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The relation between quantum and classical field theory with a classical source

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Classical field theory is about fields and how they behave in space–time. Quantum field theory, in practice, usually seems to be about particles and how they scatter. Nevertheless, classical fields must emerge from quantum field theory in appropriate limits, and Michael Duff showed how this happens for the Schwarzschild solution in perturbative quantum gravity. In a series of papers, we and others have shown how classical radiation from an accelerated charge emerges from quantum field theory when the Unruh thermal effect is taken into account. Here, we sharpen those conclusions by showing that, even at finite times, the quantum picture is meaningful and is in close agreement with the classical picture.

1. Dedication and introduction

Relativistic quantum field theory in practice has developed largely as a toolkit to calculate scattering amplitudes by perturbation theory. Conceptually, this framework does not fit really well with the classical picture of a field as a function with a value at each point in space–time, nor with the need to study quantum phenomena in gravitational backgrounds—that is, space–times that are not flat, but may have geometrical features on all scales from the subatomic to the cosmological—not to mention quantization of gravity itself. Nevertheless, our honoree, Michael J. Duff, in his dissertation work showed that the classical

Schwarzschild solution for the gravitational field of a spherically symmetric mass is contained in the perturbative development of the quantized gravitational field. More precisely, as stated in the abstract to [1],

It is verified explicitly to second order in Newton's constant, G , that the quantum-tree-graph contribution to the vacuum expectation value of the gravitational field produced by a spherically symmetric c-number source correctly reproduces the classical Schwarzschild solution.

We study here a more elementary problem: the quantized field is electromagnetic or even scalar, without the nonlinearities of gravity theory. On the other hand, the c-number source is not static, but uniformly accelerated. The purely classical theory of this configuration [2–5] is full of subtleties that remain controversial to this day. The quantum version is closely related [6] to the famous Unruh temperature [7,8]. Extracting the classical interpretation from the quantum calculations involves conceptual issues similar to those of the Schwarzschild–Duff problem.

2. The scalar model

(a) Rindler coordinates

We shall work in a four-dimensional Minkowski space with coordinates (t, x, y, z) and consider a point charge or source with constant acceleration a whose hyperbolic worldline lies in the t – z plane and is asymptotic to the light fronts $t + z = 0$ and $t - z = 0$. These light fronts divide the space–time into quadrants labelled R, F, L and P. In the region R ($z > |t|$), one introduces coordinates (τ, ξ) by

$$t = a^{-1} e^{a\xi} \sinh a\tau, \quad z = a^{-1} e^{a\xi} \cosh a\tau. \quad (2.1)$$

Then the worldline is the curve $\xi = 0$, $\mathbf{x}_\perp \equiv (x, y) = 0$, and on that curve τ is the source's proper time. The metric line element takes the forms (figure 1)

$$\begin{aligned} ds^2 &= -d\tau^2 + dx^2 + dy^2 + dz^2 \\ &= e^{2a\xi} (-d\tau^2 + d\xi^2) + dx^2 + dy^2. \end{aligned} \quad (2.2)$$

(b) The scalar field equation

We consider a scalar field satisfying

$$\square\phi \equiv \partial_\mu \partial^\mu \phi = -\rho, \quad (2.3)$$

where ρ is a prescribed source function (which we may sometimes loosely call a 'charge', as in the case of an electromagnetic field). We are primarily interested in an eternally uniformly accelerated point source,

$$\rho = q\delta(\mathbf{x}_\perp)\delta(\xi). \quad (2.4)$$

To postpone mathematical technicalities, however, it is prudent to start by developing the theory for ρ as a smooth function of compact support in space and time. (One reason for working with a scalar field is that in electromagnetism the requirement of compactness in time would conflict with gauge invariance.) The distributional source (2.4) can be obtained as a limit of a smooth source with a sausage-shaped support (figure 2).

(c) The exact solution

The retarded (causal) solution of the scalar problem (2.3) with (2.4) was given by Ren & Weinberg [9], following Rohrlich [2–4] and Boulware [5] for the classical electromagnetic problem and [6]

Figure 1. Minkowski and Rindler coordinates.

