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Author for correspondence:

Zhenya Yan

e-mail: zyyan@mmrc.iss.ac.cn

Nonlinear self-dual network equations: modulation instability, interactions of higher-order discrete vector rational solitons and dynamical behaviours

Xiao-Yong Wen¹, Zhenya Yan^{2,3} and Guoqiang Zhang⁴

¹Department of Mathematics, School of Applied Science, Beijing Information Science and Technology University, Beijing 100192, People's Republic of China

²KLMM, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, People's Republic of China

³School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, People's Republic of China

⁴Department of Mathematical Sciences, Tsinghua University, Beijing 100084, People's Republic of China

ZY, 0000-0001-6701-3664

The nonlinear self-dual network equations that describe the propagations of electrical signals in nonlinear LC self-dual circuits are explored. We firstly analyse the modulation instability of the constant amplitude waves. Secondly, a novel generalized perturbation $(M, N - M)$ -fold Darboux transform (DT) is proposed for the lattice system by means of the Taylor expansion and a parameter limit procedure. Thirdly, the obtained perturbation $(1, N - 1)$ -fold DT is used to find its new higher-order rational solitons (RSs) in terms of determinants. These higher-order RSs differ from those known results in terms of hyperbolic functions. The abundant wave structures of the first-, second-, third- and fourth-order RSs are exhibited in detail. Their dynamical behaviours and stabilities are numerically simulated. These results may be useful for understanding the wave propagations of electrical signals.

1. Introduction

Recently, rogue waves (RWs), a new type of rational soliton (RS) solutions of nonlinear wave equations, have been paid more and more attention due to depicting a unique event appearing from nowhere and disappearing without a trace [1,2]. The usual RWs are located in both space and time. The most fundamental physical model admitting exact rational RWs is the nonlinear Schrödinger (NLS) equation (see [1–5]). It should be pointed out that the RSs vary more slowly than the usual solitons such that the RSs may be more easily stable. Most of the previous studies focused on the continuous nonlinear wave models (see [1–24] and references therein), except for a few works on discrete RWs of some nonlinear lattice equations (NLEs) (see [25–30] and references therein). Moreover, RWs appearing as the collisions of solitons were observed in nonlinear optics [31,32] and in the discrete DNA models [33], and were demonstrated in the discrete non-integrable NLS models [34]. It is still an important subject to explore discrete RS solutions, even RWs, of more NLEs arising from many fields of nonlinear science.

The modulation instability (MI, alias the Benjamin–Feir instability) was first discovered for periodic surface gravity waves on deep water by Benjamin & Feir in 1967 [35]. After that, the MI was studied in both continuum and discrete nonlinear wave models (see [36–38] and references therein). It should be pointed out that the MI conditions were presented for the discrete NLS equation with both constant coefficients [39,40] and temporally periodic coefficients [41]. Moreover, the MI was applied to the light coupling between different transmission bands in optical fibres [42], and how to control the stable supercontinuum light sources [43]. Recently, the MI was embraced as a novel sensing mechanism where the shift in the gain band was tracked versus the change in a given quantity that one wants to measure, and shown to allow highly sensitive biosensing [44], refractive index sensing [45], and even strain sensing [46].

In this paper, we would like to study the nonlinear self-dual network lattice equations [47–50]:

$$\frac{dI_n}{dt} = (\sigma + I_n^2)(V_{n-1} - V_n), \quad \frac{dV_n}{dt} = (\sigma + V_n^2)(I_n - I_{n+1}), \quad (1.1)$$

where $I_n = I(n, t)$ and $V_n = V(n, t)$ are the current and voltage in the n -th capacitance and inductance of the network, respectively [47], which was constructed from commercially available circuit elements with inductance $L = 22 \mu\text{H}$ and capacitance $C(V) = 27V^{-0.48} \text{ pF}$ [51], and they are the functions of the discrete variable n and time variable t . Equation (1.1) with $\sigma = 1$ describes the propagation of electrical signals in a cascade of four-terminal nonlinear LC self-dual circuits [47,51], where the inductance and capacitance have the nonlinearities $L(I_n) = I_n^{-1} \arctan(I_n)$ and $C(V_n) = V_n^{-1} \arctan(V_n)$. Equation (1.1) also describes the wave propagation in a nonlinear, lumped and self-dual ladder-type network [48,50]. Equation (1.1) is an integrable NLE and the compatibility condition, $\varphi_{n,t} = \varphi_{t,n}$, that is, the zero curvature equation $U_{n,t} = (\Delta W_n)U_n - U_n W_n$, of the auxiliary linear problem (Lax pair) [48,49]:

$$\left. \begin{aligned} \Delta \varphi_n &= U_n \varphi_n, \quad U_n = U_n(t; \lambda) = U(n, t; \lambda) = \begin{pmatrix} \lambda + I_n V_n & -\sigma I_n + \lambda^{-1} V_n \\ I_n - \sigma \lambda V_n & \lambda^{-1} + I_n V_n \end{pmatrix}, \\ \frac{d\varphi_n}{dt} &= W_n \varphi_n, \quad W_n = W_n(t; \lambda) = W(n, t; \lambda) = \begin{pmatrix} \lambda - I_n V_{n-1} & -\sigma I_n + \lambda^{-1} V_{n-1} \\ I_n - \sigma \lambda V_{n-1} & \lambda^{-1} - I_n V_{n-1} \end{pmatrix}, \end{aligned} \right\} \quad (1.2)$$

where $\varphi_n = \varphi_n(t; \lambda) = (\phi(n, t; \lambda), \psi(n, t; \lambda))^T$ is the vector eigenfunction (T denotes the transpose), $\lambda \in \mathbb{C} \setminus \{0\}$ is the spectral parameter, the shift operator Δ is defined by $\Delta g_n = \Delta g(n, t) = g(n+1, t) \equiv g_{n+1}$, $n \in \mathbb{Z}$, $t \in \mathbb{R}$.

The single soliton of equation (1.1) has been found by the inverse scattering transform (IST) [48,49]. After that, the N -soliton solutions, breathers and other solutions in Casoratian form for equation (1.1) with $\sigma = 1$ were obtained [47,52–54]. The infinitely many conservation laws of equation (1.1) with $\sigma = 1$ were given [55,56]. The IST was used to find some shock wave and hole-type soliton solutions with non-zero asymptotic values of equation (1.1) with $\sigma = -1$ [57].

Equation (1.1) with $\sigma = 1$ and saturable nonlinearity was also discussed by the IST to find four types of solitons with non-zero asymptotic values [58]. The Darboux transform (DT) of the discrete Ablowitz–Ladik spectral problem could reduce to one of equation (1.1) [59]. A system of difference-differential equations related to equation (1.1) has been proposed, and the N -soliton solutions of equation (1.1) have been derived under the non-vanishing boundary conditions [60]. Other types of solitary wave and periodic solutions of equation (1.1) were obtained [61–64]. Recently, we constructed the N -soliton solutions for equation (1.1) with $\sigma = 1$ via the N -fold DT [65].

To the best of our knowledge, the generalized perturbation $(M, N - M)$ -fold DT and higher-order vector RS solutions for equation (1.1) were not reported before. In this paper, we would like to study these problems. The rest of this paper is organized as follows. In §2, we investigate the MI of equation (1.1), starting from its non-zero seed solutions. In §3, based on the Lax pair of equation (1.1), we present a novel idea, allowing one to derive *discrete versions* of the generalized $(M, N - M)$ -fold DT (using $M < N$ spectral parameters) for equation (1.1). In §4, the generalized $(1, N - 1)$ -fold DT is used to find discrete higher-order vector RS solutions of equation (1.1). We also analyse RS structures and their dynamical behaviours. In §5, we present a few wave features of discrete RS solutions for equation (1.1). Finally, some conclusions and discussions are given in §6.

