Research

Cite this article: Sun J, Zhu S, Tong Z, Zhou Z, Xu X. 2020 Post-buckling analysis of functionally graded multilayer graphene platelet reinforced composite cylindrical shells under axial compression. *Proc. R. Soc. A* **476**: 20200506.
<http://dx.doi.org/10.1098/rspa.2020.0506>

Received: 29 June 2020

Accepted: 13 October 2020

Subject Areas:
 mechanics

Keywords:

post-buckling, functionally graded multilayer graphene platelets reinforced composite cylindrical shell, load-shortening curves, axial compression

Author for correspondence:

Zhenhuan Zhou

e-mail: zhouzh@dlut.edu.cn

Electronic supplementary material is available online at <https://doi.org/10.6084/m9.figshare.c.5200524>.

Post-buckling analysis of functionally graded multilayer graphene platelet reinforced composite cylindrical shells under axial compression

Jiabin Sun¹, Shengbo Zhu², Zhenzhen Tong³,
 Zhenhuan Zhou² and Xinsheng Xu²

¹State Key Laboratory of Structural Analysis of Industrial Equipment and School of Ocean Science and Technology, Dalian University of Technology, Panjin 124221, People's Republic of China

²State Key Laboratory of Structure Analysis of Industrial Equipment and Department of Engineering Mechanics, Dalian University of Technology, International Center for Computational Mechanics, Dalian 116024, People's Republic of China

³College of Locomotive and Rolling Stock Engineering, Dalian Jiaotong University, Dalian 116028, People's Republic of China

ZZ, 0000-0003-0523-3746

Axially compressed composite cylindrical shells can attain multiple bifurcation points in their post-buckling procedure because of the natural transverse deformation restraint provided by their geometry. In this paper, the post-buckling analysis of functionally graded (FG) multilayer graphene platelets reinforced composite (GPLRC) cylindrical shells under axial compression is carried out to investigate the stability of such shells. Rather than the critical buckling limit, the focus of the present study is to obtain convergence post-buckling response curves of axially compressed FG multilayer GPLRC cylindrical shells. By introducing a unified shell theory, the nonlinear large deflection governing equations for post-buckling of FG multilayer GPLRC cylindrical shells with wide range of thickness are established, which can be easily changed into three widely used shell theories. Load-shortening curves for both symmetric and asymmetric post-buckling modes are obtained by Galerkin's method. Numerical results illustrate that the present solutions agree well with the existing theoretical and experimental data. The effects of geometries and

material properties on the post-buckling behaviours of FG multilayer GPLRC cylindrical shells are investigated. The differences in the three shell theories and their scopes are discussed also.

1. Introduction

Cylindrical shells have extensive applications as an important component in the fields of civil, aerospace, marine and petrochemical, etc. Due to the large span-to-thickness ratio, the carrying capability of cylindrical shells is primarily determined by the performance in resisting buckling failure which is usually sudden and catastrophic. To predict the critical loads, Lorenz [1] and Timoshenko [2] adopted the linear eigenvalue buckling theory to study the buckling behaviours of cylindrical shells under axial compression. However, the experimental evidence [3] identified that cylindrical shells could lose stability under a certain load far less than the theoretical predictions. To eliminate the significant difference between the theoretical and experimental data, Karman & Tsien [4] proposed a nonlinear post-buckling theory based on the Donnell large-deflection equations. In their theory, there exists a post-buckling equilibrium state whose critical loads are far below the ones obtained by the linear eigenvalue buckling theory. Therefore, they suggested the lower critical loads as the load-carrying capability of axial compressive cylindrical shells. Due to the limitation of computers, only a few terms of the trigonometric series were selected to represent the displacement expression in the solution procedure. Till 1960s, with the aid of the burgeoning computer technique, Hoff *et al.* [5] re-examined Karman & Tsien's work [4] and found that the lower critical loads approach to zero as the increasing terms of the trigonometric series in the displacement functions. It implies that the classical double trigonometric series cannot ensure the convergence of the post-buckling equilibrium path. Besides the large-deflection post-buckling theory, Koiter [6] established an initial post-buckling theory which only concerns the post-buckling behaviour near the critical buckling states of imperfect cylindrical shells. The proposed theory explains the knockdown of critical loads from the view point of impact of the initial geometric imperfection. However, with the development of the manufacturing technology, some experimental studies [7] on nearly perfect cylindrical shells still captured the remarkable decline of critical loads in the process of post-buckling. In view of those experimental observations, Yamaki [8] turned to the nonlinear post-buckling theory and rebuilt the displacement expressions based on the deformation measured in the experiment, and finally obtained a full post-buckling equilibrium path consisting of a series of wave-like deformations with stepwise reduced circumferential wavenumbers. After the above continuous efforts, the post-buckling models of the isotropic thin cylindrical shells which could predict the more accurate critical loads were well established.

At the end of the last century, composites were widely used in the manufacture of fundamental components in high-end equipment under complicated or extreme loading conditions. Therefore, the stability analysis of the composite cylindrical shells becomes the essential problem of the safe design of such equipment. A plenty of studies on post-buckling of composite cylindrical shells under axial compression were carried out based on the classical thin-walled shell theory (TWST). As a convenient way, some numerical methods such as finite-element method (FEM) [9–11] and finite difference method [12–14] for post-buckling of composite cylindrical shells were developed and employed to analyse the post-buckling behaviours of such shells. Besides the numerical methods, the energy method, such as the Rayleigh-Ritz method and Galerkin method are widely used in the theoretical studies of post-buckling of composite cylindrical shells. By using the energy method, Huang and Han [15,16] and Van Tung [17] investigated the post-buckling of functionally graded (FG) cylindrical shells; Dai & Zheng [18] and Reddy & Savoia [19] performed the post-buckling analysis of FG laminated cylindrical shells; Ansari *et al.* [20] analysed the post-buckling behaviours of piezoelectric carbon nanotube-reinforced composite cylindrical shells. In addition, Shen proposed a boundary layer theory to consider the dramatic changes of displacements and bending moments at the boundaries. Based on the theory

and singular perturbation method, Shen and his collaborators investigated the post-buckling behaviour of laminated [21–26] and FG [27] cylindrical shells; Li & Qiao [28] studied the post-buckling of laminated cylindrical shells; Sahmani & Aghdam [29] investigated the post-buckling of magneto-electro-elastic composite nanoshells.

However, the composite shells are usually thicker than the metal ones, especially laminated composite shells. Therefore, the conventional Kirchhoff hypothesis which neglecting the transverse shear deformations could only induce a rough approximation for the post-buckling analysis of such shells. To achieve a more accurate model, the first-order and high-order shear deformation theories which introduced two additional variables to describe the rotations of the normal of the shell [30–32] were introduced into the post-buckling of the composite cylindrical shells. Bisagni [33] and Tafreshi [34] investigated the post-buckling of composite cylindrical shells by FEM. Eslami & Shariyat [35] investigated the dynamic post-buckling of laminated cylindrical shells by the finite difference method. Khoa *et al.* [36] investigated the post-buckling of piezoelectric FGM cylindrical shells by Galerkin's method. Shen and his collaborators investigated the post-buckling behaviour of laminated [37–39], FG [40,41], textile composite [42,43] and nanotube-reinforced composite [44,45] cylindrical shells by the singular perturbation method. Using the same method, Li & Yang [46] and Sahmani & Aghdam [47] investigated the post-buckling of laminated cylindrical shells and hybrid FG nanoshells, respectively. Parts of the above work [37,38] pointed out that the critical loads of the first-/high-order shear deformation theory are smaller than those of classical TWST. Most recently, the graphene platelets reinforced composite (GPLRC) [48–57] have been widely used in many engineering structures due to its excellent mechanical properties. The stability of GPLRC plates [48–50] and shells [51–57] has received increasing the attention from academia. Among them, Ansari and his collaborator [55,56] investigated the post-buckling behaviour of FG multilayer GPLRC cylindrical shells by using the Ritz energy method and semi-analytical method.

In view of the previous literatures, there are three major limitations in the theoretical studies:

- (i) The theoretical models for post-buckling of composite cylindrical shells were concentrated on the classical TWST. The amount of research based on the higher-order shear deformation theory (HSDT) was relatively limited. Definitely, the simple TWST will benefit the mathematical derivation. However, most of the composites are thicker than the classical metal materials. Consequently, the classical TWST would cause errors in the absence of transverse shear deformations.
- (ii) The existing post-buckling analysis based on the HSDT was mainly carried out by the singular perturbation method and the energy method. For the singular perturbation method, the accuracy of solutions can only be guaranteed in a small range of deformation near the critical buckling state. According to the experimental study [7,8], the secondary buckling occurs with the further loading and the mode-jumping phenomenon appears with the decreasing circumferential wavenumber. These phenomena cannot be predicted by the singular perturbation method. For the existing energy methods, the predetermined displacement functions were usually too simple to accurately describe the real post-buckling deformation. Therefore, it will greatly reduce the accuracy and convergence of the solutions.
- (iii) In the existing literature carried out by the energy method, the displacement functions are usually assumed to be a fixed form, such as $w(x, y) = f_0 + f_1 \sin \alpha x \sin \beta y + f_2 \sin^2 \alpha x$ [15,16,18,20], $w = W^{mn} \sin \alpha_m x \cos \beta_n y$ [17] and $w = W \sin \beta_m x \sin \delta_n y$ [19,36], which are independent of the calculation parameters. However, according to the previous experimental study [7,8], there may exist two kinds of post-buckling modes (symmetric or asymmetric modes) which are highly depend on the calculation parameters. Therefore, it is necessary to develop two groups of displacement functions for the two different post-buckling modes.

