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Statistical characterization of the linear attenuation coefficient in polychromatic CT scans

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Abstract

Purpose: To provide a unifying statistical model that characterizes the integrated x-ray intensity at the detector after logarithmic transformation and can be extended to the characterization of computed tomography (CT) numbers in the reconstructed image.

Methods: We study the statistical characteristics of polyenergetic x-ray beams in the detector. Firstly, we consider the characterization of the integrated x-ray intensity at the detector through a probabilistic model (compound Poisson) that describes its statistics. We analyze its properties and derive the probabilistic distribution after the logarithmic transformation analytically. Finally, we propose a more tractable probabilistic distribution with the same features observed in the characterization, the noncentral Gamma (nc-Gamma). This distribution exhibits desirable properties for the statistical characterization across the reconstruction process. We assess the assumptions adopted in the derivation of the statistical models throughout Monte Carlo simulations and validate them with a water phantom and a lung phantom acquired in a Siemens clinical CT scan. We evaluate the statistical similarities between the theoretical distribution and the nc-Gamma using a power analysis with a Kolmogorov-Smirnov test for a 95% confidence level.

Results: The Kolmogorov-Smirnov goodness-of-fit test obtained for the Monte Carlo simulation shows an extremely high agreement between the empirical distribution of the post-logarithmic-integrated x-ray intensity and the nc-Gamma. The experimental validation performed with both phantoms confirmed the excellent match between the theoretical distribution, the proposed nc-Gamma, and sample distributions in all situations.

Conclusion: We derive an analytical model describing the post-log distribution of the linear attenuation coefficient in the sensor for polychromatic CT scans. We also demonstrate that the nc-Gamma distribution approximates well the theoretical distribution. This distribution also approximates well the CT numbers after reconstruction since it naturally extends across linear operations involved in filtered back projection reconstructions. This probabilistic model may provide the analytical foundation to define new likelihood-based reconstruction methodologies for polychromatic scans.

Keywords

CT; filtered backprojection; Gumbel distribution; iterative reconstruction; noncentral Gamma

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1. INTRODUCTION

The incorporation of noise statistics in CT reconstruction has shown important advantages in the reduction of radiation dose. However, choosing a proper approximate statistical model is still a critical step in the designing of iterative reconstruction methods. The polychromatic nature of the energy spectrum and the logarithmic transformation in the Beer-Lambert law inversion make the statistical models hard to define and their likelihood functions intractable. A tractable statistical model for the received energy can provide the basis for the development of statistical image reconstruction methods and more effective dosage-reduction techniques.¹⁻³

The commonly accepted model that describes the probabilistic distribution of energies for polyenergetic x rays is the so-called compound Poisson model, originally proposed by Whiting.⁴ This model results from assuming that the generation of individual quanta follows a Poisson distribution characterized by the mean number of quanta for a given energy, $I(E)$. The distribution of quanta generated across energies is described by its spectrum $\Phi_0(E)$, which is equivalent to its probability density function (PDF) when it is normalized.⁵

Under these conditions, the x-ray beam propagates and interacts with matter. The interaction leads to a reduction in the number of quanta reaching the detector. The random absorption caused by this interaction reduces the average amount of quanta following Beer-Lambert law for each energy:⁶

$$I(E) = I_0 \Phi_0(E) \exp\left(-\int_l \mu(x, E) dx\right) = I_0 \eta \Phi(E) \quad (1)$$

where $\mu(x, E)$ is the linear attenuation coefficient describing the fraction of x rays absorbed per unit thickness for a certain energy E , and $\Phi(E)$ is the normalized spectrum of the detected quanta, that is, the PDF of received energies. Note that the attenuation coefficient depends on the location, x , across the x-ray trajectory, denoted by l . Also, note that the spectra are normalized and, therefore, are equivalent to PDFs, so the equalities $\int_0^\infty \Phi_0(E) dE = 1$ and $\int_0^\infty \Phi(E) dE = 1$ must hold. As a consequence, the total amount of photons emitted is described by I_0 , and the received photons decrease by the *photon fluence transmission* factor, η (for simplicity, the detector efficiency is assumed to be unity).

For the sake of illustration, and to introduce the notation, we show in Fig. 1(a) the normalized spectra, $\Phi_0(E)$, of a CT scanner at $V_p = 140$ kVp and $V_p = 80$ kVp, and their corresponding attenuated spectra, $\Phi(E)$, after the interaction with 30 cm of water provided by Boone and Seibert.⁷ We also show the energies of the *equivalent monoenergetic beam* (the average energy of the spectrum), that is, $E_0 = \int_0^{V_p} E \Phi_0(E) dE$ and $E_1 = \int_0^{V_p} E \Phi(E) dE$.

In the detector, the energy received — the signal — depends on the number of quanta detected and their energies. More specifically, let us denote the signal as $S = \sum_{k=1}^N X_k$, where X_k accounts for the energy of the k th photon with probability density $\Phi(E)$. The number of received quanta, N , is in its turn a random variable that describes the number of

quanta received within a time window. This random variable is commonly modeled as a statistical counting process⁸ and follows a Poisson distribution with an average number of photons equal to $I_0 \eta$.

For each energy, the PDF of the detected signal, S , can be calculated by means of the Total Probability Theorem:⁹

$$f_S(E) = \sum_{n=0}^{\infty} f_S(E | N = n) p(N = n) \quad (2)$$

where $f_S(E|N = n)$ is the PDF of the sum of n independent and identically distributed random variables (so, the n -fold convolution of $\Phi(E)$ denoted as $\Phi^{n \otimes}(E)$):

$$f_S(E) = e^{-I_0 \eta} \sum_{n=0}^{\infty} \frac{(I_0 \eta)^n \Phi^{n \otimes}(E)}{n!} \quad (3)$$

This is the PDF of the celebrated compound Poisson model proposed by Whiting for polyenergetic spectra.⁴

Due to the difficulties in describing the probabilistic behavior of S , researchers commonly adopt simpler distributions that approximate well the compound Poisson distribution shown in Eq. (3). Xu and Tsui considered the compound Poisson distribution of Gamma distributions due to the features shared between the Gamma distribution and energy spectrum $\Phi(E)$ (lower bounded and right skewness).^{10,11} This way, the additive properties of the Gamma distribution simplify the n -fold convolution term of Eq. (3) to a Gamma density function.⁹ Probably, the most common approach is to assume a high signal-to-noise ratio (SNR) and consider a Gaussian distribution in virtue of the central limit theorem.^{4,5} This approach, however, does not meet the lower bound condition.

