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A constrained spline dynamics (CSD) method for interactive simulation of elastic rods

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Abstract

In this paper, we present constrained spline dynamics (CSD) as a unified framework for the elastodynamic simulation of elastic rods subjected to constraints at interactive rates. The geometry of the rod and its kinematics are discretized using smooth spline functions and the rod's centerline co-ordinates as degrees of freedom (DOF). Interpolating B-spline shape functions are used to take advantage of the smooth basis and the Kronecker delta property. The formulation is developed from Hamilton's principle with bending and twisting energies represented as compliant constraints. The bend-twist coupled behavior is modeled using the concept of holonomy of curves utilizing the smooth and accurate curvature and bi-normal vector fields, eliminating rotational director frames as degrees of freedom. By enforcing uniform arc-length parametrization, high accuracy is achieved in modeling bend, twist, and bend-twist coupling. Several numerical examples are presented that demonstrate the convergence behavior, computational accuracy and efficiency of the formulation.

Keywords

Elastic rods; B-splines; shape functions; arc-length parametrization; inextensibility; thread simulation

1 Introduction

Simulation of thin one-dimensional objects including hair, suture thread, and cables is common in computer animations, games, movies, and virtual design software tools [1][2]. Applications such as virtual surgical simulation [3] and virtual cable assembly design tools [4] demand physical accuracy along with computational efficiency that is needed to support interactive simulation. Most approaches model these objects as elastic rods using discrete line segments [4][5][6]. Methods that represent the rod geometry using smooth functions such as splines have been proposed [7][8] to represent accurate curvature fields to avoid fine discretization. However, such smooth methods pose challenges for interactive simulations

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including the need for rotational degrees of freedom (DOF) in addition to translational DOF for accurate modeling of bending, twisting and bend-twist coupling, complexity in the treatment of contact (or collision) with fictitious generalized coordinates as DOF and the lack of a unified framework for accurate modeling of elastodynamics under time-varying loads and constraints. In this work, a constraint-based smooth formulation is proposed with only translational degrees of freedom for interactive dynamic simulation of elastic rods.

Simple discrete methods use mass-spring systems, where the internal forces are evaluated based on empirical relations. For instance, a rigid body multi-chain mass-spring model with linear and angular springs is used to model wisps of hair and their collisions [9]. In an advanced model based on spring systems, treatment of bending and torsional internal forces is accomplished using fictitious springs [5].

A physically realistic approach is introduced in [10] for the simulation of suture thread based on the Cosserat rod theory [11], which utilizes material coordinate frames called “directors”. In this work, bending and torsion are incorporated by integrating the evolution of the director frame along the arc-length using boundary conditions from the needle end of the thread. This approach is further improved in several related works. In [4], springs are used to model tension and Cosserat directors are utilized to model bending and torsion for simulating assembly of automobile cables. A comprehensive approach is presented in [12] that uses discrete energies derived from interpolated director frames to achieve interactive rates in dynamic simulation of elastic rods. In a position-based approach [6], additional points are introduced to naturally orient the material frames on the discretized segments in order to accomplish real-time simulation of hair and rods. Further, using Kirchhoff’s elastic rod theory with the Bishop frame field [13] as reference [14], a discrete approach has been proposed to simulate elastic rods that incorporates bending and twisting to achieve physically realistic results [15].

While the Cosserat theory can account for shear deformation and can be applied for thicker rods, the treatment of the director frame as DOF in the form of quaternions and enforcement of the associated constraints is computationally intensive [12][16]. This additional effort results in redundancy in modeling slender rods. A rotation-based adapted frame is utilized to comprehensively describe the dynamics of geometrically nonlinear rods from the continuum kinetic relations for slender rods [17]. A geometrically exact approach is presented in [18] describing the kinematics of the rod using the Kirchhoff theory and using the the finite element method for numerical solution. In another, geometrically exact approach, a finite element formulation based on the Kirchhoff theory is proposed where the Frenet-Serret frame is utilized as the reference frame [19], and is later extended to address membrane locking phenomenon [20]. The advantage of the Kirchhoff’s theory as demonstrated in [15] is that the treatment of directors frames as DOF is eliminated with the material frame defined as a rotated frame by a material twist of the reference Bishop’s frame about the tangent vector. Further, uniform twist rate in rods with axisymmetric cross-sections eliminates the twist DOF, thereby eliminating all the rotational degrees of freedom. This results in centerline coordinates as the only DOF for modeling both the bend and twist behavior. Our method is inspired by this treatment.

Discrete methods require alternative definitions of curvature [21] and are often limited by the need for a finer mesh. To circumvent the shortcomings of the discrete approaches, methods representing rod geometry with higher order basis functions are introduced [7][8]. These smooth approaches naturally define the curvature fields with fewer discretization points. Early work on physically-based realistic animation of elastically deformable objects with smooth geometric definitions uses internal forces which are derived from an elastic potential [22]. In an effort to build a framework for geometric design through interactive sculpting, non-uniform rational basis splines (NURBS) have been introduced [23] to simulate elastic bodies using finite element method with the control points and weights as generalized DOF. A similar approach is used for the dynamic animation of curvilinear objects subjected to gravity and viscous drag loadings based on the Lagrange formalism [24]. This approach is later extended in [7][25] to model suture thread with empirical relations for bending forces along with point and directional constraints. A physical model using splines is proposed in [8] for the simulation of the interactive cable assembly, where internal forces are derived from the stretching, bending, and twisting energies. In another smooth approach [26], B-spline interpolation is employed to model curved rods using the Kirchhoff-Love hypothesis with control points of the centerline coordinates and geometric torsion as degrees of freedom to perform small-deformation numerical analysis of curved space rods.

Most of the smooth methods use B-splines or NURBS because of several advantages [8][16][27] despite lacking the Kronecker delta property. Interpolating splines such as Catmull-Rom splines or third order Hermitian splines, e.g., offer only first derivative continuity, leading to discontinuous curvature fields; whereas, B-splines of the same order possess second order derivative continuity [28]. However, lack of the Kronecker delta property results in nonphysical quantities as DOF and may result in increased system unknowns in the form of control points and optionally, weights. Further, conversion of control points to material co-ordinates causes inefficient treatment of collision detection and response when multiple rods are in contact or interacting with other objects. This may lead to difficulty in establishing relational constraints between the rod and other deformable objects in collision response. Moreover, such methods require specific numerical schemes to evaluate control points [8].

Models of elastic rod based on spline formulations utilize arc-length parametrization of field variables, which includes the geometry of the rod itself. For uniform and valid parametrization, the arc-length parameter must traverse the image of the curve at unit-speed [29]. Methods based on the Cosserat rod theory ensure this uniformity of the arc-length parameter through the constraints placed on the directors [12]. Other smooth methods based on Kirchhoff's theory [8] [24] [25] do not ensure the uniformity of the arc-length parameter. This may result in warping of the scale in which field quantities are measured, leading to erroneous results.

Several methods have been developed for elastodynamic simulation in interactive or real-time environments [30][31][32]. Most methods are point-mass based models in which the forces are derived from spring-damper elements. Modeling the compliant elastic deformation combined with infinite stiffness constraints such as inextensibility in such

methods poses challenges in maintaining the accuracy and stability of the solution. Methods that account for constraints of infinite stiffness in simulating rigid systems at interactive rates have been presented [33]. To enhance the stability of such methods, numerical regularization is introduced resulting in relaxed constraints [34]. Utilizing the constraints based framework, and formulating the elastic energy as potentials, a compliant constraint-based formulation is presented in [35] for dynamic simulation of elastically deformable bodies. Geometric nonlinearity is incorporated to enhance the compliant constraint formulation to achieve higher stability for large time steps. Such methods are designed for dynamic problems, while in simulation environments such as virtual cable positioning or surgical suturing, the rates of loading vary significantly. Slowly varying loads and constraints can be accurately modeled using the quasi-static approach. In such cases, performing dynamic simulation may result in unphysical motion (such as spurious vibrations under quasi-static loading) and redundant computation. On the other hand, a purely quasi-static approach does not capture dynamic loading conditions. We adapt the compliant constraint-based formulation for different loading rates suitable for such applications.

In this work, we propose a constraint-based framework using a smooth formulation with interpolatory B-splines based on Kirchhoff's rod theory to perform elastodynamic simulations of elastic rods at interactive rates. The Kronecker delta property of the B-splines enables the use of material points of the centerline as DOF as opposed to the generalized coordinates. The resulting system matrices are not fully coupled as the interpolatory spline functions have exponential decay. The framework is developed for time-varying loads, thus unifying the quasi-static or dynamic loads based on free energy. The internal forces are derived from the discretized bending and twisting energies from Kirchhoff's theory. The internal forces due to coupled bend-twist behavior are obtained using the variation of holonomy [15]. The consistency in arc-length parametrization is achieved by enforcing unit-length constraints on the tangent vector field. An attractive feature of this approach is that the scalar and vector fields can be easily represented in terms of nodal quantities. Here, we consider slender inextensible rods (based on Kirchhoff-love theory) with isotropic cross sections and of initially straight configuration. The material twist deformation in the rod is treated as quasistatic, neglecting the rotational inertia due to twist.