Motivated by the above reasons, we perform the post-buckling analysis of FG multilayer GPLRC under axial compression so as to obtain the convergence post-buckling response curves. Firstly, to deal with the wide range of shell thickness, a unified shell theory consisting of TWST, first-order shear deformation theory (FSDT) and HSDT is proposed to derive the nonlinear large deflection governing equations. Subsequently, the displacement functions for both symmetric and asymmetric post-buckling modes are proposed, which can ensure that the minimum critical load of the post-buckling response curve approaches a steady value with the increase of the number of the expansion terms. Then, the convergence and accurate large deflection post-buckling equilibrium paths are obtained by Galerkin's method. Finally, the effects of key influencing parameters on the post-buckling behaviours of the FG multilayer GPLRC cylindrical shell including the differences in the three shell theories and their scope are investigated.

The paper is organized as follows: The governing equations based on the unified shell theory are derived in §2. The Galerkin solving procedure is presented in §3. Material properties of FG multilayer GPLRC are obtained in §4. Section 5 presents several numerical examples. At last, conclusions are drawn in §6.

2. Fundamental equations based on the unified shell theory

Considering an FG multilayer GPLRC cylindrical shell under axial compression F in figure 1. The geometries are taken as length L , radius R and thickness h . An orthogonal curvilinear coordinate system is adopted and x -, y - and z -axes represent the longitudinal, circumferential and normal of the mid-surface of the shell, respectively.

To perform the post-buckling analysis of the FG multilayer GPLRC cylindrical shell with a wide range of thickness, the fundamental equations will be established in the framework of a unified shell theory, which is derived by integrating TWST, FSDT and HSDT [58–60]. In the unified shell theory, the displacement fields can be expressed by

$$u_1(x, y, z) = u(x, y) - zw(x, y)_{,x} + f(z)[\Phi_x(x, y) + w(x, y)_{,x}], \quad (2.1a)$$

$$u_2(x, y, z) = (1 + z/R)v(x, y) - zw(x, y)_{,y} + f(z)[\Phi_y(x, y) + w(x, y)_{,y}] \quad (2.1b)$$

$$\text{and} \quad u_3(x, y, z) = w(x, y), \quad (2.1c)$$

where u , v and w correspond to the mid-surface displacements of the shell in the x -, y - and z -directions, respectively; Φ_x and Φ_y represent the rotations of the transverse normal about the y - and x - axes, respectively; the comma denotes the partial differential operator hereafter. Here, it should be pointed out that equation (2.1) can be degenerated into the displacement fields of TWST, FSDT and HSDT by altering the function $f(z) = 0$, z or $z - 4z^3/3h^2$, respectively.

According to Novozhilov's nonlinear shell theory, the simplified strain–displacement relations can be written as

$$\varepsilon = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} u_{1,x} + \frac{u_{1,x}^2 + u_{2,x}^2 + u_{3,x}^2}{2} \\ \frac{u_{2,y} + u_{3/R}}{1 + z/R} + \frac{u_{1,y}^2 + (u_{2,y} + u_{3/R})^2 + (u_{2/R} - u_{3,y})^2}{2(1 + z/R)^2} \\ u_{2,x} + \frac{u_{1,y}}{1 + z/R} + u_{1,x}u_{1,y} + (u_{2,y} + u_{3/R})u_{2,x} + u_{3,x}(u_{2/R} - u_{3,y}) \end{Bmatrix} \quad (2.2a)$$

and

$$\gamma = \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} u_{1,z} + u_{3,x} + u_{1,x}u_{1,z} + u_{2,x}u_{2,z} + u_{3,x}u_{3,z} \\ u_{2,z} + \frac{u_{3,y} - u_{2/R} + u_{1,y}u_{1,z} + (u_{2,y} + u_{3/R})u_{2,z} + (u_{3,y} - u_{2/R})u_{3,z}}{1 + z/R} \end{Bmatrix}. \quad (2.2b)$$

Figure 1. Schematic of FG multilayer GPLRC cylindrical shell under axial compression. (Online version in colour.)

Substituting equation (2.1) into equation (2.2) and only retaining the nonlinear terms with respect to $w_{,x}$ and $w_{,y}$, the strain–displacement relations are

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{(0)} + z\boldsymbol{\chi}^{(0)} + f(z)\boldsymbol{\chi}^{(2)} \quad (2.3a)$$

and

$$\boldsymbol{\gamma} = f(z)_{,z}\boldsymbol{\chi}^{(1)}, \quad (2.3b)$$

where

$$\begin{aligned} \boldsymbol{\varepsilon}^{(0)} = \begin{Bmatrix} \varepsilon_x^{(0)} \\ \varepsilon_y^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} &= \begin{Bmatrix} u_{,x} + \frac{w_x^2}{2} \\ v_{,y} + \frac{w}{R} + \frac{w_{,y}^2}{2} \\ u_{,y} + v_{,x} + w_{,x}w_{,y} \end{Bmatrix}, \quad \boldsymbol{\chi}^{(0)} = \begin{Bmatrix} \chi_x^{(0)} \\ \chi_y^{(0)} \\ \chi_{xy}^{(0)} \end{Bmatrix} = - \begin{Bmatrix} w_{,xx} \\ w_{,yy} \\ 2w_{,xy} \end{Bmatrix} \\ \boldsymbol{\chi}^{(2)} = \begin{Bmatrix} \chi_x^{(2)} \\ \chi_y^{(2)} \\ \chi_{xy}^{(2)} \end{Bmatrix} &= \begin{Bmatrix} \Phi_{x,x} + w_{,xx} \\ \Phi_{y,y} + w_{,yy} \\ \Phi_{x,y} + \Phi_{y,x} + 2w_{,xy} \end{Bmatrix}, \end{aligned} \quad (2.4a)$$

and

$$\boldsymbol{\chi}^{(1)} = \begin{Bmatrix} \chi_{xz}^{(1)} \\ \chi_{yz}^{(1)} \end{Bmatrix} = \begin{Bmatrix} \Phi_x + w_{,x} \\ \Phi_y + w_{,y} \end{Bmatrix}. \quad (2.4b)$$

The constitutive equation in the k -th lamina of the FG multilayer GPLRC cylindrical shell can be given by

$$\begin{Bmatrix} \boldsymbol{\sigma}^k \\ \boldsymbol{\tau}^k \end{Bmatrix} = \mathbf{K}^k \cdot \begin{Bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\gamma} \end{Bmatrix}, \quad (2.5)$$

where $\boldsymbol{\sigma}^k = \{\sigma_x^k, \sigma_y^k, \tau_{xy}^k\}^T$ and $\boldsymbol{\tau}^k = \{\tau_{xz}^k, \tau_{yz}^k\}^T$ are the in-plane stress vector and transverse shear stress vector, respectively; $\mathbf{K}^k$ is the elastic coefficients matrix.

Integrating equation (2.5) across the thickness, the stress resultants per unit length on the mid-surface of the shell can be obtained as

$$\hat{\mathbf{n}} = \sum_{k=1}^{K_n} \int_{h_k}^{h_{k+1}} \boldsymbol{\sigma}^k dz, \quad \hat{\mathbf{m}} = \sum_{k=1}^{K_n} \int_{h_k}^{h_{k+1}} z \boldsymbol{\sigma}^k dz, \quad \hat{\mathbf{p}} = \sum_{k=1}^{K_n} \int_{h_k}^{h_{k+1}} f(z) \boldsymbol{\sigma}^k dz, \quad \hat{\mathbf{q}} = \sum_{k=1}^{K_n} \int_{h_k}^{h_{k+1}} f(z)_{,z} \boldsymbol{\tau}^k dz, \quad (2.6)$$

where $\hat{\mathbf{n}} = \{N_x, N_y, N_{xy}\}^T$, $\hat{\mathbf{m}} = \{M_x, M_y, M_{xy}\}^T$, $\hat{\mathbf{p}} = \{P_x, P_y, P_{xy}\}^T$ and $\hat{\mathbf{q}} = \{Q_x, Q_y\}^T$ represent the resultant force vector, moment vector, higher-order force vector and shear force vector,

respectively. Substituting equation (2.5) into equation (2.6) and recalling equation (2.3), the elastic equations can be obtained as

$$\begin{Bmatrix} \hat{n} \\ \hat{m} \\ \hat{p} \end{Bmatrix} = \begin{bmatrix} A & B & C \\ B & D & E \\ C & E & F \end{bmatrix} \cdot \begin{Bmatrix} \boldsymbol{\varepsilon}^{(0)} \\ \boldsymbol{\chi}^{(0)} \\ \boldsymbol{\chi}^{(2)} \end{Bmatrix} \quad (2.7a)$$

and

$$\hat{q} = \mathbf{H} \cdot \boldsymbol{\chi}^{(1)}, \quad (2.7b)$$

where the elements in the matrixes are $A_{ij} = \sum_{k=1}^{K_n} \int_{h_k}^{h_{k+1}} K_{ij}^k dz$, $B_{ij} = \sum_{k=1}^{K_n} \int_{h_k}^{h_{k+1}} z K_{ij}^k dz$, $C_{ij} = \sum_{k=1}^{K_n} \int_{h_k}^{h_{k+1}} f(z) K_{ij}^k dz$, $D_{ij} = \sum_{k=1}^{K_n} \int_{h_k}^{h_{k+1}} z^2 K_{ij}^k dz$, $E_{ij} = \sum_{k=1}^{K_n} \int_{h_k}^{h_{k+1}} z f(z) K_{ij}^k dz$, $F_{ij} = \sum_{k=1}^{K_n} \int_{h_k}^{h_{k+1}} f^2(z) K_{ij}^k dz$ and $H_{ij} = \sum_{k=1}^{K_n} \int_{h_k}^{h_{k+1}} \kappa K_{ij}^k f(z)^2 dz$ in which κ is the shear correction coefficient in the FSDT.

