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Author for correspondence:

Thomas J. Bridges

e-mail: T.Bridges@surrey.ac.uk

A proof of validity for multiphase Whitham modulation theory

Thomas J. Bridges¹, Anna Kostianko^{1,2} and Guido Schneider³

¹Department of Mathematics, University of Surrey, Guildford, Surrey GU2 7XH, UK

²School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, People's Republic of China

³Institut für Analysis, Dynamik und Modellierung, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany

TJB, 0000-0003-0855-0747; AK, 0000-0002-0950-9079

It is proved that approximations which are obtained as solutions of the multiphase Whitham modulation equations stay close to solutions of the original equation on a natural time scale. The class of nonlinear wave equations chosen for the starting point is coupled nonlinear Schrödinger equations. These equations are not in general integrable, but they have an explicit family of multiphase wavetrains that generate multiphase Whitham equations, which may be elliptic, hyperbolic, or of mixed type. Due to the change of type, the function space set-up is based on Gevrey spaces with initial data analytic in a strip in the complex plane. In these spaces a Cauchy–Kowalevskaya-like existence and uniqueness theorem is proved. Building on this theorem and higher-order approximations to Whitham theory, a rigorous comparison of solutions, of the coupled nonlinear Schrödinger equations and the multiphase Whitham modulation equations, is obtained.

1. Introduction

Given a periodic travelling wave of a conservative nonlinear wave equation, generated by a Lagrangian, Whitham modulation theory, in its simplest one-phase form, is a perturbation theory where the wavenumber k and frequency ω of the travelling wave are perturbed and allowed to vary slowly in time and space, thereby capturing modulation of the basic wave.

The theory reduces the original nonlinear wave equation to a pair of first-order quasilinear PDEs

$$\partial_T q = \partial_X \Omega \quad \text{and} \quad \partial_T \mathcal{A}(\omega + \Omega, k + q) + \partial_X \mathcal{B}(\omega + \Omega, k + q) = 0, \quad (1.1)$$

where $\Omega(X, T, \varepsilon)$ is the slowly varying frequency and $q(X, T, \varepsilon)$ is the slowly varying wavenumber. The independent variables in (1.1) are slow time and space coordinates, $T = \varepsilon t$ and $X = \varepsilon x$, with $0 < \varepsilon \ll 1$. The function $\mathcal{A}$ is the wave action and $\mathcal{B}$ is the wave action flux, and they are determined from a given Lagrangian, and satisfy $\partial_k \mathcal{A} = \partial_\omega \mathcal{B}$.

Analysis of the Whitham modulation equations (WMEs) in (1.1) then leads to deductions about the effect of perturbations on the original periodic travelling wave. For example when the pair (1.1) is elliptic (hyperbolic) the original periodic travelling wave is unstable (stable) to long wave perturbations, with appropriate hypotheses (cf. Benzoni-Gavage *et al.* [1]). There is now a vast literature on the reduction process, asymptotics and analysis of the WMEs for a wide range of nonlinear wave equations, generated by a Lagrangian, in the case where the basic state is a single-phase travelling wave (e.g. Whitham [2], Kamchatnov [3], Biondini *et al.* [4], Bridges [5] and references therein).

An obvious question is how accurate the solutions of (1.1) are when compared to solutions of the original equation. From a physical point of view, the interest in validity is that an approximation should represent the solutions of the original equation as closely as possible. There are many examples where approximate equations are *not valid* (a history with examples is in Sunny [6]), and so there is no reason to assume *a priori* that even asymptotically valid approximations are rigorously valid. A rigorous comparison between approximate and exact solutions requires introduction of a metric and an existence theory in a function space large enough to accommodate the range of solutions expected of nonlinear wave equations on the real line.

A proof of the validity of the WMEs (1.1) has been given by Düll & Schneider [7] when the original equation is the cubic nonlinear Schrödinger (NLS) equation

$$i\partial_t \Psi + \partial_x^2 \Psi + \gamma |\Psi|^2 \Psi = 0, \quad (1.2)$$

where $\Psi(x, t)$ is complex valued, $\gamma = \pm 1$, $x \in \mathbb{R}$ and $t \geq 0$. First, Ψ is expressed in the form

$$\Psi(x, t) = \exp(r(x, t) + i\phi(x, t)), \quad (1.3)$$

where r and ϕ are real-valued, and then a pair of equations for r and $v := \partial_x \phi$ is derived. The exact equations for r and v are recast in terms of the same independent variables, $X = \varepsilon x$ and $T = \varepsilon t$, as in the reduced equations,

$$r(x, t) = \check{r}(X, T, \varepsilon) \quad \text{and} \quad v(x, t) = \check{v}(X, T, \varepsilon). \quad (1.4)$$

The strategy is then to compare the solutions $\check{r}(X, T, \varepsilon)$ and $\check{v}(X, T, \varepsilon)$ of the exact equations to solutions of the WMEs for $\varepsilon > 0$ sufficiently small.

The WMEs for (1.2) are deduced by modulating the basic travelling wave solution,

$$\psi_0(x, t) = e^{r_0 + i(kx + \omega t + \theta_0)}, \quad \text{with } \omega + k^2 - \gamma e^{2r_0} = 0. \quad (1.5)$$

The modulation mapping

$$r_0 \mapsto r^*(X, T) \quad \text{and} \quad k \mapsto k + v^*(X, T), \quad (1.6)$$

leads to a form of the WMEs (1.1) in terms of $r^*(X, T)$ and $v^*(X, T)$.

The rigorous approximation result for (1.2) is as follows. Given a solution of the NLS equation in coordinates (1.4) and a solution of the WMEs in coordinates (1.6), with initial data satisfying

$$\|(\check{r}(X, T, \varepsilon), \check{v}(X, T, \varepsilon))|_{T=0} - (r^*(X, T), v^*(X, T))|_{T=0}\| = \mathcal{O}(\varepsilon), \quad (1.7)$$

in a suitably chosen norm $\|\cdot\|$, the main validity result in [7] is

$$\sup_{T \in [0, T_1]} \sup_{X \in \mathbb{R}} |(\check{r}(X, T, \varepsilon), \check{v}(X, T, \varepsilon)) - (r^*(X, T), v^*(X, T))| \leq C_2 \varepsilon \quad \text{for all } \varepsilon \in (0, \varepsilon_0), \quad (1.8)$$

where ε_0 , C_2 and T_1 are all positive constants. From this estimate we can conclude that the reduction of the NLS equation (1.2), in the neighbourhood of the family of periodic travelling waves (1.5), to the WMEs, is valid on the natural time scale, $t = \mathcal{O}(\varepsilon^{-1})$.

For the single NLS equation, there is a close connection between reduction to the WMEs and the semiclassical approximation of NLS, and this problem has been extensively studied, particularly in the case of defocusing NLS (e.g. [3,8] and references therein). A related problem is the dispersionless limit of the KdV equation [3,9–11]. The limit in both cases can be very delicate and oscillatory. The most detailed results on the limit have been obtained using integrability of the NLS equation and KdV equations.

In moving from the single NLS to coupled NLS, and moving from single-phase WMEs to multiphase WMEs, new challenges arise. Integrability is rare and so methods that do not require integrability are needed. An additional challenge is the fact that the multiphase WMEs do not, in general, preserve the characteristic type (elliptic, hyperbolic or mixed) under evolution. In addition, even if the hyperbolicity is preserved on some specific trajectory of multiphase WMEs, the linear stability of the background wavetrain may be lost during the evolution. For these reasons, a validity result in Sobolev spaces for multiphase WMEs is problematic no matter how regular these Sobolev spaces are. On the other hand, by taking initial data that is analytic in a strip about the real axis, Gevrey spaces capture all trajectories regardless of whether the characteristics are elliptic, hyperbolic or mixed.

A proof of validity requires three steps (e.g. [7,12]): (i) a local existence and uniqueness theory for the WMEs (1.1), (ii) a local existence and uniqueness theory for solutions of the original equation (1.2), and (iii) an approximation theory for the difference between the two solutions. The backbone of all three steps is the choice of function space, and the choice of coordinates, with the latter chosen to facilitate the analysis. The function space has to be large enough to include all bounded solutions on the real line, and it has to account for the fact that the WMEs may be elliptic and so not well posed. Inspired by [7], the function space to be used here is a scale of Gevrey spaces.

A two-phase wavetrain is a solution of a nonlinear wave equation of the form

$$\mathbf{u}(x, t) = \widehat{\mathbf{u}}(\theta_1, \theta_2), \quad \theta_j = k_j x + \omega_j t + \theta_j^0, \quad j = 1, 2,$$

with

$$\widehat{\mathbf{u}}(\theta_1 + 2\pi, \theta_2) = \widehat{\mathbf{u}}(\theta_1, \theta_2) \quad \text{and} \quad \widehat{\mathbf{u}}(\theta_1, \theta_2 + 2\pi) = \widehat{\mathbf{u}}(\theta_1, \theta_2).$$

The wavenumbers $\mathbf{k} = (k_1, k_2)$ and frequencies $\boldsymbol{\omega} = (\omega_1, \omega_2)$ are in general distinct and θ_1^0 and θ_2^0 are constant phase shifts. A multiphase wavetrain is the generalization of this form to N -phases with N finite.

The modulation of multiphase wavetrains, from the perspective of Whitham theory starting from a Lagrangian, was first studied by Ablowitz & Benney [13]. They derived the conservation of wave action for scalar fields with two phases in detail, and showed how the theory generalized to N phases. Examples in Ablowitz [14] show that in general one should expect small divisors, but weakly nonlinear solutions could still be obtained. However, for integrable systems, multiphase averaging and the WMEs are robust and rigorous, without small divisors, and a general theory can be obtained. There is now a vast literature on multiphase WMEs for integrable systems (see Flaschka *et al.* [15] and its citation trail). On the other hand, if the system is not integrable, but there is an N -fold toral symmetry, then again a theory for conservation of wave action and multiphase WMEs can be developed without small divisors and smoothly varying N -phase wavetrains (see Ratliff [16]). The action of an N -fold toral symmetry, $\mathbb{T}^N$, in the context of coupled NLS equations,

can be represented as

$$(\psi_1, \dots, \psi_N) \mapsto (e^{i\theta_1} \psi_1, \dots, e^{i\theta_N} \psi_N), \quad (\theta_1, \dots, \theta_N) \in S^1 \times \dots \times S^1 := \mathbb{T}^N,$$

with the $N = 2$ case needed for the coupled NLS equation introduced below. In essence the basic wavetrain is aligned with the toral symmetry, and the conservation of wave action is replaced by the conservation law generated by the symmetry. It is this latter class of multiphase WMEs whose validity is of interest here.