2. Modulation instability of non-zero seed states

To study the MI of equation (1.1), we start from the initial states of equation (1.1), $I^{(0)}(n, t) = \alpha$, $V^{(0)}(n, t) = \beta$ with α, β being real amplitudes, and consider the perturbed initial states as $I(n, t) = \alpha + \delta P_n$, $V(n, t) = \beta + \delta Q_n$ with $P_n = P(n, t)$ and $Q_n = Q(n, t)$ being the real perturbation functions of n, t , and δ an infinitesimal parameter. As a result, the real-valued linearized perturbation system is

$$\frac{dP_n}{dt} - (\sigma + \alpha^2)(Q_{n-1} - Q_n) = 0, \quad \frac{dQ_n}{dt} - (\sigma + \beta^2)(P_n - P_{n+1}) = 0. \quad (2.1)$$

Similarly to refs [29,30], we assume that equation (2.1) admits the formal solutions $P_n = F e^{g^2 t + i \omega n}$, $Q_n = G e^{g^2 t + i \omega n}$, where F, G are real-valued constant amplitudes, g denotes the gain of MI and ω the real-valued wavenumber. The assumed solutions can make equation (2.1) yield the dispersion relation for the perturbations (i.e. the existence condition of non-trivial solutions F and G): $g^2 = 4(\sigma + \alpha^2)(\sigma + \beta^2) \sin^2(\omega/2)$, from which we have the following conclusions: (i) For $\sigma > 0$, if $\omega \neq 2m\pi, m \in \mathbb{Z}$, then $g^2 > 0$, which implies that the MI occurs. Furthermore, at the point $\omega = (2m + 1)\pi, m \in \mathbb{Z}$ the function g attains its maximum $g_{\max} = 2\sqrt{(\sigma + \alpha^2)(\sigma + \beta^2)}$, so that the constant-seed states are subject to MI and rational solitons can be solutions of equation (1.1); (ii) for $\sigma < 0$, if $\omega \neq 2m\pi, m \in \mathbb{Z}$ and $\sigma < -\max\{\alpha^2, \beta^2\}$ or $0 > \sigma > -\min\{\alpha^2, \beta^2\}$, then $g^2 > 0$, which implies that the modulation instability occurs; and (iii) for the other cases, we have $g^2 \leq 0$, which implies that the MI does not occur.

3. The Darboux matrix and generalized discrete perturbation DT

In what follows we would like to present a new generalized perturbation DT of equation (1.1) with $\sigma = 1$ to find its new multi-RS solutions. To this aim, a gauge transform $\widehat{\varphi}_n = \mathcal{G}_n \varphi_n$ is introduced, and the new vector eigenfunction $\widehat{\varphi}_n = \widehat{\varphi}_n(t; \lambda) = (\widehat{\phi}(n, t; \lambda), \widehat{\psi}(n, t; \lambda))^T$ must satisfy the similar Lax pair $\widehat{\varphi}_{n+1} = \widehat{U}_n \widehat{\varphi}_n$, $d\widehat{\varphi}_n/dt = \widehat{W}_n \widehat{\varphi}_n$, where $\widehat{U}_n = \widehat{U}_n(t; \lambda) = \widehat{U}(n, t; \lambda) = U_n|_{I_n \rightarrow \widehat{I}_n = \widehat{I}(n, t), V_n \rightarrow \widehat{V}_n = \widehat{V}(n, t)}$ and $\widehat{W}_n = \widehat{W}_n(t; \lambda) = \widehat{W}(n, t; \lambda) = W_n|_{I_n \rightarrow \widehat{I}_n = \widehat{I}(n, t), V_n \rightarrow \widehat{V}_n = \widehat{V}(n, t)}$. Therefore, the gauge transform matrix (alias Darboux matrix) $\mathcal{G}_n = \mathcal{G}_n(t; \lambda) = \mathcal{G}(n, t; \lambda)$, $\widehat{U}_n$, $\widehat{W}_n$, U_n , and W_n have the relations $\widehat{U}_n \mathcal{G}_n = \mathcal{G}_{n+1} U_n$, $\widehat{W}_n \mathcal{G}_n = d\mathcal{G}_n/dt + \mathcal{G}_n W_n$.

According to ref. [65], the Darboux matrix $\mathcal{G}_n(t; \lambda)$ can be taken as the polynomial matrix of λ

$$\mathcal{G}_n(t; \lambda) = \begin{pmatrix} \mathcal{A}_n(t; \lambda) & \mathcal{B}_n(t; \lambda) \\ \mathcal{C}_n(t; \lambda) & \mathcal{D}_n(t; \lambda) \end{pmatrix}, \quad (3.1)$$

where $\mathcal{A}_n = \lambda^N + \sum_{j=0}^{N-1} A_n^{(j)}(t) \lambda^j$, $\mathcal{B}_n = \sum_{j=0}^{N-1} B_n^{(j)}(t) \lambda^j$, $\mathcal{C}_n = -\lambda^N \sum_{j=0}^{N-1} B_n^{(j)}(t) \lambda^{-j}$ and $\mathcal{D}_n = \lambda^N \sum_{j=0}^{N-1} A_n^{(j)}(t) \lambda^{-j} + 1$ with $N \in \mathbb{N}^+$ and $A_n^{(j)}(t) = A^{(j)}(n, t)$'s and $B_n^{(j)}(t) = B^{(j)}(n, t)$'s are the unknown functions of n and t .

To determine $A_n^{(j)}(t)$'s and $B_n^{(j)}(t)$'s we here consider the less M ($1 \leq M < N$) spectral parameters λ_s ($s = 1, 2, \dots, M$) and choose the perturbation of λ_s

$$\lambda_s \rightarrow \lambda_s + \varepsilon, \quad \varepsilon > 0 \quad (3.2)$$

to consider the Taylor series of $\mathcal{G}_n(t; \lambda_s + \varepsilon) \varphi(n, t; \lambda_s + \varepsilon)$ at $\varepsilon = 0$ such that $\lim_{\varepsilon \rightarrow 0} \mathcal{G}_n(t; \lambda_s + \varepsilon) \varphi(n, t; \lambda_s + \varepsilon) / \varepsilon^{k_s} = 0$ ($s = 1, 2, \dots, M$; $k_s = 0, 1, \dots, \ell_s, N = M + \sum_{s=1}^M \ell_s$) can generate the set of $2N$ linear algebraic equations with $2N$ unknowns $A_n^{(j)}(t)$ and $B_n^{(j)}(t)$ ($j = 0, 1, \dots, N-1$):

