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On the accuracy and applicability of a new implicit Taylor method and the high-order spectral method on steady nonlinear waves

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This paper presents an investigation and discussion of the accuracy and applicability of an implicit Taylor (IT) method versus the classical higher-order spectral (HOS) method when used to simulate two-dimensional regular waves. This comparison is relevant, because the HOS method is in fact an explicit perturbation solution of the IT formulation. First, we consider the Dirichlet–Neumann problem of determining the vertical velocity at the free surface given the surface elevation and the surface potential. For this problem, we conclude that the IT method is significantly more accurate than the HOS method when using the same truncation order, M , and spatial resolution, N , and is capable of dealing with steeper waves than the HOS method. Second, we focus on the problem of integrating the two methods in time. In this connection, it turns out that the IT method is less robust than the HOS method for similar truncation orders. We conclude that the IT method should be restricted to $M=4$, while the HOS method can be used with $M \leq 8$. We systematically compare these two options and finally establish the best achievable accuracy of the two methods as a function of the wave steepness and the water depth.

1. Introduction

At the heart of most numerical simulations of nonlinear water waves described by the potential flow formalism is the accurate and efficient evaluation of the vertical surface velocity, w_s , from the surface elevation and surface potential. Not only is this evaluation decisive for the accuracy and stability of the simulations, it also

happens to be the computational bottleneck of the entire numerical method. The problem is further complicated by the fact that accuracy, stability and efficiency tend to exclude each other, and finding a good compromise between these properties is necessary.

If accuracy and stability are given the highest priority, good options are the volumetric methods based on a stretching of the vertical coordinate when solving the Laplace problem. Research into such methods presumably started with the work of Li & Fleming [1]; since then Bingham & Zhang [2], Engsig-Karup *et al.* [3], Yates & Benoit [4], Raoult *et al.* [5], Nicholls [6] and Klahn *et al.* [7] have all used the volumetric approach with a wide variety of discretizations. It should be noted that this list of references is, however, by no means exhaustive. The volumetric methods excel because of their stable and accurate computation of the vertical surface velocity, but are relatively expensive in terms of computational time. In addition we stress that the high accuracy of w_s offered by the volumetric methods when using a fine spatial resolution is typically hampered by instabilities when integrating a wave field in time. For not too fine spatial resolutions they are, however, robust (albeit not necessarily highly accurate) over time when employed in connection with artificial damping. We have, for example, shown in our own recent work that steady nonlinear waves in deep water up to 99% of the limiting steepness may be integrated in time using a volumetric method (see §4 of Klahn *et al.* [7]).

If, on the other hand, efficiency (here defined as the time required to reach a given level of accuracy) is given the highest priority, there can be no doubt that the so-called high-order spectral (HOS) method is the preferred choice. Versions of this method were independently introduced by West *et al.* [8] and Dommermuth & Yue [9], while Craig & Sulem [10] rigorously derived what seemed like a third version of it. In addition, Bateman *et al.* [11] have also presented a version, but in what follows we will refer to these methods collectively as *the* HOS method since Schäffer [12] has shown that they can be executed by using almost identical computational procedures. Investigations involving the HOS method are too numerous to be listed here, but among its successful applications (in addition to the aforementioned references) we find the work of Tanaka [13], Xu & Guyenne [14] and Xiao *et al.* [15], and note that the method has been carefully analysed in a series of papers by Nicholls & Reitich (e.g. [16,17]). The efficiency of the HOS method stems from a combination of the facts that (i) it is based on an explicit procedure for the calculation of w_s in which only the horizontal part of the problem needs to be discretized and (ii) it is a spectral method, meaning that when more than a few digits of accuracy are required it is much more efficient than low-order methods (see, for example, ch. 1 in the book by Boyd [18]). The prize for efficiency is a less robust method with a smaller domain of validity than, for example, the volumetric methods. How large (or small) this domain is for steady nonlinear waves in terms of the relative water depth kh (wavenumber times water depth) and wave steepness H/L (wave height divided by wavelength) is yet to be clarified, and several authors have reported different ranges of applicability. For example, Dommermuth & Yue [9] found that in infinitely deep water the HOS method may be used up to about 80% of the limiting steepness, while Bonnefoy *et al.* [19] and Ducrozet *et al.* [20] have reported that their implementation of the HOS method could be used up to roughly 90% of the limiting steepness in deep water. We will discuss these findings in relation to our own results throughout this paper, but for now we note that no one has claimed that the HOS method is capable of handling waves steeper than 90% of the limiting steepness. As a consequence, the HOS method lags quite a bit behind, for example, the volumetric methods.

Following the above discussion, an obvious question is whether it is possible to devise a hybrid method which enjoys the best of both worlds, in the sense that it is more accurate and robust than the HOS method when used to simulate steady two-dimensional nonlinear waves, while at the same time being comparably efficient. This paper deals with exactly this question, as it presents and tests a spectral method based on a Taylor expansion of the velocity potential around the still water level, which we will denote as the implicit Taylor (IT) method. The IT method may be regarded as an implicit version of the HOS method since it replaces the explicit perturbation approach for obtaining the still water potential by an implicit exact procedure. Because of the implicit procedure, it is clear that a naive implementation of the IT method is not as efficient as even the simplest implementation of the HOS method, and the main focus throughout this paper

will therefore be on whether the IT method is actually more accurate and robust than the HOS method. The necessary improvements of the IT method that are needed to make it as efficient as the HOS method will, however, also be discussed, as will the relation of the IT method to various modern Boussinesq-type formulations such as those of Agnon *et al.* [21], Madsen *et al.* [22], Madsen & Agnon [23] and Liu *et al.* [24].

The remainder of this paper is organized as follows: §2 presents the physical system under consideration and its governing equations. In §3, the Dirichlet–Neumann (DN) operators of the new IT method as well as of the HOS method are derived, and the discretization of these methods is described. The accuracy and stability of the DN operators is tested on a single time-independent problem in §4, while the same properties are tested on a time-dependent problem in §5. Finally, conclusions are drawn in §6. It turns out that our results for the HOS method in many cases deviate substantially from results found in the literature; these differences are outlined and discussed in detail in appendix A.

2. Governing equations for the water wave problem

A Cartesian coordinate system is adopted with the x -axis and y -axis located on the still water plane and with the z -axis pointing vertically upwards. We assume the motion of the fluid to be periodic in space with period L , and assume the fluid domain to be bounded by a flat, impermeable sea bed at $z = -h$ and by the free surface at $z = \eta(x, y, t)$. We moreover assume the flow to be inviscid, incompressible and irrotational, and introduce the velocity potential Φ , which is related to the velocity components by the definitions

$$\mathbf{u} \equiv \nabla \Phi \quad \text{and} \quad w \equiv \frac{\partial \Phi}{\partial z}, \quad (2.1)$$

where $\nabla = (\partial_x, \partial_y)$ is the two-dimensional gradient operator. At all times, the velocity potential must satisfy the Laplace equation,

$$\nabla^2 \Phi + \frac{\partial^2 \Phi}{\partial z^2} = 0, \quad (2.2)$$

within the interior of the fluid domain. The kinematic bottom condition expressing that the sea bed is impermeable reads

$$w_b = 0, \quad \text{where} \quad w_b \equiv \left. \frac{\partial \Phi}{\partial z} \right|_{z=-h}. \quad (2.3)$$

At the free surface, we introduce the variables $\mathbf{u}_s \equiv \nabla \Phi|_{z=\eta}$, $w_s \equiv \partial_z \Phi|_{z=\eta}$ and $\Phi_s \equiv \Phi|_{z=\eta}$ by which the kinematic surface condition can be expressed as

$$\frac{\partial \eta}{\partial t} = w_s (1 + \nabla \eta \cdot \nabla \eta) - \nabla \Phi_s \cdot \nabla \eta, \quad (2.4)$$

while the dynamic surface condition can be expressed as

$$\frac{\partial \Phi_s}{\partial t} = -g\eta - \frac{1}{2} (\nabla \Phi_s \cdot \nabla \Phi_s) + \frac{1}{2} w_s^2 (1 + \nabla \eta \cdot \nabla \eta); \quad (2.5)$$

see, for example, Zakharov [25], West *et al.* [8] and Dommermuth & Yue [9].

