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A second order Calderón's method with a correction term and a priori information

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Abstract

Calderón's method is a direct linearized reconstruction method for the inverse conductivity problem with the attribute that it can provide absolute images of both conductivity and permittivity with no need for forward modeling. In this work, an explicit relationship between Calderón's method and the D-bar method is provided, facilitating a "higher-order" Calderón's method in which a correction term is included, derived from the relationship to the D-bar method. Furthermore, a method of including a spatial prior is provided. These advances are demonstrated on simulated data and on tank data collected with the ACE1 EIT system.

Keywords

electrical impedance tomography; inverse problems; Calderón's method

1. Introduction

Calderón's 1980 paper *On an inverse boundary value problem* [2] has inspired several decades of research on the inverse conductivity problem. In this work, he proved that the linearized problem has a unique solution and proposed a direct method of reconstruction. Electrical impedance tomography, or EIT, is the primary application of the inverse conductivity problem, with applications in medical imaging, geophysics, and nondestructive testing. Calderón's proof and reconstruction method utilized special harmonic functions, now known as *complex geometrical optics* (CGO) solutions that have been generalized in the D-bar family of direct reconstruction algorithms for EIT. Calderón's method has, however, been slow to find its way to practical implementation in EIT. The first numerical implementation for Calderón's method was in [12] for experimental EIT data collected on a saline-filled tank and on a healthy human subject reconstructed on a circular domain. This implementation was extended in [27] to account for non-circular but symmetric domains,

demonstrating the effects primarily on elliptical domains and was implemented in real time in 2-D on subject-specific domains in [28, 29]. However, as is the case with most reconstruction algorithms in EIT, due to the severe ill-posedness of the inverse problem, the reconstructions are low resolution and sensitive to noise in the data. One approach to improve the low-resolution blurred images characteristic of EIT is to include prior information in the reconstruction algorithm. This has proved successful in iterative methods [23, 33, 6, 11, 5, 7, 8, 15, 30, 34] and in the D-bar method [17, 19, 18, 20]. In the D-bar method, an effective means of including *a priori* spatial information about the unknown conductivity is to append the scattering transform computed from the data with a scattering transform computed from the conductivity prior.

In this work, we derive an explicit connection between Calderón's method and the D-bar method, beyond what was shown in [16], motivating the addition of a weighted higher order term to Calderón's method, which is both easy to compute and improves the spatial resolution and accuracy of the reconstructed conductivity values. Although the second order term is derived from the 2D D-bar method based on [1], the terms in the second order method are well-defined for complex-valued conductivities. While this does not prove that the method is valid for complex-valued conductivities, it bears investigation, and so we include examples from simulated and experimental data with a permittivity component. D-bar methods for reconstructing a complex conductivity are based on 2×2 elliptic systems of first-order PDE's [9, 24, 25], and therefore the derivation would be quite different from the method presented here, which makes use of the already established connection between the real-valued D-bar method and Calderón's method [16]. We invite others to consider such an approach and demonstrate the effectiveness of our approach on several examples.

Also introduced here is a method for including *a priori* information in the reconstruction algorithm, inspired by the technique proposed in [19] for the D-bar method. This is the first time a prior has been included in Calderón's method, and we apply it to the second-order method for both real and complex-valued conductivities. A method of including *a priori* spatial information for complex-valued conductivities in the 2-D D-bar method based on [9, 24, 25] was introduced in [18].

Also included in this paper is a technique to minimize the effect of the Gibbs' phenomena in the reconstruction of absolute images. The results of the second-order Calderón's method with a spatial prior are demonstrated on EIT data collected on a saline-filled tank with agar and cucumber targets as shown in Figure 1 and on simulated data with complex-valued conductivities. Accuracy is measured with the structural similarity index (SSIM), and it is seen that the addition of the second-order term improves accuracy, and the inclusion of a spatial prior that does not include all inhomogeneities in the experiment also improves accuracy in most cases. The paper is organized as follows. Section 2 includes the mathematical model for the inverse conductivity problem, the equations of Calderón's method, and the 2-D D-bar method for real-valued conductivities based on the constructive uniqueness proof in [1] that has been developed as a practical reconstruction algorithm (see [31, 14] and references therein). The relationship between Calderón's method and the D-bar method and the derivation of the second order Calderón's method are found in Section 3.

Calderón's method with a spatial prior is introduced in Section 4. Finally, the experimental results and discussion are found in Section 5.

2. Background

2.1. Governing equations

The inverse problem of electrical impedance tomography is modeled as follows. Let $\Omega \in \mathbb{R}^2$ be a bounded domain. The electrical potential $u(x)$ and electrical admittivity distribution $\gamma(x)$ defined on $\bar{\Omega}$ satisfy the generalized Laplace equation

$$\nabla \cdot (\gamma(x) \nabla u(x)) = 0, \quad x \in \Omega, \quad (1)$$

where $\gamma(x) = \sigma(x) + i\omega\epsilon(x)$, where σ is the conductivity, ω is the frequency of the applied alternating current, and ϵ is the permittivity of the medium. The application of a current density distribution j on the boundary corresponds to the Neumann boundary condition,

$$\gamma(x) \frac{\partial u}{\partial \nu}(x) = j(x), \quad x \in \partial\Omega,$$

where ν denotes the outward normal to Ω . The voltage f that arises on the boundary of Ω corresponds to the Dirichlet boundary condition,

$$u(x)|_{\partial\Omega} = f(x), \quad x \in \partial\Omega.$$

The current-density to voltage mapping is known as the Neumann-to-Dirichlet map. However, in the mathematical literature, it is typically the Dirichlet-to-Neumann, or voltage-to-current density map, that is used in proofs and algorithm development. Denoted by Λ_γ , it is given by

$$\Lambda_\gamma: u|_{\partial\Omega} \rightarrow \gamma \frac{\partial u}{\partial \nu}|_{\partial\Omega},$$

and the inverse problem is to find the unknown $\gamma(x)$ based on the knowledge of Λ_γ . Note that in practice, the boundary conditions in the model are usually replaced with conditions that take into account the discrete nature of the electrodes on the boundary and their interaction with the medium known as *contact impedance*. The reader is referred to [3, 32, 14] for the derivation, study, and implementation, respectively, of the Complete Electrode Model for EIT. Here, as in Calderón's method, we utilize Λ_γ as described above.

2.2. Calderón's method

Calderón assumes that the admittivity is of the form $\gamma(x) = 1 + \delta(x)$ where $\|\delta\|_{L^\infty(\Omega)}$ is small. The CGO solutions are given by $\phi_1(x; k, a) = \exp[\pi i(x \cdot k) + \pi(a \cdot x)]$ and $\phi_2(x; k, a) = \exp[\pi i(x \cdot k) - \pi(a \cdot x)]$, where $k, a \in \mathbb{R}^2$ are nonphysical frequency variables such that $|k| = |a|$ and $k \cdot a = 0$ that enable the two functions be harmonic, that is, $\Delta \phi_1 = \Delta \phi_2 = 0$. Calderón's idea is to use these two harmonic functions and the fact that when $\|\delta\|_{L^\infty(\Omega)}$ is

small, the potentials arising from applied voltages $\phi_i (i = 1, 2)$ are small perturbations from ϕ_i in $H^1(\Omega)$. Calderón defines

$$\omega_i = u_i + v_i \text{ in } \Omega, \quad \omega_i|_{\partial\Omega} = \phi_i \text{ on } \partial\Omega, \quad u_i \in H^1(\Omega), \quad v_i \in H_0^1(\Omega)$$

for $i = 1, 2$, where $u_i = \phi_i$. By the definition of Λ_γ , integration by parts, and the conductivity equation,

$$\begin{aligned} \int_{\partial\Omega} \phi_1 \Lambda_\gamma \phi_2 \, ds(x) &= \int_{\partial\Omega} w_1 \gamma \frac{\partial \omega_2}{\partial \nu} \, ds(x) \\ &= \int_{\Omega} \gamma (\nabla \omega_1 \cdot \nabla \omega_2) \, dx \\ &= \int_{\Omega} \gamma (\nabla u_1 \cdot \nabla u_2) + \gamma [(\nabla u_1 \cdot \nabla v_2) + (\nabla u_2 \cdot \nabla v_1)] + \gamma (\nabla v_1 \cdot \nabla v_2) \, dx. \\ &= - \int_{\Omega} 2\pi^2 |k|^2 \gamma(x) \exp[2\pi i x \cdot k] \, dx + \tilde{E}(k), \end{aligned}$$

where $\tilde{E}(k)$ is the last three terms in the third line. By dividing by $-2\pi^2 |k|^2$, we get

$$\int_{\Omega} \gamma(x) \exp[2\pi i x \cdot k] \, dx = - \frac{1}{2\pi^2 |k|^2} \int_{\partial\Omega} \phi_1 \Lambda_\gamma \phi_2 \, ds(x) + \frac{\tilde{E}(k)}{2\pi^2 |k|^2}. \quad (2)$$