The equilibrium equations of FG multilayer GPLRC cylindrical shells can be established by the principle of minimum potential energy. The total potential energy Π consisting of the elastic strain energy and work of external loads is expressed as

$$\begin{aligned} \Pi = & \frac{1}{2} \int_{-L/2}^{L/2} \int_0^{2\pi R} \left(\hat{n}^T \boldsymbol{\varepsilon}^{(0)} + \hat{m}^T \boldsymbol{\chi}^{(0)} + \hat{p} \boldsymbol{\chi}^{(2)} + \hat{q} \boldsymbol{\chi}^{(1)} \right) dx dy - \int_{-L/2}^{L/2} \int_0^{2\pi R} (f_x u + f_y v + p w) dx dy \\ & - \int_0^{2\pi R} [\bar{f}_x u + \bar{f}_y v + \bar{p} w - \bar{m}_x w_{,x}]_{x=-L/2}^{x=L/2} dy. \end{aligned} \quad (2.8)$$

where f_x , f_y and p indicate the distributed forces per unit area in the x -, y - and z -directions, respectively; $\bar{f}_x$, $\bar{f}_y$, $\bar{p}$ and $\bar{m}_x$ represent the external loads and bending moment per unit length along the edges of the shell. Therefore, the equilibrium equations can be derived from $\delta\Pi = 0$, i.e.

$$N_{x,x} + N_{xy,y} + f_x = 0, \quad (2.9a)$$

$$N_{xy,x} + N_{y,y} + f_y = 0, \quad (2.9b)$$

$$\begin{aligned} M_{x,xx} + M_{y,yy} + 2M_{xy,xy} - P_{x,xx} - P_{y,yy} - 2P_{xy,xy} + Q_{x,x} + Q_{y,y} \\ - \frac{N_y}{R} + (N_x w_{,x} + N_{xy} w_{,y})_{,x} + (N_y w_{,y} + N_{xy} w_{,x})_{,y} + p = 0, \end{aligned} \quad (2.9c)$$

$$P_{x,x} + P_{xy,y} - Q_x = 0, \quad (2.9d)$$

and $P_{y,y} + P_{xy,x} - Q_y = 0. \quad (2.9e)$

To simplify the solution procedure, the body forces are neglected. Define ψ as Airy's stress function which satisfies

$$N_{xx} = \psi_{,yy}, N_{yy} = \psi_{,xx}, N_{xy} = -\psi_{,xy}. \quad (2.10)$$

Substituting equation (2.10) in equation (2.9), the first two equations (equations (2.9a) and (2.9b)) are self-satisfied. Therefore, it is necessary to supplement a compatible condition, i.e.

$$\varepsilon_{xx,yy} + \varepsilon_{yy,xx} - \gamma_{xy,xy} = \frac{w_{,xx}}{R} - w_{,xx} w_{,yy} + w_{,xy}^2. \quad (2.11)$$

By means of equations (2.7), (2.9) and (2.11), equilibrium equations (2.9) can be rewritten into

$$c_1 \nabla^4 \psi + c_2 \nabla^4 w + c_3 \nabla^2 (\Phi_{x,x} + \Phi_{y,y}) - \frac{w_{,xx}}{R} + w_{,xx} w_{,yy} - w_{,xy}^2 = 0, \quad (2.12a)$$

$$\begin{aligned} c_2 \nabla^4 \psi - c_4 \nabla^4 w - c_5 \nabla^2 (\Phi_{x,x} + \Phi_{y,y}) + c_6 (\nabla^2 w + \Phi_{x,x} + \Phi_{y,y}) \\ - \frac{\psi_{,xx}}{R} + \psi_{,yy} w_{,xx} - 2\psi_{,xy} w_{,xy} + \psi_{,xx} w_{,yy} = 0, \end{aligned} \quad (2.12b)$$

$$c_3 \nabla^2 (\psi_{,x}) - c_5 \nabla^2 (w_{,x}) - c_7 \Phi_{x,xx} - c_8 \Phi_{x,yy} - c_9 \Phi_{y,xy} + c_6 (w_{,x} + \Phi_x) = 0, \quad (2.12c)$$

and $c_3 \nabla^2 (\psi_{,y}) - c_5 \nabla^2 (w_{,y}) - c_9 \Phi_{x,xy} - c_8 \Phi_{y,xx} - c_7 \Phi_{y,yy} + c_6 (w_{,y} + \Phi_y) = 0, \quad (2.12d)$

in which the coefficients c_i ($i = 1, 2, \dots, 9$) are elaborated in appendix A.

The boundary conditions at the two ends of FG multilayer GPLRC cylindrical shells are assumed to be clamped, i.e.

$$w = w_{,x} = \Phi_x = \Phi_y = u = v = 0, \int_0^{2\pi R} N_x dy = -F, \int_0^{2\pi R} N_{xy} dy = 0 \quad (\text{at } x = 0, L), \quad (2.13)$$

where $u = v = 0$ are the in-plane boundary conditions and $\Phi_x = \Phi_y = 0$ represent the rotations of the transverse normal about the y - and x -axes are zero, which only exist in the first-order/high-order shear deformation theory.

Here, it should be mentioned that it is difficult to direct describe the boundary conditions with u and v by ψ , w , Φ_x and Φ_y . Therefore, we employ the equivalent boundary conditions $u_{,yy}|_{x=0,L} = 0$ and $v_{,yy}|_{x=0,L} = 0$ to replace $u|_{x=0,L} = 0$ and $v|_{x=0,L} = 0$ in equation (2.13). The boundary conditions at $x = 0, L$ can be rewritten as

$$w = w_{,x} = \Phi_x = \Phi_y = 0, \quad (2.14a)$$

$$u_{,yy} = -c_1 \psi_{,xxx} + c_{10} \psi_{,xyy} - c_2 w_{,xxx} + c_{11} w_{,xyy} - c_3 \Phi_{x,xx} + c_{12} \Phi_{x,yy} - c_3 \Phi_{y,xy} - w_{,x} w_{,yy} + \frac{w_{,x}}{R} = 0, \quad (2.14b)$$

$$v_{,yy} = c_1 \psi_{,xx} + c_{13} \psi_{,yy} + c_2 w_{,xx} + c_{14} w_{,yy} + c_3 \Phi_{x,x} + c_{15} \Phi_{y,y} - \frac{w}{R} - \frac{1}{2} w_{,y}^2 = 0, \quad (2.14c)$$

$$\text{and} \quad [\psi_{,x}]_{y=0}^{2\pi R} = 0, [\psi_{,y}]_{y=0}^{2\pi R} = -F, \quad (2.14d)$$

where c_i ($i = 10, 11, \dots, 15$) are elaborated in appendix A.

The edge shortening δ of the shell is given by

$$\delta = - \int_0^L u_{,x} dx. \quad (2.15)$$

To simplify the manipulation, the following dimensionless parameters are defined before solving the nonlinear equilibrium equations.

$$\begin{aligned} \xi &= \frac{\pi x}{L}, \eta = \frac{N}{R} y, U = \frac{L}{\pi h^2} u, V = \frac{L}{\pi h^2} v, W = \frac{w}{h}, \Psi = \frac{\psi c_1}{h^2}, \\ \alpha &= \frac{L}{\pi h}, \beta = \frac{LN}{\pi R}, \kappa = \frac{L^2}{h\pi^2 R}, \bar{\delta} = \frac{R}{Lh} \delta, \Sigma = \frac{F}{F_{cr}}, \end{aligned} \quad (2.16)$$

where N is the circumferential wavenumber and $F_{cr} = 2\pi E_m h^2 / \sqrt{3(1-\nu^2)}$; E_m is Young's modulus of the matrix.

With these dimensionless notations (2.16), the equilibrium equations are rewritten as follows:

$$\bar{\nabla}^4 \Psi + \zeta_1 \bar{\nabla}^4 W + \zeta_2 \bar{\nabla}^2 \Phi_{x,\xi} + \zeta_2 \beta \bar{\nabla}^2 \Phi_{y,\eta} - \kappa W_{,\xi\xi} + \beta^2 W_{,\xi\xi} W_{,\eta\eta} - \beta^2 W_{,\xi\eta}^2 = 0, \quad (2.17a)$$

$$\begin{aligned} \zeta_1 \bar{\nabla}^4 \Psi - \zeta_3 \bar{\nabla}^4 W - \zeta_4 \bar{\nabla}^4 \Phi_{x,\xi} - \zeta_4 \beta \bar{\nabla}^4 \Phi_{y,\eta} + \zeta_5 \bar{\nabla}^2 W + \zeta_5 \alpha \Phi_{x,\xi} \\ + \zeta_5 \alpha \beta \Phi_{y,\eta} - \kappa \Psi_{,\xi\xi} + \Psi_{,\xi\xi} W_{,\eta\eta} - 2\beta^2 \Psi_{,\xi\eta} W_{,\xi\eta} + \beta^2 \Psi_{,\eta\eta} W_{,\xi\xi} = 0, \end{aligned} \quad (2.17b)$$

$$\zeta_2 \bar{\nabla}^2 \Psi_{,\xi} - \zeta_4 \bar{\nabla}^2 W_{,\xi} - \zeta_6 \Phi_{x,\xi\xi} - \zeta_7 \beta^2 \Phi_{x,\eta\eta} - \zeta_8 \beta \Phi_{y,\xi\eta} + \zeta_5 \alpha W_{,\xi} + \zeta_5 \alpha^2 \Phi_x = 0, \quad (2.17c)$$

$$\text{and} \quad \zeta_2 \beta \bar{\nabla}^2 \Psi_{,\eta} - \zeta_4 \beta \bar{\nabla}^2 W_{,\eta} - \zeta_8 \beta \Phi_{x,\xi\eta} - \zeta_7 \Phi_{y,\xi\xi} - \zeta_6 \beta^2 \Phi_{y,\eta\eta} + \zeta_5 \alpha \beta W_{,\eta} + \zeta_5 \alpha^2 \Phi_y = 0, \quad (2.17d)$$

in which $\bar{\nabla}^2 = \partial^2 / \partial \xi^2 + \beta^2 \partial^2 / \partial \eta^2$.