We note that (2.4) and (2.5) constitute the fully nonlinear initial value problem for η and Φ_s . When solving this problem numerically, we need, however, to construct the so-called DN operator within each time step. The DN operator is the operator mapping the pair (η, Φ_s) to the vertical surface velocity, w_s , and we address the construction of this operator in the following section. Throughout the rest of this paper, we shall ignore motions in the y -direction and hence take $\nabla = \partial_x$.

3. Construction of the Dirichlet–Neumann operator

In order to construct the DN operator, one needs to solve the Laplace equation (2.2) in the interior of the fluid domain with the boundary condition (2.3), and in the following we shall address this

problem for the IT method and the HOS method. Since we employ the same discretization scheme for both methods, we first derive the two DN operators before we describe the discretization procedure.

(a) The implicit Taylor method

Assuming that the velocity potential can be analytically extended to outside of the fluid domain, we start by making a Taylor expansion of Φ and its first z -derivative from the still water level $z = 0$ to an arbitrary z -level. Truncating the Taylor expansions at order M , this leads to the relations

$$\Phi(z) = \sum_{m=0}^M \varepsilon^m \frac{z^m}{m!} \frac{\partial^m \Phi}{\partial z^m} \Big|_{z=0} \quad \text{and} \quad w(z) = \sum_{m=0}^M \varepsilon^m \frac{z^m}{m!} \frac{\partial^{m+1} \Phi}{\partial z^{m+1}} \Big|_{z=0}, \quad (3.1)$$

from which it immediately follows that

$$\Phi_s = \sum_{m=0}^M \varepsilon^m \frac{\eta^m}{m!} \frac{\partial^m \Phi}{\partial z^m} \Big|_{z=0} \quad \text{and} \quad w_s = \sum_{n=0}^M \varepsilon^n \frac{\eta^n}{n!} \frac{\partial^{n+1} \Phi}{\partial z^{n+1}} \Big|_{z=0}. \quad (3.2)$$

Note that, throughout this paper, ε is an explicit ordering parameter, which is useful while collecting terms, but which should be set to unity once the formulae are to be used for actual computations. Second, we use that Φ should satisfy the Laplace equation (2.2), which leads to the classical recursion relations

$$\frac{\partial^m \Phi}{\partial z^m} \Big|_{z=0} = -\nabla^2 \frac{\partial^{m-2} \Phi}{\partial z^{m-2}} \Big|_{z=0}, \quad (3.3)$$

with the two starting values (for $m = 0$ and $m = 1$) being

$$\Phi|_{z=0} \equiv \Phi_0 \quad \text{and} \quad \frac{\partial \Phi}{\partial z} \Big|_{z=0} \equiv w|_{z=0} \equiv w_0. \quad (3.4)$$

Third, we combine (3.3) and (3.4) with (3.2), which allows us to express the surface variables Φ_s and w_s in terms of horizontal derivatives of Φ_0 and w_0 . The result leads to the finite-series formulations

$$\Phi_s = \sum_{m=0}^{M_1} \varepsilon^{2m} \frac{(-1)^m \eta^{2m}}{(2m)!} \nabla^{2m} \Phi_0 + \sum_{m=0}^{M_2} \varepsilon^{2m+1} \frac{(-1)^m \eta^{2m+1}}{(2m+1)!} \nabla^{2m} w_0 \quad (3.5a)$$

and

$$w_s = \sum_{m=0}^{M_1} \varepsilon^{2m} \frac{(-1)^m \eta^{2m}}{(2m)!} \nabla^{2m} w_0 - \sum_{m=0}^{M_2} \varepsilon^{2m+1} \frac{(-1)^m \eta^{2m+1}}{(2m+1)!} \nabla^{2m+2} \Phi_0, \quad (3.5b)$$

where $M_1 \equiv \lfloor M/2 \rfloor$, $M_2 \equiv \lfloor (M-1)/2 \rfloor$ and $\lfloor \cdot \rfloor$ denotes the floor function. It is straightforward to extend this formulation to cover the internal velocity field for arbitrary z , and this becomes

$$u(z) = \sum_{m=0}^{M_1} \varepsilon^{2m} \frac{(-1)^m z^{2m}}{(2m)!} \nabla^{2m} u_0 + \sum_{m=0}^{M_2} \varepsilon^{2m+1} \frac{(-1)^m z^{2m+1}}{(2m+1)!} \nabla^{2m+1} w_0 \quad (3.6a)$$

and

$$w(z) = \sum_{m=0}^{M_1} \varepsilon^{2m} \frac{(-1)^m z^{2m}}{(2m)!} \nabla^{2m} w_0 - \sum_{m=0}^{M_2} \varepsilon^{2m+1} \frac{(-1)^m z^{2m+1}}{(2m+1)!} \nabla^{2m+1} u_0, \quad (3.6b)$$

where $u_0 \equiv \nabla \Phi_0$. These expressions can be used above the still water level, i.e. for $z \geq 0$. Below the still water level, one may use the Fourier formulation (3.19) presented in §3c.

At this point, the formulation is similar to various Boussinesq-type formulations from the literature in the sense that (3.5) has been used by, for example, Agnon *et al.* [21], Madsen *et al.* [22], Madsen & Agnon [23] and Liu *et al.* [24] to describe the connection between variables at the free surface and variables at $z = 0$. However, when it comes to describing the kinematics below the still water level, i.e. for $z < 0$, the various Boussinesq-type methods deviate a great deal: Agnon

et al. [21] used (3.6) with $M = 5$ to extend the velocity profile to the sea bottom, where they Padé-enhanced the kinematic bottom condition to achieve Padé[4,4] linear dispersion characteristics. Based on a threshold of 2% error compared with the fully dispersive target, their approach was applicable up to $kh = 6.2$ (wavenumber times water depth). However, as demonstrated by Madsen & Agnon [23], (3.6) generally provides a relatively poor quality of the velocity profile below the still water level when truncated at finite order, e.g. $M = 5$. Madsen *et al.* [22] reformulated the finite-order Taylor expansion (3.6) to start from an arbitrary z -level (close to mid-depth) and introduced pseudo-velocity variables which provided Padé properties at the sea bottom as well as at $z = 0$. With this procedure and based on the threshold of 2% dispersion error, their approach was applicable up to $kh = 25$. In addition to that, the accuracy of the velocity profile was significantly improved and applicable up to $kh = 12$. As a further extension, Madsen & Agnon [23] improved the interior velocity profile to be applicable up to $kh = 20$. Liu *et al.* [24] generalized the techniques by Madsen *et al.* [22] to provide a multi-layer Padé formulation with the top layer described by (3.5). With three or four layers and with $M = 3$, they achieved outstanding accuracy in dispersion as well as in velocity profiles.

In the present approach, we shall avoid using a truncated version of (3.6) below $z = 0$. Instead, we use (3.6) with $z = -h$ and $M \rightarrow \infty$, and insert the result in the kinematic bottom condition (2.4). On a flat bottom, this leads to the following exact connection between w_0 and u_0 :

$$w_0 = -\tan(h\nabla)u_0, \quad u_0 \equiv \nabla\Phi_0, \quad (3.7)$$

where we define the operator $\tan(h\nabla)$ as the linear operator whose action on the function $\exp(ik_\eta x)$ is given by

$$\tan(h\nabla)\exp(ikx) = \tan(ikh)\exp(ikx) \quad (3.8)$$

for all values of k . We note that it makes sense to apply the operator to, for example, all functions which are periodic and differentiable on the interval $[0, L]$ since these have uniformly convergent Fourier series representations (see, for example, section 1.3.4 of Boggess & Narcowich [26]), and that to actually apply the operator in practice one must use this representation of the function. As a consequence of (3.7) and (3.8), Φ_0 is in fact the only unknown of (3.5a), and applying the DN operator of the IT method to (η, Φ_s) thus amounts to solving (3.5a) for Φ_0 and subsequently evaluating w_s using (3.5b).