Figure 2. A smooth, finite-time source between initial and final temporal hypersurfaces.

for the quantum extension. Recently [10], we have re-examined it from a point of view more conducive to making connections with the quantum problem and with the Unruh effect.

The (non-homogeneous) linearity of the equation makes it exactly solvable, even in the quantum regime. The field can be expressed in either of two forms,

$$\phi = \phi_{\text{ret}} + \phi_{\text{in}} = \phi_{\text{adv}} + \phi_{\text{out}}, \quad (2.5)$$

where ϕ_{in} and ϕ_{out} are *free* fields determined by general initial data, and

$$\phi_{\text{ret}} \equiv R(\rho) \quad \text{and} \quad \phi_{\text{adv}} \equiv A(\rho). \quad (2.6)$$

The integral kernels of the linear operators R , A and $E \equiv A - R$ are the retarded and advanced Green functions and the field commutator familiar from scalar quantum field theory. As ϕ_{in} is a free field ($\square\phi_{\text{in}} = 0$), it has the Fourier expansion

$$\phi_{\text{in}} = \int \frac{dk}{\sqrt{(2\pi)^3 2\omega_k}} [a_k^{\text{in}} e^{ix \cdot k - it\omega_k} + \text{H.c.}]. \quad (2.7)$$

This formalism holds for both the classical field (a_k are numbers) and the canonically quantized field (a_k are annihilation operators).

For eternal uniform acceleration, it turns out that

$$\phi_{\text{ret}}(x) = \frac{q}{4\pi S} \theta(t + z) \quad (2.8)$$

and

$$S \equiv \frac{a}{2} \sqrt{(X^2 - a^{-2})^2 + \frac{4x_{\perp}^2}{a^2}}, \quad X^2 \equiv \mathbf{x}_{\perp}^2 + z^2 - t^2 = \mathbf{x}_{\perp}^2 + \frac{e^{2a\xi}}{a^2}. \quad (2.9)$$

Formula (2.8) holds throughout regions R and F, while $\phi_{\text{ret}} = 0$ in regions P and L. Perhaps astonishingly, in the interior of R, $\phi_{\text{ret}} = \phi_{\text{adv}}$. In that region, the field is both static and invariant

under time reversal (more precisely, it is independent of τ and is even in t). It follows that anything worthy of the name of ‘radiation’ can exist only in F (where ‘time’ is ξ) and on the past horizon separating R from P.

3. The perturbative quantum solution

(a) Feynman diagrams

Alternatively, one can treat the quantum theory of ϕ (with a given classical source ρ) as a simple special type of interacting quantum field theory. Usually interacting quantum field theories can not be solved exactly, so instead of searching for a classical-looking field solution as in the previous section, one uses perturbation theory, implemented through Feynman diagrams, and concentrates on scattering of the quanta of the free field that would exist if the interaction were turned off. What does that procedure give here?

There is only one type of vertex,

The n -particle production amplitude is just of order n , except for higher-order loops that merely maintain unitarity (normalization):

The main question we want to ask in the present work is: what is the relation between a perturbative calculation (with its language of particles and the S -matrix) and the exact solution (a free field (2.7) plus a c -number (2.6))?

Remark. Recall that $\phi = \phi_{\text{ret}} + \phi_{\text{in}}$ is an exact Heisenberg-picture solution, whereas in perturbation theory the unperturbed free field

$$\phi_0 \equiv \int \frac{dk}{\sqrt{(2\pi)^3 2\omega_k}} [a_k e^{ix \cdot k - i t \omega_k} + \text{H.c.}] \quad (3.1)$$

is an *interaction-picture* construct. Although algebraically it looks identical to the ϕ_{in} of (2.7), it is dangerous to identify ϕ_0 with ϕ_{in} (hence a_k^{in} with a_k); they are conceptually distinct and are parts of distinct theoretical apparatus.

(b) Consequences

The principal conclusions of [10] are as follows.