In approaching the validity problem for multiphase WMEs, a general theory starting from an abstract Lagrangian is at present intractable. Therefore, we restrict attention to the case of modulation of two-phase wavetrains of coupled nonlinear Schrödinger (CNLS) equations where a rigorous and complete reduction theory can be obtained.

CNLS equations arise in a wide range of applications (e.g. models for Bose–Einstein condensates [17], the theory of water waves and rogue waves [18,19], nonlinear optics [20]). For definiteness, we take the following form for the CNLS equations as a starting point

$$\text{and} \quad \left. \begin{aligned} i\partial_t \psi_1 + \partial_x^2 \psi_1 + \gamma_1 |\psi_1|^2 \psi_1 + \alpha |\psi_2|^2 \psi_1 &= 0 \\ i\partial_t \psi_2 + \partial_x^2 \psi_2 + \alpha |\psi_1|^2 \psi_2 + \gamma_2 |\psi_2|^2 \psi_2 &= 0. \end{aligned} \right\} \quad (1.9)$$

These equations are known to be integrable for only very special values of the coefficients [21]. Here integrability is not assumed and the coefficients are free to take any values with $\alpha \in \mathbb{R} \setminus \{0\}$ and $\gamma_j \in \{-1, 1\}$ and the non-degeneracy constraint

$$\gamma_1 \gamma_2 - \alpha^2 \neq 0. \quad (1.10)$$

The CNLS equation (1.9) has an explicit four-parameter family of two-phase wavetrains. Modulation of this family of two-phase wavetrains generates a system of multiphase WMEs.

To prove validity the pair of equations (1.9) is first transformed to the form

$$\mathbf{u}_T = \mathbf{M}(\mathbf{u})\mathbf{u}_X + \varepsilon^2 \mathbf{F}(D_X^3 \mathbf{u}), \quad (1.11)$$

where $\mathbf{u}$ has four components (based on the real and imaginary parts of ψ_1 and ψ_2), $\mathbf{M}(\mathbf{u})$ is entire, $\mathbf{F}$ is a polynomial in $D_X^k \mathbf{u} = (\mathbf{u}, \partial_X \mathbf{u}, \dots, \partial_X^k \mathbf{u})$. The independent variables x, t are scaled as $T = \varepsilon t$ and $X = \varepsilon x$.

The principal advantage of the form (1.11) is that when $\varepsilon = 0$ it reduces to the multiphase WMEs. Hence if $\mathbf{u}^*$ is a solution of the WMEs

$$\partial_T \mathbf{u}^* = \mathbf{M}(\mathbf{u}^*) \partial_X \mathbf{u}^*, \quad (1.12)$$

then the validity proof is obtained by studying the difference $\|\mathbf{u}(X, T, \varepsilon) - \mathbf{u}^*(X, T)\|$ as a function of time in an appropriate function space, which in this case is a Gevrey space.

After choosing appropriate coordinates and introducing the properties of Gevrey spaces, the steps of the validity theory are set in motion. Firstly, existence is proved for the WMEs in §3a, using an abstract Cauchy–Kowalevskaya-type theorem developed in §3. Secondly, improved approximations (higher-order Whitham theory) are obtained in §4. The generated perturbation series is not necessarily convergent, and so the exact solution (perturbation series plus remainder) is studied in §5. In §5 an abstract theory is developed for general systems of the form (1.12), and then a summary for the special case of CNLS is given in §5a, thereby completing the proof of validity. In the concluding remarks section implications and generalizations are discussed. In the appendix some supplementary results on Gevrey spaces are proved.

Remarks. (a) For notational simplicity we have restricted to (1.9), although it is not the most general form of the CNLS equations. By rescaling x, t, ψ_1 and ψ_2 in general a normal form of the CNLS equations with one additional parameter in front of one of the x -derivative terms is obtained. However, for our purposes the above form (1.9) is not a restriction. (b) Throughout the paper, many different constants are denoted with the same symbol C if they can be chosen independently of the small, $0 < \varepsilon \ll 1$, perturbation parameter.

2. Formal derivation of multiphase Whitham modulation equations

The basic two-phase wavetrain of CNLS is

$$\Psi_j(x, t) = \psi_j(\theta, \omega, \mathbf{k}) := \psi_j(\omega, \mathbf{k}) e^{i\theta_j(x, t)}, \quad \theta_j(x, t) = k_j x + \omega_j t + \theta_j^0, \quad j = 1, 2, \quad (2.1)$$

with $\theta = (\theta_1, \theta_2)$, $\omega = (\omega_1, \omega_2)$ and $\mathbf{k} = (k_1, k_2)$, and the amplitudes $\psi = (\psi_1, \psi_2)$ are real-valued. Substitution into the governing equations (1.9) generates a relationship between the amplitudes and the frequencies and wavenumbers,

$$\left. \begin{aligned} \psi_1(\omega, \mathbf{k})^2 &= \frac{1}{\beta} \left(\gamma_2(\omega_1 + k_1^2) - \alpha(\omega_2 + k_2^2) \right) \\ \psi_2(\omega, \mathbf{k})^2 &= \frac{1}{\beta} \left(\gamma_1(\omega_2 + k_2^2) - \alpha(\omega_1 + k_1^2) \right) \end{aligned} \right\} \quad (2.2)$$

and

with $\beta = \gamma_1 \gamma_2 - \alpha^2$, which is non-zero due to (1.10).

The traditional approach to deriving the WMEs is to use an averaged Lagrangian (e.g. chapter 14 of Whitham [2]). The CNLS equations (1.9) are formally the Euler–Lagrange equation for

$$\delta \int_{t_1}^{t_2} \int_{x_1}^{x_2} L(\Psi_t, \Psi_x, \Psi) dx dt = 0, \quad (2.3)$$

with $\Psi := (\Psi_1, \Psi_2)$, fixed endpoint variations on $\delta\Psi$, and

$$\begin{aligned} L &= \frac{i}{2} (\overline{\Psi_1}(\Psi_1)_t - \Psi_1(\overline{\Psi_1})_t) + \frac{i}{2} (\overline{\Psi_2}(\Psi_2)_t - \Psi_2(\overline{\Psi_2})_t) \\ &\quad - |(\Psi_1)_x|^2 - |(\Psi_2)_x|^2 + \frac{1}{2} \gamma_1 |\Psi_1|^4 + \alpha |\Psi_1|^2 |\Psi_2|^2 + \frac{1}{2} \gamma_2 |\Psi_2|^4, \end{aligned} \quad (2.4)$$

with the overline indicating complex conjugate. The basic state (2.1) is substituted into (2.4) and L is averaged over the two phases reducing it to $\mathcal{L}^{avg}(\omega, \mathbf{k})$. This averaged Lagrangian is then assumed to depend slowly on $X = \varepsilon x$ and $T = \varepsilon t$, and a secondary variational principle is introduced

$$\delta \int_{T_1}^{T_2} \int_{X_1}^{X_2} \mathcal{L}^{avg}(\omega + \phi_T, \mathbf{k} + \phi_X) dX dT = 0, \quad (2.5)$$

obtained by replacing $\omega \mapsto \omega + \phi_T$ and $\mathbf{k} \mapsto \mathbf{k} + \phi_X$. Taking variations with respect to ϕ , with fixed endpoints, generates the vector-valued conservation of wave action

$$\frac{\partial}{\partial T} \left(\frac{\partial \mathcal{L}^{avg}}{\partial \omega} \right) + \frac{\partial}{\partial X} \left(\frac{\partial \mathcal{L}^{avg}}{\partial \mathbf{k}} \right) = 0.$$

Defining $\mathbf{q} = \phi_X$ and $\boldsymbol{\Omega} = \phi_T$, and adding in the integrability condition $\partial_T \mathbf{q} = \partial_X \boldsymbol{\Omega}$ gives the two-phase WMEs. This is the classic derivation that can be found in [13] and §14.9 of [2].

A direct approach, which is more amenable to rigorous analysis, is to start with the exact geometric optics ansatz,

$$\Psi(x, t) = \Psi(\theta + \varepsilon^{-1} \phi, \omega + \boldsymbol{\Omega}, \mathbf{k} + \mathbf{q}) + \varepsilon \mathbf{W}(\theta + \varepsilon^{-1} \phi, X, T, \varepsilon), \quad (2.6)$$

with

$$\psi_j(\theta + \varepsilon^{-1} \phi, \omega + \boldsymbol{\Omega}, \mathbf{k} + \mathbf{q}) := e^{i(\theta_j + \varepsilon^{-1} \phi_j)} \psi_j(\omega + \boldsymbol{\Omega}, \mathbf{k} + \mathbf{q}), \quad j = 1, 2,$$

where the function ϕ , $\boldsymbol{\Omega}$ and $\mathbf{q}$ are slowly varying functions of $X = \varepsilon x$ and $T = \varepsilon t$, and $\mathbf{W}$ is a remainder term, with the constraint $\partial_T \mathbf{q} = \partial_X \boldsymbol{\Omega}$ imposed.

The expression (2.6) is substituted into the governing equations (1.9) and expanded order by order in ε . Invoking a solvability condition at second order then generates the WMEs directly

$$\partial_T \mathcal{A}_j(\omega + \boldsymbol{\Omega}, \mathbf{k} + \mathbf{q}) + \partial_X \mathcal{B}_j(\omega + \boldsymbol{\Omega}, \mathbf{k} + \mathbf{q}) = 0, \quad j = 1, 2, \quad \text{and} \quad \partial_T \mathbf{q} = \partial_X \boldsymbol{\Omega}. \quad (2.7)$$

The first two equations in (2.7) are conservation of wave action and the second two equations are imposed as a constraint based on the conservation of waves (see Ratliff [16] for details of the geometric optics approach).