(b) The explicit perturbation method (HOS)

West *et al.* [8] and Dommermuth & Yue [9] were the first to introduce an explicit procedure to determine w_s from η and Φ_s . The first step in this procedure is to express the velocity potential as the perturbation series

$$\Phi(z) = \sum_{j=0}^M \varepsilon^j \Phi^{(j)}(z). \quad (3.9)$$

Inserting this series into (3.5) and replacing j by $p = j + m$ leads to

$$\Phi_s = \sum_{p=0}^M \sum_{m=0}^p \varepsilon^p \frac{\eta^m}{m!} \frac{\partial^m \Phi^{(p-m)}}{\partial z^m} \Big|_{z=0} \quad w_s = \sum_{p=0}^M \sum_{m=0}^p \varepsilon^p \frac{\eta^m}{m!} \frac{\partial^{m+1} \Phi^{(p-m)}}{\partial z^{m+1}} \Big|_{z=0} \quad (3.10)$$

and we note that similar expressions were presented by Schäffer [12]. As an alternative to (3.10), we may use the perturbation expansion

$$\Phi_0(x) = \sum_{j=0}^M \varepsilon^j \Phi_0^{(j)}(x), \quad (3.11)$$

and similar expressions for $w_0(x)$ and $u_0(x)$. Inserting these expansions into (3.5), while replacing j by $p = j + 2m$, we obtain

$$\Phi_s = \sum_{p=0}^{M_1} \sum_{m=0}^{F_1[p]} \varepsilon^p \frac{(-1)^m \eta^{2m}}{(2m)!} \nabla^{2m} \Phi_0^{(p-2m)} + \sum_{p=0}^{M_2} \sum_{m=0}^{F_2[p]} \varepsilon^p \frac{(-1)^m \eta^{2m+1}}{(2m+1)!} \nabla^{2m} w_0^{(p-2m-1)} \quad (3.12)$$

and

$$w_s = \sum_{p=0}^{M_1} \sum_{m=0}^{F_1[p]} \varepsilon^p \frac{(-1)^m \eta^{2m}}{(2m)!} \nabla^{2m} w_0^{(p-2m)} - \sum_{p=0}^{M_2} \sum_{m=0}^{F_2[p]} \varepsilon^p \frac{(-1)^m \eta^{2m+1}}{(2m+1)!} \nabla^{2m+2} \Phi_0^{(p-2m-1)}, \quad (3.13)$$

where $F_1[p] \equiv \lfloor p/2 \rfloor$ and $F_2[p] \equiv \lfloor (p-1)/2 \rfloor$. Note that (3.12) connects the surface potential Φ_s to horizontal derivatives of Φ_0 and w_0 . This forms a hierarchy of equations in powers of ε , which should be satisfied one by one. By using successive approximations combined with (3.7), this allows the determination of $\Phi_0^{(j)}$ and $w_0^{(j)}$ for $j = 0, 1, 2, \dots, M$. Finally, w_s can be determined from the explicit summation (3.13).

The internal velocity field, consistent with the perturbation method used above, becomes

$$u(z) = \sum_{p=0}^{M_1} \sum_{m=0}^{F_1[p]} \varepsilon^p \frac{(-1)^m z^{2m}}{(2m)!} \nabla^{2m} u_0^{(p-2m)} + \sum_{p=0}^{M_2} \sum_{m=0}^{F_2[p]} \varepsilon^p \frac{(-1)^m z^{2m+1}}{(2m+1)!} \nabla^{2m+1} w_0^{(p-2m-1)} \quad (3.14a)$$

and

$$w(z) = \sum_{p=0}^{M_1} \sum_{m=0}^{F_1[p]} \varepsilon^p \frac{(-1)^m z^{2m}}{(2m)!} \nabla^{2m} w_0^{(p-2m)} - \sum_{p=0}^{M_2} \sum_{m=0}^{F_2[p]} \varepsilon^p \frac{(-1)^m z^{2m+1}}{(2m+1)!} \nabla^{2m+1} u_0^{(p-2m-1)}. \quad (3.14b)$$

These expressions can be used above the still water level, i.e. for $z \geq 0$. Below the still water level, one may use the Fourier formulation (3.19) presented in §3c.

(c) Discretization of the Dirichlet–Neumann operators

Since we exclusively consider spatially periodic problems throughout this work, we have chosen to discretize the equations of the previous two sections using the Fourier collocation method (e.g. Kopriva [27]). In this method any function, here generically denoted by ϕ , is approximated by a truncated Fourier series, i.e.

$$\phi(x) = \sum_{n=-N}^{N-1} \hat{\phi}_n \exp(ik_n x), \quad (3.15)$$

in which $k_n = 2\pi n/L$ and L is the domain size in the x -direction. For notational convenience at later stages we here define $k = k_1$. During the computation ϕ is represented by its values at the set of grid points

$$\left\{ x_n = L \frac{n}{2N} \mid 1 \leq n \leq 2N \right\}, \quad (3.16)$$

and the values of the x -derivatives of ϕ at the grid points are calculated based on the assumption (3.15) using the fast Fourier transform algorithm to switch between the grid and coefficient representations of ϕ . It is well known that the aliasing phenomenon occurs when using the Fourier collocation method, and since the importance of dealiasing has been stressed by several authors (see, for example, the discussions by Bonnefoy *et al.* [19], Clamond & Grue [28] and Christiansen *et al.* [29]), we have carried out our computations with three different dealiasing strategies: no dealiasing, full dealiasing and quadratic dealiasing. To explain what we mean by

full and quadratic dealiasing we consider the example of computing the values of $\phi_p(x) = \phi(x)^p$ with p an arbitrary integer on the grid (3.16). We carry out full dealiasing by interpolating ϕ to the finer grid

$$\left\{ \tilde{x}_n = L \frac{n}{2pN} \mid 1 \leq n \leq 2pN \right\} \quad (3.17)$$

by zero padding its Fourier coefficients, computing $\phi_p(\tilde{x}_n) = \phi(\tilde{x}_n)^p$ for $n = 1, 2, \dots, 2pN$ and interpolating ϕ_p back to the grid (3.16) by removing its Fourier coefficients with indices larger than N . By contrast, we carry out quadratic dealiasing by computing ϕ_p as $\phi_p = (((\phi \times \phi) \times \phi) \times \dots \times \phi)$, where every sub-product is carried out using full dealiasing with interpolation to the grid (3.17) with $p = 2$. It turns out that results obtained with full dealiasing and with quadratic dealiasing are almost identical and relative differences are generally less than 1%.

In both the IT method and the HOS method w_0 must be computed from Φ_0 , and to do so we use the fact that, when combined, (3.7) and (3.8) imply that

$$w_0 = \sum_{n=-N}^{N-1} \tanh(k_n h) k_n \hat{\Phi}_n \exp(ik_n x), \quad (3.18)$$

where $\hat{\Phi}_n$ is the n th Fourier coefficient of Φ_0 .

To solve (3.5a) for the values of Φ_0 at the grid points, we use the iterative method GMRES [30] with relative tolerance 10^{-14} without any preconditioner. Compared with a direct method such as Gaussian elimination, the GMRES method has the advantage that it may be used without ever actually constructing the system matrix. Instead, the GMRES method simply requires a means of evaluating the right-hand side of the equation (i.e. the matrix vector product), and we use this in our simulations to avoid the actual construction of the matrix. Moreover, we have found that the GMRES method gives the exact same answer (in double precision) as a direct solver when used with the relative tolerance mentioned above when the system matrix is constructed by applying the routine evaluating the right-hand side of (3.5a) to the columns of the identity matrix. Now, a rough estimation of the computational work required to solve (3.5) using the unpreconditioned GMRES method (see appendix B) shows that $O(MN^2 \log(N))$ floating point operations are needed to solve the equation when using no or quadratic dealiasing and that $O(M^2 N^2 \log(N))$ operations are needed when using full dealiasing. This should be compared with the computational complexity of the DN operator of the HOS method, which is either $O(M^2 N \log(N))$ (no or quadratic dealiasing) or $O(M^3 N \log(N))$ (full dealiasing); since N is typically much larger than M , the HOS method is clearly more efficient than the IT method when no preconditioner is used. We note, however, that if a preconditioner resulting in a constant number of GMRES iterations can be found the number of operations required to solve (3.5a) reduces to $O(MN \log(N))$ or $O(M^2 N \log(N))$ depending on the dealiasing strategy, and in that case the efficiency of the IT method will be comparable to that of the HOS method. In that connection, we mention here that mainly two types of preconditioners exist in the literature for water waves: preconditioners based on the Laplace problem with a flat surface, i.e. $\eta = 0$ (e.g. the work of Fuhrman & Bingham [31], Bingham & Zhang [2] and Klahn *et al.* [32]), and preconditioners based on the multigrid method (e.g. the work of Li & Fleming [1] and Engsig-Karup *et al.* [3]). The flat surface preconditioner is of no use for the IT method since setting $\eta = 0$ transforms (3.5a) into $\Phi_s = \Phi_0$, meaning that the preconditioning operation would be to invert the identity operator. On the other hand, a preconditioner based on the multigrid method may be applied to the problem, but whether this strategy turns out to be effective or not we do not know at the present time.

