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Modification of Green's one-step-late algorithm for attenuated emission data

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Abstract

Green's one-step-late (OSL) algorithm is a popular image reconstruction algorithm in emission-data tomography, in spite of its convergence issues. One drawback of Green's algorithm is that the algorithm exhibits non-stationary regularization when the algorithm's projector and backprojector model the attenuation effects in single photon emission computed tomography (SPECT). This paper suggests a remedy to improve Green's OSL algorithm so that stationary regularization can be obtained. This paper also observes some similarities between the modified Green's OSL algorithm and the gradient descent algorithm.

Keywords

iterative image reconstruction algorithm; image reconstruction; tomography; maximum-likelihood expectation-maximization; noise regularization; emission tomography

1. Introduction

Green developed a one-step-late (OSL) expectation-maximization (EM) algorithm for emission tomography image reconstruction (Green 1987, Green 1990). Later, a line-search version of it was suggested (Lange 1990). The main drawback of Green's algorithm is that it may diverge (Fessler and Hero 1995). Green's OSL algorithm is user-friendly, easy to implement, and effective in incorporating image-domain constraints. It is thus suitable to solve penalized maximum likelihood problems when the data is corrupted with Poisson noise. Green's OSL algorithm is a popular image reconstruction algorithm. It has wide applications from positron emission tomography (PET) and single photon emission computed tomography (SPECT) to archaeological stratigraphy and x-ray computed tomography (CT) (Allum *et al* 1999, Panin *et al* 1999, Mair and Zahnen 2006, Zahnen 2006, Arya *et al* 2016, Berker *et al* 2016, Ellis and Reader 2017). Green's OSL algorithm has many other applications such as in minimization of penalized I-divergence (Choi and Lanterman 2007).

In this paper, we make an observation in section 3 that when Green's OSL algorithm is applied to SPECT imaging with attenuation correction, its regularization is not stationary throughout the image. One region of the image can be over-smoothed, while another region of the image can still be very noisy. This paper in section 2 proposes a remedy to improve Green's OSL algorithm so that the regularization is stationary. Computer simulations and comparison studies are presented to demonstrate the feasibility of the proposed revision.

2. Methods

2.1. Iterative Green's OSL algorithm

The iterative Green's OSL algorithm can be expressed as (Green 1987, Green 1990)

$$x_{i,j}^{(n+1)} = \frac{x_{i,j}^{(n)}}{\sum_k a_{(i,j)k} + \beta U_{i,j}^{(n)}} \sum_k a_{(i,j)k} \frac{p_k}{\sum_{\hat{i}, \hat{j}} a_{(\hat{i}, \hat{j})k} x_{\hat{i}, \hat{j}}^{(n)}}, \quad (1)$$

where $x_{i,j}^{(n)}$ is the reconstructed image pixel (i, j) at the n th iteration, p_k is the k th ray-sum measurement, $a_{(i,j)k}$ is the contribution of pixel $x_{i,j}$ to measurement p_k , β is a control parameter, and $U_{i,j}^{(n)}$ is the derivative of a penalty function V with respect to the image pixel $x_{i,j}^{(n)}$, at the n th iteration, that is,

$$U_{i,j}^{(n)} = \frac{\partial V}{\partial x_{(i,j)}^{(n)}}. \quad (2)$$

This penalty function V is user-defined to encourage the image to look like what the user wants. Using a penalty function to enforce the solution to have a certain property or properties is referred to as the maximum a posteriori (MAP) method.