The dimensionless boundary conditions at $\xi = 0, \pi$ are

$$W = W_{,\xi} = \Phi_x = \Phi_y = 0, \quad (2.18a)$$

$$U_{,\eta\eta} = -\frac{1}{\beta^2} \Psi_{,\xi\xi\xi} + \zeta_9 \Psi_{,\xi\eta\eta} - \frac{\zeta_1}{\beta^2} W_{,\xi\xi\xi} + \zeta_{10} W_{,\xi\eta\eta} - \frac{\zeta_2}{\beta^2} \Phi_{x,\xi\xi} + \zeta_{11} \Phi_{y,\eta\eta} - \frac{\zeta_2}{\beta} \Phi_{y,\xi\eta} - W_{,\xi} W_{,\eta\eta} + \frac{\kappa}{\beta^2} W_{,\xi} = 0, \quad (2.18b)$$

$$V_{,\eta\eta} = \frac{1}{\beta} \Psi_{,\xi\xi} + \zeta_{12} \Psi_{,\eta\eta} + \frac{\zeta_1}{\beta} W_{,\xi\xi} + \zeta_{13} W_{,\eta\eta} + \frac{\zeta_2}{\beta} \Phi_{x,\xi} + \zeta_{14} \Phi_{y,\eta} - \frac{\kappa}{\beta} W - \frac{1}{2} \beta W_{,\eta}^2 = 0, \quad (2.18c)$$

$$\text{and} \quad [\Psi_{,\xi}]_{\eta=0}^{2\pi} = 0, \quad [\Psi_{,\eta}]_{y=0}^{2\pi} = -\frac{2\pi}{\beta^2} \sigma. \quad (2.18d)$$

The dimensionless edge shortening $\bar{\delta}$ of the cylindrical shell is

$$\bar{\delta} = -\frac{1}{\pi\alpha} \int_0^\pi u_{,\xi} d\xi. \quad (2.19)$$

3. Galerkin solving procedure

In this section, the Galerkin solving procedure is presented. The first equilibrium equation (2.17a) is solved analytically first, and then the last three equilibrium equations (2.17b)–(2.17d) are solved by Galerkin's method. The existing experimental study indicated that the post-buckling of axial compressed cylindrical shells can be divided into two categories: symmetric mode and asymmetric mode [8]. Therefore, to perform the post-buckling analysis of axial compressed cylindrical shells, two types of displacement functions are adopted here, i.e.

$$W = \sum_n \sum_m d_{mn} \{\cos[(m-1)\xi] - \cos[(m+1)\xi]\} \cos(n\eta), \quad (3.1a)$$

$$\Phi_x = \sum_n \sum_m e_{mn} \{\sin[(m-1)\xi] - \sin[(m+1)\xi]\} \cos(n\eta), \quad (3.1b)$$

$$\text{and} \quad \Phi_y = \sum_n \sum_m f_{mn} \{\cos[(m-1)\xi] - \cos[(m+1)\xi]\} \sin(n\eta), \quad (3.1c)$$

in which d_{mn}, e_{mn}, f_{mn} are undetermined coefficients; m and n can be selected as $m = 1, 3, 5, \dots, n = 0, 1, 2, \dots$, for symmetric mode; $m = 1, 2, 3, \dots, n = 0, 1, 2, \dots$ ($m+n = \text{odd}$) for asymmetric mode.

Obviously, equation (3.1) satisfies the boundary conditions of equation (2.18a). Substituting equation (3.1) into equation (2.17a), we have

$$\bar{\nabla}^4 \Psi = \sum_{p=0} \sum_{q=0} \psi_{pq} \cos p\xi \cos q\eta, \quad (p, q = 0, 1, 2, \dots; \psi_{00} = 0), \quad (3.2)$$

where ψ_{pq} are constants which can be determined from equation (3.2). The general solution Ψ can be expressed as

$$\begin{aligned} \Psi = & \sum_{q=1} [A_q \cosh(q\beta\xi) + B_q q\beta\xi \sinh(q\beta\xi) + A'_q \sinh(q\beta\xi) + B'_q q\beta\xi \cosh(q\beta\xi)] \cos(q\eta) \\ & + \sum_{p=0} \sum_{q=0} \Psi_{pq} \cos(p\xi) \cos(q\eta) + p_x \eta^2 + p_y \xi^2, \end{aligned} \quad (3.3)$$

where $\Psi_{pq} = \psi_{pq} / (p^2 + q^2 \beta^2)$ and $\Psi_{00} = 0$. According to the loading conditions (2.18d), the first two items can be determined as

$$p_x = -\sigma / 2\beta^2, \quad (3.4a)$$

and

$$p_y = -\beta \zeta_{12} p_x + \frac{1}{2} \sum_{p=1} p^2 \psi_{p0} - 2 \sum_{p=1} (2p-1) d_{2p-1,0} + \sum_{p=1} (2p-1) e_{2p-1,0}. \quad (3.4b)$$

In addition, the sum of hyperbolic functions is designed for satisfying the in-plane boundary conditions (2.18b) and (2.18c) so that the coefficients A_q, B_q, A'_q, B'_q can be determined. However, these hyperbolic functions may lead to a numerical instability because of the highly nonlinear and intrinsic complexity of the described physical phenomenon. Therefore, they need to be replaced by an approximated Fourier expansion as

$$A_q \cosh(q\beta\xi) + B_q q\beta\xi \sinh(q\beta\xi) + A'_q \sinh(q\beta\xi) + B'_q q\beta\xi \cosh(q\beta\xi) = G_{0q} + \sum_{p=1} G_{pq} \cos(p\xi), \quad (3.5)$$

where

$$G_{0q} = \frac{1}{\pi} \int_0^\pi [A_q \cosh(q\beta\xi) + B_q q\beta\xi \sinh(q\beta\xi) + A'_q \sinh(q\beta\xi) + B'_q q\beta\xi \cosh(q\beta\xi)] d\xi, \quad (3.6a)$$

and

$$G_{pq} = \frac{2}{\pi} \int_0^\pi [A_q \cosh(q\beta\xi) + B_q q\beta\xi \sinh(q\beta\xi) + A'_q \sinh(q\beta\xi) + B'_q q\beta\xi \cosh(q\beta\xi)] \cos(q\eta) d\xi. \quad (3.6b)$$

The stress function can be re-written into

$$\Psi = p_x \eta^2 + p_y \xi^2 + \sum_{p=0} \sum_{q=0} H_{pq} \cos(p\xi) \cos(q\eta), \quad (3.7)$$

where $H_{pq} = \Psi_{pq} + G_{pq}$, $H_{p0} = \Psi_{p0}$, $p_x = -\sigma/2\beta^2$.

Until now, all the boundary conditions have been perfectly satisfied. To obtain the undetermined coefficients d_{mn} , e_{mn} and f_{mn} , Galerkin's method are applied to the last three equations in equation (2.17) and the following relations are obtained:

$$\int_0^\pi \int_0^{2\pi} L_1(W, \Phi_x, \Phi_y) \{\cos[(r-1)\xi] - \cos[(r+1)\xi]\} \cos(s\eta) d\xi d\eta = 0, \quad (3.8a)$$

$$\int_0^\pi \int_0^{2\pi} L_2(W, \Phi_x, \Phi_y) \{\sin[(r-1)\xi] - \sin[(r+1)\xi]\} \cos(s\eta) d\xi d\eta = 0, \quad (3.8b)$$

$$\text{and} \quad \int_0^\pi \int_0^{2\pi} L_3(W, \Phi_x, \Phi_y) \{\cos[(r-1)\xi] - \cos[(r+1)\xi]\} \sin(s\eta) d\xi d\eta = 0, \quad (3.8c)$$

where L_i ($i = 1, 2, 3$) represent the expression in equation (2.17).

By means of integration in equation (3.8), a set of nonlinear equations containing the undetermined coefficients d_{mn} , e_{mn} and f_{mn} are established. The Newton iterative method is applied to solve the nonlinear equations. Furthermore, it is worth to mention that equation (3.8b) and (3.8c) can be analytically solved before using the Newton iterative method, and the obtained analytical solutions could greatly improve the computation efficiency.

4. Material model of functionally graded multilayer graphene platelets reinforced composite

The material properties of FG multilayer GPLRC are presented in this section. A polymer matrix can be enhanced efficiently by incorporating a little amount of GPLs, which is expected to use broadly in the field of composites [61–63]. By stacking the laminas with different contents of GPLs together, we can obtain the FG multilayer GPLRC conveniently. It is assumed that the materials considered here keep a common trait that the densities of GPLs in every layer vary linearly across the thickness of the structure. The distributions of GPLs may be diverse in practice. However, we mainly concentrate on four distinct categories as illustrated in figure 2. The pattern GPL-U depicts a uniform distribution with the same content of GPLs for every layer. GPL-O and GPL-X exhibit

Figure 2. Schematic of four distribution patterns of GPLs along the thickness (darker colour means higher weight fraction of GPLs). (Online version in colour.)

Figure 3. Schematic of distribution of GPLs along the thickness for different slope factor S_g with GPL-X distribution (darker colour means higher weight fraction of GPLs). (Online version in colour.)

the symmetric GPLs distributions with an inverse relationship. GPL-A takes a linear decreased GPLs from one side to the other side of the structure. The weight fraction of GPLs M_g^k in the k -th layer, which is counted from the outer side of the plate or shell, can be depicted mathematically as [64,65]:

$$\text{GPL-U: } M_g^k = M_g, \quad (4.1a)$$

$$\text{GPL-O: } M_g^k = \frac{M_g(K_n - 2 + S_g K_n - 2S_g|1 - 2k + K_n|)}{K_n - 2}, \quad (4.1b)$$

$$\text{GPL-X: } M_g^k = \frac{M_g(K_n - 2 - S_g K_n + 2S_g|1 - 2k + K_n|)}{K_n - 2}, \quad (4.1c)$$

and

$$\text{GPL-A: } M_g^k = \frac{M_g[K_n - 1 + S_g(1 - 2k + K_n)]}{K_n - 1}, \quad (4.1d)$$

where M_g indicates the average weight fraction of GPLs over K_n laminas. In addition, a slope factor S_g ($0 \leq S_g \leq 1$) is introduced to control the steepness of the linear distribution function. Taking the GPL-X for example, the variations of the densities of GPLs with respect to S_g are displayed in figure 3. When S_g is amplified, more GPLs will gather on the exterior of the structure.

