Published in final edited form as:

J Quant Spectrosc Radiat Transf. 2017 November ; 202: 114–125. doi:10.1016/j.jqsrt.2017.07.022.

Iterative discrete ordinates solution of the equation for the surface reflected radiance Alexander Radkevich

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Abstract

This paper presents a new method of numerical solution of the integral equation for the radiance reflected from an anisotropic surface. The equation relates the radiance at the surface level with BRDF and solutions of the standard radiative transfer problems for a slab with no reflection on its surfaces. It is also shown that the kernel of the equation satisfies the condition of existence of a unique solution and converges of the successive approximations to that solution. The developed method features two basic steps: discretization on a 2D quadrature and solving the resulting system of algebraic equations with successive over-relaxation method based on Gauss – Seidel iterative process. Presented numerical examples show good coincidence between the surface reflected radiance obtained with DISORT and proposed method. Analysis of contributions of the direct and diffuse (but not yet reflected) parts of the downward radiance to the total solution is performed. Together, they represent a very good initial guess for the iterative process. This fact ensures fast convergence. The numerical evidence is given that the fastest convergence occurs with the relaxation parameter of 1 (no relaxation). An integral equation for BRDF is derived as inversion of the original equation. The potential of this new equation for BRDF retrievals is analyzed. The approach is found not viable as the BRDF equation appears to be an ill-posed problem and it requires the knowledge the surface reflected radiance on the entire domain of both Sun and viewing zenith angles.

Keywords

surface reflection; radiance; BRDF; discrete ordinates method; successive over-relaxation

1. INTRODUCTION

Radiation reflected from the Earth surface presents valuable source of information about surface properties that can be formalized in Bi-directional Reflection Distribution Function (BRDF). That information is required to specify a boundary condition for radiative transfer (RT) modeling which is used in aerosol retrievals, cloud retrievals, atmospheric modeling and other applications. Ground based measurements of reflected radiance draw increasing attention as a source of information about anisotropy of surface reflection [1 – 4], along with development of measurement techniques [5]. Atmospheric correction has to be done to derive BRDF from surface radiance, so retrieval methods were also developed [6, 7].

The retrieval methods are based on comparison of the measured and computed reflected radiance at the ground level. If yet another evaluation of the radiance is needed then a full

radiative transfer problem has to be solved anew for the new guess of BRDF. Decoupling of the atmospheric radiative transfer and anisotropic surface reflectance [8, 9] allows to avoid multiple RT computations if standard problems (no reflection on the boundaries of the atmosphere) are solved. In [8] solution was found in the form of series by the number of reflections. In [9] surface reflected radiance is presented as a solution of an integral equation relating it with BRDF and radiances transmitted through and reflected by the atmosphere. The approaches to solve that equation if standard problems are solved with the discrete ordinates method and spherical harmonics method were also considered in that study.

The first step in both cases is expansion of all functions of the relative azimuth angle into cosine Fourier series and consequent separation of the problems for the Fourier components. Once they are found, summation of the Fourier series needs to be done. King [10] studied how many terms of the Fourier expansion of the reflection function need to be retained required in the case of optically thick atmospheres. The observations of that study are: 1) “it is necessary that each term in the Fourier expansion of the phase function satisfy a normalization condition in quadraturized form,” 2) “the reflection function of a semi-infinite atmosphere can be represented by a Fourier series whose upper limit depends strongly on the angles of incidence,” 3) “for aircraft or satellite applications involving scanning radiometers for measuring the reflected intensity field at nadir angles from 0° to 45° , the number of terms required in the Fourier expansion of the reflection function for semi-infinite atmospheres will generally not exceed 16,” and 4) “Thus in order to maintain a relative accuracy of 0.1% in the reflection function of optically thick atmospheres, more terms may be required in the Fourier series expansion of the reflection function than required for a semi-infinite atmosphere.” The last indicates to the possible increase of the number of terms needed to be retained with the decrease of optical thickness of the atmosphere. In the case of optically thin atmosphere when it makes sense to perform atmospheric correction of the ground measurements, relative contribution of single scattering prevails all other orders of scattering. Thus, the number of Fourier components needed is as much as the number of Fourier components of the phase function. Taking into account King’s first finding listed above and study [11] stating that “a straightforward numerical evaluation of the Fourier coefficients of sharply peaked phase functions based on the trapezoidal rule has been shown to be more accurate and much more computationally efficient than the use of the Legendre series derived from the addition theorem,” the idea of getting rid of Fourier expansions as a first step of the solution of the equation looks promising. This paper proposes an approach of the solution of the integral equation for the surface reflected radiance based on 2D discretization on a unit hemisphere. A combination of a Gaussian quadrature for integration over cosine of the viewing zenith angle and regular (equidistant) grid with trapezoidal rule for integration over the relative angle comprises a type of 2D quadrature used in this study.