When computing the interior velocity profiles, we use the fact that the Fourier formulation employed in this work implies that

$$u(x, z) = \sum_{n=-N}^{N-1} \frac{\cosh(k_n(z+h))}{\cosh(k_n h)} \hat{u}_0 \exp(ik_n x) \quad \text{for } z < 0 \quad (3.19a)$$

and

$$w(x, z) = \sum_{n=-N}^{N-1} \frac{\sinh(k_n(z+h))}{\sinh(k_n h)} \hat{w}_0 \exp(ik_n x) \quad \text{for } z < 0. \quad (3.19b)$$

In fact, we could also use these expressions for $z \geq 0$, but this would not be consistent with (3.5) and (3.12). For this reason, we prefer to use (3.6) for the IT method and (3.14) for the HOS method when $z \geq 0$. We note that the choice of using two different formulations for the velocity field below and above $z = 0$ implies that maximally the first M z -derivatives of the velocity fields can be continuous at this location. For practical purposes this is, however, as will be seen later, much less important for the accuracy of the z variation of the velocity fields than the method's ability to accurately determine Φ_0 .

4. Accuracy of the Dirichlet–Neumann operators for steady nonlinear waves

In this section, we test the ability of the IT and HOS methods to compute the vertical velocity, w_s , at the free surface when given the surface elevation, η , and surface potential, Φ_s . The tests are carried out on steady nonlinear waves, and we compute relative errors or differences between two quantities, say, ϕ^I and ϕ^{II} , using the discrete L^2 -norm, i.e.

$$\text{Error}[\phi] = \sqrt{\frac{\sum_{n=1}^{2N} (\phi_n^I - \phi_n^{II})^2}{\sum_{n=1}^{2N} (\phi_n^I)^2}}, \quad (4.1)$$

where ϕ_n^I and ϕ_n^{II} denote the values of ϕ^I and ϕ^{II} at the n th grid point, respectively. In order to avoid confusion we will refer to (4.1) as the relative DN error when dealing with a single DN problem, and as the relative error when dealing with time integration problems (which will be considered in §5).

We obtain the input and target solutions for the nonlinear steady waves from either the (by now) classical stream function (SF) method of Rienecker & Fenton [33] or the more recent conformal mapping method (CD) of Clamond & Dutykh [34]. While both methods can be used to compute η , Φ_s and w_s , the SF method can also provide u_0 and w_0 as well as the vertical profile of the velocities. When using a relatively coarse grid, the two methods give almost the same result for the surface quantities and the absolute difference of ϕ produced by the two methods is of order $O(\hat{\phi}_N)$. Since $\hat{\phi}_N$ grows with the steepness H/L , the difference between the results of the two methods grows when increasing H/L while keeping the resolution fixed. This point is illustrated in figure 1 for $kh = 2\pi$ and $N = 32$. For large values of N , the SF method is incapable of computing an accurate result (for $kh = 2\pi$ the SF method breaks down for $N \approx 40$); for that reason, we only use the CD method when we need the surface quantities to be finely resolved.

(a) The error of w_s and w_0 as a function of M using stream function input

As a first test of the IT and HOS methods, we check their ability to compute w_s and w_0 when using the SF results for $kh = 2\pi$ and $N = 32$ as input and target. Figure 2 shows the error of w_s as a function of the order M for different values of H/L . First, we conclude that the IT method without dealiasing is consistently far more accurate than both the non-dealiased and the quadratically and fully dealiased HOS methods when using the same value of M . Second, we note that, when not dealiased, the HOS method is unstable since the error grows with M beyond a certain threshold value M_0 . The quadratically and fully dealiased HOS methods give almost the same results, and although their error does not decrease monotonically with M , they are significantly more accurate than the non-dealiased HOS method for $M > M_0$. The differences between the non-dealiased and the two dealiased HOS methods start at $M = 10$ for $H/L = 0.09$ and at $M = 6$ for $H/L = 0.12$. Third, we note that generally the relative errors gradually increase for increasing wave steepness: for $H/L = 0.08$ with $M = 6$, the error is 2×10^{-5} for the quadratically dealiased HOS method and

Figure 1. Relative difference between SF and CD results as a function of H/L for $kh = 2\pi$ and $N = 32$. Black: w_s ; blue: η ; red: Φ_s . (Online version in colour.)

Figure 2. The relative DN error of w_s as a function of M for (a) $H/L = 0.08$, (b) $H/L = 0.09$, (c) $H/L = 0.10$, (d) $H/L = 0.11$, (e) $H/L = 0.12$ and (f) $H/L = 0.13$. The figure is produced using $kh = 2\pi$ and $N = 32$, and the input and target are taken from the SF method. Red: IT method without dealiasing; blue: HOS method without dealiasing; black: HOS method with quadratic dealiasing; grey: HOS method with full dealiasing. (Online version in colour.)

Figure 3. The relative DN error of w_0 as a function of M for (a) $H/L = 0.08$, (b) $H/L = 0.09$, (c) $H/L = 0.10$, (d) $H/L = 0.11$, (e) $H/L = 0.12$ and (f) $H/L = 0.13$. The figure is produced using $kh = 2\pi$ and $N = 32$, and the input and target are taken from the SF method. Red: IT method without dealiasing; blue: HOS method without dealiasing; black: HOS method with quadratic dealiasing; grey: HOS method with full dealiasing. (Online version in colour.)

5×10^{-8} for the IT method, while for $H/L = 0.12$ with $M = 6$ the error is 1×10^{-3} for the dealiased HOS method and 2×10^{-5} for the IT method.

Figure 3 shows the error of w_0 as a function of the order M . In this case, the errors in the HOS method are far greater than those in figure 2. It is a fact that the perturbation procedure in the HOS method produces rather poor accuracy for the still water variables Φ_0 , w_0 and u_0 while these errors tend to be compensated or reduced when going back up to the free surface for the determination of w_s . It is evident that the IT method does not have this problem, and the differences in accuracy are enormous. As an example, the error on w_0 is 0.27 for the dealiased HOS method and 2×10^{-6} for the IT method when $H/L = 0.11$ and $M = 6$.

(b) Computing the vertical profile of u below the wave crest using SF input

The second test that we consider is the problem of computing the vertical profile of the horizontal velocity below the wave crest. Again, we consider the case $kh = 2\pi$ and take $N = 32$. Figure 4 shows the vertical variation of u evaluated under the wave crest within the region $-\eta_c \leq z \leq \eta_c$ (where η_c denotes the crest elevation), which is the only region where the profiles are always visibly perfect, when computed with the non-dealiased HOS and IT methods for $H/L = 0.08$, 0.09, 0.10 and 0.11. The results of the HOS method are based on (3.14) combined with (3.19), and are clearly not very accurate. This is due to the poor estimate of the still water velocities of the HOS method, and it turns out that the odd orders of M tend to underestimate u_0 while the

Figure 4. The horizontal velocity u/c (with c the wave celerity) below the wave crest as a function of z/η_c for (a) $H/L = 0.08$, (b) $H/L = 0.09$, (c) $H/L = 0.10$ and (d) $H/L = 0.11$. The figure is produced using $kh = 2\pi$ and $N = 32$, and the input is taken from the SF method. Black: IT with $M = 4$; green: HOS with $M = 6$; blue: HOS with $M = 7$. (Online version in colour.)

Figure 5. The horizontal velocity u/c (with c the wave celerity) below the wave crest as a function of z/η_c . The figure is produced using $kh = 2\pi$ and $N = 32$, and the input and target are taken from the SF method. Black: IT with $M = 4$; red: target from SF theory. (Online version in colour.)

even orders overestimate u_0 . In contrast the IT method based on (3.6) combined with (3.19) is much more accurate, and is in fact capable of computing the vertical profile of u accurately for quite large values of H/L . As an example, the vertical profile of u is compared with the SF result in figure 5 for the case of $H/L = 0.135$. It should be mentioned that the interior velocity profile of the HOS method can be improved by combining it with, for example, the IT procedure, the explicit procedure suggested by Bateman *et al.* [35] or any other more accurate method.