2.2. Modified Green's OSL algorithm

The summation $\sum_{i,j} a_{(i,j)k} x_{i,j}$ is the forward projector that maps the image $x_{i,j}$ to the sinogram domain. The summation $\sum_k a_{(i,j)k} q_k$ is the backprojector that maps the sinogram-domain quantity q_k to the image domain. It is noticed in (1) that the parameter β in $\sum_k a_{(i,j)k} + \beta U_{i,j}^{(n)}$ adjusts the level of regularization based on $U_{i,j}^{(n)}$. The first term in $\sum_k a_{(i,j)k} + \beta U_{i,j}^{(n)}$ is the backprojection of the constant one. When the attenuation is modeled in the projector and the backprojector, the backprojection of a constant one is not a uniform image. In the central region of $\sum_k a_{(i,j)k}$, the values are relatively smaller than those at the peripheral regions. Thus, $U_{i,j}^{(n)}$ will have stronger influence on the reconstruction in the central region than at the peripheral regions. As a result, the image may look too smooth in the central region, while the image may be too noisy at the peripheral regions.

We now propose a method to overcome this drawback of Green's OSL algorithm by replacing $\sum_k a_{(i,j)k} + \beta U_{i,j}^{(n)}$ by $(1 + \beta U_{i,j}^{(n)}) \sum_k a_{(i,j)k}$. In this revised form, the Bayesian

influence $\beta U_{i,j}^{(n)}$, is compared with a constant 1, instead of $\sum_k a_{(i,j)k}$ that may not be a constant. The new algorithm is shown in the formula below:

$$x_{i,j}^{(n+1)} = \frac{1}{1 + \beta U_{i,j}^{(n)}} \times \frac{x_{i,j}^{(n)}}{\sum_k a_{(i,j)k}} \sum_k a_{(i,j)k} \times \frac{p_k}{\sum_{\hat{i}, \hat{j}} a_{(\hat{i}, \hat{j})k} x_{\hat{i}, \hat{j}}^{(n)}}. \quad (3)$$

This modified Green's OSL algorithm is simply a scaled version of the famous ML-EM algorithm at each iteration. As $U_{i,j}^{(n)} \rightarrow 0$, the modified Green's OSL algorithm reduces to the ML-EM algorithm.

2.3. Comparison between the modified Green's OSL algorithm and the common gradient descent algorithm

It is straightforward to re-write (3) in the additive update form as

$$\begin{aligned} x_{i,j}^{(n+1)} &= x_{i,j}^{(n)} - \beta x_{i,j}^{(n+1)} U_{i,j}^{(n)} - \frac{x_{i,j}^{(n)}}{\sum_k a_{(i,j)k}} \\ &\times \sum_k \frac{a_{(i,j)k}}{\sum_{\hat{i}, \hat{j}} a_{(\hat{i}, \hat{j})k} x_{\hat{i}, \hat{j}}^{(n)}} \left(\sum_{\hat{i}, \hat{j}} a_{(\hat{i}, \hat{j})k} x_{\hat{i}, \hat{j}}^{(n)} - p_k \right) \\ &= x_{i,j}^{(n)} - \lambda_1 U_{i,j}^{(n)} - \lambda_2 \sum_k a_{(i,j)k} w_k \times \left(\sum_{\hat{i}, \hat{j}} a_{(\hat{i}, \hat{j})k} x_{\hat{i}, \hat{j}}^{(n)} - p_k \right), \end{aligned} \quad (4)$$

where

$$\lambda_1 = \beta x_{i,j}^{(n+1)}, \quad (5)$$

$$\lambda_2 = \frac{x_{i,j}^{(n)}}{\sum_k a_{(i,j)k}}, \quad (6)$$

$$w_k = \frac{1}{\sum_{\hat{i}, \hat{j}} a_{(\hat{i}, \hat{j})k} x_{\hat{i}, \hat{j}}^{(n)}}. \quad (7)$$

Expression (4) is in the standard form of a gradient descent algorithm that intends to minimize an objective function

$$F = V + \frac{1}{2} \sum_k w_k \left(\sum_{i,j} a_{(i,j)k} x_{i,j} - p_k \right)^2. \quad (8)$$

In fact, if we find the gradient of the objective function F in (8) we have

$$\frac{\partial F}{\partial x_{i,j}} = \frac{\partial V}{\partial x_{i,j}} + \sum_k a_{(i,j)k} w_k \left(\sum_{\hat{i}, \hat{j}} a_{(\hat{i}, \hat{j})k} x_{\hat{i}, \hat{j}} - p_k \right), \quad (9)$$

where the weighting factors w_k are assumed to be constants in the objective function.