We now derive the multiphase WMEs by a pure multiscale analysis, which is closer in spirit to the forthcoming validity proof. In this setting, the Whitham equations are obtained by setting ε to zero. Start with the CNLS equations in (1.9). Introduce

$$\psi_1 = \exp(r_1 + i(\phi_1 + \gamma_1 t + \alpha t)) \quad \text{and} \quad \psi_2 = \exp(r_2 + i(\phi_2 + \gamma_2 t + \alpha t)).$$

Here, r_1, r_2, ϕ_1, ϕ_2 are functions of (x, t) and $\gamma_1, \gamma_2, \alpha$ are the same constants as in (1.9). For notational simplicity we restrict attention to the case $\mathbf{k} = 0$ and $\omega = (\gamma_1 + \alpha, \gamma_2 + \alpha)$ although the general case is completely analogous. Note also that under this restriction the background wave train $(r_1, \phi_1, r_2, \phi_2) \equiv 0$ corresponds to a time periodic solution $\psi_i(t) = e^{i(\gamma_i + \alpha)t}$ which is constant in space.

Substitution into (1.9) gives

$$\begin{aligned} \partial_t r_1 &= -\partial_x^2 \phi_1 - 2(\partial_x r_1)(\partial_x \phi_1), \\ \partial_t \phi_1 &= \partial_x^2 r_1 - (\partial_x \phi_1)^2 + (\partial_x r_1)^2 + \gamma_1(e^{2r_1} - 1) + \alpha(e^{2r_2} - 1), \\ \partial_t r_2 &= -\partial_x^2 \phi_2 - 2(\partial_x r_2)(\partial_x \phi_2), \\ \partial_t \phi_2 &= \partial_x^2 r_2 - (\partial_x \phi_2)^2 + (\partial_x r_2)^2 + \gamma_2(e^{2r_2} - 1) + \alpha(e^{2r_1} - 1). \end{aligned}$$

At this point, the CNLS equation has just been transformed to polar coordinates with moduli (e^{r_1}, e^{r_2}) . To bring the equations into line with the perturbed Whitham equations (1.12), differentiate the second and fourth equations with respect to x and introduce the new coordinates

$$v_1 = \frac{\partial \phi_1}{\partial x} \quad \text{and} \quad v_2 = \frac{\partial \phi_2}{\partial x}.$$

Then the governing equations for $\mathbf{u} := (r_1, v_1, r_2, v_2)$ are

$$\partial_t \mathbf{u} = \mathbf{M}(\mathbf{u}) \partial_x \mathbf{u} + \mathbf{F}(D_x^3 \mathbf{u}), \quad (2.8)$$

with

$$\mathbf{M}(\mathbf{u}) = \begin{bmatrix} -2v_1 & -1 & 0 & 0 \\ 2\gamma_1 \exp(2r_1) & -2v_1 & 2\alpha \exp(2r_2) & 0 \\ 0 & 0 & -2v_2 & -1 \\ 2\alpha \exp(2r_1) & 0 & 2\gamma_2 \exp(2r_2) & -2v_2 \end{bmatrix}, \quad (2.9)$$

and

$$\mathbf{F}(D_x^3 \mathbf{u}) := \begin{pmatrix} 0 \\ \partial_x^3 r_1 + \partial_x(\partial_x r_1)^2 \\ 0 \\ \partial_x^3 r_2 + \partial_x(\partial_x r_2)^2 \end{pmatrix}. \quad (2.10)$$

Upon introducing scaled variables, $X = \varepsilon x$ and $T = \varepsilon t$, with $0 \leq \varepsilon \ll 1$, and

$$\mathbf{u}(x, t) = \check{\mathbf{u}}(X, T, \varepsilon), \quad (2.11)$$

the scaled equations are

$$\partial_T \check{\mathbf{u}} = \mathbf{M}(\check{\mathbf{u}}) \partial_X \check{\mathbf{u}} + \varepsilon^2 \mathbf{F}(D_X^3 \check{\mathbf{u}}). \quad (2.12)$$

Formally taking the limit $\varepsilon \rightarrow 0$ recovers the multiphase WMEs

$$\partial_T \mathbf{u} = \mathbf{M}(\mathbf{u}) \partial_X \mathbf{u}. \quad (2.13)$$

The characteristics of (2.13) have been studied in Bridges & Ratliff [22] and it is found numerically that the system can be either hyperbolic, elliptic or mixed depending on parameter values. The linear stability of multiphase plane waves of CNLS can be studied directly, and for parameter values when CNLS (1.9) is integrable, extensive linear stability results are obtained in Degasperis *et al.* [21]. However, in general the correlation between characteristic type in the WMEs and linear stability in CNLS is not one-to-one, as CNLS may have short wave instabilities which are missed by the WMEs approximation (cf. Benzoni-Gavage *et al.* [1]).

3. Cauchy–Kowalevskaya theory in Gevrey spaces

In this section we prove an abstract local existence theorem for quasilinear PDEs of the form

$$\partial_T \mathbf{u} = \mathbf{M}(\mathbf{u}) \partial_X \mathbf{u}, \quad \mathbf{u}|_{T=0} = \mathbf{u}_0. \quad (3.1)$$

Here $X \in \mathbb{R}$ and $T \geq 0$ are scaled variables, but since the form of the equation (3.1) is the same in scaled and unscaled variables, the result applies in either. The unknown vector-valued function $\mathbf{u} = (u_1(X, T), \dots, u_d(X, T))$ is d -dimensional. The matrix $\mathbf{M}(\mathbf{u})$ is a given entire function and $\mathbf{u}_0 \in G_\sigma^s(\mathbb{R})$, $s > 1$. Here G_σ^s is a Gevrey space defined by the norm

$$\|\mathbf{u}\|_{G_\sigma^s} := \|e^{\sigma(|\xi|+1)}(1 + |\xi|^{2s})^{1/2} \hat{\mathbf{u}}(\xi)\|_{L^2}. \quad (3.2)$$

We will use the same notation G_σ^s for scalar-valued and vector-valued functions.

The required analogue of the Cauchy–Kowalevskaya theorem for equation (3.1) is as follows. Results of this type were first proved by Ovsyannikov [23]; see also Treves [24] for an elaboration of theorems of Cauchy–Kowalevskaya type.

Theorem 3.1. *Let $s > 1$ and $\sigma_0 > 0$. Then, for every $R > 0$, there exist $\eta = \eta(R, s, \sigma_0)$ such that for every $\mathbf{u}_0 \in G_{\sigma_0}^s$, with $\|\mathbf{u}_0\|_{G_{\sigma_0}^s} \leq R$, there exists a unique local solution $\mathbf{u}(T) \in G_{\sigma(T)}^s$ of problem (3.1) with $\sigma(T) := \sigma_0 - \eta T$, $T \in [0, \sigma_0/\eta]$, and $\sup_{T \in [0, \sigma_0/\eta]} \|\mathbf{u}(T)\|_{G_{\sigma(T)}^s} \leq R$.*

Proof. We first give the formal derivation of *a priori* estimates for $\mathbf{u}(T)$ in the corresponding Gevrey spaces. This derivation can be justified *a posteriori* in a standard way, for instance, using the vanishing viscosity method (which will be briefly elaborated below).

In the proof, we will need three standard facts about Gevrey spaces:

- (1) G_σ^s is an algebra for $s > \frac{1}{2}$, i.e., if $u, v \in G_\sigma^s$ then $uv \in G_\sigma^s$ and

$$\|uv\|_{G_\sigma^s} \leq C_s \|u\|_{G_\sigma^s} \|v\|_{G_\sigma^s}, \quad (3.3)$$

where the constant C_s is independent of $\sigma > 0$. In the vector-valued case the product is replaced by an inner product on $\mathbb{R}^d$. The formula (3.3) can be generalized as follows:

$$\|uv\|_{G_\sigma^s} \leq C_s \left(\|u\|_{G_\sigma^s} \|v\|_{G_\sigma^1} + \|u\|_{G_\sigma^1} \|v\|_{G_\sigma^s} \right) \quad (3.4)$$

which holds for all $s \geq 0$. The exponent 1 here can be replaced by any $\kappa > 1/2$.

- (2) For any entire function ϕ such that $\phi(0) = 0$ there is an entire function $\phi_s(z)$, which is positive and monotone increasing for $z \in \mathbb{R}_+$ and $\phi_s(0) = 0$ such that

$$\|\phi(u)\|_{G_\sigma^s} \leq \phi_s(\|u\|_{G_\sigma^s}), \quad u \in G_\sigma^s. \quad (3.5)$$

- (3) Let $u \in G_\sigma^s$ for some s and σ . Then

$$\|u\|_{G_{\sigma-\delta}^{s+p}} \leq C(p, \delta) \|u\|_{G_\sigma^s}, \quad \delta, p > 0, \quad (3.6)$$

where $C(p, \delta)$ is independent of s and σ .

A summary of the basic properties of Gevrey spaces, in the context of parabolic PDEs, is given in Ferrari & Titi [25]. We supplement those results, with some proofs that are needed here, in appendix A.