(c) The error of w_s as a function of H/L when using stream function input

Figure 6 shows the error of w_s as a function of the steepness H/L when using SF input with $N = 32$ and $kh = 2\pi$. Results for the non-dealiased IT method are shown for $M = 4$ and $M = 6$, while

Figure 6. The relative DN error of w_s as a function of H/L . The figure is produced using $kh = 2\pi$ and $N = 32$, and the input and target are taken from the SF method. Blue dots: HOS with $M = 4$; green dots: HOS with $M = 6$; red dots: HOS with $M = 8$; blue dashed line: IT with $M = 4$; green dashed line: IT with $M = 6$; red dashed line: IT with $M = 8$. The results of the IT method are obtained without dealiasing and the results of the HOS method are obtained with quadratic dealiasing. (Online version in colour.)

the quadratically dealiased HOS method is shown with $M = 4$, $M = 6$ and $M = 8$. We conclude that for a given M the IT method is more accurate than the HOS method, even when the HOS method is given the ‘advantage’ of using a dealiasing procedure. Furthermore, the IT method can handle H/L values up to approximately 0.13 while the HOS method can handle H/L values up to approximately 0.12. On the other hand, we emphasize that figure 6 does not reveal the robustness of the individual methods during time integration (more on this issue in §5).

(d) The error of w_s as a function of M using CD input

Next, we investigate the effect of using the CD input instead of the SF input when using $kh = 2\pi$ and $N = 32$. Since the methods give very similar results for these parameters (figure 1), our expectation is that this shift should have very little influence on the results. However, for the IT method this turns out not to be the case. Figure 7 shows the error of w_s when using the CD input and the SF input. While the IT method behaves very well based on the SF input for the shown values of H/L , it is not capable of computing w_s accurately when given the CD input. When using the CD input, the IT method produces large errors on w_s for large values of M , i.e. when we include high derivatives in the formulation. As we may consider the CD input as being a perturbed SF input, the result implies that the IT method is not very robust to perturbations unless M is relatively small. This is not promising in connection with time integration where perturbations inevitably occur from time step to time step (more on this issue in §5). From figure 7, we conclude that for the IT method M should not exceed 4–6 if large values of H/L are to be considered.

We have made a similar sensitivity test of the HOS method (not shown), and it turns out that in this case it makes little difference if we use the SF or the CD input. This is a surprise considering that the HOS method involves similar high derivative terms to the IT method, but the HOS method appears to be more robust to perturbations in the input. Hence, it appears, at this point, that the HOS method is robust for higher values of M than the IT method. More on this issue in §5, where we shall discuss the time integration of the two methods.

(e) The error of w_s as a function of N based on CD input

Finally, we investigate the importance of the spatial resolution on the accuracy for shallow, intermediate and deep water conditions. In this connection, we apply the CD results as input and

Figure 7. IT results for the relative DN error on w_s as a function of M for (a) $H/L = 0.10$, (b) $H/L = 0.11$ and (c) $H/L = 0.12$. The figure is produced using $kh = 2\pi$ and $N = 32$ and without dealiasing. Black: input from CD; red: input from SF. (Online version in colour.)

target for the cases $kh = 0.3$ (shallow water), $kh = 1$ (intermediate water depth) and $kh = 2\pi$ (deep water) with $H/H_{\max} = 0.85$, where $H_{\max}$ is determined from the empirical breaking criterion of Battjes [36], i.e.

$$\frac{H_{\max}}{L} = 0.1401 \tanh(0.8864 kh). \quad (4.2)$$

Figure 8 shows the error of w_s for the IT method as a function of N for the values $M = 4, 6$ and 8 with no dealiasing and full dealiasing. From the figure, we note that the IT method is stable for $M = 4$ and $M = 6$ but not for $M = 8$. This is in agreement with our conclusions in connection with figure 7; as a consequence, we recommend that the IT method is not applied with M larger than 6. Moreover, we note that, for all values of M tested here, one effect of using full dealiasing is to slightly speed up the convergence of w_s before the error stagnates, while the minimum error does

Figure 8. IT results for the relative DN error on w_s as a function of N for $H/H_{\max} = 0.85$. The results shown with circles are computed without dealiasing and the results shown as lines are computed with full dealiasing. The legend applies to all figures. (Online version in colour.)

Figure 9. HOS results for the relative DN error on w_s as a function of N for $H/H_{\max} = 0.85$. The results shown with circles are computed without dealiasing and the results shown as lines are computed with full dealiasing. The legend applies to all figures. (Online version in colour.)

not depend on the dealiasing strategy. It should also be noted that using full dealiasing does not prevent the IT method from becoming unstable for large N , although it does alleviate the problem to some extent.

Figure 9 shows the corresponding results for the HOS method, and we note that all cases become independent of N for sufficiently large numbers of N , i.e. $N > N_0$, say. We also note that dealiasing only matters as long as the case is under-resolved, i.e. for $N < N_0$, and that N_0 tends to grow with M , i.e. the higher the derivatives we involve, the higher resolution we need. It should be emphasized that if we choose the order M larger than 8 we can no longer freely choose the resolution N without having unstable solutions with growing errors. This issue is further discussed and illustrated in appendix A.

We conclude this section with a remark about the shallow water case. Our results show that both methods are capable of computing w_s with quite small errors if M and N are chosen correctly. On the other hand, our results for the time integration of this case (see the discussion in §5d) show that neither method is capable of simulating this case with a small error regardless of how M and N are chosen. As such this case is a good example of the fact that an accurate resolution of the DN operator is a necessary, but in no way sufficient, condition for accurate time integration.

5. Time-integrating the implicit Taylor and high-order spectral methods

In this section, we shall time integrate the kinematic and dynamic boundary conditions (2.4) and (2.5), while treating the embedded DN problem with either the IT method or the HOS method. To do so, we discretize the spatial part of the boundary conditions using the Fourier collocation method, and integrate the resulting system of coupled ODEs using the classical fourth-order Runge–Kutta method with constant step size $\Delta t = T/100$, where T denotes the wave period. We have chosen this method for the time integration as it offers a good trade-off between accuracy and stability on one side and computational cost on the other side. It should be noted, however,

that other methods such as the integrating factor technique (e.g. Fructus *et al.* [37]), in which the linear parts of the equations are integrated exactly, could equally well have been applied in connection with both the IT and HOS method.

(a) Artificial damping

It turns out that, in the absence of any artificial damping, the time integration of the IT method as well as the HOS method is unstable regardless of the parameter choice and the dealiasing strategy. For that reason, we employ damping by multiplying the n th Fourier coefficients of η and Φ_s by the number

$$D_n = \frac{1}{\exp(|n| - N_{\text{cut}} - 1) + 1}, \quad N_{\text{cut}} = \frac{5}{8}N, \quad (5.1)$$

every time step, and by multiplying the n th Fourier coefficient of w_s by D_n every sub-time step in the Runge–Kutta procedure. When using this damping strategy we have found that it in general makes no difference if we use no dealiasing, quadratic dealiasing or full dealiasing. For that reason all results presented in this section have been obtained without dealiasing. On the other hand, the size of the time step plays an important role in the stability of the time integration, because it decides how many times per wave period, T , the damping is applied. This is, however, only the case because we have chosen to apply damping every time step. Had we chosen to damp the time integration a fixed number of times per period, the size of the time step would be less important.

(b) Evaluation of the kinematic and dynamic free surface boundary conditions

In the literature, two different procedures prevail for treating the nonlinear terms in the kinematic and dynamic surface conditions (2.4) and (2.5). The first one is straightforward, as w_s obtained from the DN procedure is applied directly in all products or powers. This procedure is naturally applied for all IT calculations, and it was also suggested by Dommermuth & Yue [9] that it should be used in connection with the HOS method.

The second procedure was suggested by West *et al.* [8] and it has been promoted by, for example, Bonneyfooy *et al.* [19] and Ducrozet *et al.* [38] as the so-called *consistent* approach. In this procedure, the full order of $w_s^{[M]}$ is only used in linear terms, while the order is reduced systematically in nonlinear terms. As an example, w_s^2 is not determined simply as the square of w_s but as

$$(w_s^2)^{[M]} = \sum_{n=0}^M w_s^{[n]} w_s^{[M-n]}. \quad (5.2)$$

Similarly, the w_s terms in the kinematic and dynamic equations are estimated by

$$w_s^{[M]} + w_s^{[M-2]} \nabla \eta \cdot \nabla \eta \quad (5.3)$$

and

$$(w_s^2)^{[M]} + (w_s^2)^{[M-2]} \nabla \eta \cdot \nabla \eta. \quad (5.4)$$

Bonnefooy *et al.* [19] and Ducrozet *et al.* [38] claim that, because of this treatment of the nonlinear terms involving w_s combined with a full dealiasing strategy, their HOS model can give accurate and stable results even for the steepest waves. This is remarkable, and in contrast to the achievements by, for example, Clamond & Grue [28] and Fructus *et al.* [37]. In the following, we present our conclusions on this matter.