A more suitable approach can be adopted considering the good fitting of a single Gamma distribution to lower-bounded infinitely divisible distributions such as the compound Poisson distribution.^{12,13} The Gamma approximation also fits the high SNR scenario since it converges to a Gaussian distribution as the average photon energy increases.

The estimation of the attenuation coefficient in polychromatic x-ray beams makes use of the energy of the equivalent monoenergetic beam to provide a tractable mathematical derivation.¹⁴ This approach considers a known average emitted energy $\bar{S}_0 = \mathbb{E}\{S_0\} = I_0 E_0$ and assumes an energy E_1 equivalent to the energy received from a polyenergetic source in a given material. This latter assumption is taken to obtain an inversion formula of the Beer-Lambert law as defined in Eq. (1). Therefore, the average received energy becomes $\bar{S} = I_0 E_0 \exp(-\int_l \mu(x, E_1) dx)$. With these simplifications, Eq. (1) becomes:

$$\bar{S} = \bar{S}_0 \exp\left(-\int_l \mu(x, E_1) dx\right) \quad (4)$$

Now, Eq. (4) can be rearranged to calculate the line integral of the linear attenuation coefficient: $\int_I \mu(x, E_1) dx = \log(\bar{S}_0/\bar{S})$. Therefore, given a sample of the received energy S in the sensor, the estimator for the accumulated attenuation coefficient is $P = \log(\bar{S}_0/S)$, where P is a new random variable. At this point, the statistical characterization of the signal after the inversion of Beer-Lambert law, P , is not considered due to the difficulties of obtaining an analytical derivation and a manageable distribution that can be used for likelihood-based reconstructions or noise characterization of the CT numbers.²

In this work, we provide an analytical derivation of the probabilistic distribution before and after the logarithmic transformation. Additionally, we propose a versatile and tractable statistical model that not only characterizes post-log values but also can be applied throughout linear processes involved in the reconstruction up to the statistical characterization of CT numbers.

2. MATERIALS AND METHODS

2.A. Statistical model of the accumulated attenuation coefficient

To derive the probabilistic distribution describing the random variable $P = \log(\bar{S}_0/S)$ for a polychromatic spectrum $\Phi(E)$, we assume that the signal received in the detector $S = \sum_k X_k$ follows a compound Poisson distribution given in Eq. (3). We also assume that the compound Poisson distribution can be accurately described by a single Gamma distribution,^{12,13} with the following PDF:

$$f_S(s | a, b) = \frac{s^{a-1}}{\Gamma(a)b^a} \exp\left(-\frac{s}{b}\right), \quad s > 0, a > 0, b > 0 \quad (5)$$

where $\Gamma(\cdot)$ is the Euler's Gamma function, a is the shape parameter, and b is the scale parameter. In what follows, we will denote a random variable, X , to follow a Gamma distribution with parameters a and b as $X \sim \Gamma(a, b)$. These parameters are related to the moments of the compound Poisson as follows:

$$a = \frac{\mu_{\text{CompoundPoisson}}^2}{\sigma_{\text{CompoundPoisson}}^2} = NEQ; \quad b = \frac{\sigma_{\text{CompoundPoisson}}^2}{\mu_{\text{CompoundPoisson}}} \quad (6)$$

where $\mu_{\text{CompoundPoisson}}$ and $\sigma_{\text{CompoundPoisson}}^2$ account for the mean and variance of the compound Poisson random variable.

Note that the shape parameter a is equal to the *Noise Equivalent Quanta* (NEQ), a common measure used as a figure of merit that is equivalent to the output SNR squared.⁵ This model is consistent with the fact that, as the NEQ increases, the shape parameter increases as well, enforcing the Gamma to converge to a Gaussian distribution.

The estimation of the accumulated linear attenuation coefficient obtained after the logarithmic transformation, $P = \log(\bar{S}_0/S)$, results in a new random variable that, by virtue of the Change of Variable Theorem, has the following density function:⁹

$$f_P(p \mid \bar{S}_0, a, b) = \frac{\bar{S}_0^a e^{-ap} \exp(-\bar{S}_0 e^{-p/b})}{b^a \Gamma(a)} \quad (7)$$

This PDF is known as the *Gumbel* distribution with shape parameter a .¹⁵

2.A.1. Gumbel Distribution Properties—Some properties should be noted:

1. **Mode:** A simple analysis of the log-likelihood shows that the mode of the Gumbel distribution is located at $P = \log(\bar{S}_0/ab)$. So, considering that by definition $\bar{S} = \mu_{\text{CompoundPoisson}} = ab$, as shown in Eq. (6), the mode becomes:

$$\text{Mode}\{f_P(p \mid \bar{S}_0, a, \beta)\} = \log(\bar{S}_0/\bar{S}) \quad (8)$$

2. **Skewness:** The logarithmic transformation enforces a positive skewness in the PDF. This result was shown by Macovski through a second-order Taylor series of $\log(\bar{S}_0/x)$ at $\bar{S}$ and calculating its expectation:⁶

$$\mathbb{E}\left\{\log\left(\frac{\bar{S}_0}{S}\right)\right\} = \log\left(\frac{\bar{S}_0}{\bar{S}}\right) + \frac{\sigma_S^2}{2\bar{S}} + \dots > \log\left(\frac{\bar{S}_0}{\bar{S}}\right) \quad (9)$$

where σ_S^2 is the variance of the detected signal.