2. STATEMENT OF THE PROBLEM AND NOTATION (EQUATION FOR THE SURFACE REFLECTED RADIANCE)

If a plane parallel scattering medium is illuminated on its top by light coming in direction $\mu_0 = \cos \theta_0$, $\phi_0 = 0$, then the diffuse radiance inside the and its boundaries is a solution of the radiative transfer equation (RTE):

$$\mu \frac{\partial I}{\partial \tau} + I(\tau, \mu, \phi, \mu_0) = \Lambda \int_0^{2\pi} d\phi' \int_{-1}^1 d\mu' \chi(\mu, \mu', \phi - \phi') I(\tau, \mu', \phi', \mu_0) + I_0 \mu_0 e^{-\tau/\mu_0} \chi(\mu, \mu_0, \phi) \quad (1)$$

where τ is optical depth, Λ – single scattering albedo (SSA), $\mu = \cos \theta$, θ, ϕ are the polar and azimuth angles of the direction of propagation of light, $\chi(\mu, \mu', \phi - \phi')$ is the scattering phase function normalized with condition:

$$\int_0^{2\pi} d\phi' \int_{-1}^1 d\mu' \chi(\mu, \mu', \phi - \phi') = 1 \quad (2)$$

RTE (1) has to be supplemented by appropriate boundary conditions (BCs) on the top and bottom boundaries. For further development we need to consider 3 related boundary value problems (BVPs) schematically depicted in Figures 1, a) – 1, c). Vertical axis is pointed from top to bottom surface, so that $\mu = \cos \theta > 0$ for downward radiation. The first problem is for atmosphere illuminated from its top and bounded at the bottom by a reflective surface described with BRDF ρ [12]:

$$\begin{aligned} I(\tau = 0, \mu > 0, \phi, \mu_0) &= 0 \\ I(\tau = \tau_t, \mu < 0, \phi, \mu_0) &= I_0 \mu_0 e^{-\tau_t/\mu_0} \rho(\mu_0, -\mu, \phi) \\ &+ \int_0^{2\pi} d\phi' \int_0^1 d\mu' \mu \rho(\mu', -\mu, \phi - \phi') I(\tau = \tau_t, \mu', \phi', \mu_0) \end{aligned} \quad (3)$$

We will denote this BVP1. Two other problems are similar: the medium is illuminated from its top and there is no reflection at the bottom surface

$$\begin{aligned} I(\tau = 0, \mu > 0, \phi, \mu_0) &= 0 \\ I(\tau = \tau_t, \mu < 0, \phi, \mu_0) &= 0 \end{aligned} \quad (4)$$

The BVP2 is for the same atmosphere as BVP1 while BVP3 is for the atmosphere with inverse order of layers, i.e. the atmosphere is flipped over. We will denote the solutions of the BVP1, BVP2, and BVP3 as L, I, J , respectively.

It was shown that reflected radiance at the surface level $L(\tau_t, \mu < 0, \phi, \mu_0)$ allows to express radiance $L(\tau, \mu, \phi, \mu_0)$ at any level and in any direction through radiances I and J [9]. For this reason $L(\tau_t, \mu < 0, \phi, \mu_0)$ was called Surface Resolving Kernel (SRK) in that paper. The SRK satisfies an integral equation relating it with the radiances transmitted through and reflected by the unbounded atmosphere, see eqs. (30), (36) – (38) of [9]. In the notation of this paper the equation takes form:

$$\begin{aligned} L(\tau = \tau_t, -\mu, \phi, \mu_0) &= S(\mu_0, \mu, \phi) + \int_0^{2\pi} d\phi_2 \int_0^1 d\mu_2 K(\mu, \mu_2, \phi - \phi_2) L \\ &(\tau = \tau_t, -\mu_2, \phi_2, \mu_0) \end{aligned} \quad (5)$$