First of all, we investigate the accuracy of $(w_s^2)^{[M]}$ obtained by (5.2) versus the straightforward calculation of $(w_s^{[M]})^2$. As shown in figure 10, we note that generally the straightforward calculation is the most accurate of the two. Second, we have attempted to time integrate the HOS method with the two alternative treatments of the boundary conditions, while including quadratic dealiasing and no artificial damping. For the case of $kh = 2\pi$ and $H/L = 0.08$, both

Figure 10. The relative DN error of w_s^2 as a function of M for $kh = 2\pi$ and (a) $H/L = 0.10$ and (b) $H/L = 0.12$. The figures have been produced with the HOS method with $N = 32$ with no time integration and quadratic dealiasing. Input and target results are from the SF method. Dashed black line: the order consistent approach of West *et al.* [8]. Red dots: the straightforward approach of Dommermuth & Yue [9]. (Online version in colour.)

Figure 11. The relative error of w_s after time integration up to $t = 50T$ for a steady nonlinear wave with $kh = 2\pi$. The HOS method is applied with $M = 8$, $N = 32$ and $\Delta t = T/100$ and with input and target results from the SF method. Dashed black line: the order consistent approach of West *et al.* [8]. Red dots: the approach of Dommermuth & Yue [9]. (Online version in colour.)

methods become unstable around $t \approx 1.3T$. Hence, we conclude that (5.2)–(5.4) have no stabilizing effects relative to the straightforward method. Third, we have time-stepped both HOS methods for 50 wave periods with and without quadratic dealiasing but including artificial damping. These simulations are both stable, but because of the damping the dealiasing has no effect at all (except that it slows down the computations substantially). The obtained accuracy of w_s is shown in figure 11 and we conclude that the straightforward treatment of nonlinear terms involving w_s (Dommermuth & Yue [9]) leads to more accurate results than the so-called consistent treatment involving (5.2)–(5.4).

Based on this investigation, we do not support the idea of replacing the straightforward treatment of the surface boundary conditions by the method of West *et al.* [8], and for the rest of this paper we will only use the straightforward treatment.

(c) Accuracy of the time integration for different values of M

In §4c, we concluded that, when using the SF method to provide the input, the IT method is significantly more accurate than the HOS method when using the same M for $N = 32$ and $kh = 2\pi$ (figure 6). Depending on the steepness of the wave, this conclusion does, however, not necessarily remain valid when the wave is integrated in time, since the IT method is less robust to perturbations than the HOS method. Figure 12 shows the error of w_s at time $t = 50T$ (i.e. after 5000 time steps) as a function of H/L . We note that the IT method with $M = 6$ and $M = 8$ is not

Figure 12. The relative error of w_s after time integration up to $t = 50T$ for the parameters $kh = 2\pi$, $N = 32$ and $\Delta t = T/100$. The figure is constructed with damping and without dealiasing, the input is taken from the SF method and missing data points for large values of H/L are due to break down of the time integration. Blue dots: HOS method with $M = 4$; green dots: HOS method with $M = 6$; red dots: HOS method with $M = 8$; dashed blue line: IT method with $M = 4$; dashed green line: IT method with $M = 6$; dashed red line: IT method with $M = 8$. (Online version in colour.)

Figure 13. The surface elevation at $t = 50T$ computed with the HOS method for the parameters $N = 32$, $kh = 2\pi$ and $\Delta t = T/100$. The figure is constructed with damping and without dealiasing, and the input is taken from the SF method. Green dots: $M = 4$; blue dots: $M = 6$; red dots: $M = 8$; black curve: target by SF theory. (Online version in colour.)

robust and breaks down for $H/L \approx 0.115$. By contrast, the HOS method with $M = 8$ is robust up to $H/L \approx 0.12$; at the same time, it is generally more accurate than the IT method with $M = 4$.

From figure 12, we also conclude that the IT and HOS methods are not capable of integrating waves steeper than $H/L = 0.12$ for 50 periods if the final error of w_s is required to be smaller than 1%, and $H/L = 0.12$ may therefore be considered the practical limit of the two methods for this particular case. The accuracy of the surface elevation after 50 periods for this ‘limiting’ steepness is of course of interest, and it is therefore illustrated in figures 13 and 14a when using the HOS method with $M = 4$, $M = 6$ and $M = 8$ and the IT method with $M = 4$. With $M = 4$ the difference between the HOS method and the target result is clearly visible on the scale of the figure, while the difference when using the HOS method with $M = 6$ is significantly smaller. For $M = 8$, the HOS method matches the target result seemingly without phase or shape errors. In comparison, the IT method with $M = 4$ has an accuracy corresponding to that of the HOS method with $M = 6$, meaning that a small error is visible on the figure. Although not as accurate as the HOS method with $M = 8$, the IT method with $M = 4$ manages not to break down when simulating waves with $H/L > 0.12$. For $H/L = 0.13$ the surface elevation after 50 periods is compared with the target solution of the SF method in figure 14b, and a clear difference is seen as the phase error is about 11%.

Figure 14. The surface elevation at $t = 50T$ computed with the IT method for (a) $H/L = 0.12$ and (b) $H/L = 0.13$ using the parameters $kh = 2\pi$, $N = 32$ and $\Delta t = T/100$. Purple dots: $M = 4$; black curve: target by SF theory. (Online version in colour.)

(d) The range of applicability of the two methods

In this section, we shall further investigate the accuracy of the IT method with $M = 4$ and the HOS method with $M = 8$, which were the most promising methods according to §5c. The intention is to establish a range of applicability of the two methods showing the best achievable accuracy for given values of nonlinearity $H/H_{\max}$ and relative water depth kh . In our investigation, we vary $H/H_{\max}$ from 0.10 to 0.95 and kh from 0.3 to 3.0 in discrete steps. Each combination of $H/H_{\max}$ and kh is investigated with a range of resolutions with N varying between 16 and 64 in steps of 4 in order to determine the best achievable accuracy. To avoid any potential doubt we here mention that the reference solutions with small values of N are obtained by interpolating converged reference solutions to a coarse grid by removing Fourier coefficients. All simulations propagate a steady nonlinear wave for 50 periods and determine the final relative error of w_s using either the SF or the CD method as input and target. Simulations are made with artificial damping and no dealiasing.

From the large number of simulations, the following trend can be observed: for relatively small numbers of $H/H_{\max}$, the best accuracy is always found with the highest values of N in agreement with figures 8 and 9. In principle, this should also be the case for higher nonlinearities, but here we experience the counteracting mechanism that instabilities start to occur unless N is gradually reduced. As an example the highest nonlinearities typically require that N is 32 or smaller.

Figure 15 shows the computed isolines of the minimum relative errors of w_s (i.e. the best achievable accuracy) for the two methods. The contours 0.0001, 0.001 and 0.01 are highlighted as the green, blue and red solid lines. If the limit of applicability is chosen as the red curve

Figure 15. Isolines of the minimum relative errors of w_s as a function $H/H_{\max}$ and kh after time integration up to $t = 50T$. (a) The HOS method with $M = 8$; (b) the IT method with $M = 4$. Green dashed line: 0.00005; green solid line: 0.0001; blue dashed line: 0.0005; blue solid line: 0.001; red dashed line: 0.005; red solid line: 0.01. (Online version in colour.)

(i.e. an error of 1%), it is obvious that the maximum values of $H/H_{\max}$ drop quite drastically from approximately 0.9 in deep water to only 0.3 for $kh = 0.3$. Moreover, it is evident that both the IT and the HOS methods drop to these low values in shallow water, and actually the two methods provide very similar results in the range $0.3 \leq kh \leq 0.8$. Choosing an error of 1% as the applicability limit is of course arbitrary, and we note that the IT and HOS methods are both capable of stable time integration of $H/H_{\max}$ larger than 0.3 when $kh = 0.3$ —it just results in a larger error. For example, the case $(kh, H/H_{\max}) = (0.3, 0.8)$ may be simulated by both methods, but the simulations result in a relative error of 144%, mainly because of phase errors.

