3(p, q) = \frac{f_1}{10} (10f_0^2 - 13f_1 - 1), \text{ and} \quad (\text{A5})$$

$$f_4(p, q) = \frac{-f_0 f_1}{10} (10f_0^2 - 39f_1 - 3). \quad (\text{A6})$$

To apply the asymptotic theory to $\Gamma(b + \mu + 1)/\Gamma(\delta + \mu + 1)$ in Eq. (23), we define

$$x = \mu + \delta + 1, \quad (\text{A7})$$

$$p = b - \delta, \text{ and} \quad (\text{A8})$$

$$q = 0. \quad (\text{A9})$$

Then Eq. (A1) can be expressed as

$$\begin{aligned} \left(\frac{\Gamma(\mu + b + 1)}{\Gamma(\mu + \delta + 1)} \right)^{\frac{1}{b - \delta}} &\cong (\mu + \delta + 1) + f_0 + \frac{1}{\mu + \delta + 1} f_1 + \frac{1}{(\mu + \delta + 1)^2} f_2 + \frac{1}{(\mu + \delta + 1)^3} f_3 \\ &+ \frac{1}{(\mu + \delta + 1)^4} f_4 + o((\mu + \delta + 1)^{-5}) \end{aligned} \quad (\text{A10})$$

Regarding $p - q = b - \delta$, Eqs. (A2)-(A6) become

$$f_0(b, \delta) = \frac{b - \delta - 1}{2}, \quad (\text{A11})$$

$$f_1(b, \delta) = \frac{1}{24} [1 - (b - \delta)^2], \quad (\text{A12})$$

$$f_2(b, \delta) = -f_0 f_1, \quad (\text{A13})$$

$$f_3(b, \delta) = \frac{f_1}{10}(10f_0^2 - 13f_1 - 1), \text{ and} \quad (\text{A14})$$

$$f_4(b, \delta) = \frac{-f_0f_1}{10}(10f_0^2 - 39f_1 - 3). \quad (\text{A15})$$

As $\mu + \delta + 1$ increases, the high-order terms converge to zero in Eq. (A10), and the equation can be approximated with a few terms. In other words, when $\mu + \delta + 1$ has the minimum value, ignoring the high-order terms leads the maximum uncertainty in Eq. (A10).

According to in-situ measurements [Heymsfield et al., 2002, 2013; Patade et al., 2015; Hou et al., 2014], δ is typically from -2 to 10 , and thus the minimum of $\mu + \delta + 1$ can be considered as $\delta - 1$. Table (A1) provides the magnitude of each term when the minimum value of $\mu + \delta + 1$ ($= \delta - 1$) is used. The sum of the first ($= \mu_{\min} + \delta + 1$) and second ($= f_0$) terms is $> 98\%$, $> 95\%$, $> 99\%$, $> 96\%$, and 100% of the true values of (A10), when a , b , γ , and δ are from *Brown and Francis*, plates of Yang et al. [2000], solid columns of Yang et al. [2000], ice mixtures of Yang et al. [2000], and spherical particles, respectively. This means that we can ignore the terms higher than the third orders with a less than 5% uncertainty for these m-D and A-D relationships. Neglecting terms higher than the third orders, Eq. (A10) can be approximated as

$$\left(\frac{\Gamma(\mu + b + 1)}{\Gamma(\mu + \delta + 1)}\right)^{\frac{1}{b - \delta}} \cong (\mu + \delta + 1) + \frac{b - \delta - 1}{2} = \mu + \frac{b + \delta + 1}{2}. \quad (\text{A16})$$

In a similar way, we can define x , p , and q for the approximation of $\Gamma(2b + \mu + 1)/\Gamma(\delta + \mu + 1)$ in the last term of Eq. (23), as follows:

$$x = \mu + \delta + 1, \quad (\text{A17})$$

$$p = 2b - \delta, \text{ and} \quad (\text{A18})$$

$$q = 0. \quad (\text{A19})$$

Then we get

$$\begin{aligned} \left(\frac{\Gamma(\mu + 2b + 1)}{\Gamma(\mu + \delta + 1)}\right)^{\frac{1}{2b - \delta}} &\cong (\mu + \delta + 1) + f_0 + \frac{1}{\mu + \delta + 1}f_1 + \frac{1}{(\mu + \delta + 1)^2}f_2 + \frac{1}{(\mu + \delta + 1)^3}f_3 \\ &+ \frac{1}{(\mu + \delta + 1)^4}f_4 + o((\mu + \delta + 1)^{-5}) \end{aligned} \quad (\text{A20})$$

Table A2 lists the magnitude of each term in Eq. (A20). The sum of the first ($= \mu_{\min} + \delta + 1$) and second ($= f_0$) terms is larger than the true value of (A20), and the difference is 12–16%.

