The spreading frontiers of avian-human influenza described by the free boundary

LEI ChengXia¹, KIM KwangIk² & LIN ZhiGui^{1,*}

¹*School of Mathematical Science, Yangzhou University, Yangzhou 225002, China;*

²*Department of Mathematics, Pohang University of Science and Technology,
Pohang 790-784, Korea*

Email: leichengxia001@163.com, kimki@postech.ac.kr, zglin68@hotmail.com

Received October 30, 2012; accepted April 7, 2013; published online May 27, 2013

Abstract In this paper, a reaction-diffusion system is proposed to investigate avian-human influenza. Two free boundaries are introduced to describe the spreading frontiers of the avian influenza. The basic reproduction numbers $r_0^F(t)$ and $R_0^F(t)$ are defined for the bird with the avian influenza and for the human with the mutant avian influenza of the free boundary problem, respectively. Properties of these two time-dependent basic reproduction numbers are obtained. Sufficient conditions both for spreading and for vanishing of the avian influenza are given. It is shown that if $r_0^F(0) < 1$ and the initial number of the infected birds is small, the avian influenza vanishes in the bird world. Furthermore, if $r_0^F(0) < 1$ and $R_0^F(0) < 1$, the avian influenza vanishes in the bird and human worlds. In the case that $r_0^F(0) < 1$ and $R_0^F(0) > 1$, spreading of the mutant avian influenza in the human world is possible. It is also shown that if $r_0^F(t_0) \geq 1$ for any $t_0 \geq 0$, the avian influenza spreads in the bird world.

Keywords reaction-diffusion system, avian-human influenza, free boundary, spreading frontiers

MSC(2010) 35K57, 92D30

Citation: Lei C X, Kim K I, Lin Z G. The spreading frontiers of avian-human influenza described by the free boundary. *Sci China Math*, 2014, 57: 971–990, doi: 10.1007/s11425-013-4652-7

1 Introduction

In history, epidemic diseases swept through countries and made people suffer from big disasters. The plague flourished four times in Europe, and it killed more than 15 percent population in the affected areas each time [2, 6]. In 6th century, the first outbreak caused about half of European people dead; the second outbreak occurred in 1346–1350 and the rate of the mortality was 33.33%, the third outbreak killed $\frac{1}{6}$ population of London in 1665–1666. The last happened in 1720–1722, the death rate was very high, which resulted in half population dead in Marseilles [2]. The prevalent of the disease usually leads the population declined sharply. Beside the Black Death, there are many contagious diseases, such as smallpox, malaria, yellow fever and so on.

At present, about half of the world's population is threatened with different infectious diseases [5]. Today the WHO (World Health Organization) claims that the infectious diseases are still hostile in the human world. From now on, people should be cautious about the diseases and scientists should contribute themselves to eradicate epidemic diseases. Recently, great attention has been paid to study the contagious diseases. In early 20th century, mathematical models have been constructed to study the dynamics of

*Corresponding author

infectious diseases, see the book [21] and references therein. To explore the pattern of the plague in London and the pest in Mumbai, Kermack and McKendrick constructed the well-known SIR compartment model in 1927 (see [9]), and the SIS compartment model [10] was introduced by them in 1932. The readers can see some related work in [15]. To understand the epidemiology of some infectious diseases, other compartment models were formulated, for example, the SI compartment model [19], which is constructed by the assumption that the individual cannot recover from the disease. This model is appropriate for the diseases such as herpes and HIV. If the individual first goes through a exposed period before becoming to the infectious, this can be described by the SEI compartment model (see [27] for example). The SIRS compartment model [20] supposes that the recovered individual has the temporary immunity.

In this paper, we use the SI and SIR models to describe the avian influenza spreading in the bird world and the human world, respectively. Avian influenza (AI) or “bird flu” is an infectious viral disease of birds. AI viruses are divided into high pathogenicity and low pathogenicity. The highly pathogenic avian influenza (HPAI) was first identified as a serious disease of poultry by an Italian scientist, Edoardo Perroncito, in 1878 (see [3]). It was reported that the 1918 pandemic is most possibly caused by avian influenza [26], and the pandemic is now estimated to have killed 50–100 million people [22, 26].

As we know, most avian influenza viruses do not infect humans, however, some highly pathogenic viruses, such as H5N1, can result in high death rates (up to 100 percent mortality within 48 hours [16]) in some poultry species and cause severe disease and death in humans. The H5N1 virus first infected humans in 1997 during a poultry outbreak in Hong Kong, China. After that, infection to human of avian influenza occurred successively [8]. The widespread spreading of H5N1 virus led several hundreds human to infection, many human to death and the economy to have serious impact. Continuing outbreaks of highly pathogenic avian influenza across the world, threatens to be more widespread than the SARS occurred in 2003 (see [24]). It is known from the WHO that avian influenza has now caused 359 human infections (As of 15 August 2012), with an approximate 50 percent mortality rate. Fortunately, there is still no data support that avian influenza can be propagated among humans [25].

Recently Iwami *et al.* [8] proposed an avian-human influenza epidemic model to interpret the mutation process of avian influenza:

$$\begin{cases} \dot{X}(t) = c - bX - \omega XY, \\ \dot{Y}(t) = \omega XY - (b + m)Y, \\ \dot{S}(t) = \lambda - \mu S - (\beta_1 Y + \beta_2 H)S, \\ \dot{B}(t) = \beta_1 SY - (\mu + d + \varepsilon)B, \\ \dot{H}(t) = \beta_2 SH + \varepsilon B - (\mu + \alpha + \gamma)H, \\ \dot{R}(t) = \gamma H - \mu R, \end{cases} \quad (1.1)$$

where the first two equations form the SI model, which describes the bird system, no recovered bird exists since the highly pathogenic viruses have high death rates. X and Y are the population of the susceptible birds and the infective birds, respectively. The last four equations are SIR model for humans, S , B , H and R denote, respectively, the population of the susceptible humans, infected with avian influenza, infected with mutant avian influenza and recovered humans from mutant avian influenza. The reproductive number $r_0 = \frac{c\omega}{b(b+m)}$ in the bird system and the reproductive number $R_0 = \frac{\beta_2 \lambda}{\mu(\mu + \alpha + \gamma)}$ in the human system are defined. The authors showed that when $r_0 \leq 1$ and $R_0 > 1$, the human-endemic equilibrium (i.e., the boundary equilibrium)

$$E_h := (X_0, 0, \tilde{S}, 0, \tilde{H}, \tilde{R})$$

is global asymptotically stable, while if $r_0 > 1$, the full-endemic equilibrium (i.e., the interior equilibrium)

$$E_+ := (X^*, Y^*, S^*, B^*, H^*, R^*)$$

is global asymptotically stable. In other words, mutant avian influenza spreads in the human world in the above two cases.

Considering spatial spreading and time delay, Kim et al. [12] introduced diffusion terms and extended the system (1.1) to the following reaction diffusion system:

$$\begin{cases} X_t - D_1 \Delta X = c - bX - \omega XY, \\ Y_t - D_1 \Delta Y = \omega XY - (b + m)Y, \\ S_t - D_2 \Delta S = \lambda - \mu S - S \int_{\Omega} \int_{-\infty}^t K(x, y, t-s)(\beta_1 Y + \beta_2 H)(s, y) ds dy, \\ B_t - D_2 \Delta B = \beta_1 S \int_{\Omega} \int_{-\infty}^t K(x, y, t-s) Y(s, y) ds dy - (\mu + d + \varepsilon)B, \\ H_t - D_2 \Delta H = \beta_2 S \int_{\Omega} \int_{-\infty}^t K(x, y, t-s) H(s, y) ds dy + \varepsilon B - (\mu + \alpha + \gamma)H, \\ R_t - D_2 \Delta R = \gamma H - \mu R, \end{cases} \quad (1.2)$$

for $t > 0$, $x \in \Omega$. The system (1.2) with a no-flow boundary condition is discussed, and the basic reproductive numbers r_0 and R_0 are defined the same as those in [8]. They concluded that the disease-free equilibrium $E_0 := (c/b, 0, \lambda/\mu, 0, 0, 0)$ is locally asymptotically stable if $r_0 < 1$ and $R_0 < 1$; and human-endemic equilibrium E_h is locally asymptotically stable if $r_0 < 1$ and $R_0 > 1$; while $r_0 > 1$ the full-endemic equilibrium E_+ is locally asymptotically stable. Furthermore, they proved the disease-free equilibrium E_0 is globally asymptotically stable when $r_0 < 1$ and $R_0 < 1$.

It is known that the solution to System (1.2) with Dirichlet or Neumann boundary condition in a bounded domain is always positive for any time $t > 0$ no matter what the nonnegative nontrivial initial date is. It means that the avian influenza spreads to the whole area immediately even when the infectious is confined to a small part of the area at the beginning. It does not match the observed fact that the avian influenza always spreads gradually. To describe the gradual progress of avian influenza spreading, we are attempting to consider an avian-human influenza model with free boundaries, which describe the spreading frontiers of the virus. In the next section, a mathematical model of the free boundary in both the bird world and the human world is constructed first.

In Section 3, we give the local existence and uniqueness of the solution to Problem (2.4), then prove that the right free boundary $x = h(t)$ is increasing and the left boundary $x = g(t)$ is decreasing. Finally, we conclude that the solution to (2.4) is global and unique.