The remainder of the paper is organized as follows. The elastodynamics of rods based on Kirchhoff's theory under various constraints is presented in section 2. The numerical implementation of the constrained elastodynamics of rods using interpolatory B-splines is presented in Section 3. Numerical examples are presented in Section 4 to evaluate the constrained spline dynamics framework for various constraints and loading conditions. Summary and future work are presented in Section 5.

2 Elastodynamics of constrained elastic rods

We consider the motion of inextensible elastic rods with isotropic cross section (circular or annular) in Euclidean space under the action of time-dependent forces and constraints. Let the position of the centerline of a rod be denoted by the vector field $\mathbf{x}(s, t)$ parametrized based on the arc-length parameter $s \in [0, L]$ at any given time t (Figure 1) and the forces and constraints be denoted as $\mathbf{F}(s, t)$ and $\mathbf{C}(s, t)$, respectively.

For any given centerline configuration of the rod, the Bishop frame field [13], denoted as $\mathcal{B}(s) := \{\mathbf{t}(s, t), \mathbf{u}(s, t), \mathbf{v}(s, t)\}$ can be established, where $\mathbf{t}(s, t) = \mathbf{x}'(s, t)$ is the tangent vector field while $\mathbf{u}(s, t)$ and $\mathbf{v}(s, t)$ are orthonormal vector fields. The vector fields $\mathbf{u}(s, t)$ and $\mathbf{v}(s, t)$ are uniquely determined by a chosen vector $\mathbf{u}_0$ in the plane normal to the tangent vector specified at any point in the centerline of rod, e.g., at $s = 0$. The initial Bishop frame is denoted as $\mathcal{B}(0) \equiv \mathcal{B}_0 := \{\mathbf{t}_0, \mathbf{u}_0, \mathbf{v}_0\}$, where $\mathbf{u}_0$ and $\mathbf{v}_0$ are two orthonormal vectors in the plane normal to the tangent vector $\mathbf{t}_0$ at $s = 0$. The prime notation denotes the derivative of the field with respect to the arc-length parameter s .

The material frame field, which signifies the evolution of material points away from the centerline of the rod, is denoted as $\mathcal{M}(s) := \{\mathbf{t}(s, t), \mathbf{m}(s, t), \mathbf{n}(s, t)\}$, where $\mathbf{m}(s, t)$ and $\mathbf{n}(s, t)$ are orthonormal vector fields representing the material vectors. The Bishop frame field serves as the reference for quantifying the material twist imposed on the rod. This field is unique and is naturally twist-free based on its definition [13] using parallel transport of the (initial) coordinate frame $\mathcal{B}_0$ and depends on the magnitude of the curvature vector $\boldsymbol{\kappa}(s, t) = \mathbf{t}'(s, t)$ and natural torsion $\tau(s)$ of the centerline curve when traversed from $s = 0$ to $s = L$. With the initial Bishop frame $\mathcal{B}_0$ chosen to be the same as the initial material frame $\mathcal{M}_0$ at $s = 0$, the angular deviation of the material frame (Figure 1) from the reference frame defines the material twist $\theta(s)$ of the rod at any given location s on the centerline.

The motion of the inextensible elastic rod under the influence of constraints and forces is described by Hamilton's principle for conservative systems which states that among all admissible motions, the actual motion of a conservative system subjected to constraints is such that $\delta \int_{t_1}^{t_2} [L + C] dt = 0$. Here, $L = T - V$ is the Lagrangian of the system, T and V represent the kinetic energy and potential energy of the system respectively, and C represents the constraints applied on the system.

The Euler-Lagrange equations follow from Hamilton's principle

$$\frac{\partial L(\dot{\mathbf{x}}, \mathbf{x})}{\partial \mathbf{x}} - \frac{d}{dt} \left(\frac{\partial L(\dot{\mathbf{x}}, \mathbf{x})}{\partial \dot{\mathbf{x}}} \right) + \sum_j \frac{\partial C_j(\mathbf{x})}{\partial \mathbf{x}} \lambda_j = 0 \quad (1)$$

where $C_j(\mathbf{x})$ represents the j th constraint, and λ_j denotes the corresponding Lagrange multiplier.

The kinetic energy (T) of the rod is

$$T = \frac{1}{2} \rho_L \int_{s=0}^L |\dot{\mathbf{x}}(s, t)|^2 ds \quad (2)$$

where ρ_L is mass per unit length of the rod and $\dot{\mathbf{x}}(s, t)$ is the velocity vector. The potential energy (V) of the system includes the elastic strain energy of the rod, which comprises of two components – bending and twisting. Using Kirchhoff's theory of elastic rods [14], the bending energy of the rod (V_B) can be written as

$$V_B = \frac{1}{2} \int_{s=0}^L k_B |\kappa(s, t)|^2 ds \quad (3)$$

where $\kappa(s, t)$ is the curvature vector and k_B is the bending modulus of the rod. Similarly, the elastic energy due to twisting (V_T) of the rod is given by

$$V_T = \frac{1}{2} \int_{s=0}^L \mu_T [\theta'(s, t)]^2 ds \quad (4)$$

where $\theta'(s, t)$ is the scalar field representing the twist per unit length of the rod and μ_T is the twist modulus of the rod.

The rod may be subjected to multiple constraints including positional, directional, inextensibility, and contact constraints:

- i. Positional constraint refers to the constraints imposed on the position of the centerline of the rod along single or multiple spatial directions (Figure 2a). The constraint on the position ($x_{p(\alpha)}$) at a given location $s = s_p$ in direction α , is given by

$$C_P(\mathbf{x}(s, t)) = x_{p(\alpha)}(t) - g_P(t) = 0 \quad (5)$$

where $g_P(t)$ is a specified position as a function of time.

- ii. Directional constraints are imposed on the orientation of the centerline (Figure 2b). The constraint is expressed in terms of the tangent vector as

$$C_S(\mathbf{x}(s, t)) = t_{p(\alpha)}(t) - g_S(t) = 0 \quad (6)$$

where $t_{p(\alpha)}(t)$ is the component of the tangent vector $\mathbf{t}_p(t)$ in the direction α at location $s = s_p$ and $g_S(t)$ is a specified function of time.

- iii. Inextensibility constraint is defined by the unit-length requirement of the tangent vector field for an arc-length parametrized curve as

$$C_I(\mathbf{x}(s, t)) = |\mathbf{t}(s, t)| - 1 = 0 \quad (7)$$

- iv. Contact constraints are applied when a rod interacts with another rod or other objects and a specific distance has to be maintained. This constraint is applied to the distance between a given position $\mathbf{x}_p$ on the centerline of the rod and a point $\mathbf{a}(t)$ in space, which is usually the surface of the colliding body as

$$C_C(\mathbf{x}(s, t)) = g_C - |\mathbf{x}_p(t) - \mathbf{a}(t)| = 0 \quad (8)$$

where g_C is the specified distance between the surface of the rod and the colliding body (Figure 2c).

3 Numerical implementation

In this section, we present the numerical formulation of elastic rods based on the interpolatory spline shape functions. We begin with the derivation of interpolatory B-spline shape functions. Next, we present the discretization scheme for the Euler-Lagrange equations followed by the numerical algorithm for the elastodynamic simulation of elastic rods.