In order to compare the two methods in more detail, figure 16 shows the minimum errors as a function of $H/H_{\max}$ for $kh = 0.03, 1.0, 1.5, 2.0, 2.5$ and 3 (corresponding to vertical cuts of figure 15). First, we note that the IT and HOS results are almost identical for $kh = 0.3$. Second, for $kh = 1.0$ and 1.5 the results are very similar up to $H/H_{\max} = 0.8$, beyond which the IT method starts to develop a local minimum while the error of the HOS method steadily increases. Actually, this trend continues up to $kh \simeq 1.8$. However, beyond $kh = 1.9$ the HOS method starts to develop a significant local minimum for higher nonlinearities, which makes this method the more accurate of the two for $H/H_{\max} \geq 0.8$. This trend is clearly seen for $kh = 2.0$ and also for $kh = 2.5$ and 3.0, although the single local minimum is replaced by a couple of minima in this case. On the basis of figures 15 and 16, we conclude that the IT and the HOS methods generally provide very similar accuracy for $H/H_{\max}$ up to 0.6 for any value of kh . For higher nonlinearities, the most accurate choice of method depends on the value of kh .

Figure 16. The minimum relative error of w_s as a function of $H/H_{\max}$ after time integration up to $t = 50T$ for different values of kh . Red: HOS method with $M = 8$; blue: IT method with $M = 4$. (Online version in colour.)

6. Conclusion

The background for this work has been the question of whether it is possible to devise a numerical method that is more accurate and stable than the HOS method when used to simulate steady two-dimensional nonlinear waves, while still being comparably efficient. In an attempt to find such a method, we have derived the IT method, and tested its accuracy and stability for problems dealing with steady nonlinear waves while comparing it with the HOS method. The two methods both rely on a Taylor expansion of the velocity potential around the still water level, but deviate since the HOS method computes quantities at the still water level using an explicit perturbation method while the IT method computes the still water quantities by solving the exact system of equations using an implicit method. Naturally, the IT method has several properties in common with various modern Boussinesq-type formulations, and these have been outlined in connection with its derivation.

In §4, we have considered the DN problem of determining the vertical velocity at the free surface given the surface elevation and the surface potential. For this problem, we conclude that the IT method is significantly more accurate than the HOS method (figures 2 and 6) when using the same truncation order, M , and spatial resolution, N . This difference is even more pronounced when it comes to estimating velocities at the still water level (figure 3), and it carries over to a superior determination of the interior velocity profiles (figures 4 and 5). While the original HOS formulation produces velocity profiles that are visibly inaccurate already for $H/L \approx 0.08$ when $kh = 2\pi$, the IT method has no problem in calculating the velocity profiles at least up to $H/L = 0.135$ when $kh = 2\pi$. We do acknowledge that the explicit HOS method can be combined with more advanced procedures in order to improve its interior velocity field, but this has not been our focus in this work.

In §4d, we investigated the effect of using a perturbed input to the DN problem, and much to our surprise we discovered that the IT method is sensitive to these perturbations especially for higher values of M (figure 7), while the HOS method is not. In §4e, we tested the effect of systematically increasing the resolution N with the expectation that the relative DN errors would stagnate for N larger than a certain threshold. This test has demonstrated a weakness of both methods for large values of M , in which case errors start to grow significantly. Hence, from figures 8 and 9, we have concluded that the IT method should be limited to M less than 6, while the HOS method is robust with M at least up to 8.

When integrated in time, both methods are found to be unconditionally unstable for steady nonlinear waves regardless of the dealiasing strategy if no artificial damping is applied. To that end, we have investigated the effect of the so-called order consistent approach of West *et al.* [8], and found that the approach is, in fact, slightly less accurate (figures 10 and 11) and no more stable than the straightforward approach used by, for example, Dommermuth & Yue [9]. It should be noted that this finding contradicts the results of Bonnefoy *et al.* [19] and Ducrozet *et al.* [38]. If artificial damping is used, we find that both methods can handle 50 periods of time integration accurately for waves of steepness less than $H/L \approx 0.12$ when $kh = 2\pi$. In figure 12, we have shown the relative error of w_s after integrating both methods in time for 50 periods with the time step $\Delta t = T/100$. It turns out that the HOS method with $M = 4, 6$ and 8 is robust, while the IT method is only robust for the steepest waves if the order is reduced to $M = 4$. The corresponding accuracies of the surface elevations have been presented in figures 13 and 14. On this background, we have concluded that the most promising options are the IT method with $M = 4$ and the HOS method with $M = 8$.

In §5d, we have investigated the applicability of both methods by computing the minimum errors of w_s (for a variety of resolutions N) as a function of steepness ($H/H_{\max}$) and relative water depth (kh) after 50 periods of time integration with the time step $\Delta t = T/100$ (figure 15). More details and a comparison between the two methods are provided in figure 16. In general, it turns out that the accuracies of the two methods are very similar when expressed in terms of kh and $H/H_{\max}$. For $H/H_{\max} \leq 0.6$, they are almost identical, while in other regions the choice of the most accurate method depends on the particular combination of kh and $H/H_{\max}$. Combining this with

the fact that the IT method without a preconditioner to solve (3.5a) is significantly less efficient than the HOS method as discussed in §3c, we conclude that the HOS method is the most suited of the two for practical applications for the time being.

Data accessibility. This article has no additional data.

Authors' contributions. M.K. implemented the numerical methods, carried out simulations, drafted parts of the manuscript and devised parts of the research. P.A.M. derived the implicit Taylor method, implemented the numerical methods, carried out simulations, drafted parts of the manuscript and devised parts of the research. D.R.F. critically revised the manuscript and devised parts of the research. All authors gave final approval for publication and agree to be held accountable for the work performed herein.

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Appendix A. Comparison with high-order spectral results from the literature

Since the first developments of the HOS method by Dommermuth & Yue [9] and West *et al.* [8], the method has been investigated, analysed and applied many times in the literature. Some of these investigations, however, report results and conclusions that deviate significantly from ours; since we believe that the differences are too important to be neglected, this appendix presents and discusses some of them.

(a) Application of the high-order spectral method to steep waves in deep water

Our implementation and investigation of the HOS DN operator has revealed that we cannot freely increase the resolution N and the order M in order to reduce the relative error of w_s to any desired accuracy. This is especially the case for relatively steep waves in deep water, where the error starts to grow beyond a certain threshold of N and M . Similar behaviour of the DN operator has been reported by, for example, Dommermuth & Yue [9], Nicholls & Reitich [39], Bateman *et al.* [11] and Xu & Guyenne [14], but our results are at odds with those of Bonnefoy *et al.* [19] and Ducroz *et al.* [38], who reported that their implementation of the HOS method could give exponentially decreasing errors of w_s with N and M for seemingly all values of N and M .

Bonnefoy *et al.* [19] considered a steady nonlinear wave with $H/L = 0.127$ (see their appendix C) in deep water, meaning that the steepness corresponds to about 91% of the limiting steepness given by (4.2). As mentioned above, Bonnefoy *et al.* found the error of w_s for this case to decrease exponentially with both N and M , and we have not been able to reproduce this result even when using full dealiasing. Our results for the test are compared with those of Bonnefoy *et al.* in table 1 (our N and M correspond to $N_x/2$ and $M + 1$ in their notation, respectively), which clearly shows that our errors tend to increase with N and M if these parameters are large enough, while the errors reported by Bonnefoy *et al.* are monotonically decreasing with N and M .

We have found a similar disagreement with the results of Ducroz *et al.* [38], whose fig. 8 shows the error of w_s as a function of N and M (our M corresponds to $M + 1$ in their notation) for a steady nonlinear wave with steepness $H/H_{\max} = 0.675$ (corresponding to $H/L = 0.0955$) and water depth $kh = 2\pi$. In their figure, the error of w_s is seen to decrease exponentially with both N and M , and we have not been able to reproduce this result, despite the relatively small steepness used in this case. The reason for the discrepancy is again that our implementation of the HOS method becomes unstable for large M and N as illustrated in figure 17, which has been produced with full dealiasing and by using the results of the CD method for input and target solutions. We know that for large values of N Ducroz *et al.* interpolated the SF result (G. Ducroz 2019, personal communication) to obtain input and target solutions since the SF method is unreliable when used with a fine resolution. We have tested whether the difference between our results is due to the difference between the interpolated SF results and the results of the CD method, but conclude that this is not the case. When using the interpolated SF solutions we have obtained

Figure 17. The relative DN error of w_s as a function of (a) N and (b) M using the parameters $H/L = 0.095$ and $kh = 2\pi$. The results have in both cases been obtained using full dealiasing and the input and target results are provided by the CD method. (Online version in colour.)