The effective elastic modulus of GPLRC can be derived through the Halpin-Tsai micromechanical model, which is given by [66]

$$E_{gm}^k = \frac{3}{8} \frac{1 + \xi_l \eta_l V_g^k}{1 - \eta_l V_g^k} E_m + \frac{5}{8} \frac{1 + \xi_w \eta_w V_g^k}{1 - \eta_w V_g^k} E_m, \quad (4.2)$$

where E_m is Young's modulus of the matrix and

$$\xi_l = \frac{2l_g}{h_g}, \xi_w = \frac{2w_g}{h_g}, \eta_l = \frac{E_g/E_m - 1}{E_g/E_m + \xi_l}, \eta_w = \frac{E_g/E_m - 1}{E_g/E_m + \xi_w}, \quad (4.3)$$

in which l_g , w_g and h_g indicate the average length, width and thickness of GPLs, respectively; E_g is Young's modulus of GPLs. The volume fraction V_g^k is related to the weight fraction M_g^k by the

expression

$$V_g^k = \frac{M_g^k}{M_g^k + \rho_g/\rho_m - M_g^k \rho_g/\rho_m}, \quad (4.4)$$

where ρ_g and ρ_m represent the mass densities of GPLs and matrix, respectively.

Employing the rule of mixture, Poisson's ratio ν_{gm}^k of GPLRC are obtained as

$$\nu_{gm}^k = \nu_g V_g^k + \nu_m (1 - V_g^k), \quad (4.5)$$

where ν_g and ν_m are Poisson's ratios of GPLs and matrix, respectively.

The elastic coefficients are given by $K_{11}^k = K_{22}^k = E_{gm}^k / (1 - \nu_{gm}^{k2})$, $K_{12}^k = K_{21}^k = \nu_{gm}^k E_{gm}^k / (1 - \nu_{gm}^{k2})$ and $K_{33}^k = K_{44}^k = K_{55}^k = E_{gm}^k / [2(1 + \nu_{gm}^k)]$.

5. Numerical results and discussion

In this section, the post-buckling behaviours of FG multilayer GPLRC cylindrical shells are investigated by the unified shell theory and Galerkin's method (see electronic supplementary material). Firstly, the accuracy and convergence of Galerkin's method is verified by comparing with existing data. Subsequently, a systematic parametric study of geometrical parameters on load-shortening curves is performed to explore the application of the proposed shell theory. Lastly, the influence of material properties on the post-buckling behaviours of FG multilayer GPLRC cylindrical shells are discussed. The material properties of GPLs are taken as: $E_{GPL} = 1.01 \times 10^6$ MPa, $\nu_{GPL} = 0.186$, $l_{GPL} = 2.5 \times 10^{-6}$ m, $w_{GPL} = 1.5 \times 10^{-6}$ m, $t_{GPL} = 2.5 \times 10^{-9}$ m, $\rho_{GPL} = 1.06 \times 10^3$ kg m $^{-3}$, and the corresponding parameters of the matrix are: $E_m = 3 \times 10^3$ MPa; $\nu_m = 0.34$; $\rho_m = 1.2 \times 10^3$ kg m $^{-3}$.

(a) Comparison and convergence study

To verify the accuracy and convergence of the present method, the present results are compared with the experimental and theoretical results of Yamaki [8]. A clamped isotropic cylindrical shell is considered here and the computation parameters are taken as $E = 5560$ MPa, $\nu = 0.3$, $R/h = 405$, $Z = 200$ where $Z = \sqrt{1 - \nu^2} L^2 / Rh$ is the Batdorf parameter.

The biggest barrier for post-buckling analysis is that the occurrence of multiple mode transitions and their spacing cannot be accurately predicted. To comprehensively verify the present post-buckling response predicted in the present study, the load-shortening curves are presented in figure 4 and compared with both the theoretical and experimental results [8]. Clearly, the large-deflection post-buckling behaviour of cylindrical shells are characterized by the step-wise reduced circumferential waves N . As to be expected, the results of TWST, FSDT, HSDT are almost the same due to very small shell thickness. For both the symmetric and asymmetric post-buckling modes, the curves of TWST, FSDT, HSDT and the theoretical results [8] overlap each other, are in accordance with the experimental results [8]. Furthermore, the comparison of deflection along the length direction for both the symmetric ($N = 10$) and asymmetric ($N = 14$) post-buckling modes are shown in figure 5. It is also observed that the post-buckling deformation is in good agreement with the theoretical and experimental results [8]. These observations demonstrated that the present study is capable of post-buckling analysis of FG multilayer GPLRC cylindrical shells and could provide accurate solutions.

Subsequently, a convergence study is performed in figure 6. Five cases of displacement functions W in (2.19) with different number of terms is considered in table 1 (in the computation, the numbers of terms in Φ_x and Φ_y are the same as W). The load-shortening curves of symmetric post-buckling modes for the five cases are presented in figure 6. It is observed that the load-shortening curves tend to converge with the increasing number of terms in the displacement functions. The curves for case 4 overlaps that for case 5. Therefore, to ensure the computational convergence, we take the displacement functions as case 5 in the following numerical examples.

Figure 4. Comparison of the load-shortening curves of isotropic cylindrical shells with $E = 5560$ MPa, $\nu = 0.3$, $R/h = 405$ and $Z = 200$: (a) symmetric post-buckling mode; (b) asymmetric post-buckling mode. (Online version in colour.)

Figure 5. Comparison of deflection along the length of isotropic cylindrical shells with $E = 5560$ MPa, $\nu = 0.3$, $R/h = 405$ and $Z = 200$: (a) symmetric post-buckling mode ($N = 10$); (b) asymmetric post-buckling mode ($N = 14$). (Online version in colour.)

Figure 6. Load-shortening curves of isotropic cylindrical shells for different cases of displacement functions: (a) symmetric post-buckling mode; (b) asymmetric post-buckling mode. (Online version in colour.)

Table 1. The undermined coefficients in the displacement function W for different cases.

case	the undermined coefficients	no. terms
symmetric post-buckling mode		
case 1	$d_{10}, d_{30}, d_{50}, d_{11}, d_{31}, d_{12}, d_{32}, d_{13}, d_{33}$	9
case 2	$d_{10}, d_{30}, d_{50}, d_{70}, d_{11}, d_{31}, d_{51}, d_{12}, d_{32}, d_{52}, d_{13}, d_{33}, d_{53}$	13
case 3	$d_{10}, d_{30}, d_{50}, d_{70}, d_{90}, d_{11}, d_{31}, d_{51}, d_{71}, d_{12}, d_{32}, d_{52}, d_{72}, d_{13}, d_{33}, d_{53}$	16
case 4	$d_{10}, d_{30}, d_{50}, d_{70}, d_{90}, d_{110}, d_{11}, d_{31}, d_{51}, d_{71}, d_{91}, d_{12}, d_{32}, d_{52}, d_{72}, d_{92}, d_{13}, d_{33}, d_{53}, d_{73}$	20
case 5	$d_{10}, d_{30}, d_{50}, d_{70}, d_{90}, d_{110}, d_{130}, d_{11}, d_{31}, d_{51}, d_{71}, d_{91}, d_{111}, d_{12}, d_{32}, d_{52}, d_{72}, d_{92}, d_{112}, d_{13}, d_{33}, d_{53}, d_{73}, d_{93}$	24
asymmetric post-buckling mode		
case 1	$d_{10}, d_{30}, d_{50}, d_{21}, d_{41}, d_{12}, d_{32}, d_{23}, d_{43}$	9
case 2	$d_{10}, d_{30}, d_{50}, d_{70}, d_{21}, d_{41}, d_{61}, d_{12}, d_{32}, d_{52}, d_{23}, d_{43}, d_{63}$	13
case 3	$d_{10}, d_{30}, d_{50}, d_{70}, d_{90}, d_{21}, d_{41}, d_{61}, d_{81}, d_{12}, d_{32}, d_{52}, d_{72}, d_{23}, d_{43}, d_{63}$	16
case 4	$d_{10}, d_{30}, d_{50}, d_{70}, d_{90}, d_{110}, d_{21}, d_{41}, d_{61}, d_{81}, d_{101}, d_{12}, d_{32}, d_{52}, d_{72}, d_{92}, d_{23}, d_{43}, d_{63}, d_{83}$	20
case 5	$d_{10}, d_{30}, d_{50}, d_{70}, d_{90}, d_{110}, d_{130}, d_{21}, d_{41}, d_{61}, d_{81}, d_{101}, d_{121}, d_{12}, d_{32}, d_{52}, d_{72}, d_{92}, d_{112}, d_{23}, d_{43}, d_{63}, d_{83}, d_{103}$	24

(b) Effects of geometrical parameters and shell theories

In this section, the effects of the geometrical parameters on the post-buckling behaviour of FG multilayer GPLRC cylindrical shells are investigated in detail. Consider a clamped FG multilayer GPLRC cylindrical shells with $h = 0.001$ m, $K = 20$, $M_g = 1\%$, $S_g = 1$ and GPL-X distribution. The load-shortening curves for different R/h are plotted in figure 7. It is obvious that, regardless of the post-buckling mode, the overall curves shift upward with the decreasing R/h . In other words, the critical load increases with the thickening of the shell. When the shell is thick enough, there exist a discrepancy among the three shell theories. These observations are similar to the findings in the studies on eigen-buckling problems, i.e. the increasing shell thickness improves the stability of cylindrical shells and the shear deformation should not be neglected for the shell with a certain thickness. In addition, it is interesting to find from figure 7b that the critical loads for different theories are not consistent for a very thin shell ($R/h = 500$). Clearly, the critical loads of FSDT and HSDT are different from those of TWST for the asymmetric post-buckling mode. The reason is that the TWST neglected the shear deformation effects cannot accurately describe the asymmetric post-buckling deformation of a thin shell. Consequently, the TWST predicts less accurate results than the FSDT and HSDT. This phenomenon will not happen for a very thin symmetrically buckled cylindrical shell because the shear deformation effects almost have no influence in the post-buckling process. Furthermore, it is also worth to note from figure 7 that the critical loads of asymmetric post-buckling mode increase more rapidly with the increase of thickness than those of symmetric post-buckling mode, which means that it is difficult for a thick shell to occur the asymmetric post-buckling.