Section 4 is devoted to the basic reproduction numbers $r_0^F(t)$ for avian influenza in the bird system with free boundary and $R_0^F(t)$ for the mutant avian influenza in the human system with free boundary. It is shown that $r_0^F(t)$ and $R_0^F(t)$ are strictly increasing with respect to t . Moreover, if $\lim_{t \rightarrow \infty} (h(t) - g(t)) = \infty$, then $\lim_{t \rightarrow \infty} r_0^F(t) = r_0$ and $\lim_{t \rightarrow \infty} R_0^F(t) = R_0$, where $r_0 (= \frac{\beta_h}{d_b})$ is the basic reproduction number for avian influenza in the bird system (4.1), and $R_0 (= \frac{\beta_m}{d_h + \gamma_h})$ is the basic reproduction number for the mutant avian influenza in the human system (4.1).

Sections 5–7 deal with the asymptotic behaviors of the free boundaries and the solution. In Section 5, we first prove that the free boundaries $x = h(t)$ and $x = g(t)$ are either finite or infinite at the same time. Then we show that if $r_0^F(0) < 1$ and $\|B_{i0}(x)\|_{C([-h_0, h_0])}$ is sufficiently small, we have $h_{\infty} - g_{\infty} < \infty$ and $\lim_{t \rightarrow \infty} \|B_i(\cdot, t)\|_{C([g(t), h(t)])} = 0$, i.e., the avian influenza vanishes in the bird world. Finally by constructing suitable upper solution to problem (2.4), we give the sufficient conditions for the avian influenza to vanish in the bird and human worlds, i.e., if $r_0^F(0) < 1$, $R_0^F(0) < 1$ and the initial infected data are small enough, then $h_{\infty} - g_{\infty} < \infty$ and the solution decays gradually toward zero.

In Section 6, we investigate the case that $r_0^F(0) < 1$ and

$$R_0^F(0) \geq 1 + \frac{M}{D_h(\frac{\pi}{2h_0})^2 + (d_h + \gamma_h)}.$$

In this situation, the mutant avian influenza transmits in the human. In Section 7, we show that if $r_0^F(t_0) \geq 1$ for any $t_0 \geq 0$, the avian influenza spreads in the bird world. Furthermore, we demonstrate that if $r_0^F(0) < 1 < r_0$ and $B_{i0}(x)$ is big enough, then

$$h_{\infty} - g_{\infty} = \infty \quad \text{and} \quad \liminf_{t \rightarrow +\infty} \|B_i(\cdot, t)\|_{C([g(t), h(t)])} > 0.$$

In other words, the avian influenza can spread in the whole area even that $r_0^F(0) < 1$. A short discussion is given in Section 8.

2 Model formulation

We consider a general avian-human model and classify the birds and humans as the following categories:

- Susceptible birds $B_s(t)$, infected birds $B_i(t)$ with avian influenza;
- Susceptible humans $H_s(t)$, infected humans $H_a(t)$ with avian influenza, infected humans $H_m(t)$ with mutant avian influenza and recovered ones $H_r(t)$ from mutant avian influenza.

Here, we assume that all birds infected with avian influenza are dead or remain infected and can be never recovered. Then the bird system can be described by SI model. But the human infected with mutant avian influenza can be cured, then the person has immunity against avian influenza virus in all his/her life. So we can use the SIR model to represent the human world. If we do not consider the spatial spreading of the virus, an ODE model reads

$$\begin{cases} \frac{dB_s}{dt} = N_b G(B_s, B_i) - \frac{\beta_b B_s B_i}{N_b} - d_b B_s, \\ \frac{dB_i}{dt} = \frac{\beta_b B_s B_i}{N_b} - d_b B_i, \\ \frac{dH_s}{dt} = r_h N_h - \frac{\beta_a B_i H_s}{N_h} - \frac{\beta_m H_m H_s}{N_h} - d_h H_s, \\ \frac{dH_a}{dt} = \frac{\beta_a B_i H_s}{N_h} - (d_h + \varepsilon_h) H_a, \\ \frac{dH_m}{dt} = \frac{\beta_m H_s H_m}{N_h} + \varepsilon_h H_a - (d_h + \gamma_h) H_m, \\ \frac{dH_r}{dt} = \gamma_h H_m - d_h H_r, \end{cases} \quad (2.1)$$

where $G(B_s, B_i)$ is the per capita reproduction rate of the adult birds which can be taken as $r_b(1 - \frac{B_s + B_i}{K_b})$, or simply a constant r_b ; d_b and d_h are the death rates with the bird and the human; β_a , β_b and β_m denote the contact rates of the virus from infected bird with avian influenza to human, infected bird to bird and infected human with mutant avian influenza to human, respectively; r_h is the birth rate of human; γ_h denotes the recover rate of human from the mutant avian influenza virus; ε_h is the mutant rate with the avian influenza virus. N_b and N_h denote the total numbers of the bird and the human, respectively.

In general, the individual disperses randomly in the habitat. Therefore, we consider not only the individual's activity in temporal dimension, but also the distribution of the individual in the spatial and the dynamic characteristic of the avian influenza. To describe the diffusion of the disease, we introduce the spatial diffusion terms. We start with one-dimensional case: $-\infty < x < \infty$, thus an extended version of the avian-human model can be described by

$$\begin{cases} \frac{\partial B_s}{\partial t} - D_b \frac{\partial^2 B_s}{\partial x^2} = N_b G(B_s, B_i) - \frac{\beta_b B_s B_i}{N_b} - d_b B_s, \\ \frac{\partial B_i}{\partial t} - D_b \frac{\partial^2 B_i}{\partial x^2} = \frac{\beta_b B_s B_i}{N_b} - d_b B_i, \\ \frac{\partial H_s}{\partial t} - D_h \frac{\partial^2 H_s}{\partial x^2} = r_h N_h - \frac{\beta_a B_i H_s}{N_h} - \frac{\beta_m H_m H_s}{N_h} - d_h H_s, \\ \frac{\partial H_a}{\partial t} - D_h \frac{\partial^2 H_a}{\partial x^2} = \frac{\beta_a B_i H_s}{N_h} - (d_h + \varepsilon_h) H_a, \\ \frac{\partial H_m}{\partial t} - D_h \frac{\partial^2 H_m}{\partial x^2} = \frac{\beta_m H_s H_m}{N_h} + \varepsilon_h H_a - (d_h + \gamma_h) H_m, \\ \frac{\partial H_r}{\partial t} - D_h \frac{\partial^2 H_r}{\partial x^2} = \gamma_h H_m - d_h H_r, \end{cases} \quad (2.2)$$

for $-\infty < x < \infty$ and $t > 0$, where D_b and D_h are positive diffusion coefficients for the birds and humans, respectively.

For simplicity, we take $G(B_s, B_i) = r_b$, and let $r_b = d_b$, $r_h = d_h$. In other words, we assume the total number of birds and that of humans remain constant. Let $N_b^* = B_s + B_i$ and $N_h^* = H_s + H_a + H_m + H_r$. Then (2.2) becomes

$$\begin{cases} \frac{\partial B_i}{\partial t} - D_b \frac{\partial^2 B_i}{\partial x^2} = \frac{\beta_b(N_b^* - B_i)B_i}{N_b^*} - d_b B_i, \\ \frac{\partial H_a}{\partial t} - D_h \frac{\partial^2 H_a}{\partial x^2} = \frac{\beta_a B_i(N_h^* - H_a - H_m - H_r)}{N_h^*} - (d_h + \varepsilon_h)H_a, \\ \frac{\partial H_m}{\partial t} - D_h \frac{\partial^2 H_m}{\partial x^2} = \frac{\beta_m(N_h^* - H_a - H_m - H_r)H_m}{N_h^*} + \varepsilon_h H_a - (d_h + \gamma_h)H_m, \\ \frac{\partial H_r}{\partial t} - D_h \frac{\partial^2 H_r}{\partial x^2} = \gamma_h H_m - d_h H_r, \end{cases} \quad (2.3)$$

for $-\infty < x < \infty$ and $t > 0$.

This research is devoted to the transmission of avian influenza. Despite the infected is limited in a small district, the avian influenza can spread in the whole habitat immediately. Obviously, it cannot be used to describe the real spreading of the avian influenza virus. We now use the free boundary to describe the spreading frontier of the disease. Assume that the birds and humans migrate in the whole habitat $(-\infty, \infty)$, and some birds are infected in $g(t) < x < h(t)$, there is only the susceptible birds or humans on the rest part. The right spreading frontier is represented by the free boundary $x = h(t)$. Assume that $h(t)$ grows at a rate that is proportional to the population gradient at the frontier [7, 11, 17, 18, 28]. Then the conditions on the right frontier (free boundary) is

$$B_i(h(t), t) = 0, \quad -\mu D_b \frac{\partial B_i}{\partial x}(h(t), t) = h'(t).$$

Similarly, the conditions on the left frontier (free boundary) is

$$B_i(g(t), t) = 0, \quad -\mu D_b \frac{\partial B_i}{\partial x}(g(t), t) = g'(t).$$