Thus, neglecting the remainder term $E(k) = \frac{\tilde{E}(k)}{2\pi^2 |k|}$, we get

$$\hat{\gamma}(k) \approx - \frac{1}{2\pi^2 |k|^2} \int_{\partial\Omega} \phi_1 \Lambda_\gamma \phi_2 \, ds(x) \equiv \hat{F}(k), \quad (3)$$

where $\hat{\gamma}$ denotes the Fourier transform of γ . Inverting equation (3) is the main idea behind Calderón's method. However, one can also seek to solve directly for the perturbation as follows. By the same argument, when $\gamma = 1$,

$$\int_{\partial\Omega} \phi_1 \Lambda_1 \phi_2 \, ds(x) = - \int_{\Omega} 2\pi^2 |k|^2 \exp[2\pi i x \cdot k] \, dx. \quad (4)$$

Subtracting this from equation (2) results in

$$\int_{\partial\Omega} \phi_1 (\Lambda_\gamma - \Lambda_1) \phi_2 \, ds(x) = - 2\pi^2 |k|^2 \int_{\Omega} (\gamma - 1) \exp[2\pi i x \cdot k] \, dx + \tilde{E}(k).$$

Denote

$$\mathcal{D}(k) \equiv \int_{\partial\Omega} \phi_1 (\Lambda_\gamma - \Lambda_1) \phi_2 \, ds(x).$$

Then,

$$\hat{\delta}(k) = -\frac{\mathcal{D}(k)}{2\pi^2|k|^2} + E(k), \quad (5)$$

and it is shown in [2] that $E(k)$ is bounded by $C\|\delta\|_L^\infty \exp[2\pi r|k|]$, where r is the radius of the smallest sphere containing Ω and C some constant. However, it is also shown in [2] that choosing α , $1 < \alpha < 2$, for $|k| \leq \frac{2-\alpha}{2\pi r} \log \frac{1}{\|\delta\|_L^\infty}$,

$$|E(k)| \leq C\|\delta\|_\infty^\alpha. \quad (6)$$

Now let η be a mollifier whose Fourier transform $\hat{\eta} \in C^\infty$, $\hat{\eta}(0) = 1$, and let $\eta_t(x) \equiv t^n \eta(tx)$, where t is a mollifying parameter. Then $\hat{\eta}_t(k) = \hat{\eta}(k/t)$ acts as a low pass filter, and multiplying equation (5) by $\hat{\eta}_t(k)$, we have,

$$\hat{\delta}(k)\hat{\eta}\left(\frac{k}{t}\right) = -\frac{\mathcal{D}(k)}{2\pi^2|k|^2}\hat{\eta}\left(\frac{k}{t}\right) + E(k)\hat{\eta}\left(\frac{k}{t}\right).$$

Taking the inverse Fourier transform,

$$(\delta * \eta_t)(x) = -\int_{\mathbb{R}^2} \frac{\mathcal{D}(k)}{2\pi^2|k|^2} \hat{\eta}\left(\frac{k}{t}\right) \exp[-2\pi i x \cdot k] dk + \epsilon(x),$$

where $*$ denotes the convolution, and ϵ the error term, which is the term that depends on the data $\Lambda_\gamma - \Lambda_1$ nonlinearly. It was shown in [2] that $\epsilon(x)$ has a bound

$$|\epsilon(x)| \leq C\|\delta\|_L^\infty \left[\log \frac{1}{\|\delta\|_L^\infty} \right]^2$$

which converges to zero as $\|\delta\|_L^\infty$ goes to zero. In the practical implementation, we also truncate $\mathcal{D}(k)$ to a region $|k| < R_1$ for some constant R_1 . Therefore, we take

$$\delta_{R_1}^{\text{app}}(x) \equiv -\int_{|k| < R_1} \frac{\mathcal{D}(k)}{2\pi^2|k|^2} \hat{\eta}\left(\frac{k}{t}\right) \exp[2\pi i x \cdot k] dk \quad (7)$$

as an approximation to $\delta(x)$. In this paper, we will call (7) *the first order Calderon's method*. We note that the reconstruction process

$$\Lambda_\gamma - \Lambda_1 \rightarrow \mathcal{D}(k) \rightarrow \delta_{R_1}^{\text{app}}(x)$$

is linear and holds for real or complex-valued γ .

2.3. The D-bar method background

We now review the D-bar method based on the constructive global uniqueness proof by A. Nachman [1]. As in [1], now assume that $\gamma \in C^2(\Omega)$ is a real-valued conductivity bounded away from zero, and without loss of generality that $\gamma(x) \equiv 1$ for x in a neighborhood Ω . For consistency with Calderón's method, further assume that γ is of the form $\gamma = 1 + \delta$. Define the potential $q = \frac{\Delta \sqrt{\gamma}}{\sqrt{\gamma}}$, and make the change of variables $\tilde{u} = \sqrt{\gamma}u$. Then (1) is transformed to the Schrödinger equation

$$(-\Delta + q)\tilde{u} = 0 \quad \text{in } \Omega. \quad (8)$$

Smoothly extend $\gamma \equiv 1$ and $q \equiv 0$ outside Ω and consider (8) in the entire plane. Introduce a complex parameter $k = k_1 + ik_2$ and identify it with $k = (k_1, k_2) \in \mathbb{R}^2$, depending on the context. The spatial variable $x = (x_1, x_2) \in \mathbb{R}^2$ is also identified as $x = x_1 + ix_2$. Further introduce two harmonic functions e^{ikx} and $e^{i\bar{k}\bar{x}}$ that satisfy (8) when $q = 0$. It is shown in [1] that there exist CGO solutions $\psi(x, k)$ to

$$(-\Delta + q)\psi(\cdot, k) = 0$$

satisfying the asymptotic condition

$$e^{-ikx}\psi(x, k) - 1 \in W^{1, \tilde{p}}(\mathbb{R}^2) \quad (9)$$

for any $2 < \tilde{p} < \infty$. Note that if $\gamma = \text{const.}$, so that $q \equiv 0$, then $\psi(x, k) = e^{ikx}$. Let

$$\mu(x, k) \equiv e^{-ikx}\psi(x, k),$$

and define the scattering transform $\mathbf{t}(k)$ of $q(x)$ by

$$\mathbf{t}(k) \equiv \int_{\mathbb{R}^2} 2e^{i\bar{k}\bar{x}} q(x) \psi(x, k) dx,$$

or equivalently,

$$\mathbf{t}(k) := \int_{\mathbb{R}^2} 2e_k(x) q(x) \mu(x, k) dx,$$

where

$$e_k(x) \equiv \exp[i(kx + \bar{k}\bar{x})] = \exp[-i(-2k_1, 2k_2) \cdot (x_1, x_2)].$$