- (i) Because of the triviality of the Feynman diagrams, the original first-order perturbative calculations of the early 1990s [6,9] are essentially equivalent to an *exact* calculation of the S -matrix in the interaction picture [11],

$$\hat{S} = \exp \left[-i \int d^4x \sqrt{-g} \hat{\phi}_{\text{out}}(x) \rho(x) \right]. \quad (3.2)$$

- (ii) We have introduced hypersurfaces $\Sigma_{\pm}$ in the far future and past (respectively) of the sausage that is our regularized source. The asymptotic vacua on $\Sigma_{\pm}$ are related by

$$\begin{aligned} |0_{\text{in}}\rangle &= \hat{S}|0_{\text{out}}\rangle \\ &= e^{-\|KE\rho\|^2/2} e^{-\hat{a}_{\text{out}}^{\dagger}(KE\rho)} |0_{\text{out}}\rangle. \end{aligned} \quad (3.3)$$

Here, K is an operation of extracting the positive-frequency part with respect to the inertial (Minkowski) time t [10, eqn (40)]. This equation shows the initial vacuum to be a coherent state in terms of out-particles. (That result is the analogue of a Bogolubov transformation for the case of a c-number source.)

- (iii) The particle amplitudes go together to make up precisely the classical retarded solution $R\rho$ as the field's expectation value in the far future. This strengthens the 1990s conclusion that quantum perturbation theory reproduces classical bremsstrahlung (the Larmor formula).
- (iv) A calculation in R using Rindler coordinates gives the same answer, but *only* if the presence of the Unruh thermal bath is recognized. The Unruh effect of quantum field theory is needed to deduce the classical phenomenon!
- (v) In R, the quantum emission and absorption are entirely in *zero-energy Rindler modes* (τ -independent and concentrated near the horizon, but generally with non-zero transverse momentum). This fact is necessary for consistency with the previous observation that radiation is not detectable in the interior of R. (Of course, no normalizable quantum state has exactly zero energy. In the context of physical experiments taking place in a finite time, one may always think of zero-frequency Rindler modes as being Rindler modes with arbitrarily small frequency (with respect to Rindler time).)

In summary, everything has been demonstrated to be consistent: the quantum with the classical, and the accelerated viewpoint with the inertial.

4. Fields and particles at finite times

Nevertheless, there are reasons to feel unsatisfied, to fear that something conceptual is missing.

The scattering paradigm is not truly appropriate for a source that lasts forever. The *in* and *out* operators do not (directly) describe the physics except in the *infinitely* far past and future, respectively. But we have an exact Heisenberg-picture solution that purports to represent the field at all finite times. Why are we not using it?

To grapple with this question, consider again a source ρ that lasts for only a finite time. At early times, ϕ_{ret} equals 0, and the Fock vacuum equals the *in*-vacuum; nothing is there initially. Since ϕ_{in} is free, there are no *in*-quanta present in that state at any time; the vacuum expectation value of the operator term equals 0 at all times. Therefore, the effects of ρ are all contained in the c-number ϕ_{ret} .

However, that is not the whole story. Observable particles at late times are represented by the a_k^{out} operators. From (2.5) and (2.6), we have

$$\phi_{\text{out}} = \phi_{\text{in}} + (\phi_{\text{ret}} - \phi_{\text{adv}}) \equiv \phi_{\text{in}} - E(\rho). \quad (4.1)$$

Hence the *in*-vacuum contains *out* particles. More precisely, (4.1) induces similar non-homogeneous-linear transformations on the creation and annihilation operators, and it is well known that in such a situation the vacuum of one set is a coherent state with respect to the other; for details in the case at hand, see the appendix of [10] and equation (3.3) above.

To recapitulate, the effect of the source seems to be incorporated twice: once in the c-number retarded solution (which is the vacuum expectation value of the quantized field) and once in the commutator function E that relates future particles to past particles. One might fear that we are somehow double-counting the particle creation. We need to verify that these things are not really duplicative, but rather two different ways of calculating essentially the same thing. *Note:* It is the

form of $-E(\rho)$ in the far future (and only there) that can reasonably be identified as ‘radiation’, since the subtraction of the retarded function has cancelled the static or Coulomb-like part of the field.