However, in the gradient descent algorithm (4), the weighting factors w_k are made adaptive as defined in (7). Using adaptive weighting factors is a typical strategy in EM algorithms.

In a gradient descent algorithm, the unknowns $x_{i,j}$ are updated in the negative gradient direction. This is achieved via multiplying the negative gradient by a relaxation parameter. In general, the convergence of a gradient descent algorithm depends on the objective function and on the step size (also known as the relaxation parameter). When the step size is small enough, a local convergence is guaranteed.

If the original Green's OSL algorithm is expressed in the additive form as the second line of (4), the parameter λ_1 will be modified as $\lambda_1 = \beta x_{i,j}^{(n+1)} / \sum_k a_{(i,j)k}$, which is influenced by the backprojection of constant 1. For the original Green's OSL algorithm, the regularization is stronger at locations where the value of $\sum_k a_{(i,j)k}$ smaller.

The additive form (4) serves three purposes as follows. It reveals the one-step-late nature of the algorithm because the next iteration value $x_{i,j}^{(n+1)}$ appears on the right-hand-side of (4). It clearly exhibits the gradients that the algorithm is trying to drive to zero, as well as their associated weights. It also explicitly shows the relationship between the parameter β and the relaxation parameter λ_1 in a gradient descent algorithm. The convergence of a gradient descent algorithm requires its relaxation parameters to be small enough.

If the true solution with $\sum_{\hat{i}, \hat{j}} a_{(\hat{i}, \hat{j})k} x_{\hat{i}, \hat{j}} = p_k$ and $U_{i,j} = 0$ exists, the true solution is a fixed point of the modified Green's OSL algorithm as well as the original Green's OSL algorithm. In fact, let $\sum_{\hat{i}, \hat{j}} a_{(\hat{i}, \hat{j})k} x_{\hat{i}, \hat{j}}^{(n)} = p_k$ and $U_{i,j}^{(n)} = 0$, the right-hand-side of (3) becomes

$$\begin{aligned} & \frac{x_{i,j}^{(n)}}{(1 + \beta U_{i,j}^{(n)}) \sum_k a_{(i,j)k}} \sum_k a_{(i,j)k} \frac{p_k}{\sum_{\hat{i}, \hat{j}} a_{(\hat{i}, \hat{j})k} x_{\hat{i}, \hat{j}}^{(n)}} \\ &= \frac{x_{i,j}^{(n)}}{(1 + 0) \sum_k a_{(i,j)k}} \sum_k a_{(i,j)k} \frac{p_k}{p_k} \\ &= \frac{x_{i,j}^{(n)}}{\sum_k a_{(i,j)k}} \sum_k a_{(i,j)k} \\ &= x_{i,j}^{(n)}. \end{aligned} \quad (10)$$

This implies $x_{i,j}^{(n+1)} = x_{i,j}^{(n)}$.

2.4. The total-variation (TV) penalty function

In this paper, we choose the TV-norm as the penalty function to illustrate the effectiveness of our new algorithm. The TV-norm of a two-dimensional image $x_{i,j}$ can be symbolically expressed as (Panin *et al* 1999)

$$V = \sum_{i,j} \sqrt{(x_{i,j} - x_{i,j+1})^2 + (x_{i,j} - x_{i+1,j})^2}, \quad (11)$$

and thus

$$\begin{aligned} U_{i,j} \approx & \frac{(x_{i,j} - x_{i,j+1}) + (x_{i,j} - x_{i+1,j})}{\sqrt{(x_{i,j} - x_{i,j+1})^2 + (x_{i,j} - x_{i+1,j})^2} + \epsilon} \\ & + \frac{x_{i,j} - x_{i,j-1}}{\sqrt{(x_{i,j-1} - x_{i,j})^2 + (x_{i,j-1} - x_{i+1,j-1})^2} + \epsilon} \\ & + \frac{x_{i,j} - x_{i-1,j}}{\sqrt{(x_{i-1,j} - x_{i-1,j+1})^2 + (x_{i-1,j} - x_{i,j})^2} + \epsilon}. \end{aligned} \quad (12)$$

The small value of ϵ was introduced to prevent the denominator being zero. In our implementation, we chose $\epsilon = 0.0001$.