This means that the approximation in Eq. (A20) has a larger uncertainty than the approximation in Eq. (A10). However, $\Gamma(2b + \mu + 1)/\Gamma(\delta + \mu + 1)$ has a smaller exponent $[= (b - \delta)/(2b - \delta)]$ than that $(= 1)$ of $\Gamma(b + \mu + 1)/\Gamma(\delta + \mu + 1)$ in Eq. (23). As a result, the approximation of $\Gamma(2b + \mu + 1)/\Gamma(\delta + \mu + 1)$ introduces a relatively smaller uncertainty, compared to the approximation of $\Gamma(b + \mu + 1)/\Gamma(\delta + \mu + 1)$ in Eq. (23).

To estimate total uncertainties by approximating $\Gamma(b + \mu + 1)/\Gamma(\delta + \mu + 1)$ and $\Gamma(2b + \mu + 1)/\Gamma(\delta + \mu + 1)$ in Eq. (23), we get the ratio as

$$\frac{R_{approx} - R_{true}}{R_{true}} \times 100\%, \quad (A21)$$

where

$$R_{true} = \frac{\Gamma(b + \mu + 1)}{\Gamma(\delta + \mu + 1)} \left\{ \frac{\Gamma(\delta + \mu + 1)}{\Gamma(2b + \mu + 1)} \right\}^{(b - \delta)/(2b - \delta)} \quad \text{and} \quad (A22)$$

$$R_{approx} = \left\{ \mu + \frac{b + \delta + 1}{2} \right\}^{b - \delta} \left\{ \mu + \frac{2b + \delta + 1}{2} \right\}^{-b + \delta}. \quad (A23)$$

Note that Eqs. (A22) and (A23) are used in Eqs. (23) and (25), respectively. In Fig. A1, $(r_{approx} - R_{true})/R_{true} \times 100\%$ is over 10% when $\mu = -2$. The error rapidly decreases with increasing μ , and the error is 2% for $\mu > 2$.

Figure A1.

Errors of R_{approx} relative to R_{true} as a function of the dispersion factor (μ) of a gamma particle size distribution (PSD). R_{approx} is from Eq. (A23) and R_{true} is from Eq. (A22). Different lines represent different sets of a , b , γ , and δ listed in Table 1 (cases (3)–(11)).

Table A1.

A magnitude of each term of Eq. (A10) with the minimum of $(\mu + \delta + 1)$ as $(\delta - 1)$.

Source of a, b, γ , and δ		(1) $(\mu_{\min} + \delta + 1)$	(2) f_0	(3) $f_1/(\mu_{\min} + \delta + 1)$	(4) $f_2/(\mu_{\min} + \delta + 1)^2$	(5) $f_3/(\mu_{\min} + \delta + 1)^3$	$\{(1)+(2)\} \div$ $\{\text{total sum of Eq. (A10)}\} \times 100\%$
Brown and Francis (1995) and Francis et al. (1998)	All D	0.96859	-0.08285	0.01307	0.00118	-0.00153	98.6%
Heymsfield et al. (2013)	$T = -30^\circ\text{C}$	0.61407	-0.23304	0.04851	0.01841	-0.01086	87.8%
	$T = -45^\circ\text{C}$	0.60273	-0.26787	0.05423	0.02410	-0.01056	84.4%
	$T = -60^\circ\text{C}$	0.64494	-0.32947	0.05709	0.02917	-0.00540	81.6%
Yang et al. (2000)	Plate	0.77561	-0.16327	0.02936	0.00618	-0.00502	95.3%
	Solid Column	0.86699	-0.04494	0.00825	0.00043	-0.00118	99.1%
	Bullet-6	0.71808	-0.10580	0.02196	0.00324	-0.00465	96.6%
	Mixture	1.46750	0.07443	-0.01162	0.00075	0.00068	100.9%
Sphere		1.00000	0.00000	0.00000	0.00000	0.00000	100.0%

Table A2.