3.1. Geometry discretization using interpolatory spline functions

B-spline basis functions are piecewise polynomial functions defined over a non-descending sequence of real numbers known as knot-vector $\mathcal{S} = \{s_1, \dots, s_k\}$, $s_i \in \mathbb{R}$, $s_i \leq s_{i+1}$ where the half-open interval $[s_i, s_{i+1})$ signifies the i^{th} knot span [36]. The i^{th} B-spline basis function of degree p for the given knot vector $\mathcal{S}$ is defined as

$$N_{i,0}(s) = \begin{cases} 1 & \text{if } s_i \leq s < s_{i+1} \\ 0 & \text{otherwise} \end{cases}, \quad (9)$$

$$N_{i,p}(s) = \frac{s - s_i}{s_{i+p} - s_i} N_{i,p-1}(s) + \frac{s_{i+p+1} - s}{s_{i+p+1} - s_{i+1}} N_{i+1,p-1}(s).$$

Let the space of the spline functions with an open uniform knot vector sequence be defined as $\mathbb{N}_{\mathcal{S},p} = \text{span}\{N_{i,p}, i = 1, \dots, M\}$. A B-spline curve defined by a vector of control points $\mathbf{P} \in \mathbb{R}^M$ in the spline space $\mathbb{N}_{\mathcal{S},p}$ is written as follows:

$$c(s) = \sum_{i=1}^M N_{i,p}(s) P_i \quad (10)$$

B-splines are used to approximate scalar valued functions [36]. An approximation of function $y(s): \mathbb{R} \rightarrow \mathbb{R}$, $s \in \Omega = [0, L]$ is discretized using B-spline basis functions of degree p as

$$y(s) = \sum_{i=1}^M N_{i,p}(s) P_i; \quad s \in [0, L] \quad (11)$$

For N data points, where $M = N + p - 1$, the interpolation equations are as follows

$$y(s_j) \equiv \sum_{i=1}^M N_{i,p}(s_j) P_i = y_j; \quad j = 1, 2, \dots, N \quad (12)$$

The additional $M - N$ conditions are normally derivatives set as zero at end points. For the cubic case ($p = 3$), the two additional conditions are specified at end points by setting the second derivatives as zero (natural boundary conditions).

$$\sum_{i=1}^M D^{(p-1)} N_{i,p}(s_j) P_i = 0; \quad j \in [1, 2, \dots, N] \quad (13)$$

From Eq. (12) and Eq. (13), the corresponding linear algebraic system can be expressed as

$$\mathbf{A}\mathbf{P} = \bar{\mathbf{y}} \quad (14)$$

where $\mathbf{A} \in \mathbb{R}^{M \times M}$, $\mathbf{P} \in \mathbb{R}^M$ is a vector of control points, and $\bar{\mathbf{y}} \in \mathbb{R}^M$.

The control points for any given vector $\bar{\mathbf{y}}$ can be solved using

$$\mathbf{P} = \mathbf{A}^{-1}\bar{\mathbf{y}} \quad (15)$$

We use the above approximation scheme to derive interpolatory B-spline shape functions. Using Eq. (15) and Eq. (11), we have

$$y(s) = \sum_{i=1}^M N_{i,p}(s) [\mathbf{A}^{-1}\bar{\mathbf{y}}]_i \quad (16)$$

Since we seek the shape functions to satisfy the Kronecker delta property, the corresponding

$\bar{\mathbf{y}}$ for the j^{th} shape function is given by $\bar{\mathbf{y}} \equiv \bar{\boldsymbol{\phi}}^{(j)} = \begin{bmatrix} 0 & 0 \dots 0 & 0 & \underbrace{1}_{j^{\text{th}} \text{ term}} & 0 & 0 \dots 0 \end{bmatrix}^T$, which consists

of all zeros except for a unit-value in the j^{th} entry corresponding to j^{th} interpolation equation. Note that the additional conditions also lead to zero entries. Thus, the control points required to construct the j^{th} shape function is

$$\mathbf{P}^{(j)} = \mathbf{A}^{-1}\bar{\boldsymbol{\phi}}^{(j)} = [\mathbf{A}^{-1}]_j \quad (17)$$

where $[\mathbf{A}^{-1}]_j$ represents the j^{th} column of $\mathbf{A}^{-1}$. Thus, the shape function for node ' j ' can be written as

$$\phi_j(s) = \sum_{i=1}^M N_{i,p}(s) P_i^{(j)}. \quad (18)$$

The approximation scheme using the shape functions $\phi_j(s); j = 1, 2, \dots, N$ can now be written as

$$y(s) = \sum_{j=1}^N \phi_j(s) y_j \quad (19)$$

in terms of nodal values.

We briefly point out the important properties of the shape functions:

1. Kronecker delta property:

$$\phi_i(s_j) = \delta_{ij} \quad (20)$$

Proof: From Eq. (18) at the evaluation points (s_k), we have

$$\phi_j(s_k) = \sum_{i=1}^M N_{i,p}(s_k) P_i^{(j)} = \{A\}_k P_i^{(j)} = \{A\}_k [A^{-1}]_j = \delta_{kj},$$

where $\{A\}_k$ is the k th row of A (from Eq. (12) and Eq. (14)) and the control points vector $P_i^{(j)}$ is given by Eq. (17).

2. Partition of unity property:

$$\sum_{i=1}^N \phi_i(s) = 1, \quad \forall s \in \Omega \quad (21)$$

Proof: Consider the function $y(s) = 1$. From Eq. (11),

$$y(s) = \sum_{i=1}^M N_{i,p}(s) P_i^{(y=1)} = 1 \quad (22)$$

where $P_i^{(y=1)}$ is the control points vector corresponding to $y(s) = 1$. From, Eq. (14),

$$A P^{(y=1)} = \bar{y}^{(y=1)}, \quad (23)$$

where, from Eq. (12) and Eq. (13), $\bar{y}^{(y=1)} = \begin{bmatrix} 1 & 1 \dots 1 & 0 & \dots 0 \\ \underbrace{\hspace{1cm}}_{N \text{ terms}} & \underbrace{\hspace{1cm}}_{M-N \text{ terms}} \end{bmatrix}^T$. Moreover,

from Eq. (17) and Eq. (18),

$$\begin{aligned} \sum_{j=1}^N \phi_j(s) &= \sum_{j=1}^N \sum_{i=1}^M N_{i,p}(s) \left[A^{-1} \bar{\phi}^{(j)} \right]_i \\ &= \sum_{i=1}^M N_{i,p}(s) \left[A^{-1} \sum_{j=1}^N \bar{\phi}^{(j)} \right]_i \end{aligned}$$

Note that $\sum_{j=1}^N \bar{\phi}^{(j)} = \begin{bmatrix} 1 & 1 \dots 1 & 0 & \dots 0 \\ \underbrace{\hspace{1cm}}_{N \text{ terms}} & \underbrace{\hspace{1cm}}_{M-N \text{ terms}} \end{bmatrix}^T = \bar{y}^{(y=1)}$. Therefore, from Eq.

(22) and (23)

$$\sum_{j=1}^N \phi_j(s) = \sum_{i=1}^M N_{i,p}(s) \left[A^{-1} \bar{y}^{(y=1)} \right]_i = \sum_{i=1}^M N_{i,p}(s) P_i^{(y=1)} = 1.$$

3. Consistency to p^{th} order:

$$\sum_{i=1}^N \phi_i(s) s_i^n = s^n, \quad 1 \leq n \leq p, \quad \forall s \in \Omega \quad (24)$$

Proof: This property can be shown similar to the proof for partition of unity. Consider the function $y(s) = s^n$ and the corresponding control points vector $P_i^{(sn)}$. Then, similar to the previous argument, from Eq. (12) and Eq. (13),

$$\bar{y}^{(sn)} = \left[\underbrace{s_1^n \ s_2^n \dots \ s_N^n}_{N \text{ terms}} \ \underbrace{0 \ \dots \ 0}_{M-N \text{ terms}} \right]. \text{ Further, it is noted that,}$$

$$\sum_{j=1}^N \phi_j(s) s_j^n = \sum_{i=1}^M N_{i,p(s)} \left[\mathbf{A}^{-1} \sum_{j=1}^N \bar{\phi}^{(j)} s_j^n \right]_i$$

and $\sum_{j=1}^N \bar{\phi}^{(j)} s_j^n = \left[\underbrace{s_1^n \ s_2^n \dots \ s_N^n}_{N \text{ terms}} \ \underbrace{0 \ \dots \ 0}_{M-N \text{ terms}} \right] = \bar{y}^{(sn)}$. Following the argument for partition of unity property,

$$\sum_{j=1}^N \phi_j(s) s_j^n = s^n.$$

4. $\mathbb{C}^{p-1}$ continuity:

$$\phi_i(s) \in \mathbb{C}^{p-1}, \quad i = 1, 2, \dots, N \quad (25)$$

Proof: This property follows from the $\mathbb{C}^{p-1}$ continuity of the B-spline basis functions.

5. Exponential decay :

$$|\phi_j(s)| \leq K r^{|s - s_j| - 1} \quad (26)$$

for a node span $s \in [s_k, s_{k+1}]$, where $0 < r < 1$ and K are positive constants and $c = \frac{1}{h}$, where h is the mesh parameter.