3. **Variance:** The variance of this distribution, σ_P^2 , can be calculated noting that the constant terms do not affect the variance calculation. In our specific case, where $P = \log(\bar{S}_0/S)$ with $S \sim \Gamma(a, b)$, the factors $\bar{S}_0$ and b are acting as location parameters. This is because the scale parameter, b , is multiplicative in the Gamma distribution (i.e., $\Gamma(a, b) \sim b\Gamma(a, 1)$). So, we can conclude that:

$$\log(\bar{S}_0/S) \sim \log(\bar{S}_0/b) - \log(\Gamma(a, 1)) \quad (10)$$

where $\log(\bar{S}_0/b)$ is a constant term. Therefore, the variance of a Gumbel distribution, σ_P^2 , can be calculated through the moment generating function of $\log(G)$ being $G \sim \Gamma(a, 1)$:

$$M_{\log G}(t) = \mathbb{E}\{e^{t \log G}\} = \mathbb{E}\{G^t\} = \frac{\Gamma(a+t)}{\Gamma(a)} \quad (11)$$

Finally, the variance of P is:

$$\sigma_P^2 = M''_{\log G}(0) - (M'_{\log G}(0))^2 = \psi'(a) \quad (12)$$

where $\psi'(a) = \frac{d^2}{da^2} \log \Gamma(a)$ is the so-called *trigamma* function, a monotonically decreasing function. Note that the variance depends only on the shape parameter,

which equals the NEQ. So, as the NEQ increases, the variance of the Gumbel decreases.

In this work, we propose a simpler and more manageable distribution that meets the main properties observed for the Gumbel distribution as described above: invariant mode at $\log(\bar{S}_0/\bar{S})$, right skewness, and variance equal to $\psi'(a)$.

2.B. Adopted probabilistic model: nc-Gamma

From the logarithmic transformation, all the operations involved in the reconstruction process (filtered projection and weighted backprojection) are linear. These operations imply that the final probability distribution is the result of the weighted convolution of the original ones.⁹ Although the Gumbel distribution describes the statistics of the sinogram, its PDF involves double exponential terms that make the propagation through the reconstruction process unpractical. For this reason, we seek a surrogate distribution with the same statistical features that makes the propagation of the analytical analysis feasible. We are especially interested in a distribution that is easier to model across the weighted convolutions involved within the reconstruction.

We propose to describe the sinogram intensities, P , by means of a noncentral Gamma distribution (nc- Γ), a three-parameter distribution with the following PDF:

$$f_{\Gamma}(x | \alpha, \beta, \delta) = \frac{(x - \delta)^{\alpha - 1}}{\Gamma(\alpha)\beta^{\alpha}} e^{-\frac{x - \delta}{\beta}}, \quad x > \delta, \alpha > 0, \beta > 0 \quad (13)$$

where α , β , and δ are the shape, scale, and location parameters, respectively.

This distribution does not imply a significant reduction in the descriptive power of the probabilistic model employed since it meets the conditions of right skewness, and we can impose the mode and variance of the Gumbel distribution. Our proposal also has desirable properties that make it suitable for the extension throughout the reconstruction processes:

1. The sums of nc- Γ random variables with the same scale and location parameters are nc- Γ distributed.¹⁶
2. The convolution of arbitrary Gamma variables results in a Gamma-like distribution described by a mixture of Gamma distributions.^{12,17}

Additionally, exploratory data analyses performed with phantoms in reconstructed CT scans showed that the nc- Γ distribution provides an excellent description of the statistical properties of the CT numbers for different dosages, manufacturers, and reconstruction methods.¹⁸

The described properties and the experimental evidence shown in reconstructed CT scans suggest that the nc- Γ is a versatile family of distributions that should be able to model the statistics of the sinogram intensities and the CT numbers of reconstructed images.

2.C. Simulation experiments

The hypotheses of the proposed statistical model were assessed via Monte Carlo simulations.

2.C.1. Compound Poisson model—We evaluated the goodness of fit of the compound Poisson distribution in a Monte Carlo simulation with 20 000 independent experiments for various values of NEQs. We employed the spectrum data provided by Boone and Seibert,⁷ where tungsten spectra were tabulated from 30 to 140 kV considering the measures taken by Fewell et al.¹⁹ and hardened to fit measured attenuation profiles (what Boone and Seibert denoted as *modified Fewell spectra*). The tissue attenuation was performed with a water-equivalent material for different thicknesses (0, 10, 20, and 30 cm). We employed the x-ray energy spectra after propagating through 30 cm of water depicted in Fig. 1 for 140 and 80 kVp.

Note that these data allow us to avoid any modeling of the matter-photon interactions since they are intrinsically included in the spectra. Therefore, the Monte Carlo simulations focus on the generation of photons with energies distributed as described in their corresponding spectra. The received signal was calculated as the sum of energies of all the surviving photons. The details of the simulation are given in Appendix A.

The comparison between the analytical and the empirical distribution was performed with the Kolmogorov-Smirnov goodness-of-fit test with a 0.05 significance. This test uses the uniform distance as the metric to compare the empirical cumulative distribution function (CDF) and the theoretical CDF:

$$d(F, F_N) = \sup_x \{|F(x) - F_N(x)|\} \quad (14)$$

being $F(\cdot)$ the theoretical distribution and $F_N(\cdot)$ the empirical CDF obtained from the data for N independent experiments (note that N is not related to the number of received photons $N(E)$). In our case, $N = 20\,000$).

2.C.2. Gamma approximation—After the validation of the compound Poisson model for the observed energies in the detector, we provided an analytical comparison of the Gamma distribution and the theoretical compound Poisson for an increasing NEQ using a similar approach. The Gaussian distribution was included in the comparison as the reference commonly assumed for high NEQ scenarios.^{4,5}

We used the uniform distance of Eq. (14) in the comparison since this metric is invariant to strictly monotone transformations of the random variable and, therefore, it allows us to extend the results of the analysis to the post-logarithmic distribution: the Gumbel distribution shown in Eq. (7).

2.C.3. The nc-Gamma model—Finally, we performed an analytical comparison of the nc- Γ and the Gumbel distributions through a power analysis. To do so, we normalized the post-logarithmic signal by its standard deviation given in Eq. (12). This new random variable, $P/\sqrt{\psi'(a)}$, has unitary variance, and its shape only depends on the NEQ.

The metric used for the comparison was the uniform distance defined in Eq. (14). The power analysis is accomplished by considering that $\sqrt{N}d(F, F_N)$ converges to a Kolmogorov distribution, where F_N is the sampling CDF, F is the theoretical distribution to be tested, and N is the number of samples required to provide a statistically significant difference. So, for certain confidence (95% in our case), we can calculate the critical value of the Kolmogorov distribution, $K_{95\%} = 1.3581$, and determine the number of samples as $N > (K_{95\%}/d(F, F_N))^2$.