$$S(\mu_0, \phi, \mu) = I_0 \mu_0 e^{-\tau_t/\mu_0} \rho(\mu_0, \mu, \phi) + \int_0^{2\pi} d\phi_1 \int_0^1 d\mu_1 \mu_1 \rho(\mu_1, \mu, \phi - \phi_1) I$$

$$(\tau = \tau_t, \mu_1, \phi_1, \mu_0) \quad (6)$$

$$K(\mu, \mu_2, \phi - \phi_2) = \frac{1}{I_0} \int_0^{2\pi} d\phi_1 \int_0^1 d\mu_1 \mu_1 \rho(\mu_1, \mu, \phi - \phi_1) J$$

$$(\tau = 0, -\mu_1, \phi_1 - \phi_2, \mu_2) \quad (7)$$

As one can see from eq. (5), cosine of the SZA μ_0 enters that equation as a parameter, so that solutions for different μ_0 can be found independently. Eq. (5) relates solutions of the BVPs 1 – 3 at the top and bottom boundaries of the medium, i.e. in order to find surface reflected radiance one needs only transmitted and reflected radiances from BVPs 2 and 3. These radiances are schematically shown in figures 1, b) and 1, c) with blue arrows.

We assume in this study that scattering centers are spherically symmetric thus the phase function χ depends only on the scattering angle, and, consequently, it is an even (symmetric) function of the relative azimuth angle between incoming and scattered directions:

$$\chi(\mu, \mu', \phi - \phi') = \chi(\mu, \mu', \phi' - \phi) \quad (8)$$

We also assume that surface scattering is locally isotropic, i.e. BRDF bears similar property:

$$\rho(\mu, \mu', \phi - \phi') = \rho(\mu, \mu', \phi' - \phi) \quad (9)$$

Both phase function and BRDF defined on an interval $[0, 2\pi]$. It is convenient to extend their definition so that they are periodic with respect to the relative azimuth with minimum period of 2π . Under these assumptions radiances L , I and J bear the same properties – they are periodic and symmetric with respect to the relative azimuth. This allows to reduce interval of integration over azimuth ϕ in eqs. (5) – (7):

$$\int_0^{2\pi} d\phi_1 f(\phi - \phi_1) g(\phi_1) = \int_0^\pi d\phi_1 [f(\phi - \phi_1) + f(\phi + \phi_1)] g(\phi_1) \quad (10)$$

3. 2D DISCRETIZATION OF THE EQUATION FOR THE SURFACE REFLECTED RADIANCE AND ITS ITERATIVE SOLUTION

Equation (5) can be solved with successive approximations:

$$L^{(k+1)}(\tau = \tau_t, -\mu, \phi, \mu_0) = S(\mu_0, \mu, \phi) + \int_0^{2\pi} d\phi_2 \int_0^1 d\mu_2 K(\mu, \mu_2, \phi - \phi_2) L^{(k)}$$

$$(\tau = \tau_t, -\mu_2, \phi_2, \mu_0) \quad (11)$$

$$L^{(0)}(\tau = \tau_t, -\mu, \phi, \mu_0) = S(\mu_0, \mu, \phi) \quad (12)$$

The proof of convergence of iterative process (11) is given in Appendix A.

Analytical form of the radiances I and J defining source function (6) and kernel (7) is unknown. They can be obtained numerically on certain discrete set of directions. In this case they are not readily available on other points of their domains. This leads to the need to evaluate the source and the kernel numerically and the kernel is known on set of values of μ_2 . The latter requires the use of a numerical quadrature on every iteration. Therefore, it is convenient to explicitly discretize the equation before the iterative process i.e. to use Nyström (quadrature) method. A 2D quadrature rule needs to be chosen to perform such discretization. In this study a direct product of linear quadratures is used, so that one quadrature is used for integration over μ and another for integration over ϕ . Kernel K is a difference kernel with respect to its azimuth variable. Taking into account relationship (10) it is important to choose a quadrature rule for azimuth integration so, that $\phi \pm \phi'$ belongs to the set of nodes of the rule (or its periodic extension) provided that both ϕ and ϕ' belong to the set. For this reason a regular grid of n points on $[0, \pi]$ with trapezoidal rule is used for evaluation of integrals over ϕ : $\{\phi_i, w_i\}$, $i = 1, \dots, n$, $\phi_i = \phi(i-1)$, $w_i = \phi(1 - \delta_{i0}/2 - \delta_{in}/2)$, $\phi = \pi/(n-1)$ along with a Gaussian quadrature of order m for evaluation of integrals over μ on $[0, 1]$: $\{\mu_j, \beta_j\}$, $j = 1, \dots, m$. Discretized eq. (5) takes the form of 2D system of linear equations