The function μ satisfies

$$(-\Delta - 4ik\bar{\partial})\mu(x, k) = -q(x)\mu(x, k). \quad (10)$$

Therefore, for a fundamental solution $g_k(x)$ satisfying

$$(-\Delta - 4ik\bar{\partial})g_k(x) = \delta(x),$$

where $\delta(x)$ only in this line indicates the Dirac delta function, the function μ satisfies, with the asymptotic condition (9), the Lippmann-Schwinger-type integral equation

$$\mu = 1 - g_k * (q\mu). \quad (11)$$

Applying the $\bar{\partial}_k$ operator to (11) results in the $\bar{\partial}$ equation

$$\frac{\partial}{\partial k} \mu(x, k) = \frac{1}{4\pi k} \mathbf{t}(k) e_{-k(x)} \overline{\mu(x, k)}, \quad k \in \mathbb{C} \setminus 0, \quad x \in \mathbb{R}^2,$$

which is equivalent to the weakly singular Fredholm integral equation of the second kind

$$\mu(x, k) = 1 + \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{\mathbf{t}(k')}{(k - k')\bar{k}} e_{-x}(k') \overline{\mu(x, k')} dk'_1 dk'_2. \quad (12)$$

Taking the small k limit as $k \rightarrow 0$ in (10), and recalling that $q(x) = \frac{\Delta\sqrt{\gamma(x)}}{\sqrt{\gamma(x)}}$,

$$\Delta\mu(x, 0) = \frac{\Delta\sqrt{\gamma(x)}}{\sqrt{\gamma(x)}} \mu(x, 0). \quad (13)$$

It is a direct substitution to check that $\mu(x, 0) = \sqrt{\gamma(x)}$ satisfies (13) and the asymptotic condition (9). Therefore, we can recover $\gamma(x)$ from (12) by

$$\gamma(x) = \mu(x, 0)^2. \quad (14)$$

A key step in the D-bar method is to connect the data $\Lambda_\gamma - \Lambda_1$ to the scattering transform $\mathbf{t}(k)$ and to solve (12). This is done by first solving

$$\psi(x, k) = e^{ikx} \Big|_{\partial\Omega} - \int_{\partial\Omega} G_k(\zeta - x) (\Lambda_\gamma - \Lambda_1) \psi(\zeta, k) ds(\zeta), \quad (15)$$

where $G_k(x) = e^{ikx} g_k(x)$. The scattering transform $\mathbf{t}(k)$ is related to the data as, by [1],

$$\mathbf{t}(k) = \int_{\partial\Omega} e^{i\bar{k}\bar{x}} (\Lambda_\gamma - \Lambda_1) \psi(\cdot, k) ds(x). \quad (16)$$

In the case of $\gamma \in L^\infty$, the existence and uniqueness of solutions to (12) and (15) is proven in [10]. However, (15) is very sensitive to noise in $\Lambda_\gamma - \Lambda_1$ and so are (16) and (12). In [31], approximating $\psi(x, k)$ by its asymptotic behavior e^{ikx} is suggested and

$$\mathbf{t}^{\text{exp}}(k) \equiv \int_{\partial\Omega} e^{i\bar{k}\bar{x}} (\Lambda_\gamma - \Lambda_1) e^{ikx} ds(x) \quad (17)$$

is introduced as an approximation of (16). In terms of the integral equation (15), this approximation amounts to ignoring the scattering of the incident wave e^{ikx} at the boundary Ω . Replacing $\psi(\zeta, k)$ on the right hand side of (15) by $e^{ik\zeta}$ can also be considered as a Born approximation. The approximation $\mathbf{t}^{\text{exp}}$ was further studied with simulated noisy data $\Lambda_\gamma - \Lambda_1$ in [13]. While $\mathbf{t}^{\text{exp}}(k)$ and (10) remain stable for large $|k|$ for simulated noise-free data, $\mathbf{t}^{\text{exp}}(k)$ with noisy data blows up to infinity exponentially as $|k|$ gets large. This has been overcome by truncating $\mathbf{t}(k)$ to a bounded region $|k| < R_1$ for some R_1 constant [13]. Therefore, in the practical implementation, the integral equation (12) is solved in the region $|k| < R_1$.

3. Relationship between Calde on's method and the D-bar method and the second order Calde on's method

A connection between Calde on's method and the D-bar method was first shown in [16]. Based on [16], we provide explicit correction terms to Calde on's method and suggest a modified Calde on's method with one of the correction terms.

We first note the similarity of the CGO solutions that are used in both methods. Let $k = (k_1, k_2)$ and $a = (k_2, -k_1)$. We also identify $k = k_1 + ik_2$ and $a = k_2 - ik_1$, depending on the context. By simply changing the notation from vectors in $\mathbb{R}^2$ to complex variables,

$$\begin{aligned}\phi_1(x; k, a) &= e^{\pi(ik \cdot x - a \cdot x)} \\ &= e^{\pi(i(k_1, k_2) \cdot (x_1, x_2) - (k_2, -k_1) \cdot (x_1, x_2))} \\ &= e^{\pi(i(k_1x_1 + k_2x_2) - k_2x_1 + k_1x_2)} \\ &= e^{i\pi(k_1x_1 + k_2x_2 + i(k_2x_1 - k_1x_2))} \\ &= e^{i\pi k \bar{x}},\end{aligned}$$

and similarly, $\phi_2(x; k, a) = e^{i\pi \bar{k}x}$. Therefore, the functions $e^{i\bar{k}x}$, e^{ikx} in the D-bar method and ϕ_1 , ϕ_2 are the same except for the factor of π in the exponent and the complex conjugate on the variable k . That is,

$$\mathcal{D}(k) = \int_{\partial\Omega} e^{i\pi k \bar{x}} (\Lambda_\gamma - \Lambda_1) e^{i\pi \bar{k}x} ds(x) = \mathbf{t}^{\text{exp}}(\pi \bar{k}). \quad (18)$$

Since $\mathcal{D}$ holds for complex-valued conductivities by (3), $\mathbf{t}^{\text{exp}}$ is also well-defined for complex-valued conductivities. Now, from (5),

$$\hat{\delta}(k) = -\frac{\mathbf{t}^{\text{exp}}(\pi \bar{k})}{2\pi^2 |k|^2} + E(k). \quad (19)$$

By neglecting $E(k)$, multiplying a mollifier, taking the inverse Fourier transform, and then with a simple change of variables, we arrive at the same conclusion as in [16]. In [16], starting from (17), Calderón's method is re-derived with $\mathbf{t}^{\text{exp}}(k)$ instead of $\mathcal{D}(k)$, and presented as

$$\delta^{\text{app}}(x) = - \int_{\mathbb{R}^2} \frac{\mathbf{t}^{\text{exp}}(k)}{2\pi^2 |k|^2} \hat{\eta}(k/t) e_{-x}(k) dk_1 dk_2, \quad (20)$$

and it is concluded that Calderon's method is a three-step linearized D-bar method. See section 6.1 of [16] for the full derivations. In the practical implementation, we truncate $\mathbf{t}^{\text{exp}}(k)$ to a bounded region $|k| < R_1$ and denote the reconstructed perturbation by

$$\delta_{R_1}^{\text{app}}(x) = - \int_{|k| < R_1} \frac{\mathbf{t}^{\text{exp}}(k)}{2\pi^2 |k|^2} \hat{\eta}(k/t) e_{-x}(k) dk_1 dk_2, \quad (21)$$

which is equivalent to (7). Note that the truncation radius in this paper differs by the factor π from the truncation radius in other D-bar papers due to the relation (18). We further explicitly formulate the difference between Calderon's method and the D-bar method and connect them. Note that from (12), by setting $k = 0$, (14) can be written formally as

$$\begin{aligned} \gamma(x) = 1 - \frac{1}{2\pi^2} \int_{\mathbb{R}^2} \frac{\mathbf{t}(k')}{|k'|^2} e_{-x}(k') \overline{\mu(x, k')} dk'_1 dk'_2 \\ + \left\{ \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{\mathbf{t}(k')}{|k'|^2} e_{-x}(k') \overline{\mu(x, k')} dk'_1 dk'_2 \right\}^2, \end{aligned}$$

or equivalently,

$$\begin{aligned} \delta(x) = - \frac{1}{2\pi^2} \int_{\mathbb{R}^2} \frac{\mathbf{t}(k')}{|k'|^2} e_{-x}(k') \overline{\mu(x, k')} dk'_1 dk'_2 \\ + \left\{ \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{\mathbf{t}(k')}{|k'|^2} e_{-x}(k') \overline{\mu(x, k')} dk'_1 dk'_2 \right\}^2. \end{aligned} \quad (22)$$