Table 1. The relative DN error of w_s as a function of N computed with full dealiasing. Our results are dimensionless while the dimension of the results of Bonnefoy *et al.* [19] is unknown. For our results, the CD method has been used to produce the input and target solutions.

	N	M				
		5	7	9	11	13
our results	16	1.7×10^{-2}	8.6×10^{-3}	1.5×10^{-3}	4.7×10^{-4}	4.8×10^{-4}
Bonnefoy <i>et al.</i>	16	1.5×10^{-3}	3.0×10^{-4}	6.5×10^{-5}	1.9×10^{-5}	1.0×10^{-5}
our results	32	4.4×10^{-3}	1.5×10^{-2}	3.1×10^{-2}	3.3×10^{-2}	1.9×10^{-2}
Bonnefoy <i>et al.</i>	32	1.5×10^{-3}	2.9×10^{-4}	6.0×10^{-5}	1.2×10^{-5}	2.5×10^{-6}
our results	64	2.4×10^{-3}	4.6×10^{-4}	1.7×10^{-3}	1.2×10^{-2}	9.2×10^{-2}
Bonnefoy <i>et al.</i>	64	1.5×10^{-3}	3.0×10^{-4}	6.0×10^{-5}	1.2×10^{-5}	2.5×10^{-6}

qualitatively exactly the same results (i.e. growing errors for large N and M) as when using the CD solutions.

(b) Application of the high-order spectral method to waves in shallow water

In 2017, Ducrozet *et al.* [20] investigated the range of applicability of the HOS method in deep as well as in shallow water. To this end, they have used the limiting criterion of Miche [40],

$$\left(\frac{H_{\max}}{L} \right)_{\text{Miche}} = 0.14 \tanh(kh), \quad (\text{A } 1)$$

which we note exceeds the established empirical criterion (4.2) by Battjes by approximately 13% in the limit $kh \rightarrow 0$ (and about 12% for $kh = 0.3$). Ducrozet *et al.* [20] conclude that ‘the solution is still possible up to 70% of the limiting wave height extracted from (A 1)’, and this corresponds to approximately 80% of the Battjes limiting criterion in shallow water. By contrast, our implementation of the HOS method shows errors larger than 1% after 50 periods of time integration for wave heights as low as $H/H_{\max} \geq 0.3$ when $kh \leq 0.3$, and a reasonable range of applicability for our implementation of the HOS method may thus be chosen as $H/H_{\max} \leq 0.3$ in shallow water. As such, there seems to be a large discrepancy between the

range of our implementation and that of Ducrozet *et al.* [20]. In that connection, we note that our implementation of the HOS method is capable of simulating the case of $H/H_{\max} = 0.80$ and $kh = 0.30$ for 50 periods, although the final relative error on w_s is about 144%, and so the apparent discrepancy may simply be due to different criteria for applicability.

Given that our results suggest that the HOS method is incapable of dealing with steep regular waves in shallow water, an interesting question is of course whether the HOS method may be used for the simulation of solitary waves of the Euler equations. While we have not tested this ourselves, we mention that the careful study of Craig *et al.* [41] shows that the HOS method is perfectly able to accurately resolve simulations of solitary wave collisions at least up to $S/h = 0.5$, with S being the height of the wave and h being the water depth. As such, our findings for the lack of ability of the HOS method to deal with steep regular waves in shallow water should not be confused with an inability to deal with solitary waves.

Appendix B. Computational complexity of the implicit Taylor method and the high-order spectral method

This appendix presents a rather rough estimation of the computational work required to evaluate the DN operators of the IT and HOS methods, respectively, in terms of the parameters M and N . For both methods, it holds that the computational bottleneck is the computation of Φ_0 from Φ_s , and the computational complexity of the methods may therefore be estimated from these parts of the methods. Our estimates will be based on the assumption that $M, N \gg 1$, and we will pay no attention to numerical constants in the sense that we will, for example, take

$$O\left(\sum_{m=1}^M m\right) = O(M^2) \quad \text{and} \quad O\left(\sum_{m=1}^M m^2\right) = O(M^3) \quad (\text{B } 1)$$

even though the sums evaluate to $M(M+1)/2$ and $M(M+1)(2M+1)/6$, respectively, which for moderate values of M may deviate substantially from M^2 and M^3 . We will furthermore assume that the various powers of η which are involved in the computation have been pre-computed (regardless of the dealiasing strategy), as one would do in any efficient implementation of the algorithms.

(c) The implicit Taylor method

In order to analyse the computational complexity of the IT method, we will make use of an observation from numerical experiments and an assumption about the computational cost of a single GMRES iteration. The observation is that when using no preconditioner the number of iterations grows in proportion to N , while the assumption is that the cost of a single iteration is dominated by the cost associated with evaluating the right-hand side of (3.5a) given Φ_0 . In that connection, we note that while the assumption is easily justified for iterations of small index there is no guarantee that it holds for iterations of large index since the computational cost of a single iteration increases with the iteration index. We therefore re-emphasize that the following operation count should only be taken as a rough estimate.

Now, when no dealiasing is used, evaluating the right-hand side of the equation requires $O(M)$ fast Fourier transforms of vectors of length N and $O(M)$ entry-wise products of vectors of length N . Since the fast Fourier transform requires $O(N \log(N))$ operations and the entry-wise products require $O(N)$ operations, evaluating the right-hand side is seen to have computational complexity $O(MN \log(N))$. In total, the unpreconditioned IT method therefore requires $O(MN^2 \log(N))$ operations when no dealiasing procedure is applied. If quadratic dealiasing is used, the same number of operations must be used when disregarding numerical factors, since the fast Fourier transforms will simply be on vectors of twice the length than when no dealiasing is used. When full dealiasing is used, the product $\eta^{2m} \nabla^{2m} \Phi_0$ requires a single fast Fourier transform with

$O(mN \log(N))$ operations. This implies that the computational cost per iteration becomes

$$O\left(\sum_{m=1}^M mN \log(N)\right) = O(M^2 N \log(N)) \quad (\text{B2})$$

such that the total number of operations needed to solve equation (3.5a) is $O(M^2 N^2 \log(N))$.

(d) The high-order spectral method

When computing Φ_0 from Φ_s using the HOS method, one recursively computes

$$\Phi_0^{(m)} = - \sum_{p=1}^m \frac{\eta^p}{p!} \frac{\partial^p \Phi_0^{(m-p)}}{\partial z^p} \Big|_{z=0} \quad (\text{B3})$$

for $m = 1, 2, \dots, M$ starting from $\Phi_0^{(0)} = \Phi_s$. If no dealiasing procedure is used, the computation of the p th term of $\Phi_0^{(m)}$ requires $O(N \log(N))$ operations, meaning that computing $\Phi_0^{(m)}$ must require $O(mN \log(N))$ operations. In total, the number of operations needed for the computation of Φ_0 in this case is therefore

$$O\left(\sum_{m=1}^M mN \log(N)\right) = O(M^2 N \log(N)). \quad (\text{B4})$$

If quadratic dealiasing is employed the same total number of operations is found since evaluating the p th term of (B3) requires $O(N \log(N))$ operations. When full dealiasing is used, the computation of the p th term of $\Phi_0^{(m)}$ has complexity $O(pN \log(N))$, and this result implies that the computation of Φ_0 in total requires

$$O\left(\sum_{m=1}^M \sum_{p=1}^m pN \log(N)\right) = O\left(\sum_{m=1}^M m^2 N \log(N)\right) = O(M^3 N \log(N)) \quad (\text{B5})$$

operations. We note that our result for the computational complexity of the HOS method without dealiasing is at odds with the claim of Dommermuth & Yue [9] that the computational complexity of the HOS method is ‘directly proportional to N and M ’ (see their abstract), but is in agreement with the result found by Schäffer [12], who thoroughly counted the number of fast Fourier transforms needed by the HOS method.

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