Proof: From Eq. (18), and noting that B-spline basis functions satisfy $N_{i,p} = 0, \forall i \notin [k, k+p]$ for a given span $s \in [s_k, s_{k+1}]$, we have

$$\begin{aligned} |\phi_j(s)| &= \left| \sum_{i=k}^{k+p} N_{i,p(s)} P_i^{(j)} \right| \\ &\leq \sum_{i=k}^{k+p} |N_{i,p(s)} P_i^{(j)}| = \sum_{i=k}^{k+p} |N_{i,p(s)}| |P_i^{(j)}| \end{aligned}$$

Using the boundedness of the B-spline basis functions [36], we have $0 \leq N_{i,p}(s)$

1. Noting that $P_i^{(j)} = [\mathbf{A}^{-1}]_j$ (Eq. (17)), and applying Proposition 2.1 in [37], we have,

$$|P_i^{(j)}| \leq K_p r^{|i-j|}, \quad i, j \in [1, M-p]$$

where K_p is a constant and $r \leq \|I - A\|_q^{\frac{1}{p}}$ for some $1 < q < \infty$, and $\|I - A\|_q < 1$. Thus,

$$|\phi_j(s)| \leq \sum_{i=k}^{k+p} K_p r^{|i-j|} \leq (p+1) K_p r^{|k-j|}$$

since $0 < r < 1$, and $r^{|k-j|} > r^{|k+1-j|} > \dots > r^{|k+p-j|}$. Moreover, since $h(|k-j|+1) \leq |s - s_j|$, implies $r^{|k-j|+1} \leq r^{c|s-s_j|}$ where $c = \frac{1}{h}$. Therefore,

$$|\phi_j(s)| \leq K_p r^{c|s-s_j|-1}$$

6. Exponential decay of derivatives:

$$|D^m \phi_j(s)| \leq K_m r^{c|s-s_j|-1} \quad (27)$$

for a node span $s \in [s_k, s_{k+1}]$ and $1 \leq m \leq p$, where $0 < r < 1$ and K_m are positive constants and $D^m f$ represents the m th derivative of a function f .

Proof: The exponential decay of the m th derivative of shape functions

$$D^m \phi_j(s) = \sum_{i=k}^{k+p} D^m N_{i,p}(s) P_i^{(j)}, \quad s \in [s_k, s_{k+1}]$$

for $1 \leq m \leq p$ can be shown with the same argument using the boundedness of the derivatives $|D^m N_{i,p}| \leq c_m$, $c_m \in \mathbb{R}$, $1 \leq m \leq p$ of B-spline basis functions [36].

The shape function of the mid node and its derivatives in a domain of 20 uniform intervals for $p = 3$ are shown in Figure 3. The exponential decay of the shape function and its derivatives are shown in Figure 4.

3.2. Constrained Spline Dynamics (CSD)

We discretize the Euler-Lagrange equation (Eq. (1)) using interpolatory B-spline shape functions. A scalar field $a(s, t)$ is discretized using the B-spline shape function as

$$a(s, t) = \sum_{j=1}^N \phi_j(s) a_j(t)$$

where $a_j(t)$ is the nodal values of the scalar field at node j . Similarly, a vector field $z(s, t)$ is discretized as

$$\mathbf{z}(s, t) = \sum_{j=1}^N \phi_j(s) \mathbf{z}_j(t)$$

where a vector at any given point ' j ' on the rod at s_j and time t is explicitly written as $\mathbf{z}(s_j, t) \equiv \mathbf{z}_j(t) = z_{j(\alpha)}(t) \mathbf{e}_{(\alpha)}$, where $z_{j(\alpha)}$ is the α^{th} component of the vector and $\mathbf{e}_{(\alpha)}$ is the unit vector in direction $\alpha \in \{1, 2, 3\}$. The subscripts in Latin letters denote the nodal location of the vector field and Greek letters within parenthesis denote the components of a vector. The derivative of a field $\mathbf{z}(s, t)$ with respect to the arc-length parameter is denoted as $\mathbf{z}'(s, t)$, and the temporal derivative as $\dot{\mathbf{z}}(s, t)$. Using the above notations, we can write the following discretized equations in terms of the nodal position vectors $\mathbf{x}_j(t)$:

$$\text{Position vector: } \mathbf{x}(s, t) = \sum_{j=1}^N \phi_j(s) \mathbf{x}_j(t)$$

$$\text{Tangent: } \mathbf{t}(s, t) = \mathbf{x}'(s, t) = \sum_{j=1}^N \phi_j'(s) \mathbf{x}_j(t)$$

$$\text{Curvature: } \mathbf{\kappa}(s, t) = \sum_{j=1}^N \phi_j''(s) \mathbf{x}_j(t)$$

$$\text{Velocity: } \mathbf{v}(s, t) = \dot{\mathbf{x}}(s, t) = \sum_{j=1}^N \phi_j(s) \dot{\mathbf{x}}_j(t)$$

$$\text{Acceleration: } \mathbf{a}(s, t) = \ddot{\mathbf{x}}(s, t) = \sum_{j=1}^N \phi_j(s) \ddot{\mathbf{x}}_j(t)$$

In this work, we employ the compliant constraints formulation [35] to derive the internal forces based on potentials. The potential due to bending is discretized as

$$V_B(\mathbf{x}) = \frac{1}{2} \mathbf{C}_b(\mathbf{x})^T \boldsymbol{\alpha}_b^{-1} \mathbf{C}_b(\mathbf{x}), \quad (28)$$

where $\mathbf{C}_b(\mathbf{x})$ is the vector of compliant constraints and $\boldsymbol{\alpha}_b$ is the diagonal matrix consisting of bending compliance factor for the corresponding constraint. To be able to express the bending energy (Eq. (3)) in compliant strain terms, the bending strain energy $U_B(s)$ of the rod per unit length is written as

$$U_B(s) = \frac{1}{2} \mathbf{\kappa}(s)^T \mathbf{k}_B \mathbf{\kappa}(s) \quad (29)$$

and the compliant constraint for a given interval (or element) is given by

$$\mathbf{C}_b^e(\mathbf{x}) = \sqrt{L_e} \mathbf{\kappa}(s) \quad (30)$$

where L_e is the length of the span. The internal force due to the potential is given by

$$\mathbf{f}^b = - \nabla V_B = - \nabla \mathbf{C}_b^T \boldsymbol{\alpha}_b^{-1} \mathbf{C}_b(\mathbf{x}). \quad (31)$$

Similarly, to express the twist energy (Eq. (4)) in compliant strain terms, the twist energy density can be written as $U_T(s) = \frac{1}{2} \theta'(s) \mu_T \theta'(s)$ and the corresponding compliant constraint is given by $\mathbf{C}_t^e(\mathbf{x}) = \sqrt{L_e} \theta'(s)$.

The stiff constraints on the rod are incorporated through constraint potentials in terms of the constraints as

$$V_C(\mathbf{x}) = \frac{1}{2} \mathbf{C}_c(\mathbf{x})^T \boldsymbol{\alpha}_c^{-1} \mathbf{C}_c(\mathbf{x}), \quad (32)$$

where $\mathbf{C}_c(\mathbf{x})$ is the vector of stiff constraints as defined in Section 3.3 and $\boldsymbol{\alpha}_c$ is the diagonal matrix of compliance factors corresponding to the constraints.

The total potential including the compliant and non-compliant constraints can be written as

$$V(\mathbf{x}) = \frac{1}{2} \mathbf{C}(\mathbf{x})^T \boldsymbol{\alpha}^{-1} \mathbf{C}(\mathbf{x}), \quad (33)$$

where $\mathbf{C}(\mathbf{x}) = [\mathbf{C}_b(\mathbf{x})^T \mathbf{C}_t(\mathbf{x})^T \mathbf{C}_c(\mathbf{x})^T]^T$ and $\boldsymbol{\alpha}$ is the corresponding diagonal compliance matrix.

Having defined the total potential of the system (Eq. 33), we proceed to integrate the discretized Euler-Lagrange equations given by

$$\mathbf{M}\ddot{\mathbf{x}} = -\nabla V^T(\mathbf{x}), \quad (34)$$

where $-\nabla V^T(\mathbf{x}) = -\nabla^T \mathbf{C}(\mathbf{x}) \boldsymbol{\alpha}^{-1} \mathbf{C}(\mathbf{x})$ is the internal force and, $\mathbf{M}$ is the mass matrix of the discretized system given by

$$M_{ij} = \rho_L \int_0^L \sum_{j=1}^N \phi_i(s) \phi_j(s) ds. \quad (35)$$

The central difference scheme is used to temporally evolve Eq. (34)

$$\mathbf{M} \left(\frac{\mathbf{x}^{n+1} - 2\mathbf{x}^n + \mathbf{x}^{n-1}}{h^2} \right) = -\nabla V^T(\mathbf{x}^{n+1}). \quad (36)$$

where h is the time step. Defining the Lagrange multipliers for the magnitude of the forces, following [31],

$$\boldsymbol{\Lambda} = -\tilde{\boldsymbol{\alpha}}^{-1} \mathbf{C}(\mathbf{x}), \quad (37)$$

where $\tilde{\boldsymbol{\alpha}} = \frac{1}{h^2} \boldsymbol{\alpha}$ and using Eq. (36), we have

$$\mathbf{M}(\mathbf{x}^{n+1} - 2\mathbf{x}^n + \mathbf{x}^{n-1}) - \nabla \mathbf{C}^T(\mathbf{x}^{n+1}) \boldsymbol{\Lambda}^{n+1} = 0 \quad (38)$$

$$\mathbf{C}(\mathbf{x}^{n+1}) + \tilde{\boldsymbol{\alpha}} \boldsymbol{\Lambda}^{n+1} = 0, \quad (39)$$

which can be solved through linearization and single point Newton iterations. Using the Schur complement, the equation for the i^{th} iteration is given by

$$\left[\nabla C(\mathbf{x}_{(i)}^{n+1}) \mathbf{W} \nabla C^T(\mathbf{x}_{(i)}^{n+1}) + \tilde{\alpha} \right] \Delta \Lambda = -C(\mathbf{x}_{(i)}^{n+1}) - \tilde{\alpha} \Lambda_{(i)}^{n+1}, \quad (40)$$

where $\mathbf{W} = \mathbf{M}^{-1}$. The positions are updated using

$$\Delta \mathbf{x} = \mathbf{W} \nabla C^T(\mathbf{x}_{(i)}^{n+1}) \Delta \Lambda. \quad (41)$$