2.D. Experimental validation

The assessment of the statistical distributions was performed in the sinograms acquired in two phantoms. First, we evaluate the proposed nc-Gamma distribution in a water phantom of 20 cm diameter. The water phantom was acquired with a reference current tube of 35 mA in a current modulation setting (effective current within the phantom ~ 9 mA), 120 kVp, pitch factor 1, CTDIvol 0.76, in a SOMATON Drive CT scanner (Siemens Healthcare, Germany). This scan provides 736 elements per row and 64 channels. A set of 5908 projections were acquired. The phantom was centered with the isocenter of the scanner. This way, the cylindrical shape of the phantom allows us to consider each projection as an independent realization of the photon detection for each row. Therefore, the dataset provides 5908 samples for each row, and 64 independent realizations (one per channel). The CT scan of the phantom and its sinogram are shown in Figs. 2(a)–2(b).

We perform the same analysis in a Catphan 698 lung phantom (CTP698, The Phantom Laboratory, www.phantomlab.com/catphans-copd) acquired in the same scanner with the same configuration: reference current tube of 35 mA in a current modulation setting, 120 kVp, pitch factor 1, CTDIvol 1.597. The CT scan of the phantom and its sinogram are shown in Figs. 2(b)–2(c).

We use the uniform distance of Eq. (14) to evaluate the Gumbel distribution, the proposed nc-Gamma, and the Gaussian distribution. The assessment was evaluated at different transverse thicknesses: 0, 7, 13, and 20 cm for the water phantom; and 0, 7, and 13 cm for the CTP698 phantom of the external ring. The ring is cast from a specially formulated urethane with an electron density above water. The sinogram samples corresponding to those thicknesses are represented in Figs. 2(b)–2(d). The samples for thickness = 0 cm were taken at the same distance from the isocenter to ensure the same statistical properties. Note that for the CTP698 phantom, the sampling must follow curved trajectories to ensure the same transverse thickness of the same material due to the elliptical shape. The correspondence between sinogram samples and thicknesses was calculated considering the phantom and scanner geometry, and object position. We refer the interested reader to Appendix B for more details.

3. RESULTS

3.A. Compound Poisson model

The results of the Kolmogorov-Smirnov goodness of fit test obtained for the Monte Carlo simulation are shown in Table I. The compound Poisson distribution exhibits an excellent match with the empirical distribution obtained after the simulation. The P -values obtained

for the goodness of fit test do not show any statistical difference between the empirical distribution and the compound Poisson distribution. This result confirms that the theoretical compound Poisson distribution accurately models the energies received in the detector in agreement with the results of Whiting.^{4,5}

In Figs. 3(a) and 3(b), we show the sample distribution of energies of the received x-ray spectrum after propagating through 30 cm of water along with the Gamma and Gaussian approximations for both 140 and 80 kVp. The single Gamma distribution fits well the overall behavior of the compound Poisson distribution for the whole range of NEQ, from very low SNR ($NEQ = 5$) where the distribution shows a distinctive skewness, to higher NEQ values where the Gaussian convergence can be assumed (the Gamma distribution converges to a Gaussian distribution as its shape parameter increases). Note that even for high NEQ values, the tails' behavior is better described by the Gamma distribution. This result is consistent with the literature due to the excellent approximation that a single Gamma distribution offers for the sum of independent gamma-like variables, as it is the case of the energy spectrum.¹³

Quantitatively, Fig. 3(c) shows the goodness-of-fit obtained with the uniform distance. The Gamma distribution outperforms the Gaussian approximation for both peak potentials even for higher NEQ values, where the Gaussian distribution is commonly assumed.

Additionally, the invariance to transformations provided by the uniform distance allows us to extend the conclusion to the Gumbel distribution after the logarithmic transformation. Figure 4 shows the fit of the Gumbel distribution to sample PDFs obtained for the previous Monte Carlo simulations after the logarithmic transformation and the CT number normalization to the Hounsfield scale ($CT = (p - p_{\text{water}})/p_{\text{water}} \times 1000$). Note that the mode remains invariant for any NEQ.

Interestingly, a small ripple effect can be observed in Fig. 4 for $NEQ = 5$ and $V_p = 80$ kV. This effect is the joint result of a narrow attenuated spectrum [see Fig. 1(b)] and the extremely low NEQ received: as the NEQ decreases, fewer components of the compound Poisson distribution become significant in Eq. (3), and they have a lower variance. In the extreme cases where the spectrum becomes monochromatic, the distribution would enhance these ripples until obtaining a discrete distribution: the Poisson distribution. Note, however, that the Gamma distribution models the envelope of this distribution and describes the main statistics of the received signal (mean and variance) adequately.

3.B. The nc-Gamma model

Figure 4 also shows the nc- Γ approximations (dashed lines) obtained from the same mode and variance as the Gumbel distribution. Both distributions share the main characteristics observed in the sample data concerning mode invariance and right skewness.

Finally, Fig. 5(a) shows the difference between normalized cumulative distributions, $|F_{\text{Gumbel}} - F_{\text{Gamma}}|$, for an increasing NEQ. The difference between both distributions is always below 0.003, which indicates the excellent fit of the nc- Γ CDF to the Gumbel CDF. Figure 5(b) shows the maximum distance between both $D_{KS} = \sup(|F_{\text{Gumbel}} - F_{\text{Gamma}}|)$ for

an increasing NEQ along with the estimated number of samples required to distinguish both distributions according to the Kolmogorov-Smirnov test. The monotonic decreasing nature of D_{KS} and the large number of samples required to distinguish both distributions confirm the excellent fitting of nc- Γ to the Gumbel distribution.

3.C. Experimental Validation

The goodness of fit of the Gumbel, nc-Gamma, and Gaussian distributions for both phantoms is shown in Table II. In the water phantom, the fitting was performed for the different transverse thicknesses with 5908 samples per thickness and 64 channels.

The CTP698 phantom employed 2034 samples per thickness and 64 channels. The estimated NEQ (eNEQ) was calculated by applying Eq. (12), which relates to the NEQ and the variance of the signal. Some illustrative examples of the fitting between the empirical and the proposed distributions are shown in Figs. 6 and 7 for both the post-log signal (top row) and pre-log signal (bottom row) calculated as the negative exponential of the attenuation and scaled to arbitrary units.