$$\left\{ \begin{aligned} & \phi(n, t; \lambda_s) \sum_{j=0}^{N-1} A_n^{(j)}(t) \lambda_s^j + \psi(n, t; \lambda_s) \sum_{j=0}^{N-1} B_n^{(j)}(t) \lambda_s^j = -\phi(n, t; \lambda_s) \lambda_s^N, \\ & \psi(n, t; \lambda_s) \sum_{j=0}^{N-1} A_n^{(j)}(t) \lambda_s^{N-j} - \phi(n, t; \lambda_s) \sum_{j=0}^{N-1} B_n^{(j)}(t) \lambda_s^{N-j} = -\psi(n, t; \lambda_s), \\ & \phi^{(1)}(n, t; \lambda_s) \left[\lambda_s^N + \sum_{j=0}^{N-1} A_n^{(j)}(t) \lambda_s^j \right] + \psi^{(1)}(n, t; \lambda_s) \sum_{j=0}^{N-1} B_n^{(j)}(t) \lambda_s^j \\ & \quad + \phi(n, t; \lambda_s) \left[N \lambda_s^{N-1} + \sum_{j=1}^{N-1} j A_n^{(j)}(t) \lambda_s^{j-1} \right] + \psi^{(1)}(n, t; \lambda_s) \sum_{j=1}^{N-1} j B_n^{(j)}(t) \lambda_s^{j-1} = 0, \\ & \psi^{(1)}(n, t; \lambda_s) \left[\sum_{j=0}^{N-1} A_n^{(j)}(t) \lambda_s^{N-j} + 1 \right] - \phi^{(1)}(n, t; \lambda_s) \sum_{j=0}^{N-1} B_n^{(j)}(t) \lambda_s^{N-j} \\ & \quad + \sum_{j=0}^{N-1} (N-j) \left[\psi(n, t; \lambda_s) A_n^{(j)}(t) - \phi(n, t; \lambda_s) B_n^{(j)}(t) \right] \lambda_s^{N-j-1} = 0, \\ & \dots, \\ & \sum_{k=0}^{\ell_s} \left\{ \binom{N}{k} \phi^{(\ell_s-k)}(n, t; \lambda_s) \lambda_s^{N-k} + \sum_{j=k}^{N-1} \binom{j}{k} [\phi^{(\ell_s-k)}(n, t; \lambda_s) A_n^{(j)}(t) + \psi^{(\ell_s-k)}(n, t; \lambda_s) B_n^{(j)}(t)] \lambda_s^{j-k} \right\} = 0, \\ & \psi^{(\ell_s)}(n, t; \lambda_s) + \sum_{j=0}^{N-1} [\psi^{(\ell_s)}(n, t; \lambda_s) A_n^{(j)}(t) - \phi^{(\ell_s)}(n, t; \lambda_s) B_n^{(j)}(t)] \lambda_s^{N-j} \\ & \quad + \sum_{k=0}^{\ell_s} \sum_{j=0}^{N-k} \binom{N-j}{k} [\psi^{(\ell_s-k)}(n, t; \lambda_s) A_n^{(j)}(t) - \phi^{(\ell_s-k)}(n, t; \lambda_s) B_n^{(j)}(t)] \lambda_s^{N-j-k} = 0, \end{aligned} \right. \quad (3.3)$$

where $\phi^{(j)}(n, t; \lambda_s) = (1/j!) (\partial^j / \partial \varepsilon^j) \phi(n, t; \lambda_s + \varepsilon)$, $\psi^{(j)}(n, t; \lambda_s) = (1/j!) (\partial^j / \partial \varepsilon^j) \psi(n, t; \lambda_s + \varepsilon)$.

Remark 3.1. The new Darboux matrix $\mathcal{G}_n(t; \lambda_s)$ for each λ_s can be uniquely determined by system (3.3) if the special parameters λ_s ($s = 1, 2, \dots, M$) are properly taken such that the determinant of the coefficient matrix for system (3.3) is not equal to zero. We here take the simple perturbation $\lambda_s \rightarrow \lambda_s + \epsilon$ in the expression $\mathcal{G}_n(t; \lambda_s) \varphi(n, t; \lambda_s)$. In fact, one can also choose other perturbations for λ_s such as $\lambda_s \rightarrow f_s(\lambda_s, \epsilon)$ with $\lambda_s = f_s(\lambda_s, 0)$. For the distinct $f_s(\lambda_s, \epsilon)$, the corresponding $\mathcal{G}_n^{(k)}(t; \lambda_s)$ and $\varphi^{(k)}(n, t; \lambda_s)$ are different such that the generating systems (e.g. equation (3.3)) are different, which will lead to distinct functions $A_n^{(j)}(t)$, $B_n^{(j)}(t)$ and DTs.

Theorem 3.2. Suppose that $\varphi(n, t; \lambda_s) = (\phi(n, t; \lambda_s), \psi(n, t; \lambda_s))^T$'s are solutions of the Lax pair (1.2) with spectral parameters λ_s 's ($s = 1, 2, \dots, M$) and initial solutions $(I^{(0)}(n, t), V^{(0)}(n, t))$ of system (1.1)

with $\sigma = 1$, then the generalized perturbation $(M, N - M)$ -fold DT

$$\begin{aligned}\hat{I}_n^{(N)}(t) &= \hat{I}^{(N)}(n, t) = \frac{I^{(0)}(n, t)D_N^{(M)} + \sigma \hat{B}_n^{(N-1)}(t)}{\hat{A}_n^{(0)}(t)}, \\ \hat{V}_n^{(N)}(t) &= \hat{V}^{(N)}(n, t) = \frac{\sigma \hat{B}_{n+1}^{(0)}(t) + V^{(0)}(n, t)\hat{A}_{n+1}^{(0)}(t)}{D_N^{(M)}},\end{aligned}\quad (3.4)$$

can obtain the new solutions of system (1.1), where $D_N^{(M)} = \det((\mathcal{D}^{(1)} \dots \mathcal{D}^{(M)})^T)$, $\mathcal{D}^{(s)} = (\mathcal{D}_{j,k}^{(s)})_{2(\ell_s+1) \times 2N}$ with $\mathcal{D}_{j,k}^{(s)}$ ($1 \leq j \leq 2(\ell_s + 1)$, $1 \leq k \leq 2N$, $s = 1, 2, \dots, M$; $N = M + \sum_{s=1}^M \ell_s$) being determined by

$$\mathcal{D}_{j,k}^{(s)} = \begin{cases} \sum_{l=0}^{j-1} \binom{N-k}{l} \lambda_s^{(N-k-l)} \phi^{(j-1-l)}(n, t; \lambda_s), & 1 \leq j \leq \ell_s + 1, 1 \leq k \leq N, \\ \sum_{l=0}^{j-1} \binom{2N-k}{l} \lambda_s^{(2N-k-l)} \psi^{(j-1-l)}(n, t; \lambda_s), & 1 \leq j \leq \ell_s + 1, N+1 \leq k \leq 2N, \\ \sum_{l=0}^{j-(N+1)} \binom{k}{l} \lambda_s^{(k-l)} \psi^{(j-N-1-l)}(n, t; \lambda_s), & \ell_s + 2 \leq j \leq 2(\ell_s + 1), 1 \leq k \leq N, \\ - \sum_{l=0}^{j-(N+1)} \binom{k-N}{l} \lambda_s^{(k-N-l)} \phi^{(j-N-1-l)}(n, t; \lambda_s), & \ell_s + 2 \leq j \leq 2(\ell_s + 1), N+1 \leq k \leq 2N. \end{cases} \quad (3.5)$$

$\hat{A}_n^{(0)}(t)$, $\hat{B}_n^{(0)}(t)$ or $\hat{B}_n^{(N-1)}(t)$ can be defined by the modified $D_N^{(M)}$ with its N -th, $(2N)$ -th, or $(N+1)$ -th columns replaced by the column vector $\gamma = (\gamma_j)_{2N \times 1}$ where $\gamma_j = -\sum_{l=0}^{j-1} \binom{N}{l} \lambda_s^{(N-l)} \phi^{(j-1-l)}(n, t; \lambda_s)$ for $1 \leq j \leq \ell_s + 1$ and $\gamma_j = -\psi^{(j-N-1)}(n, t; \lambda_s)$ for $\ell_s + 2 \leq j \leq 2(\ell_s + 1)$, respectively.

Proof. Similarly to refs [29,30], based on the usual N -fold DT [65] and the N -order Darboux matrix $\mathcal{G}_n(t; \lambda_s)$ with the obtained new functions $A_n^{(j)}(t)$ and $B_n^{(j)}(t)$ from system (3.3), one can complete the proof. ■

Remark 3.3. In particular, for $M = N$ ($\ell_s = 0$, $1 \leq s \leq N$), that is, one uses the N distinct spectral parameters, theorem 3.2 becomes the usual N -fold DT with $\sigma = 1$, which can be used to seek the multi-soliton solutions of equation (1.1) [65]. For the other cases, we can find the new generalized perturbation DT in theorem 3.2 to derive the new discrete RS solutions of equation (1.1).