Explicit time integration: The above implicit scheme requires an update of $\nabla C(\mathbf{x}_{(i)}^{n+1})$ in every Newton iteration, which is computationally expensive. We propose the following explicit time integration for finer time discretization suitable for real-time simulation. Substituting for $\mathbf{x}^{n+1}$ from Eq. (38) and solving for Λ^{n+1} in Eq. (39) and linearizing yields

$$\left[\nabla C(\mathbf{x}^n) \mathbf{W} \nabla C^T(\mathbf{x}^n) + \tilde{\alpha} \right] \Delta \Lambda = -C(\mathbf{x}_{(i)}^{n+1}) - \tilde{\alpha} \Lambda_{(i)}^{n+1}, \quad (42)$$

where the gradient $\nabla C(\mathbf{x}_{(i)}^n)$ is evaluated at the previous time step and is invariant in the Newton iteration. Note that in the above explicit scheme the constraints $C(\mathbf{x}_{(i)}^{n+1})$ are satisfied in the current time step. The gradients of compliant, positional and directional constraints are linear with respect to positions. The inextensibility constraints are nonlinear. With fine discretization in time, a single iterative step may be sufficient in evaluating the Λ^{n+1} that satisfies the constraint conditions. The displacements are updated using Eq. (41).

Twist and bend-twist: In this work, we assume that the twist response in the rod is instantaneous and is treated as quasistatic. This is particularly true for rods of thin cross-sections with vanishing rotational inertia [15]. Since the twist degree of freedom is eliminated in the formulation, the possible twist boundary conditions that can be applied are (i) free-to-twist ends, (ii) twist-arrested ends or (iii) applied relative twist between the ends.

To treat the bend-twist coupled phenomenon in our formulation that eliminates the twist degree of freedom, we utilize the twist-free Bishop frame field along with the uniform twist rate property of rods with isotropic cross-sections.

Consider a rod of isotropic cross-section, deformed from an initial configuration (*config1*) to a certain (non-planar) centerline configuration (*config2*) with some constraints (positional and/or directional) applied at the ends, but with the ends free to twist (see Figure 5). The rod behaves as an ‘elastic curve’ (three-dimensional elastica) with zero material twist to minimize the twist energy [14].

Before proceeding to material twist, we discuss an associated geometric property of space curves called torsion. Any (non-planar) space curve has an associated geometric torsion field, which is a measure of how sharply the curve departs from the plane of curvature at a given point and is given by

$$\tau(s) = -\hat{\mathbf{k}} \cdot \hat{\mathbf{b}}'$$

where $\hat{\mathbf{k}}$ and $\hat{\mathbf{b}}$ are the unit normal vector and unit bi-normal vector of the curve [29]. The geometric torsion integrated over the length of the rod is termed as *writhe* [38]. In the above example, the geometric torsion field and writhe in the initial (*config1*) and deformed (*config2*) configurations are different and depend on the corresponding centerline curves.

Now, consider a slightly modified scenario of the example. Here, the rod from the same initial configuration (*config1*) is deformed with the same set of end constraints as before, but with the ends arrested for a twist. The rod deforms to a centerline configuration (*config3*) different from *config2*. This is because in the modified case there is material twist on the rod, the magnitude of which is equal to the difference in the writhe between *config1* and *config2*. In other words, *config3* is a result of the application of end constraints and a material twist of difference in writhe between *config1* and *config2*.

We utilize the interaction between material twist and the geometric torsion of the centerline curve to derive the internal forces associated with the twist energy of the rod. Here, We consider three cases as twist boundary conditions of the rod; (i) twist-free ends (ii) arrested ends (iii) ends with an applied twist. The first case is trivial, where the twisting energy is not accounted for. In the case of arrested ends, the material twist is equal to the difference in writhe between the current configuration and the initial configuration. In the last case, the material twist is the sum of the applied twist and the difference in writhe. In the following, we derive the relations for the case of arrested ends, which will later be extended to applied twist.

The variation of writhe with respect to the centerline coordinates for a given variation in the tangent field $\delta \mathbf{t}$ is given by [39][38]

$$\delta W_r = \int_s \delta \mathbf{t} \cdot (\mathbf{t} \times \mathbf{t}') ds. \quad (43)$$

The Bishop frame field [13] is used to quantify writhe for a given configuration of the centerline of the rod. The material twist in the rod $\theta(s)$ signifies the rotation of material fibers with respect to the Bishop frame (see Figure 6). Using Eq. (43), the variation in material twist between any two points on the centerline of the rod is given by

$$\frac{\partial \theta_{b,a}}{\partial \mathbf{x}_j} = \int_{s_a}^{s_b} \frac{\partial \mathbf{t}}{\partial \mathbf{x}_j} \cdot (\mathbf{t} \times \mathbf{t}') ds, \quad (44)$$

where $\theta_{b,a}$ is the total twist between material points *a* and *b* on the centerline. Moreover, utilizing the fact that twist rate is homogenous [4] across the length of the rod of isotropic cross section, the internal force contribution from bend-twist coupling for a rod with an applied twist, assuming arrested end rotations, can be written as

$$\mathbf{f}_j^t = \frac{dV_T}{d\mathbf{x}_j} = \frac{\partial V_T}{\partial \theta_N} \frac{\partial \theta_N}{\partial \mathbf{x}_j}, \quad (45)$$

where $\theta_N \equiv \theta_{N,0}$ is the total twist applied between the ends of the rod. Using Eq. (4) in the above relation, we have

$$\mathbf{f}_j^t = \mu_T \frac{\theta_N}{L} \int_{s=0}^L \phi'_j(\mathbf{t} \times \mathbf{t}') ds, \quad (46)$$

where $\mathbf{t} \times \mathbf{t}'$ is the binormal vector field of the rod. In case of applied twist α , the internal force contribution will be evaluated by replacing $\theta_N + \alpha$ in place of θ_N . The integration of Eq. (46) is performed using Gauss quadrature where, the derivative of shape functions and the tangent field are quadratic functions while the curvature field is linear for the case of cubic basis functions. Similarly, the writhe is evaluated by integrating the torsion field, which is evaluated using the curvature and bi-normal vector fields. We utilize two point Gauss scheme per interval.

Discretization of the constraints: The constraints on the rod are discretized as follows:

Position constraints:

$$C_P(\mathbf{x}) = \sum_{i=1}^N \phi_i(s_p) \mathbf{x}_{i(\alpha)} - g_P = 0. \quad (47)$$

Rotational constraints:

$$C_S(\mathbf{x}) = \sum_{i=1}^N \phi'_i(s_p) \mathbf{x}_{i(\alpha)} - g_S = 0, \quad (48)$$

Inextensibility constraints:

$$C_I(\mathbf{x}) = \left| \sum_{i=1}^N \phi'_i(s_p) \mathbf{x}_i \right|^2 - 1 = 0; \quad s \in \Omega, \quad (49)$$

Note on arc-length parametrization: As the geometric curve of the centerline of the rod described by the discretized positions needs to satisfy arc-length parametrization, the magnitude of the tangent is enforced to be unity for the entire domain. Thus, enforcing the inextensibility constraint also ensures the validity of arc-length parametrization. Moreover, though the inextensibility constraint is numerically applied at specific points in the rod, the tangent field needs to satisfy unit-length property to be able to achieve uniform parametrization in the entire domain of the rod $s \in \Omega$. We further discuss the implications of this aspect in the results section.

Contact constraints:

$$C_c(\mathbf{x}) = g_c - \left| \sum_{i=1}^N \phi_i(s_p) \mathbf{x}_i - \mathbf{a} \right| = 0. \quad (50)$$

The above-discretized expressions for non-compliant constraints are applied in Eq. (32) to define the constraint potentials.