Note that line (14) is from the D-bar method while (21) is Calderon's method. Now we explicitly provide the connection between them. For $\|q(x)\|_\infty$ small, let $\tilde{\psi}(x, k)$ be a small perturbation from e^{ikx} and write

$$\psi(x, k) = e^{ikx} + \tilde{\psi}(x, k).$$

Then $\mu(x, k) \equiv 1 + \tilde{\mu}(x, k)$, and

$$\begin{aligned} \mathbf{t}(k) &= \int_{\partial\Omega} e^{i\tilde{k}\tilde{x}} (\Lambda_\gamma - \Lambda_1) \psi(x, k) ds(x) \\ &= \int_{\partial\Omega} e^{i\tilde{k}\tilde{x}} (\Lambda_\gamma - \Lambda_1) (e^{ikx} + \tilde{\psi}(x, k)) ds(x) \\ &= \mathbf{t}^{\text{exp}}(k) + \int_{\partial\Omega} e^{i\tilde{k}\tilde{x}} (\Lambda_\gamma - \Lambda_1) \tilde{\psi}(x, k) ds(x) \\ &= \mathbf{t}^{\text{exp}}(k) + \tilde{\mathbf{t}}(k), \end{aligned}$$

where $\tilde{\mathbf{t}}(k)$ and $\tilde{\mu}(x, k)$ are also small perturbations if $\|q(x)\|_\infty$ is small. We derive the second order Calderón's method as follows. Recall, the definition of the norm on $\mathcal{W}^{1, \tilde{p}}$:

$$\|u\|_{W^1, \tilde{p}} = \|u\|_{L^{\tilde{p}}} + \|\nabla u\|_{L^{\tilde{p}}} \quad (23)$$

Lemma 4.1 in [22] states the following.

Theorem 1

Fix $p \in (1, 2)$, $k \in \mathbb{C} \setminus \{0\}$, and $q_0 \in L^p(\mathbb{R}^2)$. Let $\frac{1}{\tilde{p}} + \frac{1}{p} = \frac{1}{2}$. Suppose that

$$R(q_0) := (I - S_k(q_0))^{-1}$$

exists as a bounded operator from $W^{1, \tilde{p}}(\mathbb{R}^2)$ to itself. There is a number $r > 0$ and a constant $c(p, k, q_0)$ so that for all $q \in L^p(\mathbb{R}^2)$ with $\|q_0 - q\|_p \leq r$, the estimate

$$\|\mu(\cdot, k, q) - \mu(\cdot, k, q_0)\|_{W^1, \tilde{p}} \leq c(p, k, q_0) \|q - q_0\|_p \quad (24)$$

holds.

Let $K_{R_1} = \{k \in \mathbb{C} : |k| \leq R_1\}$. Note that in Theorem 1, $\tilde{p} \in (2, \infty)$. For $\gamma = 1$, $q_0 = 0$ and $\mu(\cdot, k, q_0) = 1$, the theorem implies

$$\|\tilde{\mu}(\cdot, k, q)\|_{W^{1, \tilde{p}}(K_{R_1})} \leq c(p, R) \|q\|_p \leq c'(p, R, \Omega) \|q\|_\infty. \quad (25)$$

Corollary 1

Under the assumptions of Theorem 1,

$$\|\tilde{\psi}(\cdot, k)\|_{W^{1, \tilde{p}}(K_{R_1})} \leq C(p, R, \Omega) \|q\|_\infty. \quad (26)$$

Proof: Let $x \in \Omega$ and $k \in \mathbb{C}$ with $|k| \leq K_{R_1}$. Then $|e^{ikx}|$ is bounded by a constant, and

$$\begin{aligned} \|\tilde{\psi}(\cdot, k)\|_{W^{1, \tilde{p}}(K_{R_1})} &= \|e^{ikx} \tilde{\mu}(\cdot, k)\|_{W^{1, \tilde{p}}(K_{R_1})} \\ &= \left\{ \int_{K_{R_1}} |e^{ikx} \tilde{\mu}(x, k)|^{\tilde{p}} dk \right\}^{1/\tilde{p}} + \left\{ \int_{K_{R_1}} |\nabla(e^{ikx} \tilde{\mu}(x, k))|^{\tilde{p}} dk \right\}^{1/\tilde{p}} \\ &\leq C \|\tilde{\mu}(\cdot, k)\|_{L^{\tilde{p}}(K_{R_1})} + \left\{ \int_{K_{R_1}} |\nabla(e^{ikx} \tilde{\mu}(x, k) + e^{ikx} \nabla \tilde{\mu}(x, k))|^{\tilde{p}} dk \right\}^{1/\tilde{p}} \\ &\leq C \|\tilde{\mu}(\cdot, k)\|_{L^{\tilde{p}}(K_{R_1})} + \tilde{C} \|\tilde{\mu}(\cdot, k)\|_{L^{\tilde{p}}(K_{R_1})} + C \|\nabla \tilde{\mu}(\cdot, k)\|_{L^{\tilde{p}}(K_{R_1})} \\ &= C \|\tilde{\mu}(\cdot, k, q)\|_{W^{1, \tilde{p}}(K_{R_1})} + \tilde{C} \|\tilde{\mu}(\cdot, k)\|_{L^{\tilde{p}}(K_{R_1})} \\ &= C_1 \|\tilde{\mu}(\cdot, k, q)\|_{W^{1, \tilde{p}}(K_{R_1})} \\ &\leq C_2(p, R) \|q\|_p \\ &\leq C_2(p, R)(\text{area of } \Omega) \|q\|_\infty. \end{aligned}$$

where the third inequality follows from Minkowski's inequality and the fifth inequality from (25). $\square$

Corollary 2

Under the assumptions of Theorem 1, $\tilde{\mathbf{t}}(k) \in W^{1, \tilde{p}}(K_{R_1})$ and

$$\|\tilde{\mathbf{t}}(k)\|_{W^{1, \tilde{p}}(K_{R_1})} \leq C(p, R, \Omega) \|q\|_{\infty} \quad (27)$$

Proof: For $x \in \Omega$ and $k \in \mathbb{C}$ with $|k| \leq K_{R_1}$, $|e^{i\tilde{k}\tilde{x}}|$ is bounded by a constant, and, the perturbation of the scattering transform is bounded in the norm on $W^{1, \tilde{p}}(K_{R_1})$ as follows:

$$\begin{aligned} \|\tilde{\mathbf{t}}(k)\|_{W^{1, \tilde{p}}(K_{R_1})} &= \left\{ \int_{K_{R_1}} |e^{i\tilde{k}\tilde{x}} q(x) \tilde{\psi}(x, k)|^{\tilde{p}} dx \right\}^{1/\tilde{p}} \\ &\quad + \left\{ \int_{K_{R_1}} |\nabla(e^{i\tilde{k}\tilde{x}} q(x) \tilde{\psi}(x, k))|^{\tilde{p}} dx \right\}^{1/\tilde{p}} \\ &\leq C \|q\|_{\infty} \|\tilde{\psi}(x, k)\|_{L^{\tilde{p}}(K_{R_1})} + \|q\|_{\infty} \left\{ \int_{K_{R_1}} |\nabla(e^{i\tilde{k}\tilde{x}} \tilde{\psi}(x, k))|^{\tilde{p}} dx \right\}^{1/\tilde{p}} \\ &\leq \|q\|_{\infty} (C \|\tilde{\psi}(\cdot, k, q)\|_{W^{1, \tilde{p}}(K_{R_1})} + \tilde{C} \|\tilde{\psi}(\cdot, k, q)\|_{L^{\tilde{p}}(K_{R_1})}) \\ &\leq C' \|q\|_{\infty} \|\tilde{\psi}(\cdot, k, q)\|_{W^{1, \tilde{p}}(K_{R_1})} \\ &\leq C(p, R, \Omega) \|q\|_{\infty}, \end{aligned}$$

where the second inequality follows from Minkowski's inequality and the remainder by the analogous argument as in Corollary 1. $\square$