4. Higher-order discrete RSs and their dynamics

In the following, we would like to investigate some higher-order discrete RS solutions of equation (1.1) with $\sigma = 1$ using the generalized perturbation $(1, N-1)$ -fold DT in theorem 3.2. For the other cases $1 < M < N$, we have the other new generalized perturbation DTs, which are not discussed here. We start from the non-zero initial solutions $I^{(0)}(n, t) = \alpha \neq 0$, $V^{(0)}(n, t) = 0$ with $\alpha \in \mathbb{R}$ of equation (1.1) with $\sigma = 1$, and then substitute them into the Lax pair (1.2) to generate two sets of fundamental solutions:

$$\begin{aligned}\phi &= \begin{pmatrix} \phi_1(n, t; \lambda, \varepsilon) \\ \phi_2(n, t; \lambda, \varepsilon) \end{pmatrix} = \begin{pmatrix} \rho_1^n e^{\rho_1 t - \vartheta(\varepsilon)} \\ \alpha^{-1}(\lambda - \rho_1) \rho_1^n e^{\rho_1 t - \vartheta(\varepsilon)} \end{pmatrix}, \\ \psi &= \begin{pmatrix} \psi_1(n, t; \lambda, \varepsilon) \\ \psi_2(n, t; \lambda, \varepsilon) \end{pmatrix} = \begin{pmatrix} \rho_2^n e^{\rho_2 t + \vartheta(\varepsilon)} \\ \alpha^{-1}(\lambda - \rho_2) \rho_2^n e^{\rho_2 t + \vartheta(\varepsilon)} \end{pmatrix},\end{aligned}\quad (4.1)$$

with $\rho_1 = (\lambda^2 + 1 + \sqrt{\Lambda}) / (2\lambda)$, $\rho_2 = (\lambda^2 + 1 - \sqrt{\Lambda}) / (2\lambda)$, $\Lambda = (\lambda^2 - 1)^2 - 4\alpha^2 \lambda^2$, $\vartheta(\varepsilon) = \sqrt{\Lambda} \sum_{l=1}^N (c_l + i d_l) \varepsilon^l$, where c_l, d_l ($l = 1, 2, \dots, N$) are free real coefficients, and ε is an artificially

introduced small parameter, whose aim is to introduce more parameters c_k, d_k ($k = 1, 2, \dots, N$) in the Taylor series following vector eigenfunction $\phi(n, t; \lambda, \varepsilon)$ such that they can generate abundant wave structures of equation (1.1).

In fact, ρ_1, ρ_2 are two characteristic roots of the matrices U_n and W_n in the Lax pair (1.2) with the non-zero seed solutions $I^{(0)}(n, t) = \alpha \neq 0, V^{(0)}(n, t) = 0$. To find the RS solutions of equation (1.1) with $\sigma = 1$, we need to consider the case of double root, $\rho_1 = \rho_2$, which leads to $(\lambda^2 - 1)^2 - 4\alpha^2\lambda^2 = 0$ such that we have $\lambda = \mu_1\sqrt{\alpha^2 + 1} + \mu_2\alpha, \mu_j = \pm 1, j = 1, 2$. According to the generalized perturbation $(M, N - M)$ -fold DT in theorem 3.2, we here choose $M = 1, \lambda_1 = \mu_1\sqrt{\alpha^2 + 1} + \mu_2\alpha$, such that we assume that λ is of the perturbation form

$$\lambda = \lambda_1 + \varepsilon = \mu_1\sqrt{\alpha^2 + 1} + \mu_2\alpha + \varepsilon \quad (4.2)$$

in equation (4.1), which is consistent with one used in theorem 3.2, where the non-zero constant α is a free parameter characterizing the seed current in the electric circuits under consideration in equation (1.1).

Remark 4.1. The chosen perturbation ansatz (4.2) must agree with the ansatz (3.2) in order to use theorem 3.2 to find the RS solutions of equation (1.1) with $\sigma = 1$. Of course, the perturbation λ given by equation (4.2) can also be taken as other forms (see appendix A). It should be pointed out that since the chosen perturbation λ given by equation (A 2) in appendix A differs from equation (4.2), thus the generalized perturbation $(1, N - 1)$ -fold DT in theorem 3.2 cannot be used directly and should also be *changed*. The main reason is that the distinct perturbation λ will make these terms $\mathcal{G}_n^{(k)}(t; \lambda)$ and $\varphi^{(k)}(n, t; \lambda)$ different such that these functions $A_n^{(j)}(t)$ and $B_n^{(j)}(t)$ in the Darboux matrix $\mathcal{G}_n(t; \lambda)$, and the generalized perturbation $(1, N - 1)$ -fold DT, are also different.

For convenience, we will take equation (4.2) as the perturbation λ . Without loss of generality, we take $\mu_1 = -\mu_2 = 1$ and $\alpha = 4/3$ in equation (4.2) such that

$$\lambda_1 = \left(\sqrt{1 + \alpha^2} - \alpha \right) \Big|_{\alpha=4/3} = \frac{1}{3}, \quad \lambda = \lambda_1 + \varepsilon = \frac{1}{3} + \varepsilon. \quad (4.3)$$

In fact, we can choose the other non-zero values for α to study the RS solutions of equation (1.1) via the generalized perturbation DT in theorem 3.2.

It follows from equation (4.1) that we choose the solutions of the Lax pair (1.2) as

$$\varphi(n, t; \lambda, \varepsilon) = -\frac{i}{\sqrt{\varepsilon}} \begin{pmatrix} \phi_1(n, t; \lambda, \varepsilon) - \psi_1(n, t; \lambda, \varepsilon) \\ \phi_2(n, t; \lambda, \varepsilon) - \psi_2(n, t; \lambda, \varepsilon) \end{pmatrix}, \quad (4.4)$$

where $\phi_j(n, t; \lambda, \varepsilon), \psi_j(n, t; \lambda, \varepsilon), j = 1, 2$ are defined by equation (4.1) with λ given by equation (4.2). As a result, according to equation (4.3), we obtain the following Taylor series of $\varphi(n, t; \lambda, \varepsilon)$ at $\varepsilon = 0$

$$\varphi(n, t; \lambda, \varepsilon) \Big|_{\lambda=\lambda_1+\varepsilon=1/3+\varepsilon} = \varphi^{(0)}(n, t; 1/3) + \varphi^{(1)}(n, t; 1/3)\varepsilon + \varphi^{(2)}(n, t; 1/3)\varepsilon^2 + \varphi^{(3)}(n, t; 1/3)\varepsilon^3 + \dots, \quad (4.5)$$

where $\varphi^{(j)}(n, t; 1/3) = (\phi^{(j)}(n, t; 1/3), \psi^{(j)}(n, t; 1/3))^T$ ($j = 0, 1, 2, \dots$), whose the first two vectors are given by (with the aid of *Maple*)

$$\phi^{(0)}(n, t; 1/3) = \frac{4\sqrt{30}}{15} \left(\frac{5}{3} \right)^n e^{\frac{5}{3}t} (5t + 3n), \quad (4.6)$$

$$\psi^{(0)}(n, t; 1/3) = -\frac{\sqrt{30}}{15} \left(\frac{5}{3} \right)^n e^{\frac{5}{3}t} (20t + 12n + 15),$$

$$\begin{aligned}\phi^{(1)}(n, t; 1/3) &= \frac{\sqrt{30}}{2700} \left(\frac{5}{3}\right)^n e^{\frac{5}{3}t} (8235t + 2400c_1 - 2400d_1 + 14400t^2 + 8000t^3 + 3213n + 1728n^3 \\ &\quad + 8640nt + 14400nt^2 + 8640n^2t), \\ \psi^{(1)}(n, t; 1/3) &= -\frac{\sqrt{30}}{10800} \left(\frac{5}{3}\right)^n e^{\frac{5}{3}t} (130140t + 9600c_1 - 9600d_1 + 129600t^2 + 32000t^3 + 45252n \\ &\quad + 6912n^3 + 24705 + 25920n^2 + 120960nt + 57600nt^2 + 34560nt^2),\end{aligned}\quad (4.7)$$

and other vector functions $\varphi^{(j)}(n, t; 1/3)$'s, $j > 1$ are complicated and omitted here (of course they can be easily obtained from equations (4.4) and (4.5) with the aid of *Maple*). Note that when we consider the N th-order RSs of equation (1.1) we need to use $\varphi^{(j)}(n, t; 1/3)$, $j = 0, 1, \dots, N-1$ in equation (4.5).