In such a case, we have the problem for $B_i(x, t)$, $H_a(x, t)$, $H_m(x, t)$ and $H_r(x, t)$ with free boundaries $x = g(t)$ and $x = h(t)$ such that

$$\begin{cases} \frac{\partial B_i}{\partial t} - D_b \frac{\partial^2 B_i}{\partial x^2} = \frac{\beta_b(N_b^* - B_i)B_i}{N_b^*} - d_b B_i, \\ \frac{\partial H_a}{\partial t} - D_h \frac{\partial^2 H_a}{\partial x^2} = \frac{\beta_a B_i(N_h^* - H_a - H_m - H_r)}{N_h^*} - (d_h + \varepsilon_h)H_a, \\ \frac{\partial H_m}{\partial t} - D_h \frac{\partial^2 H_m}{\partial x^2} = \frac{\beta_m(N_h^* - H_a - H_m - H_r)H_m}{N_h^*} + \varepsilon_h H_a - (d_h + \gamma_h)H_m, \\ \frac{\partial H_r}{\partial t} - D_h \frac{\partial^2 H_r}{\partial x^2} = \gamma_h H_m - d_h H_r, \quad g(t) < x < h(t), \quad t > 0, \\ B_i(x, t) = H_a = H_m = H_r = 0, \quad x = g(t) \text{ or } x = h(t), \quad t > 0, \\ g(0) = -h_0, \quad g'(t) = -\mu D_b \frac{\partial B_i}{\partial x}(g(t), t), \quad t > 0, \\ h(0) = h_0, \quad h'(t) = \mu D_b \frac{\partial B_i}{\partial x}(h(t), t), \quad t > 0, \\ B_i(x, 0) = B_{i0}(x), \quad H_a(x, 0) = H_{a0}(x), \quad -h_0 \leq x \leq h_0, \\ H_m(x, 0) = H_{m0}(x), \quad H_r(x, 0) = H_{r0}(x), \quad -h_0 \leq x \leq h_0, \end{cases} \quad (2.4)$$

where $x = g(t)$ and $x = h(t)$ are the moving left and right boundaries to be determined, h_0 and μ are

positive constants, and the initial functions B_{i0} , H_{a0} , H_{m0} and H_{r0} are nonnegative and satisfy

$$\begin{cases} B_{i0}(x) \in C^2([-h_0, h_0]), & B_{i0}(\pm h_0) = 0 \text{ and } 0 < B_{i0}(x), \quad x \in (-h_0, h_0), \\ H_{a0}(x), H_{m0}(x), H_{r0}(x) \in C^2([-h_0, h_0]), \\ H_{a0}(\pm h_0) = H_{m0}(\pm h_0) = H_{r0}(\pm h_0) = 0, \\ 0 \leq H_{a0}(x), \quad H_{m0}(x), H_{r0}(x), x \in (-h_0, h_0). \end{cases} \quad (2.5)$$

Noting that the first equation of the system (2.4) is independent of the last three equations, we can consider the bird system as the following:

$$\begin{cases} \frac{\partial B_i}{\partial t} - D_b \frac{\partial^2 B_i}{\partial x^2} = \frac{\beta_b(N_b^* - B_i)B_i}{N_b^*} - d_b B_i, & g(t) < x < h(t), \quad t > 0, \\ B_i(g(t), t) = B_i(h(t), t) = 0, & t > 0, \\ g(0) = -h_0, \quad g'(t) = -\mu D_b \frac{\partial B_i}{\partial x}(g(t), t), & t > 0, \\ h(0) = h_0, \quad h'(t) = -\mu D_b \frac{\partial B_i}{\partial x}(h(t), t), & t > 0, \\ B_i(x, 0) = B_{i0}(x), & -h_0 \leq x \leq h_0. \end{cases} \quad (2.6)$$

In the rest part, we will consider (2.4) and (2.6), then give the properties of the solution and the free boundaries.

3 Existence and uniqueness

In this section, a contraction mapping argument is used to show the local existence and uniqueness of the solution to (2.4). Then we use standard methods (such as Hopf Lemma, the maximum principle) to derive the estimates of the solution and present the global existence and uniqueness of the solution. First, the following local existence result can be proved by the contraction mapping argument as in [4].

Theorem 3.1. *For any given $B_{i0}, H_{a0}, H_{m0}, H_{r0}$ satisfying (2.5) and any $\alpha \in (0, 1)$, there exists a positive number T such that (2.4) admits a unique solution*

$$(B_i, H_a, H_m, H_r; h, g) \in [C^{1+\alpha, (1+\alpha)/2}(\overline{D}_T)]^4 \times [C^{1+\alpha/2}([0, T])]^2;$$

moreover,

$$\begin{aligned} \|h\|_{C^{1+\alpha/2}([0, T])} + \|g\|_{C^{1+\alpha/2}([0, T])} &\leq C_1, \\ \|B_i\|_{C^{1+\alpha, (1+\alpha)/2}(\overline{D}_T)} + \|H_a\|_{C^{1+\alpha, (1+\alpha)/2}(\overline{D}_T)} &\leq C_1, \\ \|H_m\|_{C^{1+\alpha, (1+\alpha)/2}(\overline{D}_T)} + \|H_r\|_{C^{1+\alpha, (1+\alpha)/2}(\overline{D}_T)} &\leq C_1, \end{aligned}$$

where $D_T = \{(x, t) \in \mathbb{R}^2 : x \in (g(t), h(t)), t \in (0, T)\}$, C_1 and T only depend on h_0 , α , $\|B_{i0}\|_{C^2([-h_0, h_0])}$, $\|H_{a0}\|_{C^2([-h_0, h_0])}$, $\|H_{m0}\|_{C^2([-h_0, h_0])}$ and $\|H_{r0}\|_{C^2([-h_0, h_0])}$.

According to (2.5) and the equation of B_i in $(g(t), h(t)) \times [0, T_0]$, it follows from the strong maximum principle that

$$B_i(x, t) > 0, \quad \text{for } g(t) < x < h(t), \quad 0 < t < T_0.$$

By using maximum principle, it is easy to see that $0 \leq B_i \leq N_b^*$, $0 \leq H_a, H_m, H_r \leq N_h^*$ in $[g(t), h(t)] \times [0, T_0]$ as long as the solution exists for $t \in [0, T_0]$.

To explain that the local solution to (2.4) can be extended to all $t > 0$, we need the following lemma to show the free boundaries for problem (2.4) are strictly monotonous.

Lemma 3.2. *Assume $(B_i, H_a, H_m, H_r; h, g)$ is a solution to (2.4) defined for $t \in (0, T_0)$ for some $T_0 \in (0, +\infty]$. Then there exists a constant C_2 independent of T_0 such that*

$$0 < -g'(t), \quad h'(t) \leq C_2, \quad \text{for } t \in (0, T_0).$$

Proof. Using the Hopf lemma to the equation of B_i at $x = g(t)$ and $x = h(t)$ yields that

$$\frac{\partial B_i}{\partial x}(g(t), t) > 0 \quad \text{and} \quad \frac{\partial B_i}{\partial x}(h(t), t) < 0, \quad \text{for } 0 < t < T_0.$$

Hence, $g'(t) < 0$ and $h'(t) > 0$ for $t \in (0, T_0)$ by using the free boundary conditions in (2.4).

It remains to show that $-g'(t), h'(t) \leq C_2$ for $t \in (0, T_0)$ and some C_2 independent of T_0 .

Inspired by [7], define

$$B(x, t) = N_b^*[2M(h(t) - x) - M^2(h(t) - x)^2].$$

We choose

$$M = \max \left\{ \sqrt{\frac{\beta_b}{2D_b}}, \frac{4\|B_{i0}\|_{C^1([-h_0, h_0])}}{3N_b^*} \right\},$$

then it follows from the proof of [7, Lemma 2.2] that $0 < -g'(t), h'(t) \leq C_2$ for $t \in (0, T_0)$ with $C_2 = 2\mu D_b M N_b^*$. $\square$

Theorem 3.3. *For any $t \in (0, \infty)$, the solution to (2.4) exists and is unique.*

Proof. It follows from the uniqueness of the solution that there is a number $T_{\max}$ such that $[0, T_{\max})$ is the maximal time interval in which the solution exists. Now, we show that $T_{\max} = \infty$ by the contradiction argument.

Assume that $T_{\max} < \infty$. By Lemma 3.2, there exists C_2 independent of $T_{\max}$ such that

$$\begin{aligned} 0 &\leq B_i(x, t) \leq N_b^*, \quad (x, t) \in [g(t), h(t)] \times [0, T_{\max}), \\ h_0 &\leq -g(t), h(t) \leq h_0 + C_2 t, \quad 0 < -g'(t), h'(t) \leq C_2, \quad t \in (0, T_{\max}), \\ 0 &\leq H_a(x, t), H_m(x, t), H_r(x, t) \leq N_h^*, \quad (x, t) \in [g(t), h(t)] \times [0, T_{\max}). \end{aligned}$$

We now fix $\delta_0 \in (0, T_{\max})$ and $M > T_{\max}$. By standard parabolic regularity, we can find $C_3 > 0$ depending only on δ_0 , M , N_b^* , N_h^* and C_2 such that

$$\begin{aligned} \|B_i(\cdot, t)\|_{C^2([g(t), h(t)])}, \quad \|H_a(\cdot, t)\|_{C^2([g(t), h(t)])} &\leq C_3, \\ \|H_m(\cdot, t)\|_{C^2([g(t), h(t)])}, \quad \|H_r(\cdot, t)\|_{C^2([g(t), h(t)])} &\leq C_3, \end{aligned}$$

for $t \in [\delta_0, T_{\max})$. It then follows from Theorem 3.1 that there exists a $\tau > 0$ depending only on N_b^* , N_h^* and C_i ($i = 2, 3$) such that the solution to (2.4) with initial time $T_{\max} - \tau/2$ can be extended uniquely to the time $T_{\max} - \tau/2 + \tau$. Then this contradicts with the assumption. The proof is complete. $\square$

4 Basic reproduction numbers

In this section, we denote the basic reproduction numbers for different systems and present their properties which will be used in the sequel.