$$L_{j,i} = S_{j,i} + \sum_{s=1}^m \sum_{l=1}^n K_{j,l,s,i} L_{s,l}; \quad i = 1, \dots, n; \quad j = 1, \dots, m. \quad (13)$$

$$L_{j,i} = L(\tau = \tau_t, -\mu_j, \phi_i, \mu_0); S_{j,i} = S(\mu_0, \mu_j, \phi_i);$$

$$\begin{aligned} K_{j,i,s,l} = \beta_j w_i & \left(\begin{aligned} & K(\mu_j, \mu_s, \phi_{i-l+1}), 1 \leq l \leq i \\ & K(\mu_j, \mu_s, \phi_{l-i+1}), i+1 \leq l \leq n \\ & K(\mu_j, \mu_s, \phi_{i+l-1}), 1 \leq l \leq n-i+1 \\ & K(\mu_j, \mu_s, \phi_{2n-i-l+1}), n-i+2 \leq l \leq n \end{aligned} \right). \end{aligned} \quad (14)$$

The form of matrix $\mathbf{K}$ comes from the reduction of the integration over $[0, 2\pi]$ to integration over $[0, \pi]$, eq. (10) and accounting for periodicity of the kernel K . Equation (5) involves integration over a hemisphere, so specific 2D quadrature may be beneficial in the sense of reduction of the number of different directions at which functions are evaluated and, finally, reduction of the dimension of the resulting system of linear equations. Such a quadrature may have different number of nodes at different parallels. The only requirements for it are 1) if (μ_i, ϕ_j) and (μ_i, ϕ_k) are nodes then $(\mu_i, \phi_j - \phi_k)$ also belongs to the set of nodes of the rule (or its periodic extension), so that azimuth interpolation of the integrands is avoided; 2) it is also desirable to avoid equatorial plane $\mu = 0$ because radiance has discontinuity there at the surfaces of the medium.

The number of equations in system (13) is $n \times m$, so that solving it may be very time-consuming process. However, it can be solved efficiently with iterative methods. The

method given by discretized version of eq. (11) is called Richardson iterations and is known to provide slower convergence in comparison with Jacobi and Gauss-Seidel [14] iterations. Among stationary iterative methods successive over-relaxation (SOR) provides control on how far a new iterate is from the current one by blending new and current iterates with certain weights. Application of this method to 2D system of linear algebraic equations (13) in this study required the following modification of the Gauss – Seidel iterative process from its classic “textbook” description:

$$\begin{aligned}
 L_{j,i}^{(k+1)}(1 - \mathbf{K}_{j,i,j,i}) &= S_{j,i} \\
 + \sum_{s=1}^{j-1} \sum_{l=1}^n \mathbf{K}_{j,i,s,l} L_{s,l}^{(k+1)} &+ \sum_{l=1}^{i-1} \mathbf{K}_{j,i,j,l} L_{j,l}^{(k+1)} + \sum_{l=i+1}^n \mathbf{K}_{j,i,j,l} L_{j,l}^{(k)} \\
 + \sum_{s=j+1}^m \sum_{l=i+1}^n \mathbf{K}_{j,i,s,l} L_{s,l}^{(k)} & \quad (15)
 \end{aligned}$$

Then SOR method is formulated as

$$L_{j,i}^{(k+1)} = \omega \bar{L}_{j,i}^{(k+1)} + (1 - \omega) L_{j,i}^{(k)} \quad (16)$$

Where $\bar{L}_{j,i}^{(k+1)}$ is the Gauss – Seidel iterate from (15) and $0 < \omega < 2$ is a relaxation parameter that has to be chosen empirically.