Now let $A := \sqrt{-\partial_X^2}$, and multiply equation (3.1) by $e^{2\sigma(T)(1+A)}(1 + A^{2s})\mathbf{u}$ and integrate over $X \in \mathbb{R}$. After standard calculations, this gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dT} \|\mathbf{u}\|_{G_{\sigma(T)}^s}^2 + \eta \|(1 + A)^{1/2} \mathbf{u}\|_{G_{\sigma(T)}^s}^2 &= (\mathbf{M}(\mathbf{u}) \partial_X \mathbf{u}, \mathbf{u})_{G_{\sigma(T)}^s} \\ &= (\mathbf{M}(\mathbf{u}) - \mathbf{M}(0)) \partial_X \mathbf{u}, \mathbf{u})_{G_{\sigma(T)}^s} \\ &\quad + (\mathbf{M}(0) \partial_X \mathbf{u}, \mathbf{u})_{G_{\sigma(T)}^s}, \end{aligned} \quad (3.7)$$

where $(\mathbf{u}, \mathbf{v})_{G_\sigma^s}$ is the natural inner product in G_σ^s . Now use the Cauchy–Schwarz inequality

$$(\mathbf{u}, \mathbf{v})_{G_\sigma^s} \leq \|\mathbf{u}\|_{G_\sigma^{s-1/2}} \|\mathbf{v}\|_{G_\sigma^{s+1/2}},$$

together with (3.3) and (3.5) and the assumption $s - \frac{1}{2} > \frac{1}{2}$, to end up with

$$\frac{1}{2} \frac{d}{dT} \|\mathbf{u}\|_{G_{\sigma(T)}^s}^2 + \eta \|\mathbf{u}\|_{G_{\sigma(T)}^{s+1/2}}^2 \leq \|\mathbf{M}(0)\| \|\mathbf{u}\|_{G_{\sigma(T)}^{s+1/2}}^2 + \phi_s(\|\mathbf{u}\|_{G_{\sigma(T)}^{s-1/2}}) \|\mathbf{u}\|_{G_{\sigma(T)}^{s+1/2}}^2,$$

where we have used that

$$\begin{aligned} ((\mathbf{M}(\mathbf{u}) - \mathbf{M}(0))\partial_X \mathbf{u}, \mathbf{u})_{G_\sigma^s} &\leq \|(\mathbf{M}(\mathbf{u}) - \mathbf{M}(0))\partial_X \mathbf{u}\|_{G_\sigma^{s-1/2}} \|\mathbf{u}\|_{G_\sigma^{s+1/2}} \\ &\leq C \|\mathbf{M}(\mathbf{u}) - \mathbf{M}(0)\|_{G_\sigma^{s-1/2}} \|\mathbf{u}\|_{G_\sigma^{s+1/2}}^2 \leq \phi_s(\|\mathbf{u}\|_{G_\sigma^{s-1/2}}) \|\mathbf{u}\|_{G_\sigma^{s+1/2}}^2, \end{aligned}$$

for some smooth monotone increasing function ϕ_s associated with $\mathbf{M}$.

Thus, we arrive at

$$\frac{1}{2} \frac{d}{dT} \|\mathbf{u}\|_{G_{\sigma(T)}^s}^2 + \left(\eta - \|\mathbf{M}(0)\| - \phi_s(\|\mathbf{u}\|_{G_{\sigma(T)}^s}) \right) \|\mathbf{u}\|_{G_{\sigma(T)}^{s+1/2}}^2 \leq 0, \quad (3.8)$$

for a possibly different $\phi_s(\cdot)$, which for notational simplicity we denote with the same symbol, noting that $\|\mathbf{u}\|_{G_\sigma^{s-1/2}} \leq C \|\mathbf{u}\|_{G_\sigma^s}$ for some $C \geq 1$.

Now, fix η in such a way that $\eta > \|\mathbf{M}(0)\| + \phi_s(R)$. Then estimate (3.8) guarantees that

$$\|\mathbf{u}(T)\|_{G_{\sigma(T)}^s} \leq R \quad (3.9)$$

until $\sigma(T)$ vanishes. Recall that the derivation of (3.9) was formal. However, if we replace $\sigma(T)$ by $\sigma_\mu(T) := \sigma(T) - \mu$ for any positive μ , all manipulations involved in the derivation of (3.9) become rigorous and all corresponding terms make sense if the solution satisfies $\mathbf{u}(T) \in G_{\sigma(T)}^s$. Importantly, the obtained estimates are uniform with respect to μ , so passing to the limit $\mu \rightarrow 0$, we may justify estimate (3.9).

To confirm uniqueness, let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two analytic solutions of (3.1) (belonging to $G_{\sigma_0}^s$ for all $T \in [0, T_0]$ for some $\sigma_0 > 0$). Then the difference $\mathbf{v}(T) := \mathbf{u}_1(T) - \mathbf{u}_2(T)$ satisfies the analogue of equation (3.1):

$$\partial_T \mathbf{v} = \mathbf{M}(\mathbf{u}_1)\partial_X \mathbf{v} + (\mathbf{M}(\mathbf{u}_1) - \mathbf{M}(\mathbf{u}_2))\partial_X \mathbf{u}_2. \quad (3.10)$$

Arguing as before, but using in addition that

$$\|\mathbf{M}(\mathbf{u}_1) - \mathbf{M}(\mathbf{u}_2)\|_{G_\sigma^s} \leq (\psi_s(\|\mathbf{u}_1\|_{G_\sigma^s}) + \psi_s(\|\mathbf{u}_2\|_{G_\sigma^s})) \|\mathbf{v}\|_{G_\sigma^s},$$

where ψ_s is a monotone increasing function associated with $\mathbf{M}$, we get the analogue of (3.8):

$$\begin{aligned} \frac{1}{2} \frac{d}{dT} \|\mathbf{v}\|_{G_{\sigma_\mu(T)}^s}^2 + \left(\eta - \|\mathbf{M}(0)\| - \phi_s(\|\mathbf{u}_1\|_{G_{\sigma_\mu(T)}^s}) \right) \|\mathbf{v}\|_{G_{\sigma_\mu(T)}^{s+1/2}}^2 \\ \leq C_s \|\mathbf{u}_2\|_{G_{\sigma_\mu(T)}^{s+1}} \left(\psi_s(\|\mathbf{u}_1\|_{G_{\sigma_\mu(T)}^s}) + \psi_s(\|\mathbf{u}_2\|_{G_{\sigma_\mu(T)}^s}) \right) \|\mathbf{v}\|_{G_{\sigma_\mu(T)}^s}^2, \end{aligned} \quad (3.11)$$

for every $\mu > 0$. Taking the exponent η large enough, estimating the term $\|\mathbf{u}_2\|_{G_{\sigma_\mu(T)}^{s+1}}$ with the help of (3.6) and applying the Gronwall inequality to (3.11), we get the desired uniqueness on a possibly smaller interval $T \in [0, T_0]$ determined by the assumption that $\sigma_\mu(T) > 0$. Repeating these arguments on the next time intervals, we get the uniqueness on the whole existence interval $[0, T_0]$.

Let us now discuss the role of the vanishing viscosity method in the existence of a solution. Let β be a small positive parameter and consider the following semilinear parabolic equation

$$\partial_T \mathbf{u}^\beta = \mathbf{M}(\mathbf{u}^\beta)\partial_X \mathbf{u}^\beta + \beta \partial_X^2 \mathbf{u}^\beta, \quad \mathbf{u}^\beta|_{T=0} = \mathbf{u}_0.$$

Since the operator $A := \beta \partial_X^2$ generates an analytic semigroup in Gevrey spaces $G_{\sigma_0}^s$, the local solution $\mathbf{u}^\beta$ can be constructed, say, via the contraction mapping principle or the implicit function theorem (arguments like this, using the theory of analytic semigroups, can be found in Henry [26]). Crucial for the method is that all of the above estimates are *uniform* with respect

to β (the extra term $\beta \|u^\beta\|_{G_{\sigma(T)}^s}^2$ is non-negative and does not impact the estimates). Thus, we have (3.9) for $u^\beta(T)$, which is uniform with respect to β . Passing to the limit $\beta \rightarrow 0$, we get the desired solution u of (3.1). This completes the proof of the theorem. ■

Remark 3.2. Arguing analogously, it is possible to obtain a similar result for the slightly more general equation:

$$\partial_T u = M(u) \partial_X u + f(u), \quad u|_{T=0} = u_0, \quad (3.12)$$

where $f(u)$ is another entire function satisfying $f(0) = 0$.

(a) Local existence for the multiphase Whitham modulation equations

Local existence for the multiphase WMEs now follows by applying theorem 3.1 to the system (2.13). The matrix $M(u)$ in (2.9) is an entire function and $d = 4$. Hence, regardless of whether the multiphase WMEs are hyperbolic, elliptic or of mixed type, local existence will follow in Gevrey spaces.

Theorem 3.3. Let $s > 1$ and $\sigma_0 > 0$, and let

$$u^*(X, T)|_{T=0} = u_0(X) := (r_{10}^*(X), v_{10}^*(X), r_{20}^*(X), v_{20}^*(X)).$$

Then there exist $T_0 > 0$ and $C > 0$ such that for all initial data $u^*(X, 0) \in G_{2\sigma_0}^s$ with $\|u^*(X, 0)\|_{G_{2\sigma_0}^s} \leq C$, the multiphase WMEs (2.13) have a unique solution

$$u^*(X, T) \in C([0, T_0], G_{\sigma_0}^s) \quad \text{with} \quad u^*(X, T)|_{T=0} = u_0.$$

4. Approximate solutions for the perturbed problem

In this section, a continuation of the abstract theory of §3 is given for the case when there is a perturbation of the quasilinear system (3.1) of the form

$$\partial_T u = M(u) \partial_X u + \nu F(D_X^k u), \quad u|_{T=0} = u_0, \quad (4.1)$$

where $D_X^k u = \{u, \partial_X u, \dots, \partial_X^k u\}$, $\nu \in \mathbb{R}$ is a small parameter and F is a given entire function satisfying $F(0) = 0$. In this section $X \in \mathbb{R}$ and $T \geq 0$ are scaled variables. Here ν is an arbitrary small parameter but will be restored to $\nu = \varepsilon^2$ when the theory is applied to the system (1.11).