It follows from equations (3.4) and (4.5) as well as theorem 3.2 that we can derive new exact RS solutions of equation (1.1) with $\sigma = 1$. In the following we analyse the non-trivial RS solutions of equation (1.1) for $N = 1, 2, 3, 4$, and study their dynamical behaviours.

Case 4.2. *First-order discrete vector RSs and dynamics.*—For the case of $M = N = 1$, the first-order RS solutions of equation (1.1) with $\sigma = 1$ are given by

$$\hat{I}_n^{(1)}(n, t) = \frac{[\alpha D_1^{(1)} + \hat{B}_n^{(0)}(t)]}{\hat{A}_n^{(0)}(t)}, \quad \hat{V}_n^{(1)}(n, t) = \frac{\hat{B}_{n+1}^{(0)}(t)}{D_1^{(1)}}. \quad (4.8)$$

Particularly, the RS solution (4.8) with $\alpha = 4/3$ is explicitly given by

$$\begin{aligned}\hat{I}_n^{(1)}(t) &= \hat{I}_n^{(1)}(n, t) = -\frac{360}{(24n + 40t + 27)^2 + 81}, \\ \hat{V}_n^{(1)}(t) &= \hat{V}_n^{(1)}(n, t) = \frac{600}{(24n + 40t + 39)^2 + 225} - \frac{4}{3},\end{aligned}\quad (4.9)$$

from which $\hat{I}_n^{(1)}(n, t) \rightarrow 0$ and $\hat{V}_n^{(1)}(n, t) \rightarrow -4/3$ as $n \rightarrow \infty$ or $t \rightarrow \infty$.

Support that $n \in (-\infty, +\infty)$ is a continuous variable (in fact, we can only choose the discrete points). For $\alpha = 4/3$, the valleys of $\hat{I}_n^{(1)}$ in figure 1(a1–a2) are located along the line $8(3n + 5t) + 27 = 0$ with the minimal amplitude $-40/9$, and the peaks of $\hat{V}_n^{(1)}$ in figure 1(b1–b2) are located along the line $8(3n + 5t) + 39 = 0$ with the maximal amplitude $4/3$, but for $\alpha = -4/3$, the peaks of $\hat{I}_n^{(1)}$ in figure 1(c1–c2) are located along the line $8(3n + 5t) - 3 = 0$ with the maximal amplitude $40/9$, and the valleys of $\hat{V}_n^{(1)}$ in figure 1(d1–d2) are located along the line $8(3n + 5t) + 9 = 0$ with the minimal amplitude $-4/3$. The parameter α can control the first-order RS solution (4.8): if $\alpha > 0$, then $\hat{I}_n^{(1)}$ is a dark RS, and $\hat{V}_n^{(1)}$ is a bright RS, whereas if $\alpha < 0$, the results are opposite. Moreover, the amplitudes become larger as $|\alpha|$ increases.

To investigate the wave evolutions of the obtained discrete RS solutions, we mainly analyse the obtained exact RS profiles, the perturbed profiles generated numerically by using the initial conditions determined by these RS solutions without or with a small noise. Here we use the classical finite difference method to perform the numerical simulations via MATLAB. More specifically, equation (1.1) can be simulated by the iterative formula

$$\begin{aligned}I_{n,j+1} &= I_{n,j} + \tau(1 + I_{n,j}^2)(V_{n-1,j} - V_{n,j}), \\ V_{n,j+1} &= V_{n,j} + \tau(1 + V_{n,j}^2)(I_{n,j} - I_{n+1,j}),\end{aligned}\quad (4.10)$$

where τ is the time step, and chosen as 10^{-4} . The perturbing part of the initial condition can be expressed precisely as follows: if $I_n = I(n, t)$, $V_n = V(n, t)$ are exact solutions, then we take $I(n, t_0)(1 + \text{noise})$ and $V(n, t_0)(1 + \text{noise})$ as the initial conditions, where t_0 is a given time and $\text{noise} = (\text{rand}(\mathcal{N}, 1) - 0.5) \times 5\%$ denotes a 5% white noise perturbation. Moreover, $n \in [l_1, l_2]$ is regarded as a continuous variable, and the spatial step is 0.01. The boundary conditions are chosen

Figure 1. (a1–a2, b1–b2) First-order dark and bright RS solutions (4.8) with $\alpha = 4/3$; (c1–c2, d1–d2) first-order bright and dark RS solutions (4.8) with $\alpha = -4/3$. (Online version in colour.)

Figure 2. First-order RS solutions (4.9) and dynamics. The exact solution (left); the wave propagations generated by choosing the exact solutions (4.9) with $t = -3$ (middle), or one with $t = -3$ and a random 5% noise (right) as the initial conditions. (Online version in colour.)

as the corresponding exact RS solutions with $n = l_1, l_2$. Moreover, we check the so-called norms during the wave propagations by $N_I = \int_{l_1}^{l_2} |I_n(t)| dn$, $N_V = \int_{l_1}^{l_2} |V_n(t)| dn$.

Figure 2 exhibits the dynamical behaviours of the first-order RS solutions (4.9), $\hat{I}^{(1)}(n, t)$ and $\hat{V}^{(1)}(n, t)$, where figure 2 (left, middle, right) displays the exact RS solutions, ones without a small noise, and ones with a small 5% noise, which implies that three types of wave propagations are almost the same in a short time $t \in (-3, 3)$. Figure 2 (right) shows that the first-order RS solutions (4.9) are robust against a small noise. When we check the so-called norms N_I and N_V , we find that when $t \in [-3, 3]$, the norms of exact solutions without a noise and with a small noise are basically identical, but when the time becomes longer such as $t \in [-5, 5]$, the corresponding norms have the larger differences, that is, the wave propagations become unstable as time increases. Similarly

Figure 3. (a1,a2) Exact first-order RS $\hat{I}_n^{(1)}$, $\hat{V}_n^{(1)}$ and initial solution $\hat{I}_{n \rightarrow n+5, t=0}^{(1)} + 1.5\hat{I}_{n \rightarrow n-5, t=0}^{(1)}$, $\hat{V}_{n \rightarrow n+5, t=0}^{(1)} + 1.5\hat{V}_{n \rightarrow n-5, t=0}^{(1)}$; (b1,b2) the interaction of exact first-order RS solutions $\hat{I}_n^{(1)}$, $\hat{V}_n^{(1)}$ and another first-order RS solutions $1.5\hat{I}_{n \rightarrow n-5, t=0}^{(1)}$, $1.5\hat{V}_{n \rightarrow n-5, t=0}^{(1)}$. (Online version in colour.)

to ref. [66], the strong oscillations will occur in equation (1.1) by adding longer time, the blow-up phenomenon occurs and the wave may exhibit the total collapse.

Moreover, we also study the interactions of two first-order RS solutions, $P_1 = \hat{I}^{(1)}(n, t)|_{t=0, n \rightarrow n+5}$, $Q_1 = \hat{V}^{(1)}(n, t)|_{t=0, n \rightarrow n+5}$ and $P_2 = 1.5\hat{I}^{(1)}(n, t)|_{t=0, n \rightarrow n+5}$, $Q_2 = 1.5\hat{V}^{(1)}(n, t)|_{t=0, n \rightarrow n+5}$. We use $P_1 + P_2$ and $Q_1 + Q_2$ as the initial conditions to study the wave propagations such that the non-elastic collision phenomena are observed in figure 3.