Let's consider a numerical example to show the performance of the suggested method of solving eq. (5). The atmosphere is a uniform layer comprised with Rayleigh scattering and weak gas absorption (SSA $\Lambda = 0.999$) and optical thickness τ_R of 0.1 (roughly corresponds to Rayleigh optical thickness at the wavelength of 0.55 μm at mean sea level [15], Table 3 there) and transported mineral dust aerosol from OPAC [16] with optical thicknesses τ_A of 0.1 and 1.0. Surface reflection was modeled with two BRDF: bare soil model by Nilson and Kuusk [17] and MODIS operational BRDF [18]. The latter was used without any change while Nilson – Kuusk model was re-written as

$$\begin{aligned}
 \rho(\theta_s, \theta_v, \phi,) &= a(b_0 + b_1 \theta_s \theta_v \cos(\phi) + b_2(\theta_s^2 + \theta_v^2) + b_3 \theta_s^2 \theta_v^2) \\
 a &= \lim_{\theta_s \rightarrow \pi/2} a(\theta_s); b_0 = 0.31489; \quad b_1 = 0.14129; \quad b_2 = -0.082511; \quad b_3 = 0.14779 \quad (17)
 \end{aligned}$$

so, that coefficients b_i preserve the directional distribution of reflected light as given in [17] while parameter a enables variation of the overall surface brightness. In this study $a = 0.2$ was used.

Radiances L, I, J were computed with DISORT using 158 streams. The choice of the number of streams is to avoid Delta-M scaling transformation of the phase function in DISORT [19]. The radiances were computed at m_0 and $\pm m$ being nodes of the Gaussian quadrature of order $m = 24$ and ϕ on the regular grid described above with $n = 49$ as described above. The results of calculations of $L(\tau = \tau_t, -\mu, \phi, \mu_0)$ using the described method and comparison with DISORT results are given in Figures 2, 3, and 4. It is interesting to note that surface reflected radiance computed with DISORT code (middle panel in the figures) contains azimuthal

noise in the wide range of SZA from $\sim 30^\circ$ through $\sim 80^\circ$ while the solution of eq. (5) obtained with the described method (top panels) is smooth. For this reason relative difference of the solutions is highly variable. The magnitude of these variations does not exceed 3% in the wide range of viewing directions. Greater relative difference is found at VZA greater than 86° . The DISORT report [20] states: “Running DISORT on a typical 32-bit-single-precision computer usually gives results precise to at least 2–3 significant digits, although for certain special situations the precision can fall to one significant digit.” Therefore, it is hardly possible to reach better coincidence between two solutions taking into account that the precision limitation of DISORT also affects solution based on eq. (5).

It was empirically found that in the case of eq. (5) $\omega = 1$ (no relaxation) provides the fastest convergence of iterative process (16). Figure 5 shows maximum relative difference of the solution at the consecutive iterations for different relaxation parameter. Calculations were performed with machine double precision, so after reaching accuracy better than 10^{-15} further convergence cannot be achieved. One can see that such accuracy was reached with all considered values of ω but convergence slows down with the increase of absolute difference $|1 - \omega|$.

Figures (5) and (6) show that the maximum relative difference between initial approximation and the 1st iteration is ~ 0.05 . The difference decreases quickly (by orders of magnitude) with the iterative steps. This means that initial guess provides very good approximation of the final solution. In this study initial guess is always given by the source function (6) of eq. (5). The source function S is a sum of reflected direct light reaching the surface, the first term in the right-hand side of (6), and the portion of diffuse light that has not been reflected, the second term. It is interesting to compare contributions of these terms to the final solution. Figure 7 presents contribution of the reflected direct light, the whole source function, and the final solution of eq. (5) for optically thin atmosphere with $\tau_t = 0.2$, see Figure 2 for details. It is easy to see that for high Sun elevation direct light contribution dominates source function (6), however, contribution of the diffuse light may not be neglected. With the Sun elevation decreasing the contribution of the direct light decreases. At the same time, total contribution of the source function (6) to the final solution remains very high for all SZAs. It is obvious that contribution of the direct light decreases with the growth of the optical thickness of the atmosphere. Convergence pattern shown in Figure 6 is almost identical to that in Figure 5 meaning that initial guess given by the source function provides very close approximation of the final solution in the case of significantly thicker atmosphere. Two conclusions can be drawn from these observations. First, if the ground measurements of reflected radiance are used for derivation of the BRDF then atmospheric correction is necessary even for optically thin atmosphere since diffuse light contribution is not negligible. Second, since the source function gives very good approximation of the final solution, quickly converging iterative approach for BRDF derivation based on eq. (5) can potentially be developed.