Evolving quasi-static and dynamic configurations: In applications such as suturing or cable-assembly simulations, rods are subjected to time-varying loads (and constraints) such as a combination of slowly evolving and abruptly changing loads. Though the system is conservative, work is continuously added (or removed) from the system through loads and constraints with time. The nature of deformation depends on the rate of application (or release) of these time-varying loads. When subjected to loads that evolve ‘slowly’, the deformation response can be modeled using quasi-static simulation. In general, if the rate of applied load is lower than one-third of the lowest natural frequency of the structure, the effects of inertia may be neglected [40]. In such a case, the instantaneous internal forces equilibrate the applied loads and do not contribute to the dynamic motion of the system. The quasi-static internal forces correspond to quasi-static internal energy V_Q of the system.

The net internal energy $V_N = V - V_Q$ contributes to the dynamic evolution of the system, where V and V_Q are the internal energy corresponding to the current and quasi-static configurations of the system, respectively. Considering the analogous case of a mass-spring system in gravity, the oscillation of the system is described about the quasi-static configuration that results from the equilibrium of the extended spring balancing the weight of the mass-spring system. Thus, the instantaneous quasi-static equilibrium configuration is treated as the reference state for evaluating the dynamic motion of the rod. The net internal energy is used in the discretized Euler-Lagrange expression to calculate the nodal accelerations. The instantaneous quasi-static state is evaluated through constrained minimization [4] of potential energy considering the loads and constraints are applied quasi-statically.

The proposed constrained spline dynamics algorithm for the simulation of elastic rods under the action of time-dependent forces and constraints is outlined in Algorithm 1.

Algorithm 1:

Constrained spline dynamics (CSD) of elastic rods

```

procedure constrainedSplineDynamics()
1:   input: initial geometry  $\mathbf{x}$ , material properties
2:   compute shape functions and derivatives           (Section 3.1)
3:   compute mass  $\mathbf{M}$  and stiffness                 (Section 3.2)
4:   for each time-step  $t = t_1$  to  $t_n$  do
5:     evaluate non-compliant constraints and gradients (Section 3.2)
```

```

6:    compute current quasi-static positions  $\mathbf{x}_Q$           (Section 3.2)
7:    compute net energy ( $V_N = V - V_Q$ )          (Section 3.2)
8:    if (dynamic)
9:        compute dynamic positions  $\mathbf{x}$  and energy  $V$       (Section 3.2)
10:    else
11:        set  $\mathbf{x} = \mathbf{x}_Q$ ,  $V = V_Q$ 
12:    end if
13: end for
14: display time-step geometry

```

4 Numerical examples

In this section, we evaluate the constrained spline dynamics method in simulating elastic rods under various constraints and loads. We test the approximation behavior of the interpolatory B-spline shape functions in section 4.1. Next, we verify the numerical solutions with analytical results from Euler elastica in bending in section 4.2. The twist and bend-twist coupling behavior is studied in section 4.3. The dynamic behavior of the rod is evaluated in section 4.4 and compared against analytical results. Simulations with rigid body contact are performed to study the accuracy and efficacy of the proposed model for interactive applications. A computational cost analysis of the method is discussed in section 4.6.

4.1. Function approximation with interpolatory B-spline shape functions

The interpolatory B-spline functions are used to approximate the Gaussian function $f(x) = a e^{-(x-b)^2/2c^2}$, $x \in [0,1]$ with $a = 1$, $b = \frac{1}{2}$, $c = \frac{1}{10}$, using $y(x) = \sum_{i=1}^N \phi_i(x) f_i$, where the domain is divided into $N-1$ equal intervals with N nodes and $\phi_i(x)$ is the interpolatory B-spline shape function at node ' i '. The first and second derivatives of the function are respectively approximated using $y'(x) = \sum_{i=1}^N \phi'_i(x) f_i$ and $y''(x) = \sum_{i=1}^N \phi''_i(x) f_i$. Figure 7 presents the convergence in the relative L_2 norm in approximating the Gaussian function and its first and second derivatives. The rates of convergence of the function and its derivatives are consistent in comparison to the convergence rates for B-spline interpolation [28], as shown in Figure 7.

4.2. Bending

The Euler elastica problem is solved to study the deformation behavior of rods in bending using the CSD method. An elastic rod is constrained such that one end $s = 0$ is fixed in position while the other end $s = L$ is displaced towards the fixed end by $L(1 - \frac{2}{\pi})$. Both the ends are directionally constrained to make right angles to the undeformed axial direction. The problem is modeled with a rod of 10 units discretized using N_D subdomains ($N_D + 1$ nodes) in 10 time steps. The deformed geometry of the rod is a semi-circle of radius $\frac{L}{\pi}$ (Figure 8), which is consistent with the analytical solution [27].

We utilize the above benchmark problem to compare the formulation with smooth methods available in the literature. In a recent work using B-splines based on Cosserat rod theory [16], control points and quaternions are used as DOF in a primal formulation; and additionally, internal forces and moments are used as DOF in a mixed formulation. The convergence of relative error in the L_2 norm of the deformed geometry for the primal and mixed formulations based on Cosserat rod theory are compared with the CSD method in Figure 9. The CSD method shows better performance with a lesser relative error in the displacement field and a higher rate of convergence when compared to the primal formulation. The results are comparable between the CSD method and the mixed formulation, though the CSD method uses only a fraction ($3/13$) of DOF. A distinguishing feature of the CSD method over the existing smooth methods based on Kirchhoff's theory [8][25] is that the uniform arc-length parametrization is ensured by enforcement of inextensibility constraints on the entire domain of the rod. With arc-length parametrization, the scalar and vector fields are parametrized using arc-length and the derivatives and integrals are evaluated with respect to the arc-length parameter. In order to compare the proposed formulation with existing Kirchhoff's theory based methods, the benchmark problem is solved without enforcing the uniform arc-length parametrization (Eq. (7)).

The norm of the tangent vector $\mathbf{t}(s)$ with and without the enforcement of the uniform arc-length parametrization is shown in Figure 10 when 20 subdomains are utilized. From Figure 10, it can be noticed that in the case without the enforcement, the norm of the tangent deviates from unit-length, which indicates the stretching or compression of the rod. Since this benchmark problem is a case of pure bending of the rod, the solution does not involve stretching or compression in the rod. Whereas, in the case with enforcement shown in Figure 10, unit-length of the tangent field is achieved in the entire domain consistent with the unit-speed requirement for a valid arc-length parametrization.

The convergence of relative error in the L_2 norm of the displacement field and unit tangent field with and without enforcement of uniform arc-length is compared in Figure 11. It is evident that without enforcing the arc-length parametrization, the solution does not converge, whereas, the proposed formulation ensures convergence of the displacement field by ensuring accurate arc-length parametrization.

Next, we evaluate the CSD method for two complex bending cases.

CASE 1: In this case, one end of the rod of length $L = 10$ units is fixed in position and directionally rotated so that the cross-section is perpendicular to the y-axis, while the other end is displaced towards the fixed end by $\frac{3}{4}L$ units with its cross-section perpendicular to the x-axis. The rod is modeled with 10 segments (11 nodes). The deformed geometry of the rod is shown in Figure 12.

CASE 2: In this case, one end of the rod is fixed in position and directionally rotated so that its cross-section is perpendicular to the y-axis, while the other end is displaced towards the fixed end and directionally rotated so that the corresponding cross-section is perpendicular to the x-axis. The rod is modeled with 10 segments (11 nodes). The deformed geometry of the rod under general three-dimensional end conditions resulting in out-of-plane bending is

shown in Figure 13. The snapshots of the deformed rod at different time steps are presented in Figure 14. The CSD method is able to capture the complex deformation of the rod for in-plane and out-of-plane bending with few discretization points. The convergence of the CSD method in both cases is shown in Figure 15 with the error in the norm of the tangent vector.

4.3. Twist and bend-twist coupling

We utilize the constant torsion property of the helix to validate the writhe (natural twist) of the model. The writhe is defined for a given configuration of the centerline as $\theta_N = \int_s \tau ds$, where τ is the torsion field. A straight rod is bent to form a half of a helix coil with the ends free to twist. The torsion field in a helix of radius r and pitch p is constant along the arc-length and is given by $\tau_h = \frac{c}{r^2 + c^2}$, where $c = p/2\pi$, and therefore, the writhe is given by $\theta_N = \tau_h L$.

An initially straight rod is deformed to form one-half of a coil of a helix with radius $r = 10$ units and pitch $p = 2$ units by specifying the positional and directional end conditions of the helix with ends free to twist. The deformed geometry of the rod model is shown in Figure 16. The convergence of relative error in L_2 norm of torsion is shown in Figure 17.