Case 4.3. Second-order discrete vector RSs and dynamics.—For the case of $M = 1, N = 2$, we have the second-order RS solutions of equation (1.1) with $\sigma = 1$ and $\alpha = 4/3$

$$\hat{I}^{(2)}(n, t) = \frac{4/3 D_2^{(1)} + \hat{B}_n^{(1)}(t)}{\hat{A}_n^{(0)}(t)} = \frac{4}{3} \frac{F_1(n, t)}{G_1(n, t)}, \quad \hat{V}^{(2)}(n, t) = \frac{\hat{B}_{n+1}^{(0)}(t)}{D_2^{(1)}} = \frac{F_2(n, t)}{G_2(n, t)}, \quad (4.11)$$

where F_j, G_j ($j = 1, 2$) are listed in appendix B.

The two parameters c_1 and d_1 can modulate the wave structures of second-order RS solutions (4.11):

- As $c_1 = d_1 = 0$, each second-order RS solution consists of one bright RS and one dark RS, and illustrates the strong interaction (see figure 4(a1–a2, b1–b2)). The maximal peak of $\hat{I}^{(2)}(n, t)$ is $364/27$ at $(n, t) = (\frac{189}{128}, \frac{27}{128})$, and the minimal valley of $\hat{V}^{(2)}(n, t)$ is $-40/9$ at $(n, t) = (-\frac{253}{128}, \frac{27}{128})$. The two RSs both keep their shapes and speeds after the elastic interaction.
- As $c_1 \neq 0, d_1 = 0$, each second-order RS can be separated into two almost parallel first-order RSs, and exhibits the weak interaction (see figure 4(c1–c2, d1–d2)). In fact, the effects of d_1 and c_1 are similar, but when $d_1 \neq 0$, the solutions are complex, and are not discussed here.

Figure 4. (a1–a2, b1–b2) Second-order RS solutions (4.11) with $\alpha = \frac{4}{3}$, $c_1 = d_1 = 0$; (c1–c2, d1–d2) second-order RS solutions (4.11) with the same parameters except for $c_1 = 200$, $d_1 = 0$. (Online version in colour.)

Figure 5. Second-order RS solutions (4.11) with the same parameters in figure 4(a1,b1) and dynamics. The exact solution (left), the wave propagations simulated by using exact solutions (4.11) with $t = -3$ (middle), or ones with $t = -3$ and a random 5% noise (right) as the initial conditions. (Online version in colour.)

In what follows, we numerically explore the dynamical behaviours of the second-order RSs, $\hat{I}^{(2)}(n, t)$ and $\hat{V}^{(2)}(n, t)$, given by equation (4.11). For the case of strong interactions (figure 4(a1,b1)), figure 5 (left, middle, right) illustrates the wave profiles of exact second-order RS solutions (4.11) and time evolutions of $\hat{I}^{(2)}(n, -3)$, $\hat{V}^{(2)}(n, -3)$ without a small noise, and ones with a 5% small noise, respectively, which implies that three types of wave propagations are almost the same in a short time $t \in (-3, 3)$. But, for the case of the weak interaction (figure 4(c1,d1)), we find that the wave propagations (figure 6) are almost the same as ones of the strong interactions, which differ from ones in [65]. Figure 6 (right) implies that the second-order RS solutions (4.11) are also robust against a 5% small noise in a short time even if there exist small perturbations nearby $t = 3$.

Figure 6. Second-order RS solutions (4.11) with the same parameters in figure 4(c1,d1) and dynamics. The exact RS solution (left), the wave propagations simulated by using the exact solutions (4.11) with $t = -3$ (middle), or ones with $t = -3$ and a random 5% noise (right) as the initial conditions. (Online version in colour.)

Case 4.4. *Third-order discrete vector RSs and dynamics.*—For the case of $M = 1$, $N = 3$, we find the third-order RS solutions of equation (1.1) with $\sigma = 1$ and $\alpha = 4/3$:

$$\hat{I}^{(3)}(n, t) = \frac{[4/3D_3^{(1)} + \hat{B}_n^{(2)}(t)]}{\hat{A}_n^{(0)}(t)}, \quad \hat{V}^{(3)}(n, t) = \frac{\hat{B}_{n+1}^{(0)}(t)}{D_3^{(1)}}, \quad (4.12)$$

whose explicit forms are complicated and omitted here.

We analyse the wave structures of the third-order RSs (4.12) as follows:

- When $c_{1,2} = d_{1,2} = 0$, the third-order RSs exhibit the strong interactions (figures 7(a1)–(a2) and 8(a1)–(a2)), and keep their shapes and speeds after the elastic interactions. $\hat{I}^{(3)}(n, t)$ consists of one bright RS and two dark RSs, while $\hat{V}^{(3)}(n, t)$ is made up of two bright RSs and one dark RS.
- As $c_1 = 100$, $d_{1,2} = c_2 = 0$, the third-order RSs display the weak interactions, and are separated into three parallel first-order RSs (figures 7(b1)–(b2) and 8(b1)–(b2)). $\hat{I}^{(3)}(n, t)$ consists of one bright RS in the middle and two dark RSs on both sides, while $\hat{V}^{(3)}(n, t)$ is composed of one dark RS in the middle and two bright RSs on both sides.
- As $c_2 = 1000$, $d_{1,2} = c_1 = 0$, the third-order RSs also illustrate the similar wave structures of the second case. But $\hat{I}^{(3)}(n, t)$ consists of one bright RS on one side and two dark RSs on another side (figure 7(c1)–(c2)), while $\hat{V}^{(3)}(n, t)$ is composed of one dark RS on one side and two bright RSs on another side (figure 8(c1)–(c2)).

We now numerically simulate the wave propagations of third-order RS solutions (4.12). Figures 9–11(a1,b1,a2,b2) display that the unperturbed wave propagations are almost consistent with the related exact third-order RSs (4.12) for the cases of both strong and weak interactions in a short time. For the case of strong interactions, figure 9(c1,c2) shows that the third-order RSs are also robust against a 5% small noise in a short time. However, for the cases of weak interactions, figures 10–11(c1,c2) show that there exist some small oscillations only at about $t = 5$.

Figure 7. Third-order RS solution $\hat{I}^{(3)}(n, t)$ given by equation (4.12). Strong interaction with $c_{1,2} = d_{1,2} = 0$ (a1,a2) and weak interactions with $c_1 = 100, d_{1,2} = c_2 = 0$ (b1,b2) or $c_2 = 1000, c_1 = d_{1,2} = 0$ (c1,c2). (Online version in colour.)

Figure 8. Third-order RS solution $\hat{V}^{(3)}(n, t)$ given by equation (4.12). Strong interaction with $c_{1,2} = d_{1,2} = 0$ (a1,a2) and weak interactions with $c_1 = 100, d_{1,2} = c_2 = 0$ (b1,b2) or $c_2 = 1000, c_1 = d_{1,2} = 0$ (c1,c2). (Online version in colour.)

Case 4.5. Fourth-order discrete vector RSs and dynamics.—For the case of $M = 1, N = 4$, we obtain the fourth-order RS solutions of equation (1.1) with $\sigma = 1$ and $\alpha = 4/3$:

$$\hat{I}^{(4)}(n, t) = \frac{[4/3D_4^{(1)} + \hat{B}_n^{(3)}(t)]}{\hat{A}_n^{(0)}(t)}, \quad \hat{V}^{(4)}(n, t) = \frac{\hat{B}_{n+1}^{(0)}(t)}{D_4^{(1)}}, \quad (4.13)$$

Figure 9. The third-order RS solutions (4.12) with the same parameters in figures 7–8(a1)–(a2) and dynamics. The exact third-order RS solution (left), the wave propagations simulated by using the exact RS solutions (4.12) with $t = -3$ (middle), or ones with $t = -3$ and a random 5% noise (right) as the initial conditions. (Online version in colour.)

Figure 10. Third-order RS solutions (4.12) with the same parameters in figures 7–8(b1)–(b2) and dynamics. The exact RS solution (left), the wave propagations simulated by using the exact solutions (4.12) with $t = -5$ (middle), or ones with $t = -5$ and a random 5% noise (right) as the initial conditions. (Online version in colour.)

whose explicit forms are complicated and omitted here.