Rather than consider this question further in the asymptotic time setting, let us investigate what the theory says about observations at a finite Minkowski time, t . We return to the eternally accelerated point source (2.4) and consider the *in*-vacuum state. It is clear from what has been said before that the expectation values of field operators anywhere in space–time are given simply by the classical field solution $\phi_{\text{ret}} = R(\rho)$. This is the finite-time analogue of the statement that the expected field in the far future is $-E(\rho)$ (because $A(\rho)$ vanishes in that limit).

More interesting is the issue of particle observables at a time t_0 . For this, it is useful to choose the initial data at $t = t_0$ for the Heisenberg equation of motion to be standard field operators for a field and its time derivative—in other words, to choose t_0 as the time when the Heisenberg and Schrödinger pictures coincide. (This is the finite-time analogue of introducing the *out*-field, except that the phases in the latter are rolled back from $t_0 = \infty$ to time 0.) Let ϕ_{t_0} be the free field

$$\phi_{t_0}(t, \mathbf{x}) = \int \frac{d\mathbf{k}}{\sqrt{(2\pi)^3 2\omega_k}} [b_k^{t_0} e^{i\mathbf{x}\cdot\mathbf{k} - i(t-t_0)\omega_k} + \text{H.c.}]. \quad (4.2)$$

Then $\phi_{t_0}(t_0, \mathbf{x})$ and

$$\frac{\partial \phi_{t_0}}{\partial t}(t_0, \mathbf{x}) = \int \frac{d\mathbf{k}}{\sqrt{(2\pi)^3 2\omega_k}} [-i\omega_k b_k^{t_0} e^{i\mathbf{x}\cdot\mathbf{k}} + i\omega_k b_k^{t_0\dagger} e^{-i\mathbf{x}\cdot\mathbf{k}}] \quad (4.3)$$

are the field operators at t_0 . These equations can be standardly solved for b and $b^\dagger$ in terms of ϕ_{t_0} and $\partial\phi_{t_0}/\partial t$ at $t = t_0$.

The solution of (2.3) with these initial data is

$$\phi = \phi_{t_0} + R(\rho_+) + A(\rho_-), \quad \rho_+ = \theta(t - t_0)\rho, \quad \rho_- = \theta(t_0 - t)\rho. \quad (4.4)$$

But this field is also equal to $\phi_{\text{in}} + R(\rho)$, by (2.5) and (2.6), with ϕ_{in} given in (2.7). Thus

$$\phi_{t_0} = \phi_{\text{in}} + R(\rho) - [R(\rho_+) + A(\rho_-)] = \phi_{\text{in}} + R(\rho_-) - A(\rho_-) = \phi_{\text{in}} - E(\rho_-), \quad (4.5)$$

in analogy with (4.1). Therefore, one can in principle calculate the b^{t_0} operators in terms of the a^{in} operators of (2.7) plus Fourier components at t_0 of $-E(\rho_-)$ (or $R(\rho_-)$, since $A(\rho_-)$ vanishes on the $t = t_0$ hypersurface). Thus the *in*-vacuum is an eigenstate of every b^{t_0} operator, and hence, finally, the *in*-vacuum is revealed to be a coherent state of t_0 -particles, in analogy with (3.3).

5. Conclusion

In [10] and in §3, we concluded that the quantum and classical pictures of radiation from a uniformly accelerated charge are consistent, and also that the Rindler-space aspects thereof can be consistently calculated in either Rindler or Minkowski coordinate frames. In §4, we showed that the field and particle pictures are consistent; in particular, for this simple system, the particle language makes sense at finite times, unobscured by resort to either scattering theory (the need to take a limit of infinite time) or perturbation theory.

In quantum field theories with nonlinear interactions such finite-time analyses run into difficulty (infinite wavefunction renormalization, [11, pp. 202–205, 346]). Nevertheless, in their different ways, our work and Duff’s [1] provide reassurance that the local field picture of fundamental physics is not completely abandoned in contemporary theory.

Data accessibility. This article has no additional data.

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