A magnitude of each term of Eq. (A20) with the minimum of $(\mu + \delta + 1)$ as $(\delta - 1)$.

Source of a, b, γ , and δ		(1) $(\mu_{\min} + \delta + 1)$	(2) f_0	(3) $f_1/(\mu_{\min} + \delta + 1)$	(4) $f_2/(\mu_{\min} + \delta + 1)^2$	(5) $f_3/(\mu_{\min} + \delta + 1)^3$	$\{(1)+(2)\} \div$ $\{\text{total sum of (A20)}\} \times 100\%$
Brown and Francis (1995) and Francis et al. (1998)	All D	0.96859	1.31860	-0.52608	0.71619	-1.29037	112.4%
Heymsfield et al. (2013)	$T = -30^\circ\text{C}$	0.61407	0.84097	-0.42020	0.57546	-1.05045	116.1%
	$T = -45^\circ\text{C}$	0.60273	0.76564	-0.37381	0.47484	-0.80166	115.6%
	$T = -60^\circ\text{C}$	0.64494	0.66353	-0.28525	0.29347	-0.39736	113.3%
Yang et al. (2000)	Plate	0.77561	1.06130	-0.47007	0.64320	-1.17342	114.0%
	Solid Column	0.86699	1.34362	-0.60534	0.93713	-1.92279	114.1%
	Bullet-6	0.71809	1.14745	-0.57191	0.91387	-1.94153	115.9%
	Mixture	1.14675	1.72223	-0.68140	1.02335	-2.01143	112.2%
Sphere		1.00000	1.50000	-0.62500	0.93750	-1.85156	113.1%

Appendix B:: Slopes of constant lines of reff and Z/k_{ext} in a μ - Λ domain

In Section 3.2, we discuss that a unique solution of N_0 , μ , and Λ does not exist because the number of equations is smaller than the number of unknown parameters. Figure B1 further demonstrates that we cannot obtain a unique solution of r_{eff} from observed Z/k_{ext} , as a result

of multiple solutions of N_0 , μ , and Λ . In Figs. B1a and B1b, constant lines of r_{eff} and Z/k_{ext} are drawn in a $\mu - \Lambda$ domain, respectively. The m-D and A-D relationships are computed using *Brown and Francis* (case (3) of Table 1). Note that the lidar and radar measurements provide a value of Z/k_{ext} and solutions of μ and Λ exist along the constant line of Z/k_{ext} . If the contour lines of r_{eff} and Z/k_{ext} in Fig. B1 overlay in the $\mu - \Lambda$ domain, we get a single solution of r_{eff} for the given Z/k_{ext} . The slopes and intercepts of the constant lines of r_{eff} and Z/k_{ext} in Fig. B1 can be derived as follows. First, Eq. (20) can be rewritten as

$$\Lambda^{b-\delta} = \left(\frac{1}{r_{eff}} \frac{3a}{4\rho_i\gamma} \right) \frac{\Gamma(b+\mu+1)}{\Gamma(\delta+\mu+1)}. \quad (B1)$$

Using Eq. (24), Eq. (B1) can be approximated as

$$\Lambda = \left(\frac{1}{r_{eff}} \frac{3a}{4\rho_i\gamma} \right)^{\frac{1}{b-\delta}} \left(\mu + \frac{b+\delta+1}{2} \right). \quad (B2)$$

Equation (B2) is represented as $\Lambda = A_0(\mu - A_1)$ where

$$A_0 = \left(\frac{1}{r_{eff}} \frac{3a}{4\rho_i\gamma} \right)^{\frac{1}{b-\delta}} \quad \text{and} \quad (B3)$$

$$A_1 = -\frac{b+\delta+1}{2}. \quad (B4)$$

Therefore, a constant line of r_{eff} has A_0 as a slope, and A_1 as a μ -intercept in the $\mu - \Lambda$ domain (Fig. B1a). In the same way, Eq. (21) can be rewritten as

$$\Lambda = \left(\frac{k_{ext}}{Z} \frac{18f_{Mie}a^2}{\pi^2\rho_i^2\gamma} \right)^{\frac{1}{2b-\delta}} \left(\frac{\Gamma(2b+\mu+1)}{\Gamma(\delta+\mu+1)} \right)^{\frac{1}{2b-\delta}}. \quad (B5)$$