4. INVERSION FOR BRDF

It is easy to see that eq. (5) can be formally resolved with respect to BRDF:

$$\rho'(\mu_0, \mu, \phi) = S_\rho(\mu_0, \mu, \phi) + \int_0^{2\pi} d\phi_1 \int_0^1 d\mu_1 K_\rho(\mu_0, \mu_1, \phi - \phi_1) \rho'(\mu_1, \mu_0, \phi_1) \quad (18)$$

$$S_\rho(\mu_0, \phi, \mu) = e^{\tau_t/\mu_0} L(\tau = \tau_t, -\mu, \phi, \mu_0) \quad (19)$$

$$\begin{aligned} K_\rho(\mu, \mu_1, \phi_1) = & \\ - e^{\tau_t/\mu_0} & \left(I(\tau_t, \mu_1, \phi_1, \mu_0) + \int_0^{2\pi} d\phi_2 \int_0^1 d\mu_2 L(\tau = \tau_t, -\mu_2, \phi_2, \mu_0) J \right. \\ & \left. (\tau = 0, -\mu_1, \phi_1 - \phi_2, \mu_2) / I_0 \right) \end{aligned} \quad (20)$$

where $\rho'(\mu_0, \mu, \phi) = \mu_0 \rho(\mu_0, \mu, \phi)$. Eq. (18) has the same form as eq. (5), so it may be solved with the method developed in the previous section. Unlike eq. (5), μ (not μ_0) enters the equations as a parameter. This makes hardly possible to use the equation for practical retrievals of BRDF ρ : this would require knowledge of the surface reflected radiance on the entire interval $[0, 1]$ of μ_0 which cannot be achieved outside the tropical zone.

Another inconvenience of practical application of equation (18) for BRDF retrievals is that it has the form $\rho' = E^*(S_1 - S_2)$ where $\rho' \ll E$ for small μ_0 (the Sun is low) thus the difference $S_1 - S_2$ is tiny and the exponential factor E is huge. Therefore, the equation is an ill-posed problem where small change of input data i.e. surface reflected radiance $L(\tau = \tau_t, -\mu, \phi, \mu_0)$ and solutions of BVPs 2 and 3 may lead to big change in the result. Attempts to use quadrature rule as in section 3 lead to divergence of the iterations even in the case of pristine (no aerosol) atmosphere. The scheme with $m = 24$ and $n = 49$ failed because of the huge value of the factor $\exp(\tau/\mu_0) \approx 1.24 \times 10^{36}$ for the smallest $\mu_0 \approx 2.406466 \cdot 10^{-3}$ and $\tau_t = 0.2$ while the BRDF $\sim 10^{-2}$.

Attempts to perform numerical computations with an atmosphere containing aerosols were unsuccessful – iterative process did not converge for the reason discussed in the previous paragraph. Attempts to use a quadrature rule with $m = 10$ and $n = 21$ also failed because this number of nodes is inadequate for description of the radiance transmitted through the atmosphere. That radiance has sharp maximum around the direction of incidence for relatively small optical thickness. Figures 8 show a numerical example of the solution of eq. (18) obtained for pristine atmosphere of optical thickness $\tau_R = 0.1$ above a surface described by BRDF (17). The quadrature rule with $m = 10$ and $n = 21$ was used for discretization of eq. (18). Figures 9 show relative difference between two consecutive iterations for fixed viewing zenith angle. Iterative process (16) takes significantly more steps to converge comparing with the direct problem. It may not reach the threshold of numerical precision for some values of the relaxation parameter ω . The optimal value of ω is between 0.9 and 1.0 in this case. Also, at the first iteration the relative difference grows. Table 1 presents relative error of the iterative solution and exact BRDF used in the numerical modeling of the surface radiance. The error is huge for very low Sun elevation but decreases rapidly: it does not exceed 0.1% for the angle of incidence less than 80° .

CONCLUSION

The integral equation relating surface reflected radiance with surface BRDF and solutions of standard problems of atmospheric radiative transfer was considered. It was shown that the kernel of the equation satisfies the sufficient condition of uniform convergence of the successive approximations of the equation. Since the kernel of the equation is known on a certain grid while its functional form may not be easily established, the solution cannot be found in an approximate analytical form. Therefore, it is convenient to use the quadrature method instead of successive approximations.

A method of discretization of the original equation based of 2D quadrature is proposed. The quadrature rule on a unit hemisphere used in this study comprises a Gaussian quadrature for integration over zenith angle and trapezoidal rule on an equidistant grid for integration over relative azimuth. The latter was chosen because it provides convenience for discretization of the difference kernel and accuracy for periodic function integration. While the scheme of discretization leads to a linear system of great dimension that system can be efficiently solved with iterative methods. On the other hand, discretization with respect to azimuth allows to avoid the necessity to summate Fourier series after solving separate equations for the Fourier components.