We seek an approximation to the solution u in the form

$$u(X, T, \nu) = u^0(X, T) + \nu u^1(X, T) + \nu^2 u^2(X, T) + \dots \quad (4.2)$$

Inserting these expansions into equation (4.1) and equating the terms with the same powers of ν , we get at ν^0 ,

$$\partial_T u^0 = M(u^0) \partial_X u^0, \quad \text{with } u^0|_{T=0} = u_0, \quad (4.3)$$

which coincides with equation (3.1) studied earlier. The higher-order terms u^n , $n \geq 1$ can be found by solving the inhomogeneous equations of variation associated with problem (4.3):

$$\partial_T u^n - M(u^0) \partial_X u^n - D_u M(u^0) u^n \partial_X u^0 = F_n(u^0, \dots, u^{n-1}), \quad \text{with } u^n|_{T=0} = 0, \quad (4.4)$$

where F_n is an entire function of the lower-order approximations $u^0, \dots, u^{n-1}$ and their derivatives up to order k satisfying $F_n(0) = 0$. Multiplying this equation by $e^{2\sigma(T)(A+1)}(1 +$

$A^{2s})\mathbf{u}^n(T)$ and arguing as in the proof of theorem 3.1, we arrive at

$$\frac{1}{2} \frac{d}{dT} \|\mathbf{u}^n\|_{G_{\sigma(T)}^s}^2 + \left(\eta - \phi_s(\|\mathbf{u}^0\|_{G_{\sigma(T)}^s}) \right) \|\mathbf{u}^n\|_{G_{\sigma(T)}^{s+1/2}}^2 \leq \psi_s(\|\mathbf{u}^0\|_{G_{\sigma(T)}^{s+1}}) \|\mathbf{u}^n\|_{G_{\sigma(T)}^s}^2 + C \|\mathbf{F}_n\|_{G_{\sigma(T)}^s}^2, \quad (4.5)$$

where ϕ_s and ψ_s are real analytic monotone increasing functions depending only on $\mathbf{M}$ and s . We see that the solvability condition

$$\eta - \phi_s(\|\mathbf{u}^0\|_{G_{\sigma(T)}^s}) > 0, \quad T < \sigma_0/\eta \quad (4.6)$$

for this inequality depends only on the initial local solution $\mathbf{u}^0(T)$, so the lifespan of every $\mathbf{u}^n$ is determined by the properties of $\mathbf{u}^0(T)$ only. Indeed, if (4.6) is satisfied, we get the recursive estimate

$$\|\mathbf{u}^n(T)\|_{G_{\sigma(T)}^s}^2 \leq C e^{KT} \sup_{0 \leq \tau \leq T} \|\mathbf{F}_n(\mathbf{u}^0(\tau), \dots, \mathbf{u}^{n-1}(\tau))\|_{G_{\sigma(\tau)}^s}^2, \quad (4.7)$$

for some positive constants C and K depending on $\mathbf{u}^0$. Since $\mathbf{F}_n$ depends only on $\mathbf{u}^0, \dots, \mathbf{u}^{n-1}$, estimate (4.7) allows us to construct the correctors $\mathbf{u}^n$ recursively solving the linear equation (4.4) at every step (the existence of a solution for problem (4.4) can be proved exactly as in theorem 3.1). Indeed, it follows from the vector-valued version of (3.5) that

$$\|\mathbf{F}_n\|_{G_{\sigma}^s} \leq F_s \left(\|\mathbf{u}^0\|_{G_{\sigma}^{s+k}} + \dots + \|\mathbf{u}^{n-1}\|_{G_{\sigma}^{s+k}} \right) \quad (4.8)$$

for $s > \frac{1}{2}$, $\sigma \geq 0$ and some smooth monotone increasing function F_s with $F_s(0) = 0$.

However, there is still a small problem with iterating inequalities (4.7) and (4.8), namely, the number of derivatives of $\mathbf{u}^0$, which we need to control in order to estimate $\mathbf{F}_n$ grows with n , so in order to get the G_{σ}^s -norm of $\mathbf{F}_n$, we need at least a G_{σ}^{s+kn} -norm of $\mathbf{u}^0$. Therefore, performing iterations in a straightforward way will require the solvability condition (4.6) to be satisfied not only for s , but also for $s+k$, $s+2k$ and so on. Since ϕ_s is growing in s , we will have to take η depending on n and shrink the existence interval for the correctors when n is growing. To overcome this and to get the lifespan uniform with respect to n for the correctors $\mathbf{u}^n$, we will proceed in an alternative way avoiding increasing the exponent s , but decreasing slightly the analyticity exponent σ and using (3.6). This gives the following result.

Theorem 4.1. *Let $s > 1$ and let the local solution $\mathbf{u}^0(T)$ of equation (4.1) satisfying (4.6). Then, for every $\delta > 0$, the correctors $\mathbf{u}^n(T)$ satisfy the estimate*

$$\|\mathbf{u}^n(T)\|_{G_{\sigma(T)-\delta}^s} \leq Q_{\delta,n} \left(\sup_{0 \leq \tau \leq T} \|\mathbf{u}^0(\tau)\|_{G_{\sigma(\tau)}^s} \right), \quad T \leq (\sigma_0 - \delta)/\eta \quad (4.9)$$

for some monotone increasing function $Q_{\delta,n}$.

Proof. Let $0 < \delta < \sigma_0$ be arbitrary. From estimates (4.8) and (3.6), we get

$$\|\mathbf{F}_n\|_{G_{\sigma-\delta/2^n}^s} \leq F_{s,n} \left(\|\mathbf{u}^0\|_{G_{\sigma}^s} + \dots + \|\mathbf{u}^{n-1}\|_{G_{\sigma}^s} \right) \quad (4.10)$$

for some monotone increasing function $F_{s,n}$ which is independent of σ and δ . Combining this estimate with (4.7), and introducing $\delta_n := \delta \sum_{l=1}^n 2^{-l}$, we have

$$\|\mathbf{u}^n(T)\|_{G_{\sigma(T)-\delta_n}^s} \leq C F_{s,n} \left(\sup_{0 \leq \tau \leq T} \|\mathbf{u}^0(\tau)\|_{G_{\sigma(\tau)-\delta_{n-1}}^s} + \dots + \sup_{0 \leq \tau \leq T} \|\mathbf{u}^{n-1}(\tau)\|_{G_{\sigma(\tau)-\delta_{n-1}}^s} \right).$$

Iterating this estimate, we finally arrive at (4.9). ■

Now introduce the n th order approximations

$$\tilde{\mathbf{u}}^n(T) := \mathbf{u}^0(T) + \nu \mathbf{u}^1(T) + \dots + \nu^n \mathbf{u}^n(T), \quad (4.11)$$

and the corresponding residuals

$$\text{Res}^n(T) := \partial_T \tilde{\mathbf{u}}^n(T) - \mathbf{M}(\tilde{\mathbf{u}}^n(T)) \partial_X \tilde{\mathbf{u}}^n(T) - \nu \mathbf{F}(D^k \tilde{\mathbf{u}}^n(T)). \quad (4.12)$$

Since the equations for the correctors $\mathbf{u}^l$, $l=0, \dots, n$ are obtained by equating to zero the coefficients in front of ν^l in Taylor expansions of (4.12) in ν , we have

$$\text{Res}^n(T) = \nu^{n+1} \Phi_n(\nu, D^k \mathbf{u}^0(T), \dots, D^k \mathbf{u}^n(T)) \quad (4.13)$$

for some function Φ_n which is analytic with respect to all arguments. Applying the theorem and estimate (3.5) then gives the following.

Corollary 4.2. *Let $\mathbf{u}^0(T)$, $T < \sigma_0/\eta$ be the local solution of equation (4.1) satisfying the condition (4.6) for some $s > 1$. Then, for every $\delta > 0$, the approximate solutions $\tilde{\mathbf{u}}^n(T)$ and residuals $\text{Res}^n(T)$ satisfy:*

$$\begin{aligned} \|\tilde{\mathbf{u}}^n(T)\|_{G_{\sigma(T)-\delta}^s} &\leq \Phi_{n,\delta} \left(\sup_{0 \leq \tau \leq T} \|\mathbf{u}^0(\tau)\|_{G_{\sigma(\tau)}^s} \right) \\ \text{and } \|\text{Res}^n(T)\|_{G_{\sigma(T)-\delta}^s} &\leq \nu^{n+1} \Phi_{n,\delta} \left(\sup_{0 \leq \tau \leq T} \|\mathbf{u}^0(\tau)\|_{G_{\sigma(\tau)}^s} \right) \end{aligned} \quad (4.14)$$

for $T < (\sigma_0 - \delta)/\eta$ and some monotone increasing function $\Phi_{n,\delta}$ depending on n, δ and s .

Remark 4.3. Note that theorem 4.1 and corollary 4.2 are proved under the assumption that there exists a local solution $\mathbf{u}^0(T)$, $T < \sigma_0/\eta$, which satisfies condition (4.6). In this case the lifespan of the approximate solution $\tilde{\mathbf{u}}^n(T)$ remains the same as the lifespan $T = \sigma_0/\eta$ no matter how big n is. However, if we just have an analytic local solution $\mathbf{u}^0(T)$ of equation (4.1) defined on the interval $T \in [0, T_0]$ such that

$$\|\mathbf{u}^0(T)\|_{G_{\sigma_0-\eta'T}^s} \leq R, \quad T_0 \leq \frac{\sigma_0}{\eta'},$$

for some positive σ_0 and non-negative η' , then the key assumption (4.6) is not automatically satisfied. In order to satisfy it we need to increase η' till

$$\eta = \phi_s(R)$$

and this decreases the lifespan of the approximate solution $\tilde{\mathbf{u}}^n(T)$ till $T_1 := \sigma_0/\phi_s(R)$. Thus, in this general situation we can only guarantee the existence of analytic approximate solutions $\tilde{\mathbf{u}}^n(T)$ on a smaller interval $t \in [0, T_1]$ than the initial lifespan T_0 of the solution $\mathbf{u}^0(T)$.

The finite Taylor expansions, proved to exist in this section, generate higher-order corrections to multiphase Whitham modulation theory. Heretofore higher-order WMEs have only been studied in the single phase case (e.g. Luke [27] and §2 of Düll & Schneider [7]). Although these expansions have uniform lifespan with respect to n , the series (4.2) are usually divergent and do not give the exact solution of the perturbed problem no matter how small ν is. In any case, the higher-order corrections from this section are useful for generating higher order in ν estimates on the residuals (see theorem 5.1 below). We will take up filling the gap between $\tilde{\mathbf{u}}^n$ and the exact solution in the next section.

5. Exact solutions in Gevrey spaces

We now look at the validity question: How well do solutions of the multiphase WMEs approximate the solutions of CNLS? The starting point is the exact equations for CNLS in (r_j, v_j)

$$\partial_T \mathbf{u} = \mathbf{M}(\mathbf{u}) \partial_X \mathbf{u} + \nu \mathbf{F}(D_X^3 \mathbf{u}), \quad \mathbf{u} \Big|_{T=0} = \mathbf{u}_0, \quad (5.1)$$

with $\mathbf{M}(\mathbf{u})$ defined in (2.9) and $\mathbf{F}$ defined in (2.10). The independent variables X, T are scaled variables, and for notational convenience we have dropped the circumflex on $\mathbf{u}$ and used ν as the small parameter. Translation to other notations will be straightforward *a posteriori*.