Now we study the full system (2.3). The corresponding ODE system is governed by

$$\begin{cases} \frac{dB_i}{dt} = \frac{\beta_b(N_b^* - B_i)B_i}{N_b^*} - d_b B_i, \\ \frac{dH_a}{dt} = \frac{\beta_a B_i(N_h^* - H_a - H_m - H_r)}{N_h^*} - (d_h + \varepsilon_h)H_a, \\ \frac{dH_m}{dt} = \frac{\beta_m(N_h^* - H_a - H_m - H_r)H_m}{N_h^*} + \varepsilon_h H_a - (d_h + \gamma_h)H_m, \\ \frac{dH_r}{dt} = \gamma_h H_m - d_h H_r. \end{cases} \quad (4.1)$$

Similarly to that in [8], the basic reproduction number r_0 of the bird and the basic reproduction number R_0 for mutant avian influenza in the human world are represented by

$$r_0 = \frac{\beta_b}{d_b}, \quad R_0 = \frac{\beta_m}{d_h + \gamma_h}.$$

It is easy to see that System (4.1) has three equilibria. The disease-free equilibrium (DFE) $(0, 0, 0, 0)$ always exists. There exists the human-endemic equilibrium (HEE) $(0, 0, \tilde{H}_m, \tilde{H}_r)$ if $R_0 > 1$, while if $r_0 > 1$, the full-endemic equilibrium (FEE) $(B_i^*, H_a^*, H_m^*, H_r^*)$ exists uniquely, where

$$\begin{aligned}\tilde{H}_m &= \frac{\beta_m - (d_h + \gamma_h)}{\beta_m(d_h + \gamma_h)} d_h N_h^*, \quad \tilde{H}_r = \frac{\gamma_h}{d_h} \tilde{H}_m, \quad B_i^* = \left(1 - \frac{d_b}{\beta_b}\right) N_b^*, \\ H_m^* &= \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad H_a^* = \frac{\beta_a B_i^* [d_h N_h^* - (d_h + \gamma_h) H_m^*]}{d_h [\beta_a B_i^* + (d_h + \varepsilon_h) N_h^*]}, \quad H_r^* = \frac{\gamma_h}{d_h} H_m^*, \\ a &= -\beta_m(d_h + \varepsilon_h)(d_h + \gamma_h), \quad c = \beta_a \varepsilon_h d_h B_i^* N_h^*, \\ b &= (d_h + \varepsilon_h) [d_h N_h^* (\beta_m - d_h - \gamma_h) - \beta_a B_i^* (d_h + \gamma_h)].\end{aligned}$$

According to [8, Theorem 3.2], if $r_0 < 1$ and $R_0 < 1$, then DFE is locally asymptotically stable; the HEE is locally asymptotically stable if $r_0 < 1$ and $R_0 > 1$, and if $r_0 > 1$, the FEE is locally asymptotically stable.

If the environment Ω is heterogeneous and the boundary of region Ω is hostile to the population for living (see [23]), then the corresponding system is given by

$$\begin{cases} \frac{\partial B_i}{\partial t} - D_b \Delta B_i = \frac{\beta_b(N_b^* - B_i)B_i}{N_b^*} - d_b B_i, & x \in \Omega, \quad t > 0, \\ \frac{\partial H_a}{\partial t} - D_h \Delta H_a = \frac{\beta_a B_i(N_h^* - H_a - H_m - H_r)}{N_h^*} \\ \quad - (d_h + \varepsilon_h) H_a, & x \in \Omega, \quad t > 0, \\ \frac{\partial H_m}{\partial t} - D_h \Delta H_m = \frac{\beta_m(N_h^* - H_a - H_m - H_r)H_m}{N_h^*} \\ \quad + \varepsilon_h H_a - (d_h + \gamma_h) H_m, & x \in \Omega, \quad t > 0, \\ \frac{\partial H_r}{\partial t} - D_h \Delta H_r = \gamma_h H_m - d_h H_r, & x \in \Omega, \quad t > 0, \\ B_i(x, t) = H_a(x, t) = H_m(x, t) = H_r(x, t) = 0, & x \in \partial\Omega, \quad t > 0. \end{cases} \quad (4.2)$$

Let us introduce the basic reproduction numbers for the bird $r_0^D(\Omega)$ and the mutant human $R_0^D(\Omega)$, respectively, as follows:

$$\begin{aligned}r_0^D &= r_0^D(\Omega) = \sup_{\phi \in H_0^1(\Omega), \phi \neq 0} \left\{ \frac{\int_{\Omega} \beta_b \phi^2 dx}{\int_{\Omega} (D_b |\nabla \phi|^2 + d_b \phi^2) dx} \right\}, \\ R_0^D &= R_0^D(\Omega) = \sup_{\phi \in H_0^1(\Omega), \phi \neq 0} \left\{ \frac{\int_{\Omega} \beta_m \phi^2 dx}{\int_{\Omega} (D_h |\nabla \phi|^2 + (d_h + \gamma_h) \phi^2) dx} \right\}.\end{aligned}$$

Then we can have the similar result to [1, Lemma 2.3(a)].

Lemma 4.1. r_0^D and R_0^D are positive and monotonically decreasing functions of D_b , D_h , respectively.

Let λ_{r_0} be the first eigenvalue of

$$\begin{cases} -D_b \Delta \varphi = \beta_b \varphi - d_b \varphi + \lambda \varphi, & x \in \Omega, \\ \varphi(x) = 0, & x \in \partial\Omega, \end{cases} \quad (4.3)$$

and λ_{R_0} be the first eigenvalue of

$$\begin{cases} -D_h \Delta \phi = \beta_m \phi - (d_h + \gamma_h) \phi + \lambda \phi, & x \in \Omega, \\ \phi(x) = 0, & x \in \partial\Omega. \end{cases} \quad (4.4)$$

Furthermore, its corresponding eigenfunctions $\varphi_{\lambda_{r_0}}$ and $\phi_{\lambda_{R_0}}$ can be chosen to be positive on Ω .

Obviously, λ_{r_0} and λ_{R_0} can be given by variational characterization,

$$\begin{aligned}\lambda_{r_0} &= \inf \left\{ \int_{\Omega} [D_b |\nabla \varphi|^2 + (d_b - \beta_b) \varphi^2] : \varphi \in H_0^1(\Omega) \text{ and } \int_{\Omega} \varphi^2 = 1 \right\}, \\ \lambda_{R_0} &= \inf \left\{ \int_{\Omega} [D_h |\nabla \phi|^2 + (d_h + \gamma_h - \beta_m) \phi^2] : \phi \in H_0^1(\Omega) \text{ and } \int_{\Omega} \phi^2 = 1 \right\}.\end{aligned}$$

Lemma 4.2. *The following properties hold:*

- (i) $r_0^D < 1$ (> 1) if and only if $\lambda_{r_0} > 0$ (< 0);
- (ii) $R_0^D < 1$ (> 1) if and only if $\lambda_{R_0} > 0$ (< 0).

Proof. One can directly verify this statement (see [1, Lemma 2.3(d)]). □

Note that $\beta_b, \beta_m, d_b, d_h$ and γ_h are constants. Direct calculations show

$$r_0^D(\Omega) = \frac{\beta_b}{\lambda(\Omega)D_b + d_b} \quad \text{and} \quad R_0^D(\Omega) = \frac{\beta_m}{\lambda(\Omega)D_h + (d_h + \gamma_h)},$$

where $\lambda(\Omega)$ is the principle eigenvalue of the following eigenvalue problem:

$$\begin{cases} -\Delta \psi = \lambda \psi, & x \in \Omega, \\ \psi(x) = 0, & x \in \partial\Omega. \end{cases} \quad (4.5)$$

It is well known that $\lambda(\Omega)$ is a strictly decreasing continuous function and that

$$\lim_{|\Omega| \rightarrow 0^+} \lambda(\Omega) = +\infty \quad \text{and} \quad \lim_{|\Omega| \rightarrow +\infty} \lambda(\Omega) = 0.$$

From the direct calculations, the following lemma holds.

Lemma 4.3. (i) $r_0^D \rightarrow r_0$ as $D_b \rightarrow 0$ and $r_0^D \rightarrow 0$ as $D_b \rightarrow \infty$;
 (ii) $R_0^D \rightarrow R_0$ as $D_h \rightarrow 0$ and $R_0^D \rightarrow 0$ as $D_h \rightarrow \infty$;
 (iii) If $\Omega = (-h_0, h_0)$, then

$$r_0^D((-h_0, h_0)) = \frac{\beta_b}{D_b(\frac{\pi}{2h_0})^2 + d_b}, \quad R_0^D((-h_0, h_0)) = \frac{\beta_m}{D_h(\frac{\pi}{2h_0})^2 + (d_h + \gamma_h)};$$

(iv) Let B_ρ be a ball with radius ρ . Then $r_0^D(B_\rho)$ and $R_0^D(B_\rho)$ are strictly monotonically increasing function of ρ . That is to say, $r_0^D(B_{\rho_1}) < r_0^D(B_{\rho_2})$, $R_0^D(B_{\rho_1}) < R_0^D(B_{\rho_2})$ if $\rho_1 < \rho_2$.

By the free boundary problem (2.4), we know that the domain $(g(t), h(t))$ is changing with respect to t . Thus the basic reproduction number is a function with t . Now, we define the basic reproduction numbers $r_0^F(t)$ and $R_0^F(t)$ for the free boundary problem (2.4) by

$$\begin{aligned}r_0^F(t) &= r_0^D((g(t), h(t))) = \frac{\beta_b}{D_b \lambda(g(t), h(t)) + d_b}, \\ R_0^F(t) &= R_0^D((g(t), h(t))) = \frac{\beta_m}{D_h \lambda(g(t), h(t)) + (d_h + \gamma_h)}.\end{aligned}$$

Lemma 4.4. $r_0^F(t)$ and $R_0^F(t)$ are strictly monotonically increasing function of t . Moreover, if $h(t) - g(t) \rightarrow \infty$ as $t \rightarrow +\infty$, then $r_0^F(t) \rightarrow r_0$, $R_0^F(t) \rightarrow R_0$, respectively.