2.5. Computer simulations

In our computer simulations, the projector and backprojector were ray-driven, line-length weighted and with an attenuation model. Three algorithms were used to reconstruct the images. These algorithms were the famous ML-EM algorithm (Shepp and Vardi 1982, Lange and Carson 1984), the original Green's OSL algorithm, and our modified Green's OSL algorithm. The β value was 0.1 for the original Green's OSL algorithm. The β value was 0.01 for the modified Green's OSL algorithm. The β values were chosen so that the best results could be obtained for the associated reconstruction algorithm. When $\beta = 0$, the Green's algorithms reduce to the ML-EM algorithm.

There were 180 projection views over 360° . The images were reconstructed in a 128×128 (pixels) array. A parallel-hole collimation was assumed in data acquisition. The detector had 128 detection bins, and the bin size was the same as the image pixel size.

A uniform two-dimensional (2D) circular phantom with a diameter of 120.32 pixels was used in the computer simulations. The phantom, based on SPECT imaging, contained two smaller cold discs and two smaller hot discs, all with a diameter of 25.6 pixels, as shown in figure 1. Let the image intensity of the larger circular disc be 1 unit. The cold discs had an intensity value of 0.5, and the hot discs had an intensity value of 1.5. A uniform circular attenuator with a diameter of 120.32 pixels and attenuation coefficient of 0.05 per pixel was used in data generation and in image reconstruction. The projections were generated analytically without using pixels.

Noisy projections were generated using the Poisson noise model. Two noise levels were simulated: 4×10^6 counts for the higher count case and 2×10^6 counts for the lower count case. The projection sino-grams are shown in figure 2.

Two regions were selected in the emission image for total-variation (TV) norm noise evaluation. The TV norm is able to measure the image fluctuation. These two regions are indicated in figure 1 as Region 1 and Region 2. The ratio of the TV norms in Region 2 over in Region 1 was used as a figure-of-merit for noise uniformity evaluation. A line-profile was also provided for each reconstructed image. The location of the line profile is indicated in figure 1.

3. Results

Figure 3 shows the image of backprojection of constant 1 and its central horizontal profile. The minimum value of this image is 8.9 and the maximum value of this image is 110. In (1), the term $\beta U_{i,j}^{(n)}$, is compared with $\sum_k a_{(i,j)k}$. On the other hand, in (3), the term $\beta U_{i,j}^{(n)}$, is compared with 1. For similar regularization effects, the β values are different for algorithm (1) and for algorithm (3).

Figures 4 and 5 show the reconstructed images using the conventional ML-EM algorithm without penalty regularization, the original Green's OSL algorithm with $\beta = 0.1$, and the proposed algorithm with $\beta = 0.01$, respectively. In figure 4 the total acquired counts were 4×10^6 . In figure 5 the total acquired counts were 2×10^6 . The sinogram for figure 4 had lower noise level, and the sinogram for figure 5 had higher noise level. At each noise level, results from 5 different iteration numbers are shown.

The EM reconstruction images are noisy, especially in the central region, where the information is severely attenuated. As indicated by the line-profiles, the images become noisier as the iteration number increases.

The TV penalty function has the reputation of reducing the noise while maintaining the edges (Rudin *et al* 1992). Both of the Bayesian algorithms (i.e., the original Green's OSL algorithm and the proposed algorithm) use the TV penalty function for regularization. For Bayesian algorithms such as Green's original and modified algorithms, one can use large iteration numbers and run the algorithms towards convergence. The noise level in the reconstructed image depends on the β value. A larger β value gives a less-noisy image. However, if the β value is too large, the algorithm will diverge.