This system has the form of (4.1) with $d = 4$ and $k = 3$, and so the results generated in the previous sections will carry over. In particular, the limit system (5.1) with $\nu = 0$ (which is exactly the WMEs) has a unique local analytic solution $\mathbf{u}^0(T)$ and, when $\nu \neq 0$, the approximate solutions $\tilde{\mathbf{u}}^n(T)$ are well-defined and satisfy the estimates of theorem 4.1 and corollary 4.2.

We seek the desired exact solution of (5.1) in the form

$$\mathbf{u} = \tilde{\mathbf{u}}^n + \mathbf{v} := (\tilde{r}_1^n + R_1, \tilde{v}_1^n + V_1, \tilde{r}_2^n + R_2, \tilde{v}_2^n + V_2). \quad (5.2)$$

For notational convenience, the superscript n will be dropped on $\tilde{\mathbf{u}}^n$. Inserting this ansatz into equations (5.1), we end up with the following equations,

$$\begin{aligned} \partial_T R_1 &= -\partial_X V_1 - 2\tilde{v}_1 \partial_X R_1 - 2V_1 \partial_X \tilde{r}_1 - 2V_1 \partial_X R_1 + \text{Res}_1^n, \\ \partial_T V_1 &= -\partial_X (V_1)^2 - 2\partial_X (\tilde{v}_1 V_1) + 2\gamma_1 \partial_X (e^{2\tilde{r}_1} (e^{2R_1} - 1)) \\ &\quad + 2\alpha \partial_X (e^{2\tilde{r}_2} (e^{2R_2} - 1)) + \nu \partial_X^3 R_1 + \nu \partial_X (\partial_X R_1)^2 \\ &\quad + 2\nu \partial_X (\partial_X \tilde{r}_1 \partial_X R_1) + \text{Res}_2^n; \\ \partial_T R_2 &= -\partial_X V_2 - 2\tilde{v}_2 \partial_X R_2 - 2V_2 \partial_X \tilde{r}_2 - 2V_2 \partial_X R_2 + \text{Res}_3^n, \\ \partial_T V_2 &= -\partial_X (V_2)^2 - 2\partial_X (\tilde{v}_2 V_2) + 2\gamma_2 \partial_X (e^{2\tilde{r}_2} (e^{2R_2} - 1)) \\ &\quad + 2\alpha \partial_X (e^{2\tilde{r}_1} (e^{2R_1} - 1)) + \nu \partial_X^3 R_2 + \nu \partial_X (\partial_X R_2)^2 \\ &\quad + 2\nu \partial_X (\partial_X \tilde{r}_2 \partial_X R_2) + \text{Res}_4^n, \end{aligned} \quad (5.3)$$

where the residuals $\mathbf{Res}^n := (\text{Res}_1^n, \text{Res}_2^n, \text{Res}_3^n, \text{Res}_4^n)$ are defined by (4.12).

These equations are endowed with zero initial conditions. Our task is now to verify that they have a unique analytic local solution which is of order ν^{n+1} . This will be our next theorem.

Theorem 5.1. Suppose the approximate solution $\tilde{\mathbf{u}}^n(T) \in G_{\sigma(T)}^{s+1/2}$ of problem (5.1) satisfies the analogue of (4.6)

$$\eta - \phi \left(\|\tilde{\mathbf{u}}^n\|_{G_{\sigma(T)}^{s+1/2}} \right) > 0 \quad (5.4)$$

for the properly chosen constant $\eta > 0$, $s > 2$, $\sigma_0 > 0$, smooth monotone function ϕ (depending on s) and all $T < \sigma_0/\eta$. Assume also that the residual satisfies

$$\|\mathbf{Res}^n(T)\|_{E_{\sigma(T)}^s}^2 \leq Q_n(\tilde{\mathbf{u}}^0) \nu^{2(n+1)}. \quad (5.5)$$

Then, for sufficiently small $\nu > 0$, system (5.3) possesses a unique solution $\mathbf{v}(T) \in G_{\sigma(T)}^{s-1}$, satisfying $\mathbf{v}(0) = 0$, and the following estimate holds:

$$\|\mathbf{v}(T)\|_{E_{\sigma(T)}^s}^2 \leq C \nu^{2(n+1)} Q_n(\tilde{\mathbf{u}}^0), \quad (5.6)$$

for all $T < \sigma_0/\eta$, where

$$\|\mathbf{v}(T)\|_{E_{\sigma(T)}^s}^2 := \|\mathbf{v}(T)\|_{G_{\sigma(T)}^{s-1}}^2 + \nu \left(\|R_1(T)\|_{G_{\sigma(T)}^s}^2 + \|R_2(T)\|_{G_{\sigma(T)}^s}^2 \right). \quad (5.7)$$

Proof. As in the proof of theorem 3.1, we start with a formal derivation of the key estimate (5.6). Take a scalar product of the second equation in (5.3) with $(1 + A^{2(s-1)})V_1$ in the space $G_{\sigma(T)}^0$. This

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dT} \|V_1\|_{G_{\sigma(T)}^{s-1}}^2 + \eta \|(1+A)^{1/2}(1+A^{2(s-1)})^{1/2} V_1\|_{G_{\sigma(T)}^0}^2 \\
&= \left(-\partial_X (V_1)^2, (1+A^{2(s-1)}) V_1 \right)_{G_{\sigma(T)}^0} - 2 \left(\partial_X (\tilde{v}_1 V_1), (1+A^{2(s-1)}) V_1 \right)_{G_{\sigma(T)}^0} \\
&+ 2\gamma_1 \left(\partial_X \left(e^{2\tilde{r}_1} (e^{2R_1} - 1) \right), (1+A^{2(s-1)}) V_1 \right)_{G_{\sigma(T)}^0} \\
&+ 2\alpha \left(\partial_X \left(e^{2\tilde{r}_2} (e^{2R_2} - 1) \right), (1+A^{2(s-1)}) V_1 \right)_{G_{\sigma(T)}^0} \\
&+ \nu \left(\partial_X^3 R_1, (1+A^{2(s-1)}) V_1 \right)_{G_{\sigma(T)}^0} + \nu \left(\partial_X (\partial_X R_1)^2, (1+A^{2(s-1)}) V_1 \right)_{G_{\sigma(T)}^0} \\
&+ 2\nu \left(\partial_X (\partial_X \tilde{r}_1 \partial_X R_1), (1+A^{2(s-1)}) V_1 \right)_{G_{\sigma(T)}^0} + \left(\text{Res}_2^n, (1+A^{2(s-1)}) V_1 \right)_{G_{\sigma(T)}^0}. \quad (5.8)
\end{aligned}$$

The scalar products containing only first-order derivatives in X (in the left entry) can be estimated exactly as in the proof of theorem 3.1. For instance,

$$\begin{aligned}
& \left| \left(\partial_X \left(e^{2\tilde{r}_1} (e^{2R_1} - 1) \right), (1+A^{2(s-1)}) V_1 \right)_{G_{\sigma(T)}^0} \right| \\
&\leq \phi_s \left(\|\tilde{r}_1\|_{G_{\sigma(T)}^{s-1/2}} \right) \phi_s \left(\|R_1\|_{G_{\sigma(T)}^{s-1}} \right) \|R_1\|_{G_{\sigma(T)}^{s-1/2}} \|V_1\|_{G_{\sigma(T)}^{s-1/2}} \\
&\leq \phi_s \left(\|\tilde{\mathbf{u}}^n\|_{G_{\sigma(T)}^{s-1/2}} \right) \phi_s \left(\|\mathbf{v}\|_{E_{\sigma(T)}^s} \right) \|\mathbf{v}\|_{G_{\sigma(T)}^{s-1/2}}^2. \quad (5.9)
\end{aligned}$$

The remaining first-order terms are estimated analogously,

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dT} \|V_1\|_{G_{\sigma(T)}^{s-1}}^2 + (\eta - 1) \|V_1\|_{G_{\sigma(T)}^{s-1/2}}^2 \\
&\leq Q_n(\tilde{\mathbf{u}}^0) \nu^{2(n+1)} + \phi_s \left(\|\tilde{\mathbf{u}}^n\|_{G_{\sigma(T)}^{s-1/2}} \right) \phi_s \left(\|\mathbf{v}\|_{E_{\sigma(T)}^s} \right) \|\mathbf{v}\|_{G_{\sigma(T)}^{s-1/2}}^2 \\
&+ \nu \left(\partial_X^3 R_1, A^{2(s-1)} V_1 \right)_{G_{\sigma(T)}^0} \\
&+ \nu \left(\partial_X ((\partial_X R_1)^2 + \partial_X \tilde{r}_1 \partial_X R_1), A^{2(s-1)} V_1 \right)_{G_{\sigma(T)}^0}. \quad (5.10)
\end{aligned}$$

Performing the same action with the remaining equations of (5.3) and taking a sum of the obtained estimates, we arrive at

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dT} \|\mathbf{v}\|_{G_{\sigma(T)}^{s-1}}^2 + \left(\eta - 1 - \phi_s(\|\tilde{\mathbf{u}}^n\|_{G_{\sigma(T)}^{s-1/2}}) \phi_s(\|\mathbf{v}\|_{E_{\sigma(T)}^s}) \right) \|\mathbf{v}\|_{G_{\sigma(T)}^{s-1/2}}^2 \\
&\leq Q_n(\tilde{\mathbf{u}}^0) \nu^{2(n+1)} + \nu \left(\partial_X^3 R_1, A^{2(s-1)} V_1 \right)_{G_{\sigma(T)}^0} \\
&+ \nu \left(\partial_X ((\partial_X R_1)^2 + 2\partial_X \tilde{r}_1 \partial_X R_1), A^{2(s-1)} V_1 \right)_{G_{\sigma(T)}^0} \\
&+ \nu \left(\partial_X^3 R_2, A^{2(s-1)} V_2 \right)_{G_{\sigma(T)}^0} + \nu \left(\partial_X ((\partial_X R_2)^2 + 2\partial_X \tilde{r}_2 \partial_X R_2), A^{2(s-1)} V_2 \right)_{G_{\sigma(T)}^0}. \quad (5.11)
\end{aligned}$$