Subsequently, the symmetric and asymmetric post-buckling deformations are discussed by considering the load-maximum deflection curves in figure 8. It is seen that the deflection increases with the increase of R/h and the inward deflections are always larger than the outward deflections. To get a better illustration, figure 9 presents the distributions of the post-buckling deflection along the length direction when $\eta = 0$ or π . Obviously, the shapes are similar but the amplitude have some differences, especially for the asymmetric post-buckling mode. Finally,

Figure 7. Load-shortening curves of FG multilayer GPLRC cylindrical shells obtained by different shell theories with $L/R = 1$, $R/h = 50, 200, 500$ for (a) symmetric post-buckling modes, and $L/R = 1$, $R/h = 200, 500$ for (b) asymmetric post-buckling modes. The circumferential wavenumber N for each curve are also marked in the figure. (Online version in colour.)

Figure 8. Load-maximum deflection curves of FG multilayer GPLRC cylindrical shells obtained by different shell theories with $L/R = 1$, $R/h = 50, 200, 500$ for (a) symmetric post-buckling modes, and $L/R = 1$, $R/h = 200, 500$ for (b) asymmetric post-buckling modes. The circumferential wavenumber N for each curve are also marked in the figure. (Online version in colour.)

three-dimensional post-buckling deformations for different R/h at the lowest critical loads are shown in figure 10 under HSDT. As depicted in figure 10, the symmetric post-buckling modes have the shape with only one array of bulges and the asymmetric post-buckling modes have two staggered arrays of bulges.

The effects of L/R on the post-buckling behaviours are investigated by plotting the load-shortening curves and load-maximum deflection curves in figures 11 and 12. The observations in these figures are similar to that found in figures 7 and 8. The decreasing shell length will increase the critical loads and the differences between the three shell theories. The shear deformation effects cannot be neglected in the post-buckling analysis of cylindrical shells with asymmetric deformation. In addition, figure 13 presents the three-dimensional post-buckling deformations for different L/R at the lowest critical loads under HSDT.

(c) Effects of material properties for the graphene platelets reinforced composite

The effect of the GPLs distribution, slope factor and weight fraction on the post-buckling behaviours of FG multilayer GPLRC cylindrical shells are investigated in this section. The

Figure 9. Deflection along the length obtained by different shell theories at $\eta = 0$ and π with $L/R = 1$, $R/h = 50$, $N = 5$ for (a) symmetric post-buckling modes, and $L/R = 1$, $R/h = 200$, $N = 6$ for (b) asymmetric post-buckling modes. (Online version in colour.)

Figure 10. Post-buckling deformations of the lowest critical loads obtained by HSDT for different R/h . Red and green represent the outward and inward deflection, respectively. (Online version in colour.)

Figure 11. Load-shortening curves of FG multilayer GPLRC cylindrical shells obtained by different shell theories with $R/h = 200$, $L/h = 0.5, 1, 2$ for (a) symmetric post-buckling modes, and $R/h = 200$, $L/h = 1, 2$ for (b) asymmetric post-buckling modes. The circumferential wavenumber N for each curve are also marked in the figure. (Online version in colour.)

computation parameters are selected as $h = 0.001$ m and $R/h = 200$ and all results are obtained by HSDT.

Firstly, the load-shortening curves of FG multilayer GPLRC cylindrical shells with different GPLs distributions are shown in figures 14 and 15. It is obvious that the FG multilayer GPLRC

Figure 12. Load-maximum deflection curves of FG multilayer GPLRC cylindrical shells obtained by different shell theories with $R/h = 200$, $L/h = 0.5, 1, 2$ for (a) symmetric post-buckling modes, and $R/h = 200$, $L/h = 1, 2$ for (b) asymmetric post-buckling modes. The circumferential wavenumber N for each curve are also marked in the figure. (Online version in colour.)

Figure 13. Post-buckling deformations of the lowest critical loads obtained by HSDT for different L/R . Red and green represent the outward and inward deflection, respectively. (Online version in colour.)

Figure 14. Load-shortening curves of FG multilayer GPLRC cylindrical shells for different GPLs distribution with $L/R = 1$, $R/h = 200$, $M_g = 1\%$, $S_g = 1$. The grey dashed lines represent the lowest critical loads for each GPLs distribution. (Online version in colour.)

Figure 15. Load-shortening curves of FG multilayer GPLRC cylindrical shells for different GPLs distribution with $L/R = 2$, $R/h = 200$, $M_g = 1\%$, $S_g = 1$. The grey dashed lines represent the lowest critical loads for each GPLs distribution. (Online version in colour.)

Figure 16. Load-shortening curves of FG multilayer GPLRC cylindrical shells for different slope factor with $L/R = 1$, $R/h = 200$, $M_g = 1\%$, GPL-X distribution. The grey dashed lines represent the lowest critical loads for each slope factor. (Online version in colour.)

cylindrical shells with GPL-X distribution have the best anti-buckling performance while the shells with GPL-O distribution exhibit the worst performance. This indicates that the stability of FG multilayer GPLRC cylindrical shells can be improved by filling more GPLs in the outer layers of the shell, GPLs filled in the inner layers and the non-symmetric GPLs distribution will both reduce the anti-buckling performance. Comparing figure 14 with figure 15, an interesting phenomenon is found. Regardless of the GPLs distributions, the load-shortening curves of symmetric post-buckling modes are lower than those of asymmetric post-buckling modes when $L/R = 1$; while an opposite observation is found when $L/R = 2$. It is deduced that only the geometrical parameters could affect the post-buckling modes; and the material properties only change the location of the load-shortening curves. The effects of slope factor S_g and weight fraction M_g are shown in figures 15 and 16. As seen in figure 16, the curves shift upward with the increase of S_g for GPL-X distribution. It indicates that the increasing S_g improves the stability of the FG multilayer GPLRC cylindrical shells with GPL-X distribution. Moreover, it should be mentioned that, the effect of slope factor S_g on the load-shortening curves is not the same for

Figure 17. Load-shortening curves of FG multilayer GPLRC cylindrical shells for different weight fraction with $L/R = 1$, $R/h = 200$, $S_g = 1$, GPL-X distribution. The grey dashed lines represent the lowest critical loads for each GPLs weight fraction. (Online version in colour.)

all the four GPLs distributions; it is similar to that of eigen-buckling problems (Please refer to the author's previous literature [64,65]). From figure 17, it is observed that the anti-buckling performance of FG multilayer GPLRC cylindrical shells can be greatly improved by adding a small amount of GPLs. The load-shortening curve sharply shift upward when M_g increases 1%.

6. Conclusion

This paper performed the post-buckling analysis of FG multilayer GPLRC cylindrical shells under axial compression and interests on their post-buckling response curves. By introducing a unified shell theory and Galerkin's method, the governing equations for post-buckling of FG multilayer GPLRC cylindrical shells with wide range of thickness are established and solved. Numerical examples demonstrate the accuracy of the present solutions and reveal the effects of key influencing factors on the post-buckling behaviours of FG multilayer GPLRC cylindrical shells. The theoretical model and solution method proposed in this paper would be extended to other boundary conditions by introducing new displacement functions which satisfy the boundary conditions. The following conclusions were drawn from the present study: (i) The geometrical parameters are the major influencing factors on the post-buckling behaviours of FG multilayer GPLRC cylindrical shells. In other words, the load-shortening curves and post-buckling modes (symmetric or asymmetric modes) are highly dependent of the thickness and length of the shell. The symmetric post-buckling mode usually occurs for a thick and short FG multilayer GPLRC cylindrical shell. Otherwise, the asymmetric post-buckling mode occurs. (ii) The shell theories have different influences on the symmetric and asymmetric post-buckling equilibrium path of FG multilayer GPLRC cylindrical shells. For the symmetric post-buckling mode, the shell theories have no influence on the load-shortening curves when the shell is very thin. Only when h/R greater than $1/50$, the load-shortening curves obtained by different shell theories are different. For the asymmetric mode, even if the shell is very thin ($h/R = 1/200$), the post-buckling equilibrium paths obtained by the three shell theories have great difference. (iii) For an FG multilayer GPLRC cylindrical shell, the material properties do not change the post-buckling mode (symmetric or asymmetric modes). The GPL distributions, weight fraction and slope factor are three key influencing factors for the anti-buckling performance of FG multilayer GPLRC cylindrical shells.

Data accessibility. This article has no additional data.

Authors' contributions. J.S.: methodology, software, validation, writing - original draft; S.Z.: validation, data curation, writing - original draft; Z.T.: formal analysis, investigation, resources; Z.Z.: conceptualization, methodology, writing - review and editing; X.X.: supervision.

Competing interests. We declare we have no competing interests.

Funding. The supports of Dalian Innovation Foundation of Science and Technology (grant no. 2018J11CY005); Aeronautical Science Foundation of China (grant no. 2018ZC63003); the Fundamental Research Funds for the Central Universities grant no. (DUT19LK32) and State Key Laboratory of Structural Analysis for Industrial Equipment (grant nos. G19109, S18313) are gratefully acknowledged.