In the following, we consider a comparison of bend-twist interaction between writhe and applied twist by comparing three cases, viz. (i) twist-free, (ii) arrested twist and (iii) applied twist. A rod of length $L = 60$ units is constrained such that one end $s = 0$ is fixed in position while the other end $s = L$ is displaced towards the fixed end by $\frac{1}{4}L$. In the first step, both the ends are directionally constrained to make $\frac{\pi}{3}$ radians to the undeformed axial direction (see Figure 18). The torsion field until step 1 is zero as the rod is still in the planar configuration. In the second step, the end at $s = L$ is moved laterally in the positive z direction by 5 units with the twist-free ends boundary condition. The deformed geometry with number of nodes $N = 11$ is shown in Figure 18b. In case (ii), the above procedure is repeated but with arrested twist at the ends. The deformed geometry is shown in Figure 18c.

In case (iii), the first and second step is same as in case (i), with the ends free to twist. Further, as third step, an applied twist of magnitude equal to the writhe of case (i) is applied. The deformed geometry of this step is compared with the result of case (ii) as shown in Figure 18d. The L_2 error in displacement between the deformed geometries is 5.85×10^{-5} . This verifies the bend-twist behavior of the method based on the equivalence between writhe and the applied twist.

Next, a known end-twist is applied to the rod with a given centerline with zero initial writhe. The bend-twist forces due to the applied end-twist must deform the rod to a new configuration with the writhe same as the applied twist.

A rod of length $L = 100$ units and diameter $D = 1.75$ units is first deformed in-plane by arresting one end and moving the other end closer by $L/2$ units as shown in Figure 19. An end-twist of $\theta_N = 90^\circ$ in the anti-clockwise direction is applied about the x -axis in 30 steps.

The final deformed geometry is shown in Figure 19 and at different time steps during the loading is shown in Figure 20.

4.4. Oscillation of a cantilever beam

Next, we consider the free and forced vibration of a cantilever beam. An initially horizontal straight rod of 50 units length and diameter 1.75 units is fixed at one end with positional and directional constraints, as shown in Figure 21. The linear mass density of the beam is 0.012 units and Young's modulus is 3.5 units. The beam is initially displaced to its first mode shape with the tip displacement of 2.72 units at the free end along the negative y-direction and is subsequently released, allowing the beam to undergo free vibration. The simulation is performed for a duration of 25 seconds with a time step $\Delta t = 0.02s$. The evolution of the tip displacement of the beam using the CSD method is compared with the analytical results in Figure 22. The tip displacements are shown both for implicit and explicit time discretizations. The two formulations match closely as the nonlinear contribution of the Jacobian of constraints due to inextensibility does not vary between adjacent time steps.

The CSD method compares well with the analytical solution with a maximum relative L_2 error norm $\left(\frac{\|\mathbf{x}(s) - \mathbf{x}_e(s)\|_2}{\|\mathbf{x}_e(s)\|_2} \right)$ of the deformed geometry of $\sim 1.5e-3$. The comparison of potential and kinetic energies is shown in Figure 23. The convergence of error in total energy with refinement in time steps is shown in Figure 24.

The response of a cantilever beam under forced excitation is studied next. The length of the beam is chosen as 30 units with rest of the parameters chosen to be identical to the free vibration problem. An excitation force $F(t) = \hat{F} \sin(\Omega t)$ is applied at the free end of the beam, where $\hat{F} = 0.1$ units and the excitation frequency is chosen as $\Omega = 0.45\omega_1 = 0.6435Hz$, where ω_1 is the first natural frequency of the beam. The comparison of the tip displacement response using the CSD method against analytical result is shown in Figure 25. The CSD method compares well with the analytical solution with a maximum relative L_2 error norm $\left(\frac{\|\mathbf{x}(s) - \mathbf{x}_e(s)\|_2}{\|\mathbf{x}_e(s)\|_2} \right)$ of the deformed geometry of $\sim 6.e-3$ for the case of forced vibration.

4.5. Contact with rigid body

The efficacy of the CSD method in modeling contact is evaluated through simple contact tests with rigid shafts. First, we study the convergence behavior of contact with complete penetration of the shaft through the rod. A rigid shaft along the z-axis of radius 3 units is pressed against a rod of length 100 units and diameter 1.75 units, which is fixed at one end with the other end unconstrained (Figure 26). The shaft is displaced by 5 units in the negative y-direction in a single time step from the initial position with a clearance of 2 units, as shown in Figure 26.

The rate of convergence of the gap between the rod and shaft (gap function) is studied. The relative error in the gap is defined as $g = \frac{D_g - D}{D}$, where D_g is the distance between the centerlines of the shaft and the rod, while D is the sum of radii of the shaft and the rod. The convergence of relative error versus number of iterations in the gap function is shown in

Figure 27. Despite the large initial gap, the method converges to a relative error of order 10^{-3} in two iterations.

Next, we study the efficacy of the smooth formulation to study smooth sliding of the shaft across segments of the rod. A rod of diameter 3 units and length 100 units is fixed in position and direction at one end and a roller constraint is applied at the other end. A rigid shaft of radius 3 units is pressed against the rod by 8 units in the negative y-direction (Figure 28) and is subsequently slid along the rod by 20 units. The rod is modeled with 6 segments (7 nodes).

The final deformed geometry and the snap shots at different time steps during pressing and sliding of the rigid shaft against the rod are shown in Figure 29. Though the rod is modeled with only 7 nodes, the CSD method is able to capture the smoothness of the deformed rod under sliding contact. This is due to the fact that the deformed geometry of the rod is completely defined both at the nodes and internal segments accurately, despite a coarse discretization. This is in contrast to the discrete formulations where the geometry of the rod is defined by line segments connecting the nodes, and hence sliding contact will lead to a discrete response in the rod when the shaft slides from one segment to another (see Figure 30). Such sliding scenarios are common in a suturing simulation where the suture thread slides against the needle-holder tool. Moreover, segments of high curvature may lead to erroneous detection of contact or missed contact in discrete models depending on the sign of curvature.

4.6. Computational cost analysis

We analyze the cost of a simulation step using the structure of the system matrices for the CSD method. The structure of the dynamic system matrix embedding the sparsity structure is shown in Figure 31. Blank space (or blocks) represents zero entries (or sub-structures). The submatrix $\mathbf{M} \in \mathbb{R}^{3N} \times \mathbb{R}^{3N}$, is the mass matrix and is invariant during the simulation, where N is the number of nodes. $\mathbf{M}$ is a symmetric positive definite tridiagonal matrix evaluated using linear shape functions and can be resolved in $\mathcal{O}(3N)$ operations. The compliance matrices for the compliant and stiff constraints $\tilde{\alpha}_c$ and $\tilde{\alpha}_s$ are diagonal.

The Jacobian matrix corresponding to compliant constraints denoted as $\mathbf{J}_c \in \mathbb{R}^{N-1} \times \mathbb{R}^N$ remains invariant during the simulation. Since the compliance constraints are functions of curvature $\kappa(s)$ (Eq. (30)), the components of the Jacobian $[\mathbf{J}_c]_{ij}$ are the second derivative of shape function at node j evaluated at segment i . Thus, with the property of exponential decay of the shape function and its derivatives (Eq. (27)), the component $[\mathbf{J}_c]_{ij}$ diminishes exponentially with increasing $|i - j|$. Numerically, the entries below a chosen threshold $\left(\left| \frac{[\mathbf{J}_c]_{ij}}{[\mathbf{J}_c]_{ii}} \right| < 1 \cdot e^{-4} \right)$ shall be considered negligible resulting in a banded structure of $\mathbf{J}_c$ for dense mesh discretization (for instance $N > 30$). The half band-width of $\mathbf{J}_c$ for a tolerance of $1 \cdot e^{-4}$ is a 7.

The Jacobian matrix corresponding to the stiff constraints (N_s) denoted as $\mathbf{J}_s \in \mathbb{R}^{N_s} \times \mathbb{R}^{3N}$, arising from positional and directional specifications on the centerline, contact and inextensibility, varies during the simulation. For small systems (e.g. $N < 20$), the resolution

of the system is performed using Schur complement (Eq. (40)) given by $\nabla \mathbf{C}(\mathbf{x}_{(i)}^{n+1}) \mathbf{W} \nabla \mathbf{C}^T(\mathbf{x}_{(i)}^{n+1}) + \tilde{\boldsymbol{\alpha}}$, where $\nabla \mathbf{C}(\mathbf{x}_{(i)}^{n+1})$ is the block matrix containing the blocks $\mathbf{J}_c$ and $\mathbf{J}_s$ (Figure 31). The resulting reduced system is of size $3N + N_s$ and can be resolved in $\mathcal{O}((3N + N_s)^3)$ operations using an iterative solver such as the conjugate gradient method.