We use the ratio of the TV values, $TV(R2)/TV(R1)$, to characterize the noise uniformity in the image. We refer this ratio as the *quantification index*, as shown in tables 1 and 2 for the two different noise levels, respectively. After the original Green's OSL algorithm is converged, the Bayesian constraint enforcement is not stationary throughout the reconstructed images. The regularization is stronger at the central region than at the peripheral regions. On the other hand, the modified Green's OSL algorithm provides fairly stationary regularization throughout the image.

4. Conclusions

Green's one-step-late EM algorithm is popular and user-friendly. However, if the backprojection of a constant one is not uniform, the regularization is not uniform either. This drawback can cause non-uniform smoothness and noise level for different regions in the reconstructed image. This paper uses examples of the attenuation correction modeling in SPECT to illustrate this undesirable effect (see figures 4 and 5). We face the problem that in one region the β value is not large enough to control the noise, while this same β value is too large and causes over-smoothing in another region. This effect is caused by the unbalanced $\sum_k a_{(i,j)k} + \beta U_{i,j}^{(n)}$ between $\sum_k a_{(i,j)k}$ and $U_{i,j}^{(n)}$. Without the attenuation model, $\sum_k a_{(i,j)k}$ is essentially constant in the entire image and the original Green's algorithm works well. When the attenuation is modeled, $\sum_k a_{(i,j)k}$ deviates from a constant inside the image, and the variation in $\sum_k a_{(i,j)k}$ results in variation in the strength of regularization.

Our modified Green's OSL algorithm is able to reduce this undesired effect. As seen in $(1 + \beta U_{i,j}^{(n)})$, we only need to adjusting the balance between a constant 1 and $U_{i,j}^{(n)}$, which is achievable. Our modification to the original Green's OSL algorithm is minimal. Other favorable properties of Green's algorithm still maintain.

The modified Green's OSL algorithm is essentially a gradient descent algorithm if it is expressed in an additive form. This observation makes the convergence analysis easier. If the penalty function is unimodal, the only requirement for local convergence is β being small enough and positive.

Our modification strategy can apply to other applications, where the backprojection of constant one is not uniform such as in pin-hole imaging and convergent-beam imaging.

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Figure 1.
Left: an emission phantom. The dotted horizontal line indicates the location of line-profiles. Square Region 1 and rectangular Region 2 are used for TV norm evaluation. Right: a uniform attenuator.

Figure 2. Sinograms. Left: noiseless case. Middle: the case with 4×10^6 counts. Right: the case with 2×10^6 counts.

Figure 3. Backprojection image of constant 1 and its associated line-profile along the central horizontal line.

Figure 4. Reconstruction results from the sinogram of a lower noise level (4×10^6 counts).

Figure 5.
Reconstruction results from the sinogram of a higher noise level (2×10^6 counts).

Table 1.

Quantification index (TV(R2)/TV(R1)) for the lower noise level (4×10^6 counts).

Algorithm	10 iterations	100 iterations	500 iterations	1000 iterations	10 000 iterations
ML-EM	1.1692	0.7815	0.4824	0.4514	0.3663
Green ($\beta = 0.1$)	1.4678	4.6211	5.3305	5.3996	5.3505
Proposed ($\beta = 0.01$)	0.9724	0.9566	0.9141	0.9207	0.9624

Table 2.

Quantification index(TV(R2)/TV(R1)) for the higher noise level (2×10^6 counts).

Algorithm	10 iterations	100 iterations	500 iterations	1000 iterations	10 000 iterations
ML-EM	1.2274	0.8287	0.5530	0.5323	0.4738
Green ($\beta = 0.1$)	2.2609	8.1766	9.6805	9.8039	9.8522
Proposed ($\beta = 0.01$)	0.9899	0.9500	0.8508	0.8780	0.8433