The resulting system of linear algebraic equations has large dimension. In the considered numerical examples it consists of 1176 (24 zenith by 49 azimuthal nodes) equations. Gauss – Seidel iteration method along with possibility of over-relaxation was used in this paper because it has close relationship with discretized version of the successive approximations method for integral equations. However, some other methods of solving linear algebraic system, e.g. singular value decomposition, can be considered in the future. The optimal value of the relaxation parameter was empirically found: the fastest convergence occurred without the use of relaxation technique.

Contributions of the different terms of the equation for the reflected radiance were analyzed. It was shown that direct light contribution does not provide good estimate to the full solution while the contribution of both direct and diffuse but not yet reflected light coming to the surface does. This enables development of an iterative fitting algorithm to derive BRDF from the surface reflected radiance. Such development constitutes the future work direction.

The approach for computation of the surface reflected radiance considered here requires knowledge of solutions of two standard problems of the atmospheric radiative transfer: radiance transmitted through the atmosphere and radiance reflected by the atmosphere with flipped over order of layers. In this study these problems were solved with discrete ordinates method by means of DISORT code. Surface reflected radiance was also computed with DISORT. The DISORT solutions were compared against the iterative solutions. Good coincidence was found between them. Significant azimuthal noise was observed in the DISORT solutions in the middle range of the Sun zenith angles. The analysis of such behavior is beyond the scope of this study. However, clear azimuthal structure of the noise allows to presume that it is caused by inaccurate summation of the Fourier series in DISORT.

The studied equation for the reflected radiance allows inversion with respect to BRDF. The equation for BRDF was derived and analyzed. It has similar form to the original equation, so it can be solved with the same method developed for radiance. However, that new equation appeared to be an ill-posed problem and its potential for BRDF retrievals is limited because it requires measurements of the reflected radiance over the entire domain of the Sun zenith angle which is physically possible only in tropical zone. The equation for BRDF was solve for the case of pristine atmosphere with. The convergence is much slower and unstable in this case.

APPENDIX A.: Convergence of successive approximations for the equation for surface radiance

Corollary 2.10 of [13] gives sufficient condition of convergence of successive approximations for integral equation of the second kind. In terms of eq. (5) that condition takes the form

$$\max \int_0^{2\pi} d\phi_2 \int_0^1 d\mu_2 K(\mu, \mu_2, \phi - \phi_2) < 1 \quad (\text{A1})$$

Consider integral

$$N = \int_0^{2\pi} d\phi_2 \int_0^1 d\mu_2 K(\mu, \mu_2, \phi - \phi_2) \quad (\text{A2})$$

using definition (7) we obtain:

$$N = \int_0^{2\pi} d\phi_2 \int_0^1 d\mu_2 \int_0^{2\pi} d\phi_1 \int_0^1 d\mu_1 \mu_1 \rho(\mu_1, \mu, \phi - \phi_1) \frac{J(\tau = 0, -\mu_1, \phi_1 - \phi_2, \mu_2)}{I_0} \quad (\text{A3})$$

Consider now the fraction in (A3). $J(\tau = 0, -\mu_1, \phi_1 - \phi_2, \mu_2)$ is radiance reflected by the flipped atmosphere in direction defined by $(-\mu_1, \phi_1)$ if it is illuminated solely from the direction defined by (μ_2, ϕ_2) with irradiance $\mu_2 I_0$. Then the ratio

$$\frac{J(\tau = 0, -\mu_1, \phi_1 - \phi_2, \mu_2)}{\mu_2 I_0} \quad (\text{A4})$$

is BRDF of the atmosphere $\rho_{atm}(\mu_2, \mu_1, \phi_1 - \phi_2)$. Strictly speaking, BRDF of a vertically inhomogeneous slab does depend on the side of the slab being illuminated. So, fraction (A4) is the BRDF of the atmosphere illuminated on its bottom surface. However, the side of illumination does not matter in the sense of condition (A1). Term “BRDF of the atmosphere” will be used for further development. Integral N can now be re-written as

$$N = \int_0^{2\pi} d\phi_1 \int_0^1 d\mu_1 \mu_1 \rho(\mu_1, \mu, \phi - \phi_1) \int_0^{2\pi} d\phi_2 \int_0^1 d\mu_2 \mu_2 \rho_{atm}(\mu_2, \mu_1, \phi_1 - \phi_2) \quad (A5)$$