Proof. This conclusion follows directly from Lemmas 3.2 and 4.3. □

5 Vanishing of the avian influenza

In this section, we will consider the vanishing of the avian influenza in the bird and in the human world. First, we show that the double free boundary frontiers $x = g(t)$ and $x = h(t)$ are either finite or infinite at the same time.

Theorem 5.1. Suppose that $(B_i, H_a, H_m, H_r; h, g)$ is a solution to (2.4) defined for $x \in [g(t), h(t)]$ and $t \in [0, \infty)$. Then we have

$$-2h_0 < g(t) + h(t) < 2h_0, \quad \text{for } t \in [0, +\infty).$$

Proof. For small $t > 0$, we have $g(t) + h(t) > -2h_0$ by the continuity. Define

$$T = \sup\{\tau : g(t) + h(t) > -2h_0, \text{ for all } t \in [0, \tau)\}.$$

Now, we show that $T = \infty$ by the contradiction argument. Assume that there exists T with $0 < T < \infty$ such that

$$g(t) + h(t) > -2h_0, \quad \text{for } t \in [0, T), \quad g(T) + h(T) = -2h_0.$$

Furthermore, we get

$$g'(T) + h'(T) \leq 0. \quad (5.1)$$

Define

$$B(x, t) = B_i(x, t) - B_i(-x - 2h_0, t), \quad \text{for } x \in [g(t), -h_0] \text{ and } t \in [0, T].$$

Direct calculation shows that

$$B_t - D_b B_{xx} = C(x, t)B, \quad \text{for } g(t) < x < -h_0, \quad 0 < t \leq T.$$

Noting that $C(x, t) \in L^\infty$ and

$$B(-h_0, t) = 0, \quad B(g(t), t) \leq 0, \quad \text{for } 0 < t < T,$$

then by using the comparison principle, we obtain

$$B(x, t) \leq 0, \quad \text{for } (x, t) \in [g(t), -h_0] \times (0, T].$$

Applying the strong maximum principle gives that

$$B(x, t) < 0, \quad \text{for } (x, t) \in (g(t), -h_0) \times (0, T].$$

However,

$$B(g(T), T) = B_i(g(T), T) - B_i(h(T), T) = 0.$$

By the Hopf lemma, we have $B_x(g(T), T) < 0$. Moreover,

$$B_x(g(T), T) = \frac{\partial B_i}{\partial x}(g(T), T) + \frac{\partial B_i}{\partial x}(h(T), T) = -\frac{g'(T) + h'(T)}{\mu D_b}.$$

Then we have

$$g'(T) + h'(T) > 0,$$

which is a contradiction to (5.1). Hence we have shown

$$g(t) + h(t) > -2h_0, \quad \text{for all } t \geq 0,$$

and $g(t) + h(t) < 2h_0$ for all $t \geq 0$ can be proven by the same argument. $\square$

It follows from Lemma 3.2 that $-g(t)$, $h(t)$ are monotonic increasing and there exist $-g_\infty$, $h_\infty \in (0, +\infty]$ such that $\lim_{t \rightarrow +\infty} g(t) = g_\infty$ and $\lim_{t \rightarrow +\infty} h(t) = h_\infty$.

Lemma 5.2. If $h_\infty - g_\infty < \infty$, then $\lim_{t \rightarrow +\infty} \|B_i(\cdot, t)\|_{C([g(t), h(t)])} = 0$.

Proof. Suppose $\limsup_{t \rightarrow +\infty} \|B_i(\cdot, t)\|_{C([g(t), h(t)])} = \sigma > 0$ by contradiction. Then there exists a sequence (x_n, t_n) in $(g(t), h(t)) \times (0, \infty)$ such that $B_i(x_n, t_n) \geq \frac{\sigma}{2}$ for all $n \in \mathbb{N}$, and $t_n \rightarrow \infty$ as $n \rightarrow +\infty$. Noting that $-\infty < g_\infty < g(t_n) < x_n < h(t_n) < h_\infty < \infty$, there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \rightarrow x_0$ and $x_0 \in (g_\infty, h_\infty)$. Without loss of generality, we take $x_n \rightarrow x_0$ as $n \rightarrow \infty$.

Define

$$B_n(x, t) = B_i(x, t_n + t), \quad \text{for } x \in (g(t_n + t), h(t_n + t)), \quad t \in (-t_n, \infty).$$

It follows from the regularity of the parabolic equation that $\{B_n\}$ has a subsequence $\{B_{n_i}\}$ such that $B_{n_i} \rightarrow \hat{B}$ as $i \rightarrow \infty$, where $\hat{B}$ satisfies

$$\hat{B}_t - D_b \hat{B}_{xx} = \frac{\beta_b \hat{B}(N_b^* - \hat{B})}{N_b^*} - d_b \hat{B}, \quad \text{for } g_\infty < x < h_\infty, \quad t \in (-\infty, +\infty).$$

Moreover, $\hat{B}(x_0, 0) \geq \frac{\sigma}{2}$, and thus we can get

$$\hat{B}(x, t) > 0, \quad \text{for } (x, t) \in (g_\infty, h_\infty) \times (-\infty, +\infty).$$

Applying the Hopf lemma yields that there exists $\epsilon > 0$ such that $\hat{B}_x(h_\infty, 0) \leq -\epsilon$.

It follows from Theorem 3.1 and Lemma 3.2 that $-g(t)$ and $h(t)$ are monotonically increasing and bounded. By the standard L^p theory and the Sobolev imbedding theorem (see [13]), for any $0 < \alpha < 1$, there exists a constant C which depends on α , h_0 , $\|B_{i0}\|_{C^{1+\alpha}([-h_0, h_0])}$, g_∞ and h_∞ such that

$$\|B_i\|_{C^{1+\alpha, (1+\alpha)/2}([g(t), h(t)] \times [0, +\infty))} + \|h\|_{C^{1+\alpha/2}([0, \infty))} \leq C.$$

On the other hand, since $h(t)$ is bounded, we have $h'(t) \rightarrow 0$ as $t \rightarrow +\infty$. So $\frac{\partial B_i}{\partial x}(h(t_n), t_n) \rightarrow 0$ as $n \rightarrow +\infty$ by the free boundary condition.

Noting that $\|B_i\|_{C^{1+\alpha, (1+\alpha)/2}([g(t), h(t)] \times [0, +\infty))} \leq C$, we have

$$\frac{\partial B_i}{\partial x}(h(t_n), t_n) = (B_n)_x(h(t_n), 0) \rightarrow \hat{B}_x(h_\infty, 0) \quad \text{as } n \rightarrow +\infty.$$

This is a contradiction to that $\hat{B}_x(h_\infty, 0) \leq -\epsilon$. Hence, $\lim_{t \rightarrow +\infty} \|B_i(\cdot, t)\|_{C([g(t), h(t)])} = 0$. $\square$

Lemma 5.2 implies the following result.

Corollary 5.3. *If $\liminf_{t \rightarrow +\infty} \|B_i(\cdot, t)\|_{C([g(t), h(t)])} > 0$, then $h_\infty - g_\infty = \infty$.*

To give the sufficient conditions for the avian influenza to vanish, we first give the definition of the upper and the lower solutions to the bird system, then give the comparison principle.

Definition 5.4. Suppose that $T \in (0, \infty)$, $\bar{g}, \bar{h} \in C^1([0, T])$, $\bar{B}_i \in C^{2,1}((\bar{g}, \bar{h}) \times (0, T]) \cap C([\bar{g}, \bar{h}] \times [0, T])$, and

$$\begin{cases} \frac{\partial \bar{B}_i}{\partial t} - D_b \frac{\partial^2 \bar{B}_i}{\partial x^2} \geq \frac{\beta_b \bar{B}_i(N_b^* - \bar{B}_i)}{N_b^*} - d_b \bar{B}_i, & \bar{g} < x < \bar{h}, \quad t > 0, \\ \bar{B}_i(x, t) = 0, & x = \bar{g} \text{ or } \bar{h}, \quad t > 0, \\ \bar{g}(0) \leq -h_0, \quad \bar{g}'(t) \leq -\mu D_b \frac{\partial \bar{B}_i}{\partial x}(\bar{g}(t), t), & t > 0, \\ \bar{h}(0) \geq h_0, \quad \bar{h}'(t) \geq -\mu D_b \frac{\partial \bar{B}_i}{\partial x}(\bar{h}(t), t), & t > 0, \\ B_{i0} \leq \bar{B}_i(x, 0), & -h_0 \leq x \leq h_0. \end{cases} \quad (5.2)$$

Then we call $(\bar{B}_i; \bar{g}, \bar{h}) \in [C^{2,1}((\bar{g}, \bar{h}) \times (0, T]) \cap C([\bar{g}, \bar{h}] \times [0, T])] \times [C^1([0, T])]^2$ is an upper solution to the bird system (2.6).

Remark 5.5. We can give the lower solution $(\underline{B}_i; \underline{g}, \underline{h}) \in [C^{2,1}((\underline{g}, \underline{h}) \times (0, T]) \cap C([\underline{g}, \underline{h}] \times [0, T])] \times [C^1([0, T])]^2$ to the bird system (2.6) by reversing the inequalities in (5.2).

Next, we give the comparison principle. The proof is almost the same as [7, Lemma 3.5].

Lemma 5.6. Let $(\overline{B}_i; \overline{g}, \overline{h})$ and $(\underline{B}_i; \underline{g}, \underline{h})$ be the upper and lower solution to the bird system with the free boundary, respectively. Then the solution $(B_i; g, h)$ to (2.6) satisfies

$$\begin{aligned} \underline{h} &\leq h(t) \leq \overline{h}(t), \quad \overline{g} \leq g(t) \leq \underline{g}(t), \quad t \in (0, T], \\ B_i(x, t) &\leq \overline{B}_i(x, t), \quad (x, t) \in (g(t), h(t)) \times (0, T], \\ \underline{B}_i(x, t) &\leq B_i(x, t), \quad (x, t) \in (\underline{g}(t), \underline{h}(t)) \times (0, T]. \end{aligned}$$

Now, we give the sufficient condition for the avian influenza to vanish in the bird world.