The fourth-order RSs (4.13) illustrate distinct wave structures for the chosen parameters:

- As $c_{1,2,3} = d_{1,2,3} = 0$, the fourth-order RSs display the strong and elastic interactions (figures 12–13(a1,a2)). $\hat{I}^{(4)}(n, t)$ and $\hat{V}^{(4)}(n, t)$ are both made up of two bright RSs and two dark RSs.
- For $c_1 = 100$, $c_{2,3} = d_{1,2,3} = 0$, the fourth-order RSs illustrate the weak interactions, and are composed of four parallel first-order RSs containing two bright RSs and two dark RSs (figures 12(b1,b2) and 13(b1,b2)).

Figure 11. Third-order RS solutions (4.12) with the same parameters in figures 7–8(c1)–(c2) and dynamics. The exact RS solution (left), the wave propagations simulated by using the exact solutions (4.12) with $t = -5$ (middle), or ones with $t = -5$ and a random 5% noise (right) as the initial conditions. (Online version in colour.)

Figure 12. Fourth-order RS solution $\hat{I}^{(4)}(n, t)$ given by equation (4.13). Strong interaction with $c_{1,2} = d_{1,2} = 0$ (a1,a2), weak interaction with $c_1 = 100, c_{2,3} = d_{1,2,3} = 0$ (b1,b2) or $c_2 = 10^3, c_{1,3} = d_{1,2,3} = 0$ (c1,c2), and strong-weak interaction with $c_3 = 10^3, c_{1,2} = d_{1,2,3} = 0$ (d1,d2). (Online version in colour.)

- When $c_2 = 1000, c_{1,3} = d_{1,2,3} = 0$, the fourth-order RSs also exhibit similar weak interactions as the above-mentioned second case (figures 12(c1,c2) and 13(c1,c2)).
- As $c_3 = 1000, c_{1,2} = d_{1,2,3} = 0$, the fourth-order RSs show the wave structures of strong-weak interactions with a second-order RS with the strong interaction in the middle and two almost parallel first-order RSs on both sides (figures 12–13(d1,d2)).

We now numerically investigate the wave propagations of the fourth-order RS solutions (4.13). For the cases of strong (figures 12–13(a1,a2)) and strong-weak (figures 12–13(d1,d2)) interactions,

Figure 13. Fourth-order RS solution $\widehat{V}^{(4)}(n, t)$ given by equation (4.13). Strong interaction with $c_{1,2} = d_{1,2} = 0$ (a1,a2), weak interaction with $c_1 = 100, c_{2,3} = d_{1,2,3} = 0$ (b1,b2) or $c_2 = 10^3, c_{1,3} = d_{1,2,3} = 0$ (c1,c2) and strong-weak interaction with $c_3 = 10^3, c_{1,2} = d_{1,2,3} = 0$ (d1,d2). (Online version in colour.)

Figure 14. Fourth-order RS solutions (4.13) with the same parameters in figures 12–13(a1)–(a2) and dynamics. The exact RS solution (left), the wave propagations simulated by using the exact solutions (4.13) with $t = -2$ (middle), or ones with $t = -2$ and a random 5% noise (right) as the initial conditions. (Online version in colour.)

figure 14 (middle, right) and figure 15 (middle, right) show that the wave propagations of fourth-order RSs without a noise and with a small 5% noise are almost the same, but they are different from the profiles of exact fourth-order RSs (4.13) (figure 14 (left) and figure 15 (left)). These results imply that the fourth-order RSs with strong or strong-weak interactions are not robust against a small noise or even a zero noise. For the other two types of weak interactions (figures 12–13(a1,a2) and (c1,c2)), figures 16–17 show that the fourth-order RSs are almost robust against a 5% small noise in a short time except that there exist some small oscillations, only nearby $t = 5$ and $t = 4$, respectively.

Similarly, it follows from theorem 3.2 with $N \geq 5$ that the higher-order RS solutions with abundant wave structures can be found for the equation (1.1) with $\sigma = 1$.

Figure 15. Fourth-order RS solutions (4.13) with the same parameters in figures 12–13(d1)–(d2) and dynamics. The exact RS solution (left), the wave propagations simulated by using the exact RS solutions (4.13) with $t = -3$ (middle), or ones with $t = -3$ and a random 5% noise (right) as the initial conditions. (Online version in colour.)

Figure 16. Fourth-order RS solutions (4.13) with the same parameters in figures 12–13(b1)–(b2) and dynamics. The exact RS solution (left), the wave propagations simulated by using the exact solutions (4.13) with $t = -5$ (middle), or ones with $t = -5$ and a random 5% noise (right) as the initial conditions. (Online version in colour.)

5. Wave features of higher-order discrete vector RSs

We now summarize the wave features of general discrete vector RS solutions of equation (1.1) in table 1. The first column of table 1 displays the order of the RS solutions, while the second, third, fourth and fifth columns exhibit the powers of the polynomials involved in each pair of solutions. The last two columns denote the background levels of the RS solutions. For the RS solution $I_n(t)$ of order m , (i) if m is odd, the maximal degree of the numerator polynomial is $m(m+1) - 2$, while the highest degree of the denominator polynomial is $m(m+1)$, and the background levels are all zero; and (ii) if m is even, the highest degrees of the numerator and denominator polynomials are both

Figure 17. Fourth-order RS solutions (4.13) with the same parameters in figures 12–13(c1)–(c2) and dynamics. The exact RS solution (left), the wave propagations simulated by using the exact solutions (4.13) with $t = -4$ (middle), or ones with $t = -4$ and a random 5% noise (right) as the initial conditions. (Online version in colour.)

Table 1. RS solutions $I_n(t)$, $V_n(t)$ of order m : HPN, HPD and BG stand for the highest powers in the numerator and denominator, and backgrounds of the solutions, respectively.

m	HPN of $I_n(t)$	HPN of $V_n(t)$	HPD of $I_n(t)$	HPD of $V_n(t)$	BG of $I_n(t)$	BG of $V_n(t)$
1	0	2	2	2	0	$-4/3$
2	6	4	6	6	$4/3$	0
3	10	12	12	12	0	$-4/3$
4	20	18	20	20	$4/3$	0
odd	$m(m+1)-2$	$m(m+1)$	$m(m+1)$	$m(m+1)$	0	$-4/3$
even	$m(m+1)$	$m(m+1)-2$	$m(m+1)$	$m(m+1)$	$4/3$	0

$m(m+1)$, and the background level is $4/3$. However, for the RS solution $V_n(t)$ of order m , (i) if j is odd, the highest degrees of numerator and denominator polynomials are both $m(m+1)$, and the background level is $-4/3$; and (ii) if m is even, the highest degree of the numerator polynomial is $m(m+1)-2$, while the highest degree of the denominator polynomial is $m(m+1)$, and the background level is zero.

6. Conclusions and discussion

In conclusion, we have presented a new generalized perturbation discrete $(M, N-M)$ -fold DT of equation (1.1), which with $M=1$ is used to find the discrete higher-order vector RS solutions in terms of determinants. In particular, the wave propagations and interaction structures of first-, second-, third- and fourth-order RS solutions are analysed in detail. By using numerical simulations, we find that the wave evolutions of some strong interactions of RS solutions are robust against a small noise, and the wave propagations of weak interactions of RS solutions illustrate the almost stable evolutions in a short time expected for some small perturbations. For the more M ($1 < M < N$) spectral parameters, we can also find the other new types of vector RS solutions of equation (1.1), which will be considered in another paper.

Nonlinear electrical lattices are the very convenient models studying the wave propagations in the one-dimensional nonlinear dispersive systems, and have been used to study the properties of solitons on lattices [67–69]. Moreover they also have a range of applications in sampling oscilloscopes, network analysers, pulse compression devices and so on [70]. These found higher-order vector RS solutions may be useful for understanding the wave propagations of electrical signals in circuits or have some other practical applications. Of course, this idea used in this paper can also be extended to study the novel wave structures of other coupled NLEs and their dynamical behaviours.