Using reciprocity of BRDF, i.e. $\rho(\mu_2, \mu_1, \phi_1 - \phi_2) = \rho(\mu_1, \mu_2, \phi_2 - \phi_1)$, and the definition of directionalhemispherical reflectance, see line 3 in Table 1 of [12] (also called black sky albedo), the inner integral is the albedo of the atmosphere under unidirectional illumination:

$$\int_0^{2\pi} d\phi_2 \int_0^1 d\mu_2 \mu_2 \rho_{atm}(\mu_2, \mu_1, \phi_1 - \phi_2) = a_{atm}(\mu_1) \leq 1 \quad (A6)$$

Using (A6) we obtain

$$N \leq \int_0^{2\pi} d\phi_1 \int_0^1 d\mu_1 \mu_1 \rho(\mu_1, \mu, \phi - \phi_1) = a_{sfc}(\mu) \leq 1 \quad (A7)$$

So, we obtained that $N < 1$ unless both atmosphere albedo a_{atm} and surface albedo a_{sfc} are exactly unity for any direction of illumination. Thus, condition (A1) is satisfied proving that eq. (5) has a unique solution and successive approximations (11) converge to that solution.

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Highlights

- Convergence of the successive approximations of the integral equation for the surface reflected radiance is proven.
- 2D discretization of the equation is proposed.
- Modification of the successive over-relaxation method is applied to the discretized equation.
- The value of the relaxation parameter for the fastest convergence is found empirically.
- Inversion of the equation to derive BRDF is considered.

Figure 1.

Schemes of the BVPs: a – 1, b – 2, c – 3. Thick green line in a) depicts reflecting surface. Darker green arrows are for surface reflected radiance. Blue arrows in b) are for diffuse transmitted radiance. Blue arrows in c) are for radiance reflected by the flipped atmosphere.

Figure 2.

Reflected radiance at the surface level for two SZAs for total optical thickness of 0.2 ($\tau_A = \tau_R = 0.1$) and surface BRDF given by eq. (17). Top panel – iterative solution with eqs. (15), (16); middle pannel – DISORT solution; bottom panel – relative difference, %. Positive part of the horizontal axis represents viewing directions with relative azimuth $\phi = 0^\circ$, negative part – $\phi = 180^\circ$.

Figure 3.

The same as Figure 2 but for total optical thickness of 1.1 ($\tau_A = 1.0$, $\tau_R = 0.1$).

Figure 4.

The same as Figure 3 but for MODIS BRDF model with parameters $p_1 = 0.265$, $p_2 = 0.066$, $p_3 = 0.0$ (actual values retrieved over Sahara desert).

Figure 5.

Maximum relative difference between iterative solutions at the consecutive steps for different relaxation parameter ω : red dots – 0.8, blue – 0.9, black – 1.0, cyan – 1.1, magenta – 1.2. Atmosphere, surface, and illumination conditions are from Figure 2. Black dashed line – desirable accuracy – iterative process stops by reaching relative difference better than 10^{-15} .

Figure 6.

The same as Figure 5 but for atmosphere, surface, and illumination conditions from Figure 4.

Figure 7. Reflected radiance at the surface level due to direct light reaching the surface, top panels; direct and diffuse but not reflected reaching the surface, middle panels; and the solution of eq. (5), bottom panels.

Figure 8.

BRDF ($\times 100$) obtained with iterative solution of eq. (18) for the case of Rayleigh atmosphere, $\tau_R = 0.1$. Solutions of the BVPs 1 – 3 defining source function (19) and kernel (20) were computed with DISORT for BRDF (17).

Figure 9.

The same as Figure 5 but for BRDF solutions presented in Figure 8 for different viewing zenith angles θ_v (not to be confused with θ_i values in Figure 8): *a)* 24.17° , *b)* 44.22° , *c)* 64.81° , *d)* 86.13° . Relaxation parameter ω : red dots – 0.8, blue – 0.9, black – 0.95, cyan – 1.0, magenta – 1.1.

Table 1.

Relative error of the BRDF obtained from eq. (18) with the use of modeled surface reflected radiance with BRDF (17) as a function of SZA.

Polar angle of incidence, degree	Maximum relative error, %
89.25	70.
86.13	0.31
80.78	0.12
73.54	0.075
64.81	0.065
54.94	0.065
44.22	0.072
32.89	0.080
21.17	0.077
9.265	0.057