For the water phantom, the Gumbel distribution provides the best fit to the data with a uniform distance always lower than the ones obtained for the Gaussian distribution. The worst case obtained for the Gumbel distribution reaches a distance of 0.0136. A power analysis calculated for this distance shows that at least 9,972 samples would be needed to find a statistical difference between the empirical and the theoretical distribution for 95% confidence. In the best case (0.008), this number increases up to 28 819 samples.

The nc-Gamma distribution also exhibits an excellent approximation. Although the uniform distance is slightly higher, the skewness and the distribution are also well described. The only observable difference between the nc-Gamma and the Gumbel distributions lies in the slightly higher densities close to the mode. This effect is probably due to small differences in the kurtosis, a fourth-order moment of the distribution. Nevertheless, this difference decays as the transverse thickness increases. This fact is due to the more symmetric shape of the spectrum after propagating through water, leading to a better fitting of more symmetric distributions such as the Gaussian. Even in these more symmetric scenarios, the nc-Gamma distribution outperforms the Gaussian due to its asymptotic characteristics, which includes convergence to a Gaussian distribution from a positively skewed distribution.

The increase in the skewness as the attenuation grows is probably a consequence of the logarithmic transformation in the attenuated signal, which has a narrower PDF. Note that the reduction in the number of detected photons reduces the variance of the pre-log signal significantly. Therefore, the logarithmic transformation induces a lower skewness.

Results for the CTP698 phantom show a similar behavior: the Gumbel distribution gets the best fit with a uniform distance always lower than the Gaussian, and the nc-Gamma distribution approximates the Gumbel distribution. Compared to the water phantom, the uniform distances increase due to the lower number of samples available for the fitting. The examples shown in Fig. 7 exhibit the same trends in both post-log and pre-log signals.

4. DISCUSSION AND CONCLUSIONS

In this work, we provide a statistical characterization of the distributions involved in the estimation of the linear attenuation coefficient in polychromatic CT scans. Our derivation assumes the widely accepted compound Poisson process that describes the current-integrating process in x-ray detectors.

Using the compound Poisson process as a starting model, we demonstrate that the compound Poisson obtained from conventional x-ray spectra can be accurately approximated by Gamma distributions.^{12,13} This reasonable assumption allows us to extend the statistical characterization to the post-log transformation of the linear attenuation coefficient, where the post-log values are known to follow a non-Poisson distribution. As a result, we show that the post-log distribution of the sinogram intensities follows a Gumbel distribution. This distribution is characterized by a positive skewness, and its mode is invariant to the number of equivalent quanta.

An additional contribution of our work is the proposal of the noncentral Gamma (nc-Gamma) as distribution to characterize CT sinograms. The distribution shares multiple statistical properties with the Gumbel distribution, and it is more tractable. Our results, using Monte Carlo simulations of the x-ray statistical process and two different phantoms, show that the Gumbel distribution is in good agreement with the empirical samples. Results indicate that the Gumbel and the nc-Gamma are virtually indistinguishable, and they both share the main characteristics observed in the sample data (mode invariance and right skewness).

Our data also support the fact that the Gamma distribution is a good approximation of the well-known compound Poisson distribution for the detected x-ray energy before the logarithmic transformation. Since the uniform distance between CDFs used as a figure of merit in this work is invariant to transformations, the excellent match obtained for the Gumbel distribution also demonstrates that the Gamma approximation assumed for the compound Poisson distribution is accurate.

Since both the Gamma and nc-Gamma converge to a Gaussian distribution for high NEQ, they naturally generalize the Gaussian approximation usually adopted in high NEQ scenarios. Furthermore, the Monte Carlo simulations with increasing NEQ showed that the Gamma and nc-Gamma approximations outperform the Gaussian simplification even in for high NEQs. This fact confirms the suitability of the approximations used in the derivation of the Gumbel distribution and the characterization of post-log sinogram intensities.

The nc-Gamma is also manageable and leads to the extension of the statistical characterization across the linear processes involved in the reconstruction, such as the backprojection filtering. In particular, the convolution of arbitrary Gamma variables results in a Gamma-like distribution.^{12,17} This property is in agreement with the exploratory data analysis performed to characterize the linear attenuation coefficient in CT phantoms for different doses, reconstruction kernels, and manufacturers.¹⁸

The nc-Gamma could be postulated as a general distribution that works across the acquisition stages. This could enable the development of likelihood-based iterative reconstruction methods and the noise characterization in the reconstructed CT scans.
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Probably, the most straightforward application of these results is by considering the log-likelihood function, $\mathcal{L}$, of the post-log signal PDF (the Gumbel distribution) shown Eq. (7):

$$\mathcal{L} = \sum_i a_i \log\left(\frac{\bar{S}_0}{b_i}\right) - a_i p_i - \bar{S}_0 \frac{e^{-p_i}}{b_i} - \log \Gamma(a_i) \quad (15)$$

where we assume independence across detector bins (indexed by i), a common assumption in signal formation models.²² The calculation of the maximum likelihood estimates from Eq. (15) is no trivial task. However, given the already mentioned properties of the Gumbel distribution, some approaches might be applied. For instance, following the methodology introduced by Xu and Tsui,¹⁰ one could link Eq. (15) to the attenuation coefficients through the Gumbel distribution moments, and find a surrogate function that meets the principle of monotonic increase to find a local optimal.²³

In contrast, a remarkable result obtained in our theoretical derivation of the Gumbel distribution is the mode invariance to the number of equivalent quanta. This implies that the mode of the data can be used as an estimate for the detected signal. The nc-Gamma distribution offers an excellent parametric model to provide a robust estimator of the mode.¹⁸ This property can be incorporated in signal estimation approaches in the sinogram and reconstructed domains to harmonize the data across different reconstructions with varying levels of radiation dose.

It is worth mentioning that our methodology is not considering the beam-hardening correction stage in the statistical characterization. Beam-hardening correction in clinical scanners transforms the post-log intensities through a polynomial $f(p) = \sum_k w_k p^k$. The coefficients, w_k , are determined by a least-squares fit performed on samples taken in a water phantom for known transverse thickness.²⁴ This transformation introduces an additional nonlinear stage in the statistical characterization that changes the signal's probability distribution and might compromise the extension of the nc-Gamma distribution across the reconstruction. Nevertheless, the coefficients w_k decay rapidly, minimizing the impact in the shape of the statistical distribution. As an example, Herman reported in²⁵ the following cubic beam-hardening correction formula: $f(p) = -1.635 \cdot 10^{-4} + 0.943p + 0.0138 p^2 - 0.000527 p^3$.