Data accessibility. This article has no additional material.

Authors' contributions. All authors contributed to the overall theoretical analysis and numerical simulations. X.-Y.W. and Z.Y. wrote and improved the manuscript. All authors gave final approval for publication and agree to be held accountable for the work performed therein.

Competing interests. We declare we have no competing interests.

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Appendix A. The other form of the perturbation λ

It follows from the expressions for ρ_1 and ρ_2 given in §3 that $\rho_1\rho_2 = 1 + \alpha^2$. In order to facilitate the analysis, one can introduce the parametrizations $\alpha = \sinh \theta$, $\rho_1 = z \cosh \theta$, $\rho_2 = 1/z \cosh \theta$, where θ , z are new parameters. According to the expression $\rho_1 + \rho_2$, we come to the relations between the old and new spectral parameters: $\lambda^2 - \lambda(z + 1/z) \cosh \theta + 1 = 0$, which yields

$$2\lambda = \left(z + \frac{1}{z}\right) \cosh \theta - \sqrt{(z - 1/z)^2 \cosh^2 \theta + 4 \sinh^2 \theta}. \quad (\text{A } 1)$$

For $\theta = 0$, we return to the trivial relationships $\lambda = z$.

Now let us define λ_1 as one of the roots of equation $(\lambda^2 - 1)^2 - 4\alpha^2\lambda^2 = 0$, then we have $\lambda_1 = \eta(\sqrt{1 + \alpha^2} + \nu\alpha)$, where $\eta^2 = \nu^2 = 1$. The same result in terms of parameter θ looks as follows $\lambda_1 = \eta e^{\nu\theta}$.

Taking into account the relationship $\sqrt{(\lambda^2 - 1)^2 - 4\alpha^2\lambda^2} = \lambda(\rho_1 - \rho_2) = \lambda(z - 1/z) \cosh \theta = 0$, we can obtain $z_1 = \eta$. For practical purposes it is sufficient to consider the case $\eta = 1$, $\nu = -1$ such that $z_1 = 1$, $\lambda_1 = e^{-\theta}$, which are consistent with equation (A 1). Notice that θ is a free parameter characterizing the seed current in the electric circuits under consideration. To apply the expansion scheme and the limiting procedure, we need to fix the new spectral parameter z by $z = (1 - \varepsilon)/\sqrt{1 - \varepsilon^2}$. Consequently for the old spectral parameter λ , we have

$$\lambda = \left(\cosh \theta - \sqrt{\sinh^2 \theta + \varepsilon^2}\right) / \sqrt{1 - \varepsilon^2}. \quad (\text{A } 2)$$

According to the above-mentioned perturbation λ and theoretical analysis in §3, we can obtain another form of the generalized perturbation $(1, N - 1)$ -fold DT of equation (1.1), which obviously differs from theorem 3.2. We here do not give the detailed derivation about the new generalized perturbation $(1, N - 1)$ -fold DT due to the page limit.

Appendix B

For $M = 1$, $N = 2$,

$$\begin{aligned} F_1(n, t) = & -36905625 - 827117568n^4 - 1017847296n^3 - 591411456n^2 - 6013440000t^4 \\ & - 3965760000t^3 + 174960000c_1 - 1049760000t^2 - 125971200n + 552960000nt^2c_1 \end{aligned}$$

$$\begin{aligned}
& + 331776000n^2tc_1 + 746496000ntc_1 - 47775744n^6 - 1024000000t^6 + 23040000d_1^2 \\
& - 23040000c_1^2 - 4147200000t^5 - 322486272n^5 - 4820497920n^2t - 5434490880n^3t \\
& - 13387161600n^2t^2 - 1595635200nt - 14653440000nt^3 - 7586265600nt^2 + 313113600nc_1 \\
& + 466560000tc_1 - 3686400000nt^5 - 4423680000n^3t^3 - 1990656000n^4t^2 - 8957952000n^3t^2 \\
& - 477757440n^5t + 307200000t^3c_1 + 622080000t^2c_1 - 5529600000n^2t^4 - 14929920000n^2t^3 \\
& - 2687385600n^4t - 12441600000nt^4 + 223948800n^2c_1 + 66355200n^3c_1 + i(66355200n^3 \\
& + 331776000n^2t + 552960000nt^2 + 307200000t^3 + 223948800n^2 + 746496000nt \\
& + 622080000t^2 + 313113600n + 466560000t - 46080000c_1 + 174960000)d_1, \\
G_1(n, t) = & -332150625 + 331776000n^2tc_1 - 1024000000t^6 - 939091968n^4 - 1521732096n^3 \\
& - 1592182656n^2 - 6877440000t^4 - 6298560000t^3 - 4140720000t^2 - 47775744n^6 \\
& - 1062882000n - 1771470000t + 552960000nt^2c_1 + 746496000ntc_1 - 23040000c_1^2 \\
& - 4147200000t^5 - 322486272n^5 - 6180986880n^3t - 7339921920n^2t - 15253401600n^2t^2 \\
& - 5118163200nt - 16727040000nt^3 - 11785305600nt^2 + 157593600nc_1 + 207360000tc_1 \\
& - 3686400000nt^5 - 4423680000n^3t^3 - 1990656000n^4t^2 - 8957952000n^3t^2 \\
& - 477757440n^5t + 307200000t^3c_1 + 622080000t^2c_1 - 5529600000n^2t^4 - 14929920000n^2t^3 \\
& - 2687385600n^4t - 12441600000nt^4 + 223948800n^2c_1 + 66355200n^3c_1 + 23040000d_1^2 \\
& + i(66355200n^3 + 331776000n^2t + 552960000nt^2 + 307200000t^3 + 223948800n^2 \\
& + 746496000nt + 622080000t^2 + 157593600n + 207360000t - 46080000c_1)d_1, \\
F_2(n, t) = & 89579520n^4 + 597196800n^3t + 1492992000n^2t^2 + 1658880000nt^3 + 691200000t^4 \\
& + 582266880n^3 + 2911334400n^2t + 4852224000nt^2 + 2695680000t^3 + 1539648000n^2 \\
& + 5281459200nt + 124416000nc_1 + 4525632000t^2 + 207360000tc_1 + 1897266240n \\
& + 3404721600t + 202176000c_1 + 850305600 + i(124416000n + 207360000t \\
& + 202176000)d_1, \\
G_2(n, t) = & 1888634880n^4 + 4092664320n^3 + 5088583296n^2 + 14204160000t^4 + 17869248000t^3 \\
& - 223948800c_1 + 13205980800t^2 + 3529398096n + 5613591600t + 20075143680n^2t \\
& + 12511272960n^3t + 16384654080nt + 34311168000nt^3 + 32811609600nt^2 \\
& - 481075200nc_1 - 746496000tc_1 - 331776000n^2tc_1 + 465813504n^5 + 5990400000t^5 \\
& + 17971200000nt^4 + 23040000c_1^2 - 23040000d_1^2 - 323481600n^2c_1 + 1024000000t^6 \\
& - 552960000nt^2c_1 + 4423680000n^3t^3 + 12939264000n^3t^2 + 21565440000n^2t^3 \\
& + 47775744n^6 + 477757440n^5t + 1990656000n^4t^2 + 1099787625 - 307200000t^3c_1 \\
& - 66355200n^3c_1 - 1078272000ntc_1 + 3881779200n^4t - 898560000t^2c_1 \\
& + 552960000n^2t^4 + 3686400000nt^5 + 31079116800n^2t^2 + i(-66355200n^3 \\
& - 331776000n^2t - 552960000nt^2 - 307200000t^3 - 323481600n^2 - 1078272000nt \\
& - 898560000t^2 - 481075200n - 746496000t + 46080000c_1 - 223948800)d_1.
\end{aligned}$$

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