Using Eq. (24), Eq. (B5) is approximated as

$$\Lambda = \left(\frac{k_{ext}}{Z} \frac{18f_{Mie}a^2}{\pi^2\rho_i^2\gamma} \right)^{\frac{1}{2b-\delta}} \left(\mu + \frac{(2b+\delta+1)}{2} \right). \quad (B6)$$

Equation (B6) can be represented as $\Lambda = B_0(\mu - B_1)$ where

$$B_0 = \left(\frac{k_{ext}}{Z} \frac{18f_{Mie}a^2}{\pi^2\rho_i^2\gamma} \right)^{\frac{1}{2b-\delta}} \quad \text{and} \quad (B7)$$

$$B_1 = -\frac{(2b + \delta + 1)}{2}. \quad (\text{B8})$$

Above indicates that a constant line of Z/k_{ext} has B_0 as a slope, and B_1 as a μ -offset in the $\mu - \Lambda$ domain (Fig. B1b). Note that $|A_1| \leq |B_1|$ by comparing between Eqs. (B4) and (B8). Therefore, a constant line of Z/k_{ext} has a larger μ -offset than r_{eff} in the $\mu - \Lambda$ domain, as also shown in Fig. B1. In addition, by rearranging Eq. (25),

$$\left(\frac{k_{ext}}{Z} \frac{18f_{Mie} a^2}{\pi^2 \rho_i^2 \gamma} \right)^{\frac{1}{2b-\delta}} = \left(\frac{3a}{4\rho_i \gamma} \frac{1}{r_{eff}} \right)^{\frac{1}{b-\delta}} \left(\frac{2\mu + b + \delta + 1}{2\mu + 2b + \delta + 1} \right)^{\frac{1}{b-\delta}}. \quad (\text{B9})$$

Combining Eqs. (B3), (B7), and (B9), we get

$$B_0 = A_0 \left(\frac{2\mu + b + \delta + 1}{2\mu + 2b + \delta + 1} \right)^{\frac{1}{b-\delta}}. \quad (\text{B10})$$

For $b \neq 0$ and $b \neq \delta$, $\left(\frac{2\mu + b + \delta + 1}{2\mu + 2b + \delta + 1} \right)^{\frac{1}{b-\delta}} < 1$, and thus $A_0 > B_0$ in Eq. (B10). Therefore, a slope of the constant line of r_{eff} is larger than that of Z/k_{ext} , which is also found in Fig. B1. If $b = \delta$, $r_{eff} = 3a/(4\rho_i \gamma)$ from Eq. (23). In this case, r_{eff} is constant regardless of the choice of μ and Λ , which is the same condition for Eq. (32) = $\partial(\ln r_{eff})/\partial\mu = 0$ or Eq. (48) = $\partial(\ln r_{eff})/\partial\omega = 0$. For $b = 0$, $A_1 = B_1$ from Eqs. (B4) and (B8). Also $A_0 = B_0$ from Eq. (B10). This means that r_{eff} and Z/k_{ext} have the same slope and offset, and r_{eff} has a single solution for the given Z/k_{ext} regardless of the choice of μ and Λ . This is also consistent with Eq. (32) = 0 or Eq. (48) = 0. However, the ideal case of $b = 0$ or $b = \delta$ would not practically happen, because mass increases with the maximum dimension ($b > 0$), and also mass increases faster than projected area with the maximum dimension ($b > \delta$).

Figure B1.

The contour of constant values of (a) r_{eff} and (b) Z/k_{ext} in a $\mu - \Lambda$ domain. μ is a dispersion and Λ is a slope factor in a gamma particle size distribution (PSD) (Eq. 14). Equations (20) and (21) are used to compute r_{eff} and Z/k_{ext} , respectively. The particle shape of *Brown and Francis* for all D (case (3) of Table 1) is used for a , b , γ , and δ .