Theorem 5.7. Assume that $r_0^F(0)(:=r_0^D((-h_0, h_0))) < 1$. Then

$$h_\infty - g_\infty < \infty \quad \text{and} \quad \lim_{t \rightarrow +\infty} \|B_i(\cdot, t)\|_{C([g(t), h(t)])} = 0,$$

if $\|B_{i0}(x)\|_{C([-h_0, h_0])}$ is sufficiently small.

Proof. We construct suitable upper solution to the bird system. Since $r_0^D((-h_0, h_0)) < 1$, we then have that the first eigenvalue $\lambda_{r_0} > 0$ by Lemma 4.2(i) and its corresponding eigenfunction $\varphi(x) > 0$ such that (4.3) holds in $\Omega = (-h_0, h_0)$. Therefore, we can choose a sufficiently small δ such that $\delta(1+\delta)^2 + [(1+\delta)^2 - 1](\beta_b - d_b) \leq \lambda_{r_0}$.

Inspired by [7], we define

$$\begin{aligned} \sigma(t) &= h_0 \left(1 + \delta - \frac{\delta}{2} e^{-\delta t} \right), \quad t \geq 0, \\ \overline{B}_i(x, t) &= \varepsilon e^{-\delta t} \varphi(xh_0/\sigma(t)), \quad -\sigma(t) \leq x \leq \sigma(t), \quad t \geq 0, \end{aligned}$$

where $0 < \varepsilon \leq -\frac{\delta^2 h_0}{2\mu D_b \varphi'(h_0)}$. Direct computations yield that

$$\begin{aligned} \frac{\partial \overline{B}_i}{\partial t} - D_b \frac{\partial^2 \overline{B}_i}{\partial x^2} - \frac{\beta_b(N_b^* - \overline{B}_i)\overline{B}_i}{N_b^*} + d_b \overline{B}_i &= -\delta \overline{B}_i - \frac{xh_0\sigma'}{\sigma^2} \varepsilon e^{-\delta t} \varphi' + \frac{h_0^2}{\sigma^2} (\beta_b \overline{B}_i - d_b \overline{B}_i + \lambda_{r_0} \overline{B}_i) \\ &\quad - \beta_b \overline{B}_i + \frac{\beta_b}{N_b^*} \overline{B}_i^2 + d_b \overline{B}_i \\ &\geq \overline{B}_i \left[-\delta + (\beta_b - d_b) \left(\frac{h_0^2}{\sigma^2(t)} - 1 \right) + \frac{h_0^2}{\sigma^2(t)} \lambda_{r_0} \right] \\ &\geq 0, \end{aligned}$$

for all $-\sigma(t) < x < \sigma(t)$ and $t > 0$.

It is easy to see that

$$\begin{aligned} -(\overline{B}_i)_x(-\sigma(t), t) &= -\frac{h_0}{\sigma(t)} \varepsilon e^{-\delta t} \varphi'(-h_0), \\ -(\overline{B}_i)_x(\sigma(t), t) &= -\frac{h_0}{\sigma(t)} \varepsilon e^{-\delta t} \varphi'(h_0) \quad \text{and} \quad \sigma'(t) = \frac{\delta^2 h_0}{2} e^{-\delta t}. \end{aligned}$$

But since $-\varphi'(-h_0) = \varphi'(h_0)$, we get

$$\begin{aligned} \sigma'(t) &\geq -\mu D_b (\overline{B}_i)_x(\sigma(t), t) = -\mu D_b \frac{h_0}{\sigma(t)} \varepsilon e^{-\delta t} \varphi'(h_0), \quad \text{for } t > 0, \\ -\sigma'(t) &\leq -\mu D_b (\overline{B}_i)_x(-\sigma(t), t) = -\mu D_b \frac{h_0}{\sigma(t)} \varepsilon e^{-\delta t} \varphi'(-h_0), \quad \text{for } t > 0. \end{aligned}$$

Therefore, we have

$$\left\{ \begin{array}{ll} \frac{\partial \overline{B}_i}{\partial t} - D_b \frac{\partial^2 \overline{B}_i}{\partial x^2} \geq \frac{\beta_b(N_b^* - \overline{B}_i)\overline{B}_i}{N_b^*} - d_b \overline{B}_i, & -\sigma(t) < x < \sigma(t), \quad t > 0, \\ \overline{B}_i(-\sigma(t), t) = \overline{B}_i(\sigma(t), t) = 0, & t > 0, \\ -\sigma(0) < -h_0, \quad -\sigma'(t) \leq -\mu D_b \frac{\partial \overline{B}_i}{\partial x}(-\sigma(t), t), & t > 0, \\ \sigma(0) > h_0, \quad \sigma'(t) \geq -\mu D_b \frac{\partial \overline{B}_i}{\partial x}(\sigma(t), t), & t > 0. \end{array} \right.$$

Assume that $\|B_{i0}(x)\|_{C([-h_0, h_0])}$ is sufficiently small so that $B_{i0} \leq \varepsilon\varphi(\frac{h_0}{1+\frac{1}{2}})$, then $B_{i0} \leq \overline{B}_i(x, 0)$ for $-h_0 \leq x \leq h_0$. Using Lemma 5.6 gives that

$$g(t) \geq -\sigma(t) \quad \text{and} \quad h(t) \leq \sigma(t), \quad \text{for } t > 0.$$

Hence,

$$h_\infty - g_\infty \leq \lim_{t \rightarrow \infty} 2\sigma(t) = 2h_0(1 + \delta) < \infty.$$

It follows from Lemma 5.2 that $\lim_{t \rightarrow +\infty} \|B_i(\cdot, t)\|_{C([g(t), h(t)])} = 0$. $\square$

The following comparison principle for the full system can be obtained similarly to [7, Lemma 3.5].

Lemma 5.8. Suppose that the upper solution $(\overline{B}_i, \overline{H}_a, \overline{H}_m, \overline{H}_r; \overline{g}, \overline{h}) \in [C^{2,1}(D_{1T}) \cap C(\overline{D}_{1T})]^4 \times [C^1([0, T])]^2$ and the lower solution $(\underline{B}_i, \underline{H}_a, \underline{H}_m, \underline{H}_r; \underline{g}, \underline{h}) \in [C^{2,1}(D_{2T}) \cap C(\overline{D}_{2T})]^4 \times [C^1([0, T])]^2$ satisfy

$$\left\{ \begin{array}{ll} \frac{\partial \overline{B}_i}{\partial t} - D_b \frac{\partial^2 \overline{B}_i}{\partial x^2} \geq \frac{\beta_b(N_b^* - \overline{B}_i)\overline{B}_i}{N_b^*} - d_b \overline{B}_i, & \overline{g}(t) < x < \overline{h}(t), \quad t > 0, \\ \frac{\partial \underline{B}_i}{\partial t} - D_b \frac{\partial^2 \underline{B}_i}{\partial x^2} \leq \frac{\beta_b(N_b^* - \underline{B}_i)\underline{B}_i}{N_b^*} - d_b \underline{B}_i, & \underline{g}(t) < x < \underline{h}(t), \quad t > 0, \\ \frac{\partial \overline{H}_a}{\partial t} - D_h \frac{\partial^2 \overline{H}_a}{\partial x^2} \geq \frac{\beta_a \overline{B}_i(N_h^* - \overline{H}_a - \underline{H}_m - \underline{H}_r)}{N_h^*} - (d_h + \varepsilon_h) \overline{H}_a, & \overline{g}(t) < x < \overline{h}(t), \quad t > 0, \\ \frac{\partial \underline{H}_a}{\partial t} - D_h \frac{\partial^2 \underline{H}_a}{\partial x^2} \leq \frac{\beta_a \underline{B}_i(N_h^* - \underline{H}_a - \overline{H}_m - \overline{H}_r)}{N_h^*} - (d_h + \varepsilon_h) \underline{H}_a, & \underline{g}(t) < x < \underline{h}(t), \quad t > 0, \\ \frac{\partial \overline{H}_m}{\partial t} - D_h \frac{\partial^2 \overline{H}_m}{\partial x^2} \geq \frac{\beta_m(N_h^* - \underline{H}_a - \overline{H}_m - \underline{H}_r)\overline{H}_m}{N_h^*} + \varepsilon_h \overline{H}_a - (d_h + \gamma_h) \overline{H}_m, & \overline{g}(t) < x < \overline{h}(t), \quad t > 0, \\ \frac{\partial \underline{H}_m}{\partial t} - D_h \frac{\partial^2 \underline{H}_m}{\partial x^2} \leq \frac{\beta_m(N_h^* - \overline{H}_a - \underline{H}_m - \overline{H}_r)\underline{H}_m}{N_h^*} + \varepsilon_h \underline{H}_a - (d_h + \gamma_h) \underline{H}_m, & \underline{g}(t) < x < \underline{h}(t), \quad t > 0, \\ \frac{\partial \overline{H}_r}{\partial t} - D_h \frac{\partial^2 \overline{H}_r}{\partial x^2} \geq \gamma_h \overline{H}_m - d_h \overline{H}_r, & \overline{g}(t) < x < \overline{h}(t), \quad t > 0, \\ \frac{\partial \underline{H}_r}{\partial t} - D_h \frac{\partial^2 \underline{H}_r}{\partial x^2} \geq \gamma_h \underline{H}_m - d_h \underline{H}_r, & \underline{g}(t) < x < \underline{h}(t), \quad t > 0, \end{array} \right.$$

and when $t = 0$ and $-h_0 \leq x \leq h_0$, the initial functions satisfy

$$\left\{ \begin{array}{l} \overline{g}(0) \leq -h_0 \leq \underline{g}(0), \quad \underline{h}(0) \leq h_0 \leq \overline{h}(0), \\ \underline{B}_i(x, 0) \leq B_{i0}(x) \leq \overline{B}_i(x, 0), \quad \underline{H}_a(x, 0) \leq H_{a0}(x) \leq \overline{H}_a(x, 0), \\ \underline{H}_m(x, 0) \leq H_{m0}(x) \leq \overline{H}_m(x, 0), \quad \underline{H}_r(x, 0) \leq H_{r0}(x) \leq \overline{H}_r(x, 0). \end{array} \right.$$