Note that the nonlinear terms' contribution is about 1.5%, so we can expect a minimal change in the shape of the Gumbel and noncentral distributions. This was confirmed in the experimental validation section since the clinical CT scanner implements beam-hardening correction, and the Gumbel and nc-Gamma distributions show an excellent fit.

Bowtie filter normalization also modifies the spectrum for each x-ray trajectory across the field of view. However, unlike the beam-hardening correction, the spectrum's transformation

stems from the photon-matter interaction within the bowtie filter and, therefore, all the assumptions adopted in the detector for the compound Poisson and the subsequent derivations hold.

Similarly, the heel effect varies the x-ray intensity along the direction perpendicular to the CT gantry plane. The effect is the result of the different x-ray path length through the source.²⁴ Therefore, the spectrum modification is the consequence of photon absorption in the x-ray source and, as in the bowtie filter normalization, all the derivations provided in our work are applicable.

Quantization of ADC measurement and storage of logarithmic data also modify the statistics. However, we believe the quantization effects are negligible, considering the precision used in clinical CT scanners. In our case, the data were recorded in 32-bit floating-point format (precision = $1.19 \cdot 10^{-7}$).

In summary, our work paves the way for a set of methodologies that can exploit the properties of the nc-Gamma as a core statistical model. We believe that results obtained in this work may contribute to the development of both pre-log and post-log model-based iterative reconstruction methods.

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APPENDIX A

MONTE CARLO SIMULATION

The energies of received photons are simulated as independent events with energies distributed according to a normalized polychromatic tube spectrum $\Phi_0(E)$, its corresponding attenuated normalized spectrum $\Phi(E)$ after propagating across matter, and the photon fluence transmission factor η (i.e., the ratio between traversed photon fluence and emitted photon fluence). In our specific case, we employed the spectra shown in Fig. 1 for 140 and 80 kVp provided by Boone and Seibert⁷ with photon fluence transmission factors after 30 cm of water attenuation: $\eta^{140kVp,30cm} = 1.845 \cdot 10^{-3}$ and $\eta^{80kVp,30cm} = 6.738 \cdot 10^{-4}$.

The average number of emitted photons, I_0 , to achieve a certain NEQ after photon reduction by a factor η is calculated by considering the expected energy received and its variance of a compound Poisson:

$$\mathbb{E} \left\{ \sum_{k=1}^N X_k \mid \mathbb{E}\{N\} = \eta I_0 \right\} = \eta I_0 E_1, \quad (A1)$$

$$\text{Var}\left\{\sum_{k=1}^N X_k \mid \mathbb{E}\{N\} = \eta I_0\right\} = \eta I_0 E_2, \quad (\text{A2})$$

where N is the number of received quanta, X_k is the energy of the k th photon with probability density $\phi(E)$, and E_1 is the average energy of the normalized attenuated spectrum (see Fig. 1), and E_2 is the second-order moment of the normalized attenuated spectrum, that is, $E_2 = \int_0^V E^2 \phi(E) dE$. Note that the average number of received quanta, I_0 , is reduced by the photon fluence transmission factor η . For the spectra shown in Fig. 1, the moments before attenuation are the following (we overload the notation to emphasize the tube voltage and the length of transverse water):

$$\begin{aligned} E_1^{140kVp, 0cm} &= 55.471 & E_2^{140kVp, 0cm} &= 3615.714 \\ E_1^{80kVp, 0cm} &= 40.801 & E_2^{80kVp, 0cm} &= 1853.763 \end{aligned}$$

and after 30 cm of water attenuation:

$$\begin{aligned} E_1^{140kVp, 30cm} &= 79.642 & E_2^{140kVp, 30cm} &= 6859.080 \\ E_1^{80kVp, 30cm} &= 57.369 & E_2^{80kVp, 30cm} &= 3387.238 \end{aligned}$$

Fig. B1.

Schematic representation of the geometric properties of the CT scanner and the phantoms for the calculation of the transverse thickness per detector. The symmetric properties of the water phantom makes the transverse thickness rotational-invariant and each detector corresponds to a different thickness. The CTP698 phantom's elliptical geometry makes the thickness dependent on the source location.

The moments were calculated with the rectangle quadrature rule from the spectrum samples. Samples were provided in 1 keV steps.⁷

Now, since NEQ is the output signal-to-noise ratio squared,⁵

$$NEQ = \frac{(\eta I_0 E_1)^2}{\eta I_0 E_2} = \eta I_0 \frac{(E_1)^2}{E_2}, \quad (A3)$$

the number of emitted photons becomes:

$$I_0 = \frac{NEQ}{\eta} \frac{E_2}{(E_1)^2} \quad (A4)$$

Therefore, for a fixed NEQ, we can simulate the number of photons per specific energy level $E_i = \{0, \dots, Vp\}$ in 1 keV steps, $N_0(E_i)$. This is done considering that $N_0(E_i)$ follows as a Poisson distribution whose average is defined by the transmitting energy spectrum, that is, $N_0(E_i) \sim \text{Pois}(I_0 \phi_0(E_i))$. The simulation involves the following three steps:

1. For each energy, E_i , we simulate the number of emitted photons: $N_0(E_i) \sim \text{Pois}(I_0 \phi_0(E_i))$
2. The number of photons received for each energy, $N(E_i)$, is calculated by a binomial process with survival probability $\eta \phi(E_i) / \phi_0(E_i)$, that is, $N(E_i) \sim \text{Bin}(N_0(E_i), \eta \phi(E_i) / \phi_0(E_i))$.
3. Finally, the received signal was calculated as the sum of energies of all the surviving photons, $S = \sum_i E_i N(E_i)$.

This process is repeated for each NEQ to get the desired number of independent experiments (20 000 in our case).

APPENDIX B

TRANSVERSE THICKNESS CALCULATION IN SINOGRAMS

B.1. Water phantom

The transverse thickness is calculated considering the geometrical properties of the CT scanner and the phantom. A scheme of the phantom geometry, the source, and the detector row is depicted in Fig. B1(a), where the main parameters are given: source-to-isocenter distance ($d = 59.5$ cm), fan angle ($\Theta = 49.95^\circ$), number of detectors ($N = 736$), and the phantom radius ($r = 10$ cm). Note that phantom is assumed to be centered with respect to the isocenter.