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Key points:

1. Ice particle shape determines m-D and A-D relations, which is used for radar-lidar retrievals.
2. Effective radius is a function of coefficients in m-D and A-D relations.
3. The conversion method of an effective radius is derived when different m-D and A-D are used.

Figure 1. Mass [$m(D)$] and projected area [$A(D)$] as a function of diameter (maximum linear dimension, D). Different lines represent nine sets of a , b , γ , and δ provided by cases (3)–(11) of Table 1.

Figure 2.

Relationships between $m(D)^2/A(D)$ and $m(D)/A(D)$. The radar-reflectivity-to-extinction-ratio with a particle size D is proportional to $m(D)^2/A(D)$, while the effective radius is proportional to $m(D)/A(D)$. Therefore, the relationship between $m(D)^2/A(D)$ and $m(D)/A(D)$ approximately equals to the relationship between radar-reflectivity-to-extinction-ratio and effective radius for a particle size D . Different lines represent nine sets of a , b , γ , and δ provided by cases (3)–(11) of Table 1.

Figure 3.

An example of Z/k_{ext} from CloudSat CPR and CALIPSO CALIOP measurements on 3 March 2011 20 UTC. (a) Gas-attenuation-corrected radar reflectivity (Z_{dB}) (dB) from CloudSat 2B-GEOPROF product. Equivalent radar reflectivity (Z_e) in Eq. (8) is related to Z_{dB} as $Z_{dB} = 10 \log Z_e$. (b) Cloud extinction coefficient k_{ext} (km^{-1}) from CALIPSO CPROF product. (c) Distribution of $\log(Z/k_{ext})$ for ice clouds, where the ice clouds are defined for $k_{ext} > 0.01 \text{ km}^{-1}$ and air temperature < 253 K. Z in (c) is estimated from Z_e using Eq. (9).

(d) Scatter plot between Z_{dB} and $\log(Z/k_{ext})$ for ice clouds. (e) Histogram of Z/k_{ext} for ice clouds.

Figure 4.

Retrieved r_{eff} as a function of four parameters (a , b , γ , and δ) expressing m-D and A-D relationships (Eqs. (1) and (2)). Reference values of a , b , γ , and δ are set using *Brown and Francis* for all D (case (3) of Table 1). In each panel, two of four parameters (a , b , γ , and δ) are perturbed by 10%. All panels use a gamma particle size distribution (PSD) with the dispersion factor (μ) of -1 . Z/k_{ext} is set as 10^{-10} (black lines) and 10^{-6} cm^4 (red lines). f_{Mie} is fixed as 1 for this figure but note that $f_{Mie} = 0.9$ derives 1.5–3% larger r_{eff} than those with $f_{Mie} = 1$.

Figure 5.

Retrieved r_{eff} as a function of Z/k_{ext} for nine sets of a , b , γ , and δ provided in cases (3)–(11) in Table 1. A gamma particle size distribution (PSD) is used with assuming dispersion parameter (μ) as -1 . f_{Mie} is fixed as 1 for this figure but note that $f_{Mie} = 0.9$ derives 1.5–3% larger r_{eff} than those with $f_{Mie} = 1$.

Figure 6.

Ratio of r_{eff} derived with $\mu = \mu_0 + 2$ to r_{eff} derived with $\mu = \mu_0$, i.e. $r_{eff}(\mu_0 + 2)/r_{eff}(\mu_0)$. Different lines represent nine sets of a , b , γ , and δ provided by cases (3)–(11) of Table 1. Z/k_{ext} is fixed as 10^{-7} cm^4 . Note that f_{Mie} does not change $r_{eff}(\mu_0 + 2)/r_{eff}(\mu_0)$, and thus is fixed as 1.

Figure 7.

A priori ω (red solid line) and retrieved ω (frequency in color and average in black line) as a function of temperature from CloudSat 2B-CWC RO R04_E04 products. One track of CloudSat 2B-CWC RO data observed on 2 October 2008 19.00 UTC is used.

Figure 8.

Ratio of r_{eff} derived with $\omega = \omega_0 + 1$ to r_{eff} derived with $\omega = \omega_0$, i.e. $r_{eff}(\omega_0 + 0.1)/r_{eff}(\omega_0)$.