Appendix A

$$c_1 = \frac{A_{11}}{(A_{11}^2 - A_{12}^2)}, \quad (\text{A } 1)$$

$$c_2 = \frac{[A_{12}(C_{11} - B_{11}) + A_{11}(B_{12} - C_{12})]}{(A_{11}^2 - A_{12}^2)}, \quad (\text{A } 2)$$

$$c_3 = \frac{(A_{12}C_{11} - A_{11}C_{12})}{(A_{11}^2 - A_{12}^2)}, \quad (\text{A } 3)$$

$$c_4 = [-A_{11}(B_{11}^2 + B_{12}^2 - 2B_{11}C_{11} + C_{11}^2 - 2B_{12}C_{12} + C_{12}^2) + A_{11}^2(D_{11} - 2E_{11} + F_{11}) - A_{12}(-2B_{11}B_{12} + 2B_{12}C_{11} + 2B_{11}C_{12} - 2C_{11}C_{12} + A_{12}D_{11} - 2A_{12}E_{11} + A_{12}F_{11})]/ \times (A_{11}^2 - A_{12}^2), \quad (\text{A } 4)$$

$$c_5 = \{A_{11}[B_{11}C_{11} - C_{11}^2 + (B_{12} - C_{12})C_{12}] + A_{11}^2(-E_{11} + F_{11}) + A_{12}(-B_{12}C_{11} - B_{11}C_{12} + 2C_{11}C_{12} + A_{12}E_{11} - A_{12}F_{11})\}/(A_{11}^2 - A_{12}^2), \quad (\text{A } 5)$$

$$c_6 = H_{11}, \quad (\text{A } 6)$$

$$c_7 = \frac{[-A_{11}(C_{11}^2 + C_{12}^2) + A_{11}^2F_{11} + A_{12}(2C_{11}C_{12} - A_{12}F_{11})]}{(A_{11}^2 - A_{12}^2)}, \quad (\text{A } 7)$$

$$c_8 = \frac{1}{2} \left[\frac{-(C_{11} - C_{12})^2}{(A_{11} - A_{12}) + F_{11} - F_{12}} \right], \quad (\text{A } 8)$$

$$c_9 = \frac{[-C_{11}^2 + 2C_{11}C_{12} + C_{12}^2 - (A_{11} + A_{12})(F_{11} + F_{12})]}{[2(A_{11} + A_{12})]}, \quad (\text{A } 9)$$

$$c_{10} = -\frac{(2A_{11} + A_{12})}{(A_{11}^2 - A_{12}^2)}, \quad (\text{A } 10)$$

$$c_{11} = \frac{[A_{12}(2B_{11} - B_{12} - 2C_{11} + C_{12}) + A_{11}(B_{11} - 2B_{12} - C_{11} + 2C_{12})]}{(A_{11}^2 - A_{12}^2)}, \quad (\text{A } 11)$$

$$c_{12} = \frac{-(C_{11} - C_{12})}{(A_{11} - A_{12})}, \quad (\text{A } 12)$$

$$c_{13} = \frac{-A_{12}}{(A_{11}^2 - A_{12}^2)}, \quad (\text{A } 13)$$

$$c_{14} = \frac{(A_{11}B_{11} - A_{12}B_{12} - A_{11}C_{11} + A_{12}C_{12})}{(A_{11}^2 - A_{12}^2)}, \quad (\text{A } 14)$$

$$\text{and } c_{15} = \frac{(-A_{11}C_{11} + A_{12}C_{12})}{(A_{11}^2 - A_{12}^2)}. \quad (\text{A } 15)$$

References

1. Lorenz R. 1908 Achsensymmetrische verzerrungen in dünnwandigen hohlzylindern. *Zeitschrift des Vereines Deutscher Ingenieure* **52**, 1706–1713.

2. Timoshenko SP. 1910 Einige stabilitätsprobleme der elastizitätstheorie. *Zeitschrift für Mathematik und Physik* **58**, 337–385.
3. Donnell LH. 1934 A new theory for the buckling of thin cylinders under axial compression and bending. *Trans. ASME* **56**, 795–806.
4. Von Karman T, Tsien HS. 1941 The buckling of thin cylindrical shells under axial compression. *J. Aeronaut. Sci.* **8**, 303–312. (doi:10.2514/8.10722)
5. Hoff NJ, Madsen WA, Mayers J. 1966 Postbuckling equilibrium of axially compressed circular cylindrical shells. *AIAA J.* **4**, 126–133. (doi:10.2514/3.3395)
6. Koiter W. 1963 The effect of axisymmetric imperfections on the buckling of cylindrical shells under axial compression. *Proc. Koninklijke Nederlandse Akademie van Wetenschappen* **66**, 265–279.
7. Yamaki N, Otomo K, Matsuda K. 1975 Experiments on the postbuckling behavior of circular cylindrical shells under compression. *Exp. Mech.* **15**, 23–28. (doi:10.1007/BF02318521)
8. Yamaki N. 1984 *Elastic stability of circular cylindrical shells*. Amsterdam, The Netherlands: Elsevier.
9. Xu GH, Huang HW, Han Q. 2018 Study on postbuckling of axial compressed elastoplastic functionally graded cylindrical shells. *Mech. Adv. Mater. Struct.* **25**, 820–828. (doi:10.1080/15376494.2017.1308589)
10. Wang SS, Srinivasan S, Hu HT, Hajali R. 1995 Effect of material nonlinearity on buckling and postbuckling of fiber composite laminated plates and cylindrical shells. *Compos. Struct.* **33**, 7–15. (doi:10.1016/0263-8223(95)00096-8)
11. Tafreshi A. 2002 Buckling and post-buckling analysis of composite cylindrical shells with cutouts subjected to internal pressure and axial compression loads. *Int. J. Pressure Vessels Pip.* **79**, 351–359. (doi:10.1016/S0308-0161(02)00026-1)
12. Eslami MR, Shariyat M, Shakeri M. 1998 Layerwise theory for dynamic buckling and postbuckling of laminated composite cylindrical shells. *AIAA J.* **36**, 1874–1882. (doi:10.2514/2.281)
13. Shariyat M, Eslami MR. 1999 Dynamic buckling and post-buckling of imperfect orthotropic cylindrical shells under mechanical and thermal loads, based on the three-dimensional theory of elasticity. *J. Appl. Mech.-Trans. ASME* **66**, 476–484. (doi:10.1115/1.2791072)
14. Sheinman I, Jabareen M. 2005 Postbuckling of laminated cylindrical shells in different formulations. *AIAA J.* **43**, 1117–1123. (doi:10.2514/1.11557)
15. Huang HW, Han Q. 2009 Nonlinear buckling and postbuckling of heated functionally graded cylindrical shells under combined axial compression and radial pressure. *Int. J. Non-Linear Mech.* **44**, 209–218. (doi:10.1016/j.ijnonlinmec.2008.11.016)
16. Huang HW, Han Q. 2009 Nonlinear elastic buckling and postbuckling of axially compressed functionally graded cylindrical shells. *Int. J. Mech. Sci.* **51**, 500–507. (doi:10.1016/j.jimecsci.2009.05.002)
17. Van Tung H. 2014 Postbuckling of functionally graded cylindrical shells with tangential edge restraints and temperature-dependent properties. *Acta Mech.* **225**, 1795–1808. (doi:10.1007/s00707-013-1011-2)
18. Dai HL, Zheng HY. 2011 Buckling and post-buckling analyses for an axially compressed laminated cylindrical shell of FGM with PFRC in thermal environments. *Eur. J. Mech. A-Solids* **30**, 913–923. (doi:10.1016/j.euromechsol.2011.05.009)
19. Reddy JN, Savoia M. 1992 Layer-wise shell theory for postbuckling of laminated circular cylindrical shells. *AIAA J.* **30**, 2148–2154. (doi:10.2514/3.11193)
20. Ansari R, Pourashraf T, Gholami R, Shahabodini A. 2016 Analytical solution for nonlinear postbuckling of functionally graded carbon nanotube-reinforced composite shells with piezoelectric layers. *Compos. Pt. B-Eng.* **90**, 267–277. (doi:10.1016/j.compositesb.2015.12.012)
21. Shen HS, Xiang Y. 2008 Buckling and postbuckling of anisotropic laminated cylindrical shells under combined axial compression and torsion. *Compos. Struct.* **84**, 375–386. (doi:10.1016/j.compstruct.2007.10.002)
22. Shen HS. 2008 Boundary layer theory for the buckling and postbuckling of an anisotropic laminated cylindrical shell. Part I: prediction under axial compression. *Compos. Struct.* **82**, 346–361. (doi:10.1016/j.compstruct.2007.01.024)
23. Shen HS. 1997 Post-buckling analysis of imperfect stiffened laminated cylindrical shells under combined external pressure and axial compression. *Comput. Struct.* **63**, 335–348. (doi:10.1016/S0045-7949(96)00341-0)