Then, expanding the integrands in (22) by using the perturbation for μ and $\mathbf{t}$ and folding out the square in the second term, one sees that compared to (20), (22) has three more first order terms and 16 more second order terms which depend on the data $\Lambda_{\gamma} - \Lambda_1$ nonlinearly. All integrals in practice are over K_{R_1} , and

$$\begin{aligned} \delta(x) &= - \int_{K_{R_1}} \frac{\mathbf{t}^{\exp}(k) + \tilde{\mathbf{t}}(k)}{2\pi^2 |k|^2} e_{-x}(k) \overline{1 + \tilde{\mu}(x, k)} dk \\ &\quad + \left\{ \frac{1}{(2\pi)^2} \int_{K_{R_1}} \frac{\mathbf{t}^{\exp}(k) + \tilde{\mathbf{t}}(k)}{|k|^2} e_{-x}(k) \overline{1 + \tilde{\mu}(x, k)} dk \right\}^2 \\ &= - \int_{K_{R_1}} \frac{\mathbf{t}^{\exp}(k)}{2\pi^2 |k|^2} e_{-x}(k) dk - \int_{K_{R_1}} \frac{\mathbf{t}^{\exp}(k)}{2\pi^2 |k|^2} e_{-x}(k) \overline{\tilde{\mu}(x, k)} dk \end{aligned} \quad (28)$$

$$- \int_{K_{R_1}} \frac{\tilde{\mathbf{t}}(k)}{2\pi^2 |k|^2} e_{-x}(k) dk - \int_{K_{R_1}} \frac{\tilde{\mathbf{t}}(k)}{2\pi^2 |k|^2} e_{-x}(k) \overline{\tilde{\mu}(x, k)} dk \quad (29)$$

$$+ \left\{ \frac{1}{(2\pi)^2} \int_{K_{R_1}} \frac{\mathbf{t}^{\exp}(k)}{|k|^2} e_{-x}(k) dk \right\}^2 + \left\{ \frac{1}{(2\pi)^2} \int_{K_{R_1}} \frac{\mathbf{t}^{\exp}(k)}{|k|^2} e_{-x}(k) \overline{\tilde{\mu}(x, k)} dk \right\}^2 \quad (30)$$

$$\begin{aligned}
& + \left\{ \frac{1}{(2\pi)^2} \int_{K_{R_1}} \frac{\tilde{\mathbf{t}}(k)}{|k|^2} e_{-x}(k) dk \right\}^2 + \left\{ \frac{1}{(2\pi)^2} \int_{K_{R_1}} \frac{\tilde{\mathbf{t}}(k)}{|k|^2} e_{-x}(k) \overline{\tilde{\mu}(x, k)} dk \right\}^2 \\
& + 2 \left\{ \frac{1}{(2\pi)^2} \int_{K_{R_1}} \frac{\mathbf{t}^{\text{exp}}(k)}{|k|^2} e_{-x}(k) dk \right\} \left\{ \frac{1}{(2\pi)^2} \int_{K_{R_1}} \frac{\mathbf{t}^{\text{exp}}(k)}{|k|^2} e_{-x}(k) \overline{\tilde{\mu}(x, k)} dk \right\} \\
& + 2 \left\{ \frac{1}{(2\pi)^2} \int_{K_{R_1}} \frac{\mathbf{t}^{\text{exp}}(k)}{|k|^2} e_{-x}(k) dk \right\} \left\{ \frac{1}{(2\pi)^2} \int_{K_{R_1}} \frac{\tilde{\mathbf{t}}(k)}{|k|^2} e_{-x}(k) dk \right\} \\
& + 2 \left\{ \frac{1}{(2\pi)^2} \int_{K_{R_1}} \frac{\mathbf{t}^{\text{exp}}(k)}{|k|^2} e_{-x}(k) dk \right\} \left\{ \frac{1}{(2\pi)^2} \int_{K_{R_1}} \frac{\tilde{\mathbf{t}}(k)}{|k|^2} e_{-x}(k) \overline{\tilde{\mu}(x, k)} dk \right\} \\
& + 2 \left\{ \frac{1}{(2\pi)^2} \int_{K_{R_1}} \frac{\mathbf{t}^{\text{exp}}(k)}{|k|^2} e_{-x}(k) \overline{\tilde{\mu}(x, k)} dk \right\} \left\{ \frac{1}{(2\pi)^2} \int_{K_{R_1}} \frac{\tilde{\mathbf{t}}(k)}{|k|^2} e_{-x}(k) dk \right\} \\
& + 2 \left\{ \frac{1}{(2\pi)^2} \int_{K_{R_1}} \frac{\mathbf{t}^{\text{exp}}(k)}{|k|^2} e_{-x}(k) \overline{\tilde{\mu}(x, k)} dk \right\} \left\{ \frac{1}{(2\pi)^2} \int_{K_{R_1}} \frac{\tilde{\mathbf{t}}(k)}{|k|^2} e_{-x}(k) \overline{\tilde{\mu}(x, k)} dk \right\} \\
& + 2 \left\{ \frac{1}{(2\pi)^2} \int_{K_{R_1}} \frac{\tilde{\mathbf{t}}(k)}{|k|^2} e_{-x}(k) dk \right\} \left\{ \frac{1}{(2\pi)^2} \int_{K_{R_1}} \frac{\tilde{\mathbf{t}}(k)}{|k|^2} e_{-x}(k) \overline{\tilde{\mu}(x, k)} dk \right\}.
\end{aligned} \tag{31}$$

Assume $\|q\|_\infty$ is sufficiently small so that $\|\tilde{\mu}(\cdot, k, q)\|_{W^1, \tilde{p}(K_{R_1})} \leq 1$ and

$\|\tilde{\mathbf{t}}(k)\|_{W^1, \tilde{p}(K_{R_1})} \leq \|\mathbf{t}^{\text{exp}}(k)\|_{W^1, \tilde{p}(K_{R_1})}$. Note [13] that $|\mathbf{t}^{\text{exp}}(k)| \leq C|k|^2$ for $k \in K_{R_1}$ and $|e_{-x}(k)| = 1$. Then $\mathbf{t}^{\text{exp}}(k)/|k|^2 \in L^r(K_{R_1})$, $r = 2 - \varepsilon$, where here ε is a small value less than 1, and r is the Hölder conjugate of $\tilde{p}$. Thus, by Hölder's inequality, the following shows that the first term in (28) dominates the second term:

$$\begin{aligned}
\left| \int_{K_{R_1}} \frac{\mathbf{t}^{\text{exp}}(k)}{2\pi^2 |k|^2} e_{-x}(k) \overline{\tilde{\mu}(x, k)} dk \right| & \leq \left\{ \int_{K_{R_1}} \left| \frac{\mathbf{t}^{\text{exp}}(k)}{2\pi^2 |k|^2} \right|^r dk \right\}^{1/r} \left\{ \int_{K_{R_1}} |\overline{\tilde{\mu}(x, k)}|^{\tilde{p}} dk \right\}^{1/\tilde{p}} \\
& \leq \|\tilde{\mu}(\cdot, k, q)\|_{W^1, \tilde{p}(K_{R_1})} \left\{ \int_{K_{R_1}} \left| \frac{\mathbf{t}^{\text{exp}}(k)}{2\pi^2 |k|^2} \right|^r dk \right\}^{1/r} \\
& \leq \left\{ \int_{K_{R_1}} \left| \frac{\mathbf{t}^{\text{exp}}(k)}{2\pi^2 |k|^2} \right|^r dk \right\}^{1/r}
\end{aligned}$$

Also since $\|\tilde{\mathbf{t}}(k)\|_{W^1, \tilde{p}(K_{R_1})} \leq \|\mathbf{t}^{\text{exp}}(k)\|_{W^1, \tilde{p}(K_{R_1})}$, the first term in (29) is smaller than the first term in (28), and likewise, the second term in (29) is smaller than the first term in (28). Thus, among the first order terms in the expansion of $\delta(x)$, the first term is dominant. Analogous arguments show that the first term in (30) is larger than all subsequent second order terms.