In contrast to the first-order terms, estimates for higher-order terms are a bit more delicate and cannot be pre-formed on the level of G_{σ}^{s-1} -norms. To handle them we use the special structure of equations (5.1), namely, the possibility to get anisotropic estimates where the R_i components are taken in G_{σ}^s -norms and V_i components remain in the G_{σ}^{s-1} -norm. With this strategy, the estimates

for the second-order terms in (5.11) also become straightforward. For instance, using the Cauchy–Schwarz inequality, we get

$$\begin{aligned} \nu \left(\partial_X ((\partial_X R_1)^2, A^{2(s-1)} V_1) \right)_{G_{\sigma(T)}^0} &\leq 2\nu \|\partial_X R_1\|_{G_{\sigma}^{s-3/2}} \|\partial_X^2 R_1\|_{G_{\sigma}^{s-1/2}} \|V_1\|_{G_{\sigma}^{s-1/2}} \\ &\leq C\nu^{1/2} \|R_1\|_{G_{\sigma}^s} \left(\nu \|R_1\|_{G_{\sigma}^{s+1/2}}^2 + \|V_1\|_{G_{\sigma}^{s-1/2}}^2 \right). \end{aligned} \quad (5.12)$$

Estimating the other second-order terms in (5.11) analogously, we arrive at

$$\begin{aligned} &\frac{1}{2} \frac{d}{dT} \|\mathbf{v}\|_{G_{\sigma(T)}^{s-1}}^2 + \left(\eta - 1 - \phi_s(\|\tilde{\mathbf{u}}^n\|_{G_{\sigma(T)}^{s-1/2}}) \phi_s(\|\mathbf{v}\|_{E_{\sigma(T)}^s}) \right) \|\mathbf{v}\|_{G_{\sigma(T)}^{s-1/2}}^2 \\ &\leq Q_n(\tilde{\mathbf{u}}^0) \nu^{2(n+1)} + \nu \left(\partial_X^3 R_1, A^{2(s-1)} V_1 \right)_{G_{\sigma(T)}^0} + \nu \left(\partial_X^3 R_2, A^{2(s-1)} V_2 \right)_{G_{\sigma(T)}^0} \\ &\quad + C \left(\|\mathbf{v}\|_{E_{\sigma}^s} + \|\tilde{\mathbf{u}}^n\|_{E_{\sigma}^{s+1/2}} \right) \|\mathbf{v}\|_{E_{\sigma}^{s+1/2}}^2, \end{aligned} \quad (5.13)$$

where the constant C is independent of ν .

To complete this estimate, we need to analyse the G_{σ}^s -norm of the solutions R_1 and R_2 . To this end, we multiply the first equation of (5.3) by

$$\nu(1 + A^{2s})e^{2\sigma(T)A}R_1 = \nu(1 - \partial_X^2 A^{2(s-1)})e^{2\sigma(T)A}R_1$$

and integrate over X . This gives

$$\begin{aligned} &\nu \frac{1}{2} \frac{d}{dT} \|R_1\|_{G_{\sigma(T)}^s}^2 + \eta \nu \|R_1\|_{G_{\sigma(T)}^{s+1/2}}^2 \\ &= \nu(\partial_X V_1, \partial_X^2 A^{2(s-1)} R_1)_{G_{\sigma(T)}^0} - \nu(\partial_X V_1, R_1)_{G_{\sigma(T)}^0} \\ &\quad - \nu \left(2\tilde{v}_1 \partial_X R_1 + 2V_1 \partial_X \tilde{r}_1 + 2V_1 \partial_X R_1, (1 + A^{2s}) R_1 \right)_{G_{\sigma(T)}^0} \\ &\quad + \nu(\text{Res}_1'', (1 + A^{2s}) R_1)_{G_{\sigma(T)}^0}. \end{aligned} \quad (5.14)$$

Using the Cauchy–Schwarz inequality together with the estimate (3.4), we arrive at

$$\begin{aligned} 2\nu(V_1 \partial_X R_1, (1 + A^{2s}) R_1)_{G_{\sigma(T)}^0} &\leq 2\nu \|V_1 \partial_X R_1\|_{G_{\sigma(T)}^{s-1/2}} \|R_1\|_{G_{\sigma(T)}^{s+1/2}} \\ &\leq 2C\nu (\|V_1\|_{G_{\sigma(T)}^1} \|R_1\|_{G_{\sigma(T)}^{s+1/2}} + \|V_1\|_{G_{\sigma(T)}^{s-1/2}} \|R_1\|_{G_{\sigma(T)}^2}) \|R_1\|_{G_{\sigma(T)}^{s+1/2}} \\ &\leq 2C\nu \|V_1\|_{G_{\sigma(T)}^1} \|R_1\|_{G_{\sigma(T)}^{s+1/2}}^2 + C\nu^{1/2} \|R_1\|_{G_{\sigma(T)}^2} (\|V_1\|_{G_{\sigma(T)}^{s-1/2}}^2 + C\nu \|R_1\|_{G_{\sigma(T)}^{s+1/2}}^2) \\ &\leq C\|\mathbf{v}\|_{E_{\sigma(T)}^s} \|\mathbf{v}\|_{E_{\sigma(T)}^{s+1/2}}^2. \end{aligned} \quad (5.15)$$

Estimates for all other terms on the right-hand side of (5.14) except for the first one can be performed exactly as in (5.12). This gives

$$\begin{aligned} &\nu \frac{1}{2} \frac{d}{dT} \|R_1\|_{G_{\sigma(T)}^s}^2 + \nu(\eta - 1) \|R_1\|_{G_{\sigma(T)}^{s+1/2}}^2 \leq \nu(\partial_X V_1, \partial_X^2 A^{2(s-1)} R_1)_{G_{\sigma(T)}^0} \\ &\quad + C \left(\|\tilde{\mathbf{u}}^n\|_{E_{\sigma(T)}^{s+1/2}} + \|\mathbf{v}\|_{E_{\sigma(T)}^s} \right) \|\mathbf{v}\|_{E_{\sigma(T)}^{s+1/2}}^2 + Q_n(\tilde{\mathbf{u}}^0) \nu^{2(n+1)}. \end{aligned} \quad (5.16)$$

Note finally that due to integration by parts

$$\left(\partial_X V_1, \partial_X^2 A^{2(s-1)} R_1 \right)_{G_{\sigma(T)}^0} + \left(\partial_X^3 R_1, A^{2(s-1)} V_1 \right)_{G_{\sigma(T)}^0} = 0.$$

With this identity, the first term in (5.16) will be cancelled after the summation with equation (5.11). Performing the analogous estimates with the equation for R_2 , and taking a sum,

we cancel all third-order terms in the estimate (5.11) and arrive at the desired estimate

$$\begin{aligned} & \frac{1}{2} \frac{d}{dT} \|\mathbf{v}\|_{E_{\sigma(T)}^s} + \left(\eta - C - \phi_s \left(\|\tilde{\mathbf{u}}^n\|_{E_{\sigma(T)}^{s+1/2}} \right) \phi_s \left(\|\mathbf{v}\|_{E_{\sigma(T)}^s} \right) \right) \|\mathbf{v}\|_{E_{\sigma(T)}^{s+1/2}}^2 \\ & \leq C\nu^{2(n+1)} Q_n(\tilde{\mathbf{u}}^0), \quad \mathbf{v}|_{T=0} = 0, \end{aligned}$$

for some monotone smooth function ϕ_s and positive constant C which are independent of ν . This gives the desired estimate (5.5) if the constant η is large enough so that

$$\eta - C - \phi_s \left(\|\tilde{\mathbf{u}}^n\|_{E_{\sigma(T)}^{s+1/2}} \right) \phi_s \left(\|\mathbf{v}\|_{E_{\sigma(T)}^s} \right) > 0, \quad T < \sigma_0/\eta.$$

In this case we will have $\|\mathbf{v}\|_{E_{\sigma(T)}^s} = O(\nu^{n+1})$. By this reason, the condition will be satisfied for sufficiently small ν if, say,

$$\eta - C - 2\phi_s(0)\phi_s \left(\|\tilde{\mathbf{u}}^n\|_{E_{\sigma(T)}^{s+1/2}} \right) := \eta - \phi \left(\|\tilde{\mathbf{u}}^n\|_{E_{\sigma(T)}^{s+1/2}} \right) > 0.$$

This estimate finishes the derivation of the desired estimate (5.6). Exactly as in theorem 3.1, the above derivation becomes rigorous if we replace $\sigma(T)$ by $\sigma_\mu(T) := \sigma(T) - \mu$ for any small positive μ . Passing after that to the limit $\mu \rightarrow 0$ we get the desired estimate for $\mu = 0$ as well.

The uniqueness of the solution $\mathbf{v}$ in the class of analytic functions can be obtained similarly to the proof of theorem 3.1 by writing out the equation for the difference of two solutions and deriving the analogue of estimate (5.6) for the obtained equation. Since this proof follows almost word for word the derivation given above, we leave it to the reader.

Finally, the existence of a solution $\mathbf{v}$ can be obtained using the vanishing viscosity method (also similar to the proof of theorem 3.1). The only difference here is that adding the viscosity of the form $-\beta \partial_X^2 \mathbf{v}$ is *not sufficient* to make the extended equation parabolic and semilinear (due to the presence of third-order dispersion terms), so the viscosity term should be replaced by $\beta \partial_X^4 \mathbf{v}$. Then the analogue of estimate (5.6) will obviously be uniform with respect to β and passage to the limit $\beta \rightarrow 0$ gives the desired solution. Thus, the theorem is proved. ■

(a) Summary of CNLS to WMEs reduction

The main approximation theorem for (2.12) is as follows, stated in terms of ε using $\nu = \varepsilon^2$.