24. Shen HS. 1998 Postbuckling analysis of stiffened laminated cylindrical shells under combined external liquid pressure and axial compression. *Eng. Struct.* **20**, 738–751. (doi:10.1016/S0141-0296(97)00069-2)
25. Shen HS, Li Q. 2002 Postbuckling of cross-ply laminated cylindrical shells with piezoelectric actuators under complex loading conditions. *Int. J. Mech. Sci.* **44**, 1731–1754. (doi:10.1016/S0020-7403(02)00056-5)
26. Shen HS. 2001 Postbuckling analysis of axially-loaded laminated cylindrical shells with piezoelectric actuators. *Eur. J. Mech. A-Solids* **20**, 1007–1022. (doi:10.1016/S0997-7538(01)01176-7)
27. Shen HS. 2002 Postbuckling analysis of axially-loaded functionally graded cylindrical shells in thermal environments. *Compos. Sci. Technol.* **62**, 977–987. (doi:10.1016/S0266-3538(02)00029-5)
28. Li ZM, Qiao P. 2015 Buckling and postbuckling of anisotropic laminated cylindrical shells under combined external pressure and axial compression in thermal environments. *Compos. Struct.* **119**, 709–726. (doi:10.1016/j.compstruct.2014.09.039)
29. Sahmani S, Aghdam MM. 2018 Nonlocal strain gradient shell model for axial buckling and postbuckling analysis of magneto-electro-elastic composite nanoshells. *Compos. Pt. B-Eng.* **132**, 258–274. (doi:10.1016/j.compositesb.2017.09.004)
30. Reddy JN. 1984 Exact solutions of moderately thick laminated shells. *J. Eng. Mech.-ASCE* **110**, 794–809. (doi:10.1061/(ASCE)0733-9399(1984)110:5(794))
31. Reddy JN, Chandrashekhara K. 1985 Geometrically non-linear transient analysis of laminated, doubly curved shells. *Int. J. Non-Linear Mech.* **20**, 79–90. (doi:10.1016/0020-7462(85)90002-2)
32. Reddy JN, Liu CF. 1985 A higher-order shear deformation theory of laminated elastic shells. *Int. J. Eng. Sci.* **23**, 319–330. (doi:10.1016/0020-7225(85)90051-5)
33. Bisagni C. 2000 Numerical analysis and experimental correlation of composite shell buckling and post-buckling. *Compos. Pt. B-Eng.* **31**, 655–667. (doi:10.1016/S1359-8368(00)00031-7)
34. Tafreshi A. 2006 Delamination buckling and postbuckling in composite cylindrical shells under combined axial compression and external pressure. *Compos. Struct.* **72**, 401–418. (doi:10.1016/j.compstruct.2005.01.009)
35. Eslami MR, Shariyat M. 1999 A high-order theory for dynamic buckling and postbuckling analysis of laminated cylindrical shells. *J. Press. Vess.-Trans. ASME* **121**, 94–102. (doi:10.1115/1.2883673)
36. Khoa ND, Thiem HT, Duc ND. 2019 Nonlinear buckling and postbuckling of imperfect piezoelectric S-FGM circular cylindrical shells with metal-ceramic-metal layers in thermal environment using Reddy's third-order shear deformation shell theory. *Mech. Adv. Mater. Struct.* **26**, 248–259. (doi:10.1080/15376494.2017.1341583)
37. Shen HS. 2001 Postbuckling of shear deformable cross-ply laminated cylindrical shells under combined external pressure and axial compression. *Int. J. Mech. Sci.* **43**, 2493–2523. (doi:10.1016/S0020-7403(01)00058-3)
38. Li ZM, Shen HS. 2008 Postbuckling of shear-deformable anisotropic laminated cylindrical shells under axial compression. *Int. J. Struct. Stab. Dyn.* **8**, 389–414. (doi:10.1142/S0219455408002715)
39. Shen HS. 2013 Postbuckling of axially-loaded laminated cylindrical shells surrounded by an elastic medium. *Mech. Adv. Mater. Struct.* **20**, 130–150. (doi:10.1080/15376494.2011.584141)
40. Shen HS, Noda N. 2005 Postbuckling of FGM cylindrical shells under combined axial and radial mechanical loads in thermal environments. *Int. J. Solids Struct.* **42**, 4641–4662. (doi:10.1016/j.ijsolstr.2005.02.005)
41. Shen HS. 2005 Postbuckling of axially loaded FGM hybrid cylindrical shells in thermal environments. *Compos. Sci. Technol.* **65**, 1675–1690. (doi:10.1016/j.compscitech.2005.02.008)
42. Li ZM, Shen HS. 2008 Postbuckling analysis of three-dimensional textile composite cylindrical shells under axial compression in thermal environments. *Compos. Sci. Technol.* **68**, 872–879. (doi:10.1016/j.compscitech.2007.08.009)
43. Li ZM, Shen HS. 2008 Postbuckling of 3D braided composite cylindrical shells under combined external pressure and axial compression in thermal environments. *Int. J. Mech. Sci.* **50**, 719–731. (doi:10.1016/j.ijmecsci.2007.12.001)
44. Shen HS. 2011 Postbuckling of nanotube-reinforced composite cylindrical shells in thermal environments, Part I: axially-loaded shells. *Compos. Struct.* **93**, 2096–2108. (doi:10.1016/j.compstruct.2011.02.011)

45. Shen HS, Xiang Y. 2013 Postbuckling of nanotube-reinforced composite cylindrical shells under combined axial and radial mechanical loads in thermal environment. *Compos. Pt. B-Eng.* **52**, 311–322. (doi:10.1016/j.compositesb.2013.04.034)
46. Li ZM, Yang DQ. 2011 Postbuckling of shear deformable stiffened anisotropic laminated cylindrical shell under axial compression. *Ocean Eng.* **38**, 1246–1255. (doi:10.1016/j.oceaneng.2011.05.008)
47. Sahmani S, Aghdam MM. 2017 Size dependency in axial postbuckling behavior of hybrid FGM exponential shear deformable nanoshells based on the nonlocal elasticity theory. *Compos. Struct.* **166**, 104–113. (doi:10.1016/j.compstruct.2017.01.051)
48. Gholami R, Ansari R. 2019 Asymmetric nonlinear bending analysis of polymeric composite annular plates reinforced with graphene nanoplatelets. *Int. J. Multiscale Comput. Eng.* **17**, 45–63. (doi:10.1615/IntJMultCompEng.2019029156)
49. Ansari R, Hassani R, Gholami R, Rouhi H. 2020 Nonlinear bending analysis of arbitrary-shaped porous nanocomposite plates using a novel numerical approach. *Int. J. Non-Linear Mech.* **126**, 103556. (doi:10.1016/j.ijnonlinmec.2020.103556)
50. Gholami R, Ansari R. 2019 On the nonlinear vibrations of polymer nanocomposite rectangular plates reinforced by graphene nanoplatelets: a unified higher-order shear deformable model. *Iran J. Sci. Technol.-Trans. Mech. Eng.* **43**, 603–620. (doi:10.1007/s40997-018-0182-9)
51. Mao JJ, Zhang W. 2019 Buckling and post-buckling analyses of functionally graded graphene reinforced piezoelectric plate subjected to electric potential and axial forces. *Compos. Struct.* **216**, 392–405. (doi:10.1016/j.compstruct.2019.02.095)
52. Ansari R, Hassani R, Torabi J. 2020 Mixed-type formulation of higher-order shear deformation theory for vibration and buckling analysis of FG-GPLRC plates using VDQFEM. *Compos. Struct.* **235**, 111738. (doi:10.1016/j.compstruct.2019.111738)
53. Gholami R, Ansari R. 2019 Nonlinear stability and vibration of pre/post-buckled multilayer FG-GPLRPC rectangular plates. *Appl. Math. Model.* **65**, 627–660. (doi:10.1016/j.apm.2018.08.038)
54. Ansari R, Torabi J, Hasrati E. 2020 Postbuckling analysis of axially-loaded functionally graded GPL-reinforced composite conical shells. *Thin-Walled Struct.* **148**, 106594. (doi:10.1016/j.tws.2019.106594)
55. Ansari R, Torabi J. 2019 Semi-analytical postbuckling analysis of polymer nanocomposite cylindrical shells reinforced with functionally graded graphene platelets. *Thin-Walled Struct.* **144**, 106248. (doi:10.1016/j.tws.2019.106248)
56. Blooriyan S, Ansari R, Darvizeh A, Gholami R, Rouhi H. 2019 Postbuckling analysis of functionally graded graphene platelet-reinforced polymer composite cylindrical shells using an analytical solution approach. *Appl. Math. Mech.-Engl. Ed.* **40**, 1001–1016. (doi:10.1007/s10483-019-2498-8)
57. Phuong NT, Nam VH, Trung NT, Duc VM, Loi NV, Thinh ND, Tu PT. 2019 Thermomechanical postbuckling of functionally graded graphene-reinforced composite laminated toroidal shell segments surrounded by Pasternak's elastic foundation. *J. Thermoplast. Compos. Mater.* (doi:10.1177/0892705719870593)
58. Amabili M, Reddy JN. 2010 A new non-linear higher-order shear deformation theory for large-amplitude vibrations of laminated doubly curved shells. *Int. J. Non-Linear Mech.* **45**, 409–418. (doi:10.1016/j.ijnonlinmec.2009.12.013)
59. Viola E, Tornabene F, Fantuzzi N. 2013 General higher-order shear deformation theories for the free vibration analysis of completely doubly-curved laminated shells and panels. *Compos. Struct.* **95**, 639–666. (doi:10.1016/j.compstruct.2012.08.005)
60. Dong YH, Li YH. 2017 A unified nonlinear analytical solution of bending, buckling and vibration for the temperature-dependent FG rectangular plates subjected to thermal load. *Compos. Struct.* **159**, 689–701. (doi:10.1016/j.compstruct.2016.10.001)
61. Ji XY, Cao YP, Feng XQ. 2010 Micromechanics prediction of the effective elastic moduli of graphene sheet-reinforced polymer nanocomposites. *Model. Simul. Mater. Sci. Eng.* **18**, 045005. (doi:10.1088/0965-0393/18/4/045005)
62. Ahmad SR, Xue C, Young RJ. 2017 The mechanisms of reinforcement of polypropylene by graphene nanoplatelets. *Mater. Sci. Eng. B-Adv. Funct. Solid-State Mater.* **216**, 2–9. (doi:10.1016/j.mseb.2016.10.003)
63. Young RJ, Liu M, Kinloch IA, Li S, Zhao X, Vallés C, Papageorgiou DG. 2018 The mechanics of reinforcement of polymers by graphene nanoplatelets. *Compos. Sci. Technol.* **154**, 110–116. (doi:10.1016/j.compscitech.2017.11.007)

64. Sun JB, Ni YW, Gao HY, Zhu SB, Tong ZZ, Zhou ZH. 2019 Torsional buckling of functionally graded multilayer graphene nanoplatelet-reinforced cylindrical shells. *Int. J. Struct. Stab. Dyn.* **20**, 2050005. (doi:10.1142/S0219455420500054)
65. Zhou ZH, Ni YW, Tong ZZ, Zhu SB, Sun JB, Xu XS. 2019 Accurate nonlinear stability analysis of functionally graded multilayer hybrid composite cylindrical shells subjected to combined loads. *Mater. Des.* **182**, 108035. (doi:10.1016/j.matdes.2019.108035)
66. Rafiee MA, Rafiee J, Wang Z, Song H, Yu ZZ, Koratkar N. 2009 Enhanced mechanical properties of nanocomposites at low graphene content. *ACS Nano* **3**, 3884–3890. (doi:10.1021/nn9010472)