Fig. B2.

Calculation of the transverse thickness with both the geometric and empirical approaches. Both approaches describe the same profile of transverse thickness.

As described in Fig. B1(a), the distance between the chord and the isocenter is $h = d \sin(\theta)$.

Therefore, considering that $r^2 = h^2 + \left(\frac{l}{2}\right)^2$ holds, the transverse thickness per detector is:

$$l_{\text{geometric}} = 2\sqrt{r^2 - (d\sin(\theta))^2} \quad (\text{B1})$$

where θ is the angle between the detector and the isocenter. In our specific case, the angles are defined in the interval $[-\Theta/2, \Theta/2]$ in steps of $\theta = \Theta/(N-1) = 0.068^\circ$. Note that Eq. (B1) only has real solutions if $d \sin(\theta)^2 > r^2$. Otherwise, the x ray does not intersect the phantom.

We also estimated the transverse thickness by applying the Beer-Lambert law. This empirical approach serves us to validate if the phantom is centered to the isocenter. Note that the post-log sinogram value is $p_i = \log(\bar{S}_0/S_i) = \mu \cdot l_i$, where μ is the linear attenuation

coefficient and l_i is the transverse thickness for the i th detector. Therefore, since the maximum attenuation is achieved for the maximum transverse thickness, $2r$, and the minimum corresponds to 0 cm of transverse thickness, one can easily normalize the sinogram values and estimate the thickness as:

$$l_{\text{empirical}} = 2r \cdot \frac{\hat{p}_i - p_{\min}}{p_{\max} - p_{\min}} \quad (\text{B2})$$

where $\hat{p}_i$ is the mode of the sinogram values for the i th detector, and $p_{\max}$ and $p_{\min}$ are the maximum and minimum of $\hat{p}_i$, respectively.

The empirical estimation should fit the geometrical calculation if the phantom is centered on the scanner's isocenter. In Fig. B2, we show the results of both approaches applied to the sinogram depicted in Fig. 2. Note that the results of both methods describe the same profile of transverse thickness.

Fig. B3.

Example of the fitted curve for a first-order correction term. Although the geometrical model follows the sinusoidal profile derived from the phantom's elliptical shape, the profile

trajectory deviates from samples with similar attenuation due to misalignment in the gantry axis. The proposed correction adapts the trajectory and follows the samples.

B.2. CTP698 phantom

In this case, due to the phantom's elliptical shape, the geometrical transverse thickness depends on the source location, φ . In Fig. B1(b), we show the schematic representation of the CTP698 phantom geometry and the calculation of the same transverse thickness, l , for different source locations, $A = (x_0, y_0)$ and $A' = (x'_0, y'_0)$, with reference angle φ and φ' , respectively; that is, $(x_0, y_0) = (d \cos \varphi, d \sin \varphi)$ and $(x'_0, y'_0) = (d \cos \varphi', d \sin \varphi')$.

The dimensions of the external ellipse are $a = 17.5$ cm, $b = 12.5$ cm. For the internal ellipse, $a' = 12.5$ cm and $b' = 7.5$ cm. The transverse thickness, l , with respect the source A at a given detector located at angle θ from the isocenter is given by $\|C-B\|$. The coordinates of the intersection points B and C are easily calculated considering the ellipse equation, $x^2/a^2 + y^2/b^2 = 1$, and the x-ray line equation, $y = mx + c$, where $m = \tan(\varphi + \theta - \pi)$ and $c = y_0 - mx_0$:

$$B, C = \left(\frac{-a^2mc \pm ab\sqrt{a^2m^2 + b^2 - c^2}}{a^2m^2 + b^2}, \frac{b^2c \pm abm\sqrt{a^2m^2 + b^2 - c^2}}{a^2m^2 + b^2} \right) \quad (B3)$$

Therefore, the transverse thickness becomes:

$$l = \|C - B\| = \frac{2ab\sqrt{D(m^2 + 1)}}{a^2m^2 + b^2} \quad (B4)$$

where $D = a^2m^2 + b^2 - c^2$ is the discriminant. Note that when $D < 0$, the x ray does not intersect the phantom and Eqs. (B3) and (B4) have no real solutions.

So, for a given length, l , and a given source location, φ , we can find the detector angular coordinate, θ , by solving Eq. (B4):

$$\theta_1, \theta_2 = \operatorname{argmin}_{\theta} \left| l - \frac{2ab\sqrt{D(m^2 + 1)}}{a^2m^2 + b^2} \right| \quad (B5)$$

where both D and m depend on θ . Note that, although this equation is nonlinear, it can be efficiently solved by any numerical method since the solution is bounded by the angular coordinates of the tangent x rays to the external and internal ellipses. Additionally, there can be two solutions, one for positive θ and the other for negative θ .

Finally, the detector is assigned by considering the full fan angle, $\theta \in [-\theta/2, \theta/2]$, and number of detectors as:

$$\text{Num. Detector} = \frac{\theta + \Theta/2}{\Theta}(N - 1) \quad (\text{B6})$$

where the N detectors are enumerated from 0 to $N-1$.

The trajectories obtained are oscillatory curves that become closer as the x rays align to the ellipse long-axis. This is an expected behavior considering that a similar angle span covers a wider transverse thickness interval. However, it implies that the sampling error is not homogeneous and is greater when the trajectories are packed near the phantom boundary. Therefore, it is difficult to provide an accurate sampling of small transverse thicknesses.

Although centering the phantom with the isocenter is an easy task, misalignment results in deviations in the geometrical assumptions. To overcome errors due to small misalignment, we introduce a polynomial correction term, $g(\phi)$, in the trajectory as follows:

First, we consider the family of parametric trajectories that can be derived for each thickness l from Eq. (B4):

$$f_l(\varphi) = \begin{cases} l(a^2m^2 + b^2) = 2ab\sqrt{(a^2m^2 + b^2 - c^2)(m^2 + 1)} \\ c = d\sin\varphi + m\cos\varphi \\ m = \tan(\varphi + \theta - \pi) \end{cases} \quad (\text{B7})$$

where the parameter φ varies in $[0, 2\pi)$. Note that $f_l(\varphi)$ is continuous in $[0, 2\pi)$ as long as $l \in [0, 2b\sqrt{1 - (a'/a)^2}]$. This is because the elliptical shape imposes a maximum thickness for each source location, which in the smallest case, is the x-ray projection tangent to the internal ellipse at $x = \pm a'$, which gives rise to a thickness equal to $2b\sqrt{1 - (a'/a)^2} = 17.496\text{ cm}$. So, to ensure the same number of samples, we perform the goodness of fit analysis for 0, 7, and 13 cm.