Therefore, we suggest a modified Calderón's method by adding the known second order term and define

$$\delta_{R_1}^{\text{app, sec}}(x) = \delta_{R_1}^{\text{app}}(x) + (\beta \delta_{R_1}^{\text{app}}(x))^2, \tag{32}$$

where β is a weighting parameter for the second order term. One can change the value β to adjust the impact of the second order term. For the results in Figure 3, we chose $\beta = 0.5$, and for all the other examples we chose $\beta = 0.7$. We will call (32) *the second order Calderón's method*, and compare the reconstructions with (32) to the first order Calderón's method (7). Since we use (7) for the first order Calderón's method, from equation (18), the truncation radius $R_1 = 1.5$, for example, in our paper is equivalent to the truncation radius $1.5 \cdot \pi \approx 4.712$ for the D-bar method with $t^{\text{exp}}(k)$ in other papers.

The experimental tank data for Cases (i) and (ii) were also used in [20], and are included here for the purpose of comparison with the D-bar method. It should be noted that the results in [20] are difference images, and here we include both difference and absolute images. The data was collected using the ACE1 EIT system at Colorado State University [21] on a circular tank with diameter 30 cm. The electrodes were 2.54 cm wide, and the level of saline bath was 1 cm. Adjacent current patterns were applied at 125 kHz with amplitude 3.3 mA. The phantoms are made of agar with a "heart" with conductivity 0.238 S/m, and right and left "lungs" with conductivity 0.136 S/m, and a saline bath of conductivity 0.190 S/m. Case (i) includes a copper pipe 1.6 cm in diameter inserted in the right lower lung region, and Case (ii) is the same agar targets with a PVC pipe 2.2 cm in diameter inserted in the same place as in Case (i). The background conductivity γ_0 was estimated empirically from a preliminary reconstruction.

Figure 1 (D), (E), (F) and (G) show the reference images used to compute the SSIM measures in the figures for Cases (i), (ii), and the real and imaginary parts of Case (iii), respectively. The spatial organ boundaries were obtained from the photos in Figure 1 (A), (B), and (C). The conductivity values assigned to the saline and agar were the experimentally measured values. The copper pipe in the reference was assigned the value 0.76 S/m, which was chosen because although copper is infinitely conductive, Calderón's method cannot come close to approximating a perfect conductor, and so we chose a value that is significantly higher than that of the agar heart without making it completely out of reach for the reconstruction algorithm. Similarly, the PVC pipe is a perfect resistor, and we approximated its conductivity by 0.01 S/m. For the cucumber, the conductivity was assigned to be 0.2 S/m and the susceptibility (imaginary part) $9 \times 10^{-5} \text{ S/m}$ based on measured data for cucumber in [26], and the background was assigned the true value of 0.18 S/m.

In Figure 2, we illustrate the effect of the second order term in the second order Calderón's method on reconstructions for case (i). Figure 2 shows the absolute images for the case (i), where (A) and (C) depict the image with the first order Calderón's method, and (B) and (D) depict the images with the second order Calderón's method, with the truncation radius $R_1 = 1.125$ and $R_1 = 1.35$, respectively, with no prior inclusion. Compared to (A) and (C), the organs are sharper and the artifacts near the boundary are reduced in (B) and (D). We use Figure 1 (D) as the reference image for SSIM values given below each reconstruction. This kind of improvement was observed in all examples to varying degrees. In order to investigate the second order Calderón's method for complex-valued admittivities, 2D data was simulated for complex-valued conductivities using the Finite Element Method (FEM) with the Complete Electrode Model [32]. Trigonometric current patterns with current amplitude 0.85 mA were applied at 28.8 kHz in the simulation. The contact impedance was simulated

to be $0.0024 \Omega - m^2$. The conductivity of the heart is $\sigma = 2$ Siemens/m (S/m) and the permittivity is $\epsilon = 0.88 \times 10^{-6}$ Farads/m (F/m). The conductivity of the lung is 0.5 Siemens/m (S/m) and the permittivity is 0.22×10^{-6} Farads/m (F/m). The conductivity of the background is 1 Siemens/m (S/m) and the permittivity is 0.55×10^{-6} Farads/m (F/m). The susceptibility is the imaginary component, which is plotted in the reconstructions, given by $\omega\epsilon$, where the angular frequency $\omega = 2\pi \times 28.8$, with units S/m. The results in Figure 3 show that the second order Calderón's method improves the reconstruction. Therefore, in the results section, we only present the reconstructions by the second order Calderón's method, whether or not those are absolute images, difference images, images without a prior or images with a prior.

3.1. Computational considerations

The results shown in this work are on experimental data and so the assumption of Calderón's method that the conductivity distribution is a small perturbation from the background conductivity 1 is violated by our phantoms. This is overcome by rescaling as follows in an approach also taken in [4]. Let $\gamma(x) = \gamma_0 + \delta(x)$, where $\gamma_0 > 0$ is the background constant conductivity and $\delta(x)$ is the perturbation from γ_0 . Scale γ by γ_0 and denote

$$\tilde{\gamma} = \frac{\gamma}{\gamma_0} = 1 + \frac{\delta}{\gamma_0}.$$

Since $\Lambda_{c\gamma} = c\Lambda_\gamma$ for c constant, $\Lambda_{\tilde{\gamma}} = \frac{1}{\gamma_0}\Lambda_\gamma$. We use this map to compute the scaled perturbation $\tilde{\delta} = \frac{\delta}{\gamma_0}$. Once we reconstruct $\tilde{\delta}$, we obtain $\delta = \gamma_0\tilde{\delta}$ and γ . An approximation of γ_0 was obtained by first reconstructing $\gamma(x)$ without any scaling of the DN map, and then choosing an appropriate value in the region of background in the images. In the following lines, we simply denote $\Lambda_{\tilde{\gamma}}$ by Λ_γ for the simplicity of the notation.

Most of the images in this paper are *absolute images*. That is, they are computed using only one experimental data set, without an experimental reference. *Difference images* show the conductivity change from a reference image, which in this paper is the homogeneous tank containing only saline. For the difference images, $\mathcal{D}(k)$ is replaced by $\mathcal{D}_{\text{dif}}$ defined by

$$\mathcal{D}_{\text{dif}}(k) \equiv \int_{\partial\Omega} \phi_1(\Lambda_\gamma - \Lambda_{\text{ref}})\phi_2 ds(x), \quad (33)$$

where Λ_{ref} is the DN map computed from the reference data.

For practical computation of the absolute images, first write

$$\begin{aligned} \mathcal{D}(k) &= \int_{\partial\Omega} \phi_1(\Lambda_\gamma - \Lambda_1)\phi_2 ds(x) \\ &= \int_{\partial\Omega} \phi_1\Lambda_\gamma\phi_2 ds(x) - \int_{\partial\Omega} \phi_1\Lambda_1\phi_2 ds(x). \end{aligned} \quad (34)$$

Now, let $x|_{\partial\Omega} = Re^{i\theta}$, where R is the radius of the tank. Then, by using the Taylor expansion of the exponential function, we get the Fourier type expansion of ϕ_1, ϕ_2 as

$$\phi_1 = e^{i\pi k \bar{x}} = \sum_{m=0}^{\infty} a_m(k) e^{-im\theta}, \phi_2 = e^{i\pi \bar{k} x} = \sum_{n=0}^{\infty} b_n(k) e^{in\theta},$$

where $a_m(k) = \frac{(i\pi k R)^m}{m!}$ and $b_n(k) = \frac{(i\pi \bar{k} R)^n}{n!}$. Let L be the number of electrodes, $\theta_l = \frac{2\pi l}{L}$ be the angle of the center of the l^{th} electrode, and let the m^{th} trigonometric current pattern on the l^{th} electrode be defined by

$$T_l^m = \begin{cases} M \cos m \theta_l, & m = 1, \dots, \frac{L}{2} - 1 \\ M \cos \pi l, & m = \frac{L}{2} \\ M \sin((m - L/2)\theta_l), & m = \frac{L}{2} + 1, \dots, L - 1, \end{cases}$$

where M is the amplitude of the applied current, and let the normalized current pattern