In systems of larger dimension, the hierarchical matrix (H-matrix) algorithm [41] is utilized, taking advantage of the structure of the system matrix. H-matrices utilize low rank factors of off-diagonal blocks and offers the advantage of factoring a $n \times n$ matrix with operational complexity $\mathcal{O}(nk^\alpha \log(n)^\beta)$, where k is the chosen parameter for accuracy of the low rank approximation. For instance, k is chosen to be the rank of the off-diagonal block for no loss in accuracy while a lower value may be chosen for computational speed [42]. In this work, full rank factorization is utilized.

We utilize a two-level hierarchy in structuring the dynamic system matrix (Figure 31) as H-matrix. In the first level, the block structure with combined diagonal ($\mathbf{M}$ and $\tilde{\boldsymbol{\alpha}}_c$) is treated as one diagonal block ($H_{1,11}$) and $\tilde{\boldsymbol{\alpha}}_s$ as the second diagonal block ($H_{1,22}$). The structure with $\mathbf{J}_s$ is the off-diagonal block ($H_{1,12}$). The $H_{1,11}$ block, which remains invariant during the simulation, is further divided into blocks with diagonal $H_{2,11} \equiv \mathbf{M}$ and $H_{2,22} \equiv \tilde{\boldsymbol{\alpha}}_c$. The inversion of the hierarchical block $H_{1,11}$ is performed before the start of the time step loop (Algorithm 1). The off-diagonal block $H_{1,12}$ is factorized in $\mathcal{O}(3N \times N_s)$ operations. The inverse of the dynamic system is obtained using H-block multiplication of $H_{1,12}$ and $H_{1,11}$ and inversion of the complement matrix of size $N_s \times N_s$ [42]. The comparison of the computational cost of the Schur complement and the H-matrix approaches on a desktop computer with an Intel Core i7-4930 K, 3.4 GHz CPU is shown in Figure 32.

5 Concluding remarks

In this paper, we present the constrained spline dynamics method as a novel constraints based smooth framework for elastodynamic simulation of elastic rods with only centerline coordinates as DOF. Interpolatory B-spline shape functions are derived that enable the use of real-world coordinates as DOF. Enforcing inextensibility constraints using consistent arc-length parametrization enables accurate simulation of complex deformed configurations of the rod with low computational cost. Bend-twist coupling is modeled in the formulation using the smooth curvature and bi-normal vector fields, thus ensuring accuracy. The elastodynamic formulation using compliant constraints enables stable simulation with large time steps. Using only the centerline coordinates as DOF results in a highly efficient formulation that supports interactive simulations.

Several benchmark tests are performed to study the performance of the CSD method in comparison with other existing methods. The proposed method demonstrates superior convergence in modeling the elastic rod against existing smooth methods while using fewer DOF. The benchmark problems based on the vibration of the cantilever beam demonstrate accuracy and stability of the method in handling quasistatic and dynamic loading conditions.

The contact examples show the suitability of the method for continuous collision and large sliding application often encountered in interactive simulations.

The CSD formulation is developed for the case of isotropic cross-sections, which can be extended to simulate general asymmetric cross-sections by accounting for the material twist as an additional DOF. The imposition of consistent arc-length parametrization restricts the model to treat inextensible rods. A general treatment is needed to extend the application of the CSD method to extensible rods. The rotational inertia due to twist is neglected, enabling the elimination of twist degrees of freedom. Consequently, the formulation can only handle free-to-twist ends, twist-arrested ends or an applied relative twist between the ends, as possible twist boundary conditions. Contact is modeled with constraints. A more general contact formulation satisfying the Kuhn-Tucker condition is needed for interactions with deformable objects. Moreover, the current implementation is limited to elastic materials. This can be extended to nonlinear materials, exhibiting plastic and rate-dependent behavior.

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Figure 1:

Arc-length parametrization of rod geometry defined by the position vector field $\mathbf{x}(s, t)$ of the centerline at time t . Inset: Section view of the normal plane at a location $s = s_i$ with Bishop vectors $(\mathbf{u}_i, \mathbf{v}_i)$ signifying the reference frame and material frame vectors $(\mathbf{m}_i, \mathbf{n}_i)$ of the rod.

Figure 2: Constraints on the rod: (a) positional constraint (b) directional constraint (c) contact constraint.

Figure 3:

B-spline shape functions and derivatives for degree $p = 3$ and $s \in [0, 10]$ using a uniform open knot vector sequence with 20 uniform intervals. (a) $\phi_j(s)$ (b) $\phi_j'(s)$ (c) $\phi_j''(s)$ and (d) $\phi_j'''(s)$ at mid node.

Figure 4:

Exponential decay of the shape function and its derivatives for degree $p = 3$ and $s \in [0,10]$. using a uniform open knot vector sequence with 50 uniform intervals.

Figure 5:

Bend-twist behavior of a rod: (a) initial centerline configuration, b) deformed configuration with one end shifted away from the plane and ends free to twist, and c) comparison of deformed configurations with twist-free and twist-arrested ends.

Figure 6:

Bend-twist behavior on the application of end-twist on the rod (a) initial configuration with ends free of twist b) deformed configuration (of centerline due to bend-twist forces) with applied end-twist.

Figure 7:

Convergence in relative L_2 error norm $\left(\frac{\|f - y\|_2}{\|f\|_2}\right)$ in approximation of $f(x) = a e^{-(x-b)^2/2c^2}$, $x \in [0,1]$ and derivatives using shape functions of degree $p = 3$.

Figure 8:
Deformed geometry of the rod for the Euler elastica benchmark problem.

Figure 9:

Convergence in relative L_2 error norm $\left(\frac{\|\mathbf{x}(s) - \mathbf{x}_e(s)\|_2}{\|\mathbf{x}_e(s)\|_2} \right)$ of the deformed geometry ($\mathbf{x}_d(s)$) for the Euler elastica problem. The exact geometry from analytical results is denoted as $\mathbf{x}_e(s)$.

Figure 10:

Effect of uniform arc-length parametrization. Comparison of the norm of the tangent $t(s)$ with and without enforcement of uniform parametrization.

Figure 11:

Effect of uniform arc-length parametrization. Comparison of convergence of relative error in the L_2 norm of displacement field and unit-tangent length $|\mathbf{t}(s)|^2 - 1$ with and without enforcement with increasing number of sub-domains.

Figure 12: In-plane deformed geometry of the rod under general end conditions modeled with 11 nodes.

Figure 13:
Out-of-plane deformed geometry of the rod under general end conditions modeled with 11 nodes.

Figure 14:

Different stages of the deformed rod under out-of-plane general end conditions modeled with 11 nodes in 10 time steps.

Figure 15:

Error in arc-length parametrization (unit-tangent) of the deformed rod under in-plane and out-of-plane general end conditions.

Figure 16:
Deformed geometry of the rod into a half segment of a helix coil

Figure 17:
Convergence of relative error in L_2 norm of torsion in a segment of a helix.

Figure 18:

Bend-twist behavior with different cases of twist constraints (a) deformed geometry with in-plane positional and directional constraints (common step) (b) with lateral displacement (twist-free) (c) with lateral displacement (arrested twist) (d) comparison of twist-free and applied twist.

Figure 19:
Deformed geometry of the rod under end-twist loading modeled with 15 nodes.

Figure 20:

Different stages of the deformed geometry of the rod under the application of the end-twist modeled with 15 nodes in 30 time steps.

Figure 21:
Cantilever beam with dynamic loading.

Figure 22:
Comparison of tip displacement of the cantilever beam under free vibration with $t = 0.02s$.

Figure 23:
Comparison of potential, kinetic and total energy in the free vibration of a cantilever beam.

Figure 24:
Convergence of error in total energy in the free vibration of a cantilever beam with refinement in time-steps.

Figure 25:

Comparison of tip displacement response and the relative error of a cantilever beam subjected to forced excitation $F(t) = \widehat{F} \sin(\Omega t)$ (a) Tip displacement (b) Relative L_2 error

$\text{norm} \left(\frac{\|\mathbf{x}(s) - \mathbf{x}_e(s)\|_2}{\|\mathbf{x}_e(s)\|_2} \right)$ of the deformed geometry.

Figure 26:
Contact of the rod with a rigid shaft.

Figure 27:

Convergence of relative error in the gap function $\left(g = \frac{D_g - D}{D}\right)$ in contact of the rod with a rigid rod.

Figure 28:
Cantilever beam model for studying dynamic behavior modeled with 9 nodes.

Figure 29:

Different stages of the deformed geometry of the rod modeled with 7 nodes under sliding contact with a rigid shaft in 100 time steps.

Figure 30:

Comparison of sliding of a rigid shaft on the rod modeled using a piece-wise linear discretization and the CSD method. Modeling of sliding with smooth discretization allows for a smooth response while the discrete case results in a discontinuous response.

Figure 31:

Block structure of the dynamic system matrix embedded in the sparsity structure for $N=31$.

Figure 32: Computational cost in solution of dynamic system using Schur complement and H-matrix approaches.