Suppose, next, that when $t > 0$, $(\overline{B}_i, \overline{H}_a, \overline{H}_m, \overline{H}_r)$ and $(\underline{B}_i, \underline{H}_a, \underline{H}_m, \underline{H}_r)$ satisfy, respectively,

$$\overline{B}_i(x, t) \geq 0, \quad \overline{H}_a(x, t) \geq 0, \quad \overline{H}_m(x, t) \geq 0, \quad \overline{H}_r(x, t) \geq 0$$

on the boundary $x = \overline{g}(t)$ or on the boundary $x = \overline{h}(t)$, and

$$\underline{B}_i(x, t) = 0, \quad \underline{H}_a(x, t) = 0, \quad \underline{H}_m(x, t) = 0, \quad \underline{H}_r(x, t) = 0,$$

for $x \leq \underline{g}(t)$ or $x \geq \underline{h}(t)$. Suppose, furthermore, that for any $t > 0$, $\underline{g}$, $\overline{g}$, $\underline{h}$ and $\overline{h}$ satisfy

$$\underline{g}'(t) \geq -\mu D_b \frac{\partial \underline{B}_i}{\partial x}(\underline{g}(t), t), \quad \overline{g}'(t) \leq -\mu D_b \frac{\partial \overline{B}_i}{\partial x}(\overline{g}(t), t),$$

$$\underline{h}'(t) \leq -\mu D_b \frac{\partial \underline{B}_i}{\partial x}(\underline{h}(t), t), \quad \bar{h}'(t) \geq -\mu D_b \frac{\partial \bar{B}_i}{\partial x}(\bar{h}(t), t).$$

Then the solution $(B_i, H_a, H_m, H_r; g, h)$ to the free boundary problem (2.4) satisfies

$$\begin{aligned} B_i &\leq \bar{B}_i, \quad H_a \leq \bar{H}_a, \quad H_m \leq \bar{H}_m, \quad H_r \leq \bar{H}_r, \quad \text{for } g(t) \leq x \leq h(t) \quad \text{and } 0 < t \leq T, \\ B_i &\geq \underline{B}_i, \quad H_a \geq \underline{H}_a, \quad H_m \geq \underline{H}_m, \quad H_r \geq \underline{H}_r, \quad \text{for } \underline{g}(t) \leq x \leq \underline{h}(t) \quad \text{and } 0 < t \leq T, \\ \underline{g}(t) &\geq g(t) \geq \bar{g}(t), \quad \underline{h}(t) \leq h(t) \leq \bar{h}(t), \quad \text{in } [0, T], \end{aligned}$$

where $T \in (0, \infty)$, $D_{1T} = (\bar{g}(t), \bar{h}(t)) \times (0, T]$ and $D_{2T} = (\underline{g}(t), \underline{h}(t)) \times (0, T]$.

Theorem 5.9. Suppose that $r_0^F(0) < 1$ and $R_0^F(0) := R_0^D((-h_0, h_0)) < 1$, then

$$\begin{aligned} h_\infty - g_\infty &< \infty, \\ \lim_{t \rightarrow +\infty} \|B_i(\cdot, t)\|_{C([g(t), h(t)])} &= 0, \quad \lim_{t \rightarrow +\infty} \|H_a(\cdot, t)\|_{C([g(t), h(t)])} = 0, \\ \lim_{t \rightarrow +\infty} \|H_m(\cdot, t)\|_{C([g(t), h(t)])} &= 0, \quad \lim_{t \rightarrow +\infty} \|H_r(\cdot, t)\|_{C([g(t), h(t)])} = 0, \end{aligned}$$

if $\|B_{i0}(x)\|_{C([-h_0, h_0])}$, $\|H_{a0}(x)\|_{C([-h_0, h_0])}$, $\|H_{m0}(x)\|_{C([-h_0, h_0])}$ and $\|H_{r0}(x)\|_{C([-h_0, h_0])}$ are small enough.

Proof. Since $R_0^D((-h_0, h_0)) < 1$, it follows from Lemma 4.2(ii) that $\lambda_{R_0} > 0$ and there exists $\phi(x) > 0$ satisfying (4.4) over $\Omega = (-h_0, h_0)$. Since $r_0^F(0) < 1$, we have $h_\infty - g_\infty < \infty$ by Theorem 5.7.

Let the definition of $\sigma(t)$ and $\bar{B}_i$ be the same as those in Theorem 5.7. Now, we define

$$\begin{aligned} \bar{H}_a &= \varepsilon_a \bar{B}_i, \quad \bar{H}_m = e^{-\delta t} \phi\left(\frac{x h_0}{\sigma(t)}\right) \quad \text{and} \quad \bar{H}_r = \varepsilon_r \bar{H}_m, \quad -\sigma(t) \leq x \leq \sigma(t), \quad t \geq 0, \\ \underline{B}_i &= \underline{H}_a = \underline{H}_m = \underline{H}_r = 0, \quad -h(t) \leq x \leq h(t), \quad t \geq 0, \end{aligned}$$

where

$$\varepsilon_a = \frac{8\beta_a h_0^2 (1+\delta)^2}{\pi^2 D_h - 4\delta(1+\delta)^2 h_0^2}, \quad \varepsilon_r = \frac{8\gamma_h h_0^2 (1+\delta)^2}{\pi^2 D_h - 4\delta(1+\delta)^2 h_0^2}$$

and δ satisfies

$$\begin{aligned} \pi^2 D_h - 4\delta(1+\delta)^2 h_0^2 &> 0, \\ \delta(1+\delta)^2 + [(1+\delta)^2 - 1](\beta_b - d_b) &\leq \lambda_{r_0}, \\ \delta(1+\delta)^2 + [(1+\delta)^2 - 1](\beta_m - (d_h + \gamma_h)) &< \lambda_{R_0}. \end{aligned}$$

Direct computations yield that

$$\begin{aligned} &\frac{\partial \bar{H}_a}{\partial t} - D_h \frac{\partial^2 \bar{H}_a}{\partial x^2} - \frac{\beta_a \bar{B}_i (N_h^* - \bar{H}_a)}{N_h^*} + (d_h + \varepsilon_h) \bar{H}_a \\ &= \varepsilon_a \frac{\partial \bar{B}_i}{\partial t} - \varepsilon_a D_h \frac{\partial^2 \bar{B}_i}{\partial x^2} - \frac{\beta_a \bar{B}_i (N_h^* - \bar{H}_a)}{N_h^*} + (d_h + \varepsilon_h) \bar{H}_a \\ &\geq \varepsilon_a \frac{\partial \bar{B}_i}{\partial t} - \varepsilon_a D_h \frac{\partial^2 \bar{B}_i}{\partial x^2} - \beta_a \bar{B}_i \\ &= -\varepsilon_a \delta \bar{B}_i - \varepsilon_a \frac{x h_0 \sigma'(t)}{\sigma^2(t)} \varepsilon e^{-\delta t} \varphi' - \varepsilon_a D_h \varepsilon \frac{h_0^2}{\sigma^2(t)} e^{-\delta t} \varphi'' - \beta_a \bar{B}_i \\ &\geq \bar{B}_i \left[-\delta \varepsilon_a + \varepsilon_a \frac{D_h}{D_b} \frac{h_0^2}{\sigma^2(t)} (\beta_b - d_b + \lambda_{r_0}) - \beta_a \right] \geq 0, \\ &\frac{\partial \bar{H}_m}{\partial t} - D_h \frac{\partial^2 \bar{H}_m}{\partial x^2} - \frac{\beta_m (N_h^* - \bar{H}_m) \bar{H}_m}{N_h^*} - \varepsilon_h \bar{H}_a + (d_h + \gamma_h) \bar{H}_m \\ &= -\delta e^{-\delta t} \phi - \frac{x h_0 \sigma'(t)}{\sigma^2(t)} e^{-\delta t} \phi' - D_h \frac{h_0^2}{\sigma^2(t)} e^{-\delta t} \phi'' \end{aligned}$$

$$\begin{aligned}
 & -\frac{\beta_m(N_h^* - \overline{H}_m)\overline{H}_m}{N_h^*} - \varepsilon_h \overline{H}_a + (d_h + \gamma_h)\overline{H}_m \\
 & \geq \overline{H}_m \left\{ -\delta + \left(\frac{h_0^2}{\sigma^2(t)} - 1 \right) [\beta_m - (d_h + \gamma_h)] + \frac{h_0^2}{\sigma^2(t)} \lambda_{R_0} \right\} - \varepsilon_h \overline{H}_a \\
 & \geq 0,
 \end{aligned}$$

where we have used the fact that $\overline{H}_a = \varepsilon \varepsilon_a e^{-\delta t} \varphi(x h_0 / \sigma(t))$ and ε is sufficiently small, and we also have

$$\begin{aligned}
 & \frac{\partial \overline{H}_r}{\partial t} - D_h \frac{\partial^2 \overline{H}_r}{\partial x^2} - \gamma_h \overline{H}_m + d_h \overline{H}_r \\
 & = -\delta \varepsilon_r e^{-\delta t} \phi - \varepsilon_r \frac{x h_0 \sigma'(t)}{\sigma^2(t)} e^{-\delta t} \phi' - D_h \varepsilon_r \frac{h_0^2}{\sigma^2(t)} e^{-\delta t} \phi'' - \gamma_h \overline{H}_m + d_h \overline{H}_r \\
 & \geq \overline{H}_m \left\{ -\delta \varepsilon_r + \varepsilon_r \frac{h_0^2}{\sigma^2(t)} [\beta_m - (d_h + \gamma_h) + \lambda_{R_0}] - \gamma_h \right\} \geq 0.
 \end{aligned}$$