Theorem 5.2. Let $(r_1^*, v_1^*, r_2^*, v_2^*) \in C([0, T_0], G_{2\sigma}^0)$, for some $T_0 > 0$ and a $\sigma > 0$, be a solution of the multiphase WMEs (2.13), with

$$\sup_{T \in [0, T_0]} \|(r_1^*, v_1^*, r_2^*, v_2^*)(\cdot, T)\|_{G_{2\sigma}^0} \leq C_{wh},$$

for some positive constant C_{wh} . Then for all $C_1 > 0$ there exist positive constants C_2, T_1, ε_0 , and solutions $(\check{r}_1, \check{v}_1, \check{r}_2, \check{v}_2)$ of (2.12), such that

$$\sup_{T \in [0, T_1]} \|(\check{r}_1, \check{v}_1, \check{r}_2, \check{v}_2)(X, T, \varepsilon) - (r_1^*, v_1^*, r_2^*, v_2^*)(X, T)\|_{G_\sigma^0} \leq C_2 \varepsilon^2 \quad \text{for all } \varepsilon \in (0, \varepsilon_0).$$

For the subsequent discussion of the phase in the next section we need the following formulation of our approximation result.

Theorem 5.3. Fix $n \in \mathbb{N}_0$ and let $\tilde{\mathbf{u}}^n(T) = (r_1^{*,n}, v_1^{*,n}, r_2^{*,n}, v_2^{*,n})(T)$ be the higher-order approximation constructed in §4. Then there exist positive constants C_2, T_1, ε_0 , and solutions $(\check{r}_1, \check{v}_1, \check{r}_2, \check{v}_2)$ of (2.12) with

$$\sup_{T \in [0, T_1]} \|(\check{r}_1, \check{v}_1, \check{r}_2, \check{v}_2)(X, T, \varepsilon) - (r_1^{*,n}, v_1^{*,n}, r_2^{*,n}, v_2^{*,n})(X, T, \varepsilon)\|_{G_\sigma^0} \leq C_2 \varepsilon^{2n+2} \quad \text{for all } \varepsilon \in (0, \varepsilon_0).$$

6. Concluding remarks

Validity of multiphase WMEs deduced from CNLS equations has been proved. As far as we are aware this is the first proof of validity for multiphase WMEs. The theory shows that multiphase

modulation, at least in the case where the underlying equation has a toral symmetry, is robust. Only two-phase solutions have been considered, but extension to any finite number of phases, at least in the context of N -coupled NLS equations reduced to N -phase WMEs, is conceivable.

The proof is independent of the phases, but the role of phases can be seen by going back to the original Ψ_1, Ψ_2 variables in the CNLS equations (1.9). Integration is necessary for the reconstruction of the phases ϕ_j^β from $v_j^\beta = \partial_X \phi_j^\beta$, and so only a local in space approximation result is possible. In particular, we have the following corollary of theorem 5.3.

Corollary 6.1. *For all $b \in [0, 2n + 1]$ we have*

$$\sup_{t \in [0, T_1/\varepsilon]} \sup_{|x| \leq \varepsilon^{-b}} \left| (\Psi_1, \Psi_2)(x, t) \exp(-i\phi(0, \varepsilon t)) \right. \\ \left. - \left(\exp \left(r_1^{*,n}(\varepsilon x, \varepsilon t) + i \int_0^x v_j^{*,n}(\varepsilon x', \varepsilon t) dx' + i\omega t \right) \right)_{j=1,2} \right| \leq C\varepsilon^{2n+1-b},$$

where ϕ satisfies $\sup_{T \in [0, T_1]} |\phi(0, T)| = \mathcal{O}(1/\varepsilon)$.

Further detail on phase estimates in the case of one-phase wavetrains can be found in §2 of [7].

The Whitham (geometric optics) approximation is also used in dissipative systems with underlying conservation laws, and a validity proof has been given for the case of one-phase wavetrains (e.g. Johnson *et al.* [28] and references therein). However, an approximation theorem in the sense of theorem 5.2 for dissipative systems is still work in progress [29].

Finally, the Cauchy–Kowalevskaya theorem and spaces of analytic functions are essential when the WMEs are mixed or elliptic. However, a question arises in the case when the WMEs are hyperbolic. Can the use of analytic functions, and the additional restriction on the approximation time $T_1 < T_0$, be avoided in the hyperbolic case, where the WMEs are locally well posed? In particular, can the validity result be proved in Sobolev spaces? These questions, in the case of multiphase hyperbolic WMEs, remain open.

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Appendix A. Auxiliary estimates in Gevrey spaces

In this appendix we record the auxiliary estimates in Gevrey spaces which are used in the paper. Although all of these estimates are more or less standard, there is not a standard reference, and so we sketch their proof here for the convenience of the reader. We start with the estimate for the point-wise product.

Lemma A.1. *Let $u, v \in C_\sigma^s(\mathbb{R})$ for some $s, \sigma \geq 0$ and $\kappa > 1/2$. Then the following estimate holds:*

$$\|uv\|_{C_\sigma^s} \leq C_{s,\kappa} (\|u\|_{C_\sigma^s} \|v\|_{C_\sigma^s} + \|u\|_{C_\sigma^s} \|v\|_{C_\sigma^s}), \quad (\text{A } 1)$$

for some positive constant $C_{s,\sigma}$.

Proof. We use Young's convolution inequality in the form

$$\|f * g\|_{L^2} \leq \|f\|_{L^2} \|g\|_{L^1},$$

together with the following inequalities, which follow in turn from the triangle inequality,

$$(1 + |\xi|^{2s})^{1/2} \leq C_s \left((1 + |\eta|^{2s})^{1/2} + (1 + |\xi - \eta|^{2s})^{1/2} \right), \quad e^{\sigma(1+|\xi|)} \leq e^{\sigma(1+|\eta|)} e^{\sigma(1+|\xi-\eta|)}$$

to get

$$\begin{aligned} & |e^{\sigma(|\xi|+1)}(1+|\xi|^{2s})^{1/2}(\hat{u} * \hat{v})(\xi)| \\ & \leq C_s \int_{\mathbb{R}} \left(e^{\sigma(|\eta|+1)}(1+|\eta|^{2s})^{1/2}|\hat{u}(\eta)| \right) \left(e^{\sigma(1+|\xi-\eta|)}|\hat{v}(\xi-\eta)| \right) d\eta \\ & \quad + C_s \int_{\mathbb{R}} \left(e^{\sigma(|\xi-\eta|+1)}(1+|\xi-\eta|^{2s})^{1/2}|\hat{v}(\xi-\eta)| \right) \left(e^{\sigma(1+|\eta|)}|\hat{u}(\eta)| \right) d\eta \end{aligned} \quad (\text{A } 2)$$

and

$$\begin{aligned} \|uv\|_{G_\sigma^s} &= \|e^{\sigma(1+|\xi|)}(1+|\xi|^{2s})^{1/2}(\hat{u} * \hat{v})(\xi)\|_{L^2} \\ &\leq C_s \|u\|_{G_\sigma^s} \|e^{\sigma(1+|\xi|)}\hat{v}(\xi)\|_{L^1} + C_s \|v\|_{G_\sigma^s} \|e^{\sigma(1+|\xi|)}\hat{u}(\xi)\|_{L^1}. \end{aligned} \quad (\text{A } 3)$$

Finally, due to Cauchy–Schwarz inequality and the fact that $\kappa > \frac{1}{2}$, we arrive at

$$\begin{aligned} \int_{\mathbb{R}} e^{\sigma(1+|\xi|)}|\hat{u}(\xi)| d\xi &= \int_{\mathbb{R}} \left(e^{\sigma(1+|\xi|)}(1+|\xi|^{2\kappa})^{1/2}|\hat{u}(\xi)| \right) \left(1+|\xi|^{2\kappa} \right)^{-1/2} d\xi \\ &\leq \|u\|_{G_\sigma^s} \|(1+|\xi|^{2\kappa})^{-1}\|_{L^1} \leq C_\kappa \|u\|_{G_\sigma^s}. \end{aligned} \quad (\text{A } 4)$$

This estimate, together with (A 2) and with the analogous estimate for v finish the proof of the lemma. ■

Corollary A.2. Let $u \in G_\sigma^s(\mathbb{R})$ for some $\sigma > \frac{1}{2}$ and let $\varphi : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function such that $\varphi(0) = 0$. Then, $\varphi(u) \in G_\sigma^s$ and there exists an entire function $\varphi_s(z)$, $z \in \mathbb{C}$ such that $\varphi_s(0) = 0$, $\varphi_s(x)$ is positive and monotone increasing for $x \in \mathbb{R}_+$ such that

$$\|\varphi(u)\|_{G_\sigma^s} \leq \varphi_s(\|u\|_{G_\sigma^s}).$$

Proof. Since φ is entire, it can be expressed in terms of the convergent Taylor expansion

$$\varphi(z) = \sum_{n=1}^{\infty} a_n z^n, \quad z \in \mathbb{C}$$

($a_0 = 0$ since $\varphi(0) = 0$). Thus, using (A 1) with $\kappa = s$, we get

$$\|\varphi(u)\|_{G_\sigma^s} \leq \sum_{n=1}^{\infty} |a_n| \|u^n\|_{G_\sigma^s} \leq \sum_{n=1}^{\infty} |a_n| (2C_{s,s})^n \|u\|_{G_\sigma^s}^n := \varphi_s(\|u\|_{G_\sigma^s}).$$

Since φ is entire, we have $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 0$ and therefore $\varphi_s(z)$ is also an entire function. Obviously $\varphi_s(0) = 0$ and $\varphi'_s(x) \geq 0$ if $x \geq 0$. This finishes the proof of the corollary. ■

To conclude, we state one more useful estimate which compares Gevrey norms with different exponents.

Lemma A.3. Let $u \in G_\sigma^s$ for some $\sigma > 0$, $s \geq 0$. Then

$$\|u\|_{G_{\sigma-\delta}^{s+p}} \leq C(p, \delta) \|u\|_{G_\sigma^s},$$

where $C(p, \delta)$ is independent of s and σ .

Proof. Indeed, by definition

$$\|u\|_{G_{\sigma-\delta}^{s+p}}^2 = \int_{\mathbb{R}} e^{2(\sigma-\delta)(1+|\xi|)} (1+|\xi|^{2(s+p)}) |\hat{u}(\xi)|^2 d\xi \leq \max_{\xi \in \mathbb{R}} \{e^{-2\delta(1+|\xi|)} (1+|\xi|^{2p})\} \|u\|_{G_\sigma^s}^2.$$

Thus, the inequality holds with

$$C(p, \delta) := \max_{\xi \in \mathbb{R}} \{e^{-\delta(1+|\xi|)} (1+|\xi|^{2p})^{1/2}\}. \quad \blacksquare$$

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