Different lines represent nine sets of a , b , γ , and δ provided in cases (3)–(11) of Table 1. Z/k_{ext} is fixed as 10^{-7} cm^4 . Note that f_{Mie} does not change $r_{eff}(\omega_0 + 0.1)/r_{eff}(\omega_0)$, and thus is fixed as 1.

Figure 9.

Relationships between $r_{\text{eff},1}$ and $r_{\text{eff},2}$. $r_{\text{eff},1}$ is the effective radius retrieved with a , b , γ , and δ of *Brown and Francis* for all D (case (3) of Table 1), and $r_{\text{eff},2}$ is the effective radius retrieved from other sets of a , b , γ , and δ in cases (4)-(11) of Table 1. Both $r_{\text{eff},1}$ and $r_{\text{eff},2}$ are retrieved using the same gamma particle size distribution (PSD). Two values of μ are considered at $T = -75^\circ\text{C}$ (solid line) and -5°C (dashed line) using Eq. (31). It is assumed that two algorithms use the same Mie correction factor ($f_{\text{Mie},1} = f_{\text{Mie},2}$). Grey solid line indicates the one-to-one line.

Figure 10.

The ratio of $r_{\text{eff,Gam}}$ to $r_{\text{eff,LN}}$, where $r_{\text{eff,Gam}}$ is an effective radius retrieved from a gamma PSD and $r_{\text{eff,LN}}$ is an effective radius retrieved from a lognormal PSD. The ratio is provided as a function of dispersion (μ) of the gamma particle size distribution (PSD) and width parameter (ω) of the lognormal PSD. Both $r_{\text{eff,Gam}}$ and $r_{\text{eff,LN}}$ use the same a , b , γ , and δ from *Brown and Francis* for all D (case (3) of Table 1). It is assumed that two algorithms use the same Mie correction factor ($f_{\text{Mie,Gam}} = f_{\text{Mie,LN}}$).

Figure 11.

Same as Fig. 9 except that $r_{\text{eff},2}$ uses a lognormal particle size distribution (PSD) instead of a gamma PSD. μ of the gamma PSD is computed with Eq. (31), and ω of the lognormal PSD is computed with Eq. (47) for temperatures at -75°C (solid lines) and -5°C (dashed lines). It is assumed that two algorithms use the same Mie correction factor ($f_{\text{Mie},1}$ $f_{\text{Mie},2}$).

Table 1.

Coefficients (a , b , γ , and δ) of m-D and A-D relationships derived in earlier studies. All variables are in cgs units; D in cm, $m(D)$ in gram, and $A(D)$ in cm^2 . Small D for Brown Francis corresponds to $D < 97 \times 10^{-4}$ cm for $m(D)$, and $D < 128 \times 10^{-4}$ cm for $A(D)$. Large D for Brown and Francis corresponds to $D > 97 \times 10^{-4}$ cm for $m(D)$, and $D > 128 \times 10^{-4}$ cm for $A(D)$.

	Ice habit/shape	$m(D) = aD^b$		$A(D) = \gamma D^\delta$		Case Number
		a (g cm^{-b})	b (unitless)	γ ($\text{g cm}^{2-\delta}$)	δ (unitless)	
Brown and Francis [1995] and Francis et al. [1998]	Small D	0.480140	3.00000	0.785398	2.00000	(1)
	Large D	0.002938	1.90000	0.026240	1.26667	(2)
	All D	0.145666	2.80290	0.650146	1.96859	(3)
Heymsfield et al. [2013]	$T = -30^\circ\text{C}$	0.005484	2.14800	0.116804	1.61407	(4)
	$T = -45^\circ\text{C}$	0.004513	2.06700	0.106844	1.60273	(5)
	$T = -60^\circ\text{C}$	0.003713	1.98600	0.125475	1.64494	(6)
Yang et al. [2000]	Plate	0.008210	2.44908	0.159987	1.77561	(7)
	Solid Column	0.086534	2.77712	0.313698	1.86699	(8)
	Bullet-6	0.004834	2.50649	0.076765	1.71809	(9)
	Mixture	0.497345	3.29561	0.847120	2.14675	(10)
Sphere		0.480140	3.00000	0.785398	2.00000	(11)