The correction term is performed by fitting the residual with respect to the expected value:

$$\hat{g} = \underset{g}{\text{argmin}} \|l\mu - P(f_l(\varphi) - g(\varphi))\| \quad (\text{B8})$$

where P is the sinogram and μ is the linear attenuation coefficient of the material.

In our specific case, we estimate μ as the maximum attenuation coefficient within the external ring. This value is reached in the projection with maximum transverse thickness, which is the x-ray projection tangent to the internal ellipse at $y = \pm b'$, and its thickness is $l = 2\sqrt{a^2(1 - b^2/b'^2)} = 28\text{ cm}$. The value obtained is $\mu = 0.1903\text{ cm}^{-1}$. It is important to note that, although this estimate of μ may have a systematic bias, the samples taken along the estimated trajectories will take independent and identically distributed samples, which is our main purpose.

In Fig. B3, we show the parametric curve of Eq. (B7) for $l = 7\text{ cm}$, the curve fit was performed with a first-order polynomial and the samples in the sinogram that are $\mu(I \pm 0.2)$.

Note that the geometrical model given in Eq. (B7) follows the phantom sinusoidal profile in the sinogram. However, a closer look shows a decay in the profile trajectory due to misalignment in the gantry axis. The fitted model adapts the trajectory and follows the samples.

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Fig. 1. Normalized spectra before and after propagating across 30 cm of water, and their equivalent monoenergetic beam energies for different peak potentials.

Fig. 2.

Siemens water phantom, CTP698 phantom, and their sinograms. (a) Water phantom axial slice. (b) sinogram acquired for several spins of the helical trajectory, where the rows represent the intensities per sensor (736) and the columns represent different projections along the helical trajectory. The intensities given by each sensor (rows) are invariant across the trajectory (columns) due to the cylindrical shape and the centering on the isocenter. The horizontal lines are the corresponding transverse thickness employed in the experimental validation. (c) CTP698 axial slice. (d) CTP698 sinogram and transverse thickness used in the experimental validation. The sampling must follow curved trajectories to ensure the same transverse thickness of the same material due to the elliptical shape.

Fig. 3.

(a) Sampling PDFs of the received energies for different NEQ, the theoretical PDF and its Gamma and Gaussian approximates for a 140 keV spectrum. (b) Sampling PDFs of the received energies for different NEQ, the theoretical PDF and its Gamma and Gaussian approximates for a 80 keV spectrum. (c) Goodness of fit analysis of the Gamma and Gaussian models to the compound Poisson obtained from samples distributed as $\Phi(E)$ shown in Fig. 1.

Fig. 4. Sample PDFs of the estimated attenuation coefficient in Hounsfield scale and the log-transformed Gamma models for increasing NEQ (columns) and tube peak voltage (rows).

Fig. 5. Goodness of fit of between the nc- Γ distribution and the Gumbel distribution. (a) Absolute difference between both cumulative distributions. (b) Maximum distance between cumulative distributions and the number of samples required to distinguish both distributions in a Kolmogorov-Smirnov test with 95% of confidence.

Fig. 6.

Goodness of fit examples for different thicknesses in the water phantom. Top row: Gumbel, nc-Gamma, and Gaussian fitting of the post-log signal. Bottom row: Gamma distribution in the pre-log signal. The Gumbel distribution accurately fits the post-log empirical distribution. The nc-Gamma provides a good approximation of the Gumbel distribution. Similarly, the empirical transmission signal is well described by the Gamma distribution from which the Gumbel distribution is derived.

Fig. 7.

Goodness of fit examples for different thicknesses in the CTP698 phantom. Top row: Gumbel, nc-Gamma, and Gaussian fitting of the post-log signal. Bottom row: Gamma distribution in the pre-log signal. The Gumbel distribution accurately fits the post-log empirical distribution. The nc-Gamma provides a good approximation of the Gumbel distribution. Similarly, the empirical transmission signal is well described by the Gamma distribution from which the Gumbel distribution is derived.

Table I.

Kolmogorov-Smirnov goodness of fit test of the Monte Carlo simulation with 20 000 samples and 0.05 significance

$V_p[kV]$	NEQ	$d(F, F_n)$	P -value	Accepted hypothesis
140	5	0.008446	0.1153	Compound Poisson
140	25	0.007634	0.1942	Compound Poisson
140	50	0.003866	0.9260	Compound Poisson
140	100	0.004125	0.8854	Compound Poisson
80	5	0.008917	0.0831	Compound Poisson
80	25	0.009255	0.0650	Compound Poisson
80	50	0.007703	0.1861	Compound Poisson
80	100	0.007133	0.2608	Compound Poisson

Table II.

Goodness of fit of the Gumbel, nc-Gamma, and Gaussian distributions calculated with the uniform distance for different thickness

	Thickness	eNEQ	Gumbel	nc-Gamma	Gaussian
<i>Water phantom</i>	0 cm	427	0.0136 ± 0.0028	0.0183 ± 0.0020	0.0672 ± 0.0046
	7 cm	246	0.0109 ± 0.0018	0.0170 ± 0.0017	0.0392 ± 0.0026
	13 cm	201	0.0095 ± 0.0010	0.0124 ± 0.0009	0.0322 ± 0.0020
	20 cm	106	0.0080 ± 0.0532	0.0086 ± 0.0373	0.0242 ± 0.0028
<i>CTP698 phantom</i>	0 cm	860	0.0176 ± 0.0025	0.0212 ± 0.0028	0.0364 ± 0.0048
	7 cm	228	0.0322 ± 0.0068	0.0375 ± 0.0081	0.0712 ± 0.0196
	13 cm	72	0.0281 ± 0.0115	0.0359 ± 0.0132	0.1030 ± 0.0347