$t^m \equiv \frac{T^m}{\|T^m\|_{L^2}}$. Define an inner product $(u(\cdot), w(\cdot))_{L^2} = \sum_{l=1}^L \overline{u(\theta_l)} w(\theta_l)$. Let $\mathbf{L}$ be the

discretized DN map with entries $\mathbf{L}_{m,n} = \left(t_l^m, \left(\frac{R_Y}{A} \right)^{-1} t_l^n \right)_{L^2}$. The finite-dimensional discrete

approximation to the DN map has been presented in previous works, and the reader is referred to, for example, [12, 14, 28] for details of its computation. We model the current density j on the boundary by the ave-gap model as

$$j^m(x) = \begin{cases} \frac{T_l^m}{A_l}, & x \in e_l, \\ 0, & x \notin e_l, \end{cases}$$

where e_l denotes the l^{th} electrode. Then,

$$\begin{aligned}
\int_{\partial\Omega} \phi_1 \Lambda_\gamma \phi_2 ds(x) &\approx R \sum_{m=1}^{L-1} \sum_{n=1}^{L-1} a_m b_n \int_0^{2\pi} e^{-im\theta} \Lambda_\gamma e^{in\theta} d\theta, \\
&= R \sum_{m=1}^{L-1} \sum_{n=1}^{L-1} a_m b_n \int_0^{2\pi} (\cos m\theta - i \sin m\theta) \Lambda_\gamma (\cos n\theta + i \sin n\theta) d\theta \\
&= \frac{R\Delta\theta}{A} \sum_{m=1}^{L-1} \sum_{n=1}^{L-1} a_m b_n \left\{ \left(\cos m \cdot \left(\frac{R_\gamma}{A} \right)^{-1} \cos n \right)_{L^2} + i \left(\cos m \cdot \left(\frac{R_\gamma}{A} \right)^{-1} \sin n \right)_{L^2} \right. \\
&\quad \left. - i \left(\sin m \cdot \left(\frac{R_\gamma}{A} \right)^{-1} \cos n \right)_{L^2} + \left(\sin m \cdot \left(\frac{R_\gamma}{A} \right)^{-1} \sin n \right)_{L^2} \right\} \\
&= \frac{R\Delta\theta}{A} \left\{ \sum_{m=1}^{L/2} \sum_{n=1}^{L/2} a_m b_n \left(\mathbf{L}_{m, n} + \mathbf{L}_{\frac{L}{2}+m, \frac{L}{2}+n} + i \left(\mathbf{L}_{m, \frac{L}{2}+n} - \mathbf{L}_{\frac{L}{2}+m, n} \right) \right) \right. \\
&\quad \left. + \sqrt{2} \sum_{n=1}^{L/2-1} a_{\frac{L}{2}} b_n \left(\mathbf{L}_{\frac{L}{2}, n} + i \mathbf{L}_{\frac{L}{2}, \frac{L}{2}+n} \right) + \sqrt{2} \sum_{m=1}^{L/2-1} a_m b_{\frac{L}{2}} \left(\mathbf{L}_{m, \frac{L}{2}} - i \mathbf{L}_{\frac{L}{2}+m, \frac{L}{2}} \right) \right. \\
&\quad \left. + 2(-1)^{\frac{L}{2}} a_{\frac{L}{2}} b_{\frac{L}{2}} \mathbf{L}_{\frac{L}{2}, \frac{L}{2}} \right\}.
\end{aligned}$$

Here $\Delta\theta = \frac{2\pi}{L}$ is the angle between the center of electrodes. For the difference images, $\int_{\partial\Omega} \phi_1 \Lambda_{ref} \phi_2 ds(x)$ is computed with the same expansion technique as above. On the other hand for absolute images, from (4), $\int_{\partial\Omega} \phi_1 \Lambda_1 \phi_2 ds(x) = -2\pi^2 |k|^2 \int_{\Omega} e^{2\pi i x \cdot k} dx$, and we compute the integral $\int_{\Omega} e^{2\pi i x \cdot k} dx$ by using a quadrature rule. Note that this differs computationally from the methods of [12, 27, 28, 29] where γ is approximated by computing the inverse Fourier transform of $\hat{F}(k)$ from (3). The reason why we include the term $\int_{\partial\Omega} \phi_1 \Lambda_1 \phi_2 ds(x)$, while one could simply ignore it and reconstruct γ directly from the first term only (which is $\hat{F}(k)$), is to compensate for the artifact along the boundary caused by the Gibbs phenomena. This method improves the absolute images significantly, as illustrated in Figure 4 on the first order Calderón's method. Interestingly, the SSIM measure does not correlate in this case with the visual improvement. We conjecture that the blue ring of low conductivity near the boundary is the reason for this, since it more accurately approximates the background conductivity value than the background in the image with subtraction.

Figure 4 illustrates the absolute images for Case (i). (A) is the images computed from $\hat{F}(k)$ and (B) is the images with the subtraction of the synthesized homogeneous term. The truncation radius is $R_1 = 1.125$ for both images. As seen in Figure 4, the subtraction of the synthesized homogeneous term reduces the artifact along the boundary and therefore gives a better reconstruction.

4. Calderón's method with a spatial prior

We introduce a method of inclusion of *a priori* information to the first and second order Calderón's methods in the same spirit as [19] for the D-bar method. We assume that we know the approximate location of the inhomogeneous inclusions in Ω and approximate constant conductivity for each inclusion. We denote this *a priori* information by $\delta_{pr}(x)$ of which we will make use as follows. Define

$$\widehat{H}_{R_1, R_2}(k) = \begin{cases} \frac{1}{2\pi^2|k|^2} \mathcal{D}(k) \widehat{\eta}\left(\frac{k}{t}\right), & |k| \leq R_1, \\ \widehat{\delta}_{\text{pr}}(k) \widehat{\eta}(k/t), & R_1 < |k| \leq R_2, \\ 0, & |k| > R_2, \end{cases}$$

by appending $\widehat{\delta}_{\text{pr}}(k) \widehat{\eta}(k/t)$ to the outer region of $|k| < R_1$ and inside of $|k| = R_2$ for some R_1 and R_2 . Since $\widehat{\delta}_{\text{pr}}(k) \widehat{\eta}(k/t)$ is numerically stable, and it decreases to zero as $|k|$ tends to infinity, the outer truncation radius R_2 can be far bigger than R_1 , giving more weight to the prior as R_2 increases or as R_1 decreases. Therefore, by taking the inverse Fourier transform of (4), we define

$$\delta_{R_1, R_2}^{\text{app}}(x) = \mathcal{F}^{-1}(\widehat{H}_{R_1, R_2}(k))(x).$$

Since the inverse Fourier transform is a linear operation, this can be written as

$$\delta_{R_1, R_2}^{\text{app}}(x) = \delta_{R_1}^{\text{app}}(x) + \int_{R_1 < |k| < R_2} \widehat{\delta}_{\text{pr}}(k) \widehat{\eta}(k/t) e^{2\pi i k \cdot x} dk. \quad (35)$$

Therefore, we can think of $\delta_{R_1, R_2}^{\text{app}}(x)$ as the Calderón's method with a high pass filtered prior. We call (35) *the first order Calderon's method with a prior*. We apply this idea to (32) also and define

$$\delta_{R_1, R_2}^{\text{app,sec}}(x) = \delta_{R_1}^{\text{app,sec}}(x) + \int_{R_1 < |k| < R_2} \widehat{\delta}_{\text{pr}}(k) \widehat{\eta}(k/t) e^{2\pi i k \cdot x} dk,$$

and call it *the second order Calderon's method with a prior*.

5. Experimental results and discussion

5.1. Cases (i) and (ii)

Spatial priors without the pipes were constructed and used in the algorithm. To construct a prior conductivity γ_{pr} , we open the photo of Case (i) in Matlab, click points along the boundary of the heart and lungs to generate boundary data, assign approximate conductivity values to each organ and apply a mollification by using convolution to have a smooth prior. Since we never know the accurate conductivity distribution in practical applications, we assign rather incorrect values for each organ. We used a prior with conductivity values 0.323, 0.190, 0.123, and 0.123 for the heart, background, left lung, and right lung, respectively.

5.2. Results and discussion for Cases (i) and (ii)

The absolute and difference images computed by the second order Calderón's method with and without a prior for Cases (i) and (ii) are found in Figures 5 and 6. The first column shows the absolute images without a prior, the second column shows the absolute images

with a prior, the third column shows the difference images without a prior, and the last column shows the difference images with a prior. It is important to note that the prior does not include the pipes in the lung, mimicking the case of an unknown pathology in the lung. The inner truncation radius R_1 for each row is 1.125, 1.35, and 1.575 from the top to the bottom row, and for the mollification function $\hat{\eta}(k/t)$ in (4), we used $\exp(-\pi|k|^2/t)$, where t is 500, 50, and 5 for each row. As we increased the truncation radius R_1 , we also increased the degree of the mollification which results in more suppression on $\hat{H}(k)$ for high frequency $|k|$. Otherwise, $\hat{H}(k)$ blows up very quickly as the truncation radius increases.

The outer truncation radius R_2 is 3 for all reconstructions. Because we fixed R_2 , within which almost all of the practical information is included, a larger R_1 means that relatively more information from the measurement data is used and the influence of the prior is smaller. Therefore, there is a trade off between the measured data and the prior with increasing R_1 . With $R_1 = 1.125$ in the second column in Figure 5, the heart and lung regions are almost perfect, and we still can see the copper pipe. With bigger truncation radius R_1 , the copper pipe appears more clear. Note that the information of the copper pipe is included in the measured data only, but not in the prior. At the same time, with bigger R_1 , there are more artifacts in general as expected. The use of a prior with bigger R_1 doesn't dramatically reduce the artifacts since the influence of the prior is reduced as R_1 increases. On the other hand, with moderate R_1 values, the effect of the inclusion of a prior is significant. Even though the conductivity values for each organ differ in the prior from the true values and don't include the pipes, the use of the prior sharpens the organ boundaries and reduces the artifacts, and even helps to see the presence of the pipe. Therefore, choosing the right value for R_1 is of importance. For SSIM values, we used Figure 1 (D) and (E) as the reference images. SSIM values for Figure 5 indicate that reconstructions with a prior are closer to the reference image Figure 1 (D) than reconstructions without a prior. This is the same for the SSIM values of the first two columns of Figure 6. However, the SSIM values of the last two columns of Figure 6 demonstrate that the images with a prior are not closer to the reference image. This could be because the reference image Figure 1 (E) depicts the PVC pipe too sharply, while in the reconstructed images, the sharply contrasted conductivity inclusion, the PVC pipe in this case, appears very blurred.

5.3. Case (iii)

We also test our method on experimental tank data with three pieces of cucumber in the tank. The picture of the tank is shown in Figure 1 (C). Each slice of cucumber is about 4.9 cm in diameter. The data was collected using the ACE 1 system at 125 kHz with adjacent current patterns with amplitude 3.3 mA. The level of saline bath was 1.6 cm. Since cucumber has a cellular structure, and therefore a nonzero permittivity, $\gamma(x) = \sigma(x) + i\omega\epsilon(x)$ is now complex valued, and we reconstruct both the conductivity and the permittivity distributions at the same time. The conductivity of the saline bath was 0.180 S/m, and the conductivity and susceptibility of the cucumber were estimated to be 0.2 S/m and $9 * 10^{-5} S/m$, respectively, based on [26].

5.4. Results and discussion for Case (iii)

Second order difference images of the cucumber data are shown in Figure 7. The real part of the prior is depicted in (B) and the imaginary part is in (C). The first two columns are for the real part of the reconstructions without and with a prior, respectively. The last two columns are the imaginary part of the reconstructions without and with a prior, respectively. We used the prior (B) and (C) as the reference images for the SSIM values. The effect of a prior inclusion can be interpreted as the same way as in the other cases.

6. Conclusions

We have presented a higher order Calderón's method based on its relationship to the D-bar method that is easy to compute. Its improvement over the original Calderón's method is demonstrated on experimental data. Furthermore, a method of including *a priori* spatial information is introduced and demonstrated on experimental tank data. The methods and results provide new insight into Calderón's method and suggest that the higher order method with a prior could be a competitive alternative to existing direct and iterative methods for the computation of absolute images in electrical impedance tomography.

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Figure 1:

The three experimental data sets and reference images. Case (i): an agar heart and lungs in a saline bath with a copper pipe in one lung, Case (ii): an agar heart and lungs in a saline bath with a PVC pipe in one lung, Case (iii): three slices of cucumber in a saline bath. Figure (D) is a reference image for Case (i), Figure (E) is a reference image for Case (ii). Figure (F) and Figure (G) are the real part and the imaginary part reference images for Case (iii). The reference images are used for the SSIM measures.

Figure 2:

Absolute images for case (i) with $R_1 = 1.125$ for (A) and (B), $R_1 = 1.35$ for (C) and (D). (A) and (C) are the images by the 1st order Calderón's method. (B) and (D) are the images by the 2nd order Calderón's method. The values that are provided are the SSIM measures.

Figure 3:

The high contrast complex valued admittivity example where the admittivity of the heart, lungs, and the background are $2 + i(5.5 \times 10^{-6}) S/m$, $0.5 + i(1.4 \times 10^{-6}) S/m$ and $1 + i(3.55 \times 10^{-6}) S/m$, respectively. (A) is the ground truth for the real part, (B) is the ground truth for the imaginary part. R_1 is 1.125, 1.35 and 1.575 from the top row to the bottom row. 1st column is the real part of the first order reconstruction. 2nd column is the real part of the second order reconstruction. 3rd column is the imaginary part of the first order reconstruction. 4th column is the imaginary part of the second order reconstruction. The values that are provided are the SSIM measures computed against (A) or (B).

Figure 4:

The absolute images for Case (i) with $R_1 = 1.125$. Image (A) is the reconstruction without subtracting the synthesized homogeneous term. Image (B) is the reconstruction with subtracting the synthesized homogeneous term. The values that are provided are the SSIM measures.

(A) Case (i)
: Copper pipe

(B) Conductivity prior
with no pipe

R_1	Absolute image		Difference image	
	w/o a prior	with prior (B)	w/o prior	with prior (B)
1.125	 0.43693	 0.59878	 0.58095	 0.68145
1.35	 0.44307	 0.52666	 0.65459	 0.65969
1.575	 0.39746	 0.47348	 0.61054	 0.60626

Figure 5:

The reconstructed γ for Case (i). The background conductivity is added to the reconstruction. (A) is the picture of the Case (i), (B) is the prior, which does not include the presence of the pipe. R_1 is 1.125, 1.35 and 1.575 from the top row to the bottom row. $R_2 = 3$ for all cases. 1st column is 2nd order absolute images without a prior. 2nd column is 2nd order absolute images with a prior. 3rd column is 2nd order difference images without a prior. 4th column is 2nd order difference images with a prior. The values that are provided are the SSIM measures.

(A) Case (ii)
:PVC pipe

(B) Conductivity prior
no pipe

R	Absolute image		Difference image	
	w/o prior	with prior (B)	w/o prior	with prior (B)
1.125	 0.75173	 0.81248	 0.76211	 0.75227
1.35	 0.75600	 0.78277	 0.73502	 0.72791
1.575	 0.62665	 0.66040	 0.76713	 0.78067

Figure 6:

The reconstructed γ for Case (ii). The background conductivity is added to the reconstruction. (A) is the picture of the tank, (B) is the prior, which does not include the presence of the pipe. R_1 is 1.125, 1.35 and 1.575 from the top row to the bottom row. $R_2 = 3$ for all cases. 1st column is 2nd order absolute images without a prior. 2nd column is 2nd order absolute images with a prior. 3rd column is 2nd order difference images without a prior. 4th column is 2nd order difference images with a prior. The values that are provided are the SSIM measures.

Figure 7:

The 2nd order difference images of Case (iii) with and without a prior. The background conductivity is added to the reconstructions. (A) is the picture of case (iii), (B) is the real part of the prior and (C) is the imaginary part of the prior. R_1 is 1.125, 1.35 and 1.575 from the top row to the bottom row. $R_2 = 3$ for all cases. 1st column is the real part of the images without a prior. 2nd column is the real part of the images with a prior. 3rd column is the imaginary part of the images without a prior. 4th column is the imaginary part of the images with a prior. The values that are provided are the SSIM measures.