Then since $\|B_{i0}(x)\|_{C([-h_0, h_0])}$, $\|H_{a0}(x)\|_{C([-h_0, h_0])}$, $\|H_{m0}(x)\|_{C([-h_0, h_0])}$ and $\|H_{r0}(x)\|_{C([-h_0, h_0])}$ are small enough for $-h_0 < x < h_0$, we have

$$\begin{aligned}
 B_{i0}(x) & \leq \varepsilon \varphi\left(\frac{h_0}{1 + \delta/2}\right) \leq \overline{B}_{i0}(x, 0), \quad H_{a0} \leq \varepsilon_a \varphi\left(\frac{h_0}{1 + \delta/2}\right) \leq \overline{H}_{a0}(x, 0), \\
 H_{m0} & \leq \phi\left(\frac{h_0}{1 + \delta/2}\right) \leq \overline{H}_{m0}(x, 0), \quad H_{r0} \leq \varepsilon_r \phi\left(\frac{h_0}{1 + \delta/2}\right) \leq \overline{H}_{r0}(x, 0).
 \end{aligned}$$

Now, it follows from Lemma 5.8 that

$$0 \leq B_i \leq \overline{B}_i, \quad 0 \leq H_a \leq \overline{H}_a, \quad 0 \leq H_m \leq \overline{H}_m, \quad 0 \leq H_r \leq \overline{H}_r,$$

for $g(t) < x < h(t)$ and $0 < t \leq T$. Hence, we have

$$\begin{aligned}
 \lim_{t \rightarrow +\infty} \|B_i(\cdot, t)\|_{C([g(t), h(t)])} & = 0, \quad \lim_{t \rightarrow +\infty} \|H_a(\cdot, t)\|_{C([g(t), h(t)])} = 0, \\
 \lim_{t \rightarrow +\infty} \|H_m(\cdot, t)\|_{C([g(t), h(t)])} & = 0 \quad \text{and} \quad \lim_{t \rightarrow +\infty} \|H_r(\cdot, t)\|_{C([g(t), h(t)])} = 0.
 \end{aligned}$$

□

6 Spreading in the human world

Theorem 5.9 shows that if $r_0^F(0) < 1$ and $R_0^F(0) < 1$, avian influenza vanishes in the bird world and the human world for small initial infection. The following result shows that if $R_0^F(0)$ is big, the mutant avian influenza will spread in the human world.

Theorem 6.1. Assume that $r_0^F(0) < 1$ and $R_0^F(0) \geq 1 + \frac{M}{D_h(\frac{\pi}{2h_0})^2 + (d_h + \gamma_h)}$, where $M = \beta_m \max\{\frac{\gamma_h}{d_h}, 2 - \frac{\gamma_h}{d_h}\}$. Then $h_\infty - g_\infty < \infty$ and

$$\begin{aligned}
 \lim_{t \rightarrow +\infty} \|B_i(\cdot, t)\|_{C([g(t), h(t)])} & = 0, \quad \lim_{t \rightarrow +\infty} \|H_a(\cdot, t)\|_{C([g(t), h(t)])} = 0, \\
 \liminf_{t \rightarrow +\infty} \|H_m(\cdot, t)\|_{C([g(t), h(t)])} & > 0 \quad \text{and} \quad \liminf_{t \rightarrow +\infty} \|H_r(\cdot, t)\|_{C([g(t), h(t)])} > 0
 \end{aligned}$$

provided that $\|B_{i0}(x)\|_{C([-h_0, h_0])}$ and $\|H_{a0}(x)\|_{C([-h_0, h_0])}$ are small enough.

Proof. Let $\overline{B}_i$ and $\sigma(t)$ be the same as those in Theorem 5.7, and $\overline{H}_a(x, t)$ is defined in Theorem 5.9. Let

$$\begin{aligned}
 \underline{g}(t) & = -h_0, \quad \underline{h}(t) = h_0, \quad \underline{B}_i = 0, \\
 \underline{g}(t) & = -\sigma(t), \quad \underline{h}(t) = \sigma(t), \quad \underline{H}_a = 0, \\
 \underline{H}_m(x, t) & = \delta_m \phi(x), \quad \underline{H}_r(x, t) = \frac{\gamma_h}{2(\beta_m + d_h)} \underline{H}_m(x, t), \quad \text{for } -h_0 \leq x \leq h_0, \quad t \geq 0,
 \end{aligned}$$

$$\overline{H}_m(x, t) = N_h^*, \quad \overline{H}_r = \frac{\beta_m \gamma_h - \lambda_{R_0} d_h}{2\beta_m d_h} N_h^*, \quad \text{for } -\sigma(t) \leq x \leq \sigma(t), \quad t \geq 0,$$

where $\phi(x) > 0$ and $\lambda_{R_0} \leq -M$ satisfy (4.4) on $\Omega = (-h_0, h_0)$. Without loss of generality, assume that $H_{m0}(x) > 0$ and $H_{r0}(x) > 0$ for $x \in (-h_0, h_0)$, otherwise we replace the initial time 0 by any $t_0 > 0$. Therefore, there exists $\delta_m > 0$ such that

$$\delta_m \leq \min_{x \in (-h_0, h_0)} \left\{ \frac{H_{m0}(x)}{\phi(x)}, \frac{2(\beta_m + d_h)H_{r0}(x)}{\gamma_h \phi(x)} \right\}.$$

Direct calculations give that

$$\begin{aligned} \frac{\partial \underline{B}_i}{\partial t} - D_b \frac{\partial^2 \underline{B}_i}{\partial x^2} - \frac{\beta_b(N_b^* - \underline{B}_i)\underline{B}_i}{N_b^*} + d_b \underline{B}_i &\leq 0, \\ \frac{\partial \underline{H}_a}{\partial t} - D_h \frac{\partial^2 \underline{H}_a}{\partial x^2} - \frac{\beta_a \underline{B}_i(N_h^* - \underline{H}_a - \overline{H}_m - \overline{H}_r)}{N_h^*} + (d_h + \varepsilon_h) \underline{H}_a &\leq 0. \end{aligned}$$

According to Theorems 5.7 and 5.9, we have

$$\begin{aligned} \frac{\partial \overline{B}_i}{\partial t} - D_b \frac{\partial^2 \overline{B}_i}{\partial x^2} - \frac{\beta_b(N_b^* - \overline{B}_i)\overline{B}_i}{N_b^*} + d_b \overline{B}_i &\geq 0, \\ \frac{\partial \overline{H}_a}{\partial t} - D_h \frac{\partial^2 \overline{H}_a}{\partial x^2} - \frac{\beta_a \overline{B}_i(N_h^* - \overline{H}_a - \underline{H}_m - \underline{H}_r)}{N_h^*} + (d_h + \varepsilon_h) \overline{H}_a \\ &= \varepsilon_a \frac{\partial \overline{B}_i}{\partial t} - \varepsilon_a D_h \frac{\partial^2 \overline{B}_i}{\partial x^2} - \frac{\beta_a \overline{B}_i(N_h^* - \overline{H}_a - \underline{H}_m - \underline{H}_r)}{N_h^*} + (d_h + \varepsilon_h) \overline{H}_a \\ &\geq \varepsilon_a \frac{\partial \overline{B}_i}{\partial t} - \varepsilon_a D_h \frac{\partial^2 \overline{B}_i}{\partial x^2} - \beta_a \overline{B}_i \geq 0. \end{aligned}$$

Now, we can chose δ_m and ε small enough such that

$$\begin{aligned} \frac{\partial \overline{H}_m}{\partial t} - D_h \frac{\partial^2 \overline{H}_m}{\partial x^2} - \frac{\beta_m(N_h^* - \underline{H}_a - \overline{H}_m - \underline{H}_r)\overline{H}_m}{N_h^*} - \varepsilon_h \overline{H}_a + (d_h + \gamma_h) \overline{H}_m \\ &= \beta_m \underline{H}_r - \varepsilon_h \overline{H}_a + (d_h + \gamma_h) N_h^* \geq 0, \\ \frac{\partial \underline{H}_m}{\partial t} - D_h \frac{\partial^2 \underline{H}_m}{\partial x^2} - \frac{\beta_m(N_h^* - \overline{H}_a - \underline{H}_m - \overline{H}_r)\underline{H}_m}{N_h^*} - \varepsilon_h \underline{H}_a + (d_h + \gamma_h) \underline{H}_m \\ &= \underline{H}_m \left\{ \beta_m - (d_h + \gamma_h) + \lambda_{R_0} - \beta_m + \frac{\beta_m(\overline{H}_a + \underline{H}_m + \overline{H}_r)}{N_h^*} + (d_h + \gamma_h) \right\} \leq 0. \end{aligned}$$

On the other hand,

$$\frac{\partial \overline{H}_r}{\partial t} - D_h \frac{\partial^2 \overline{H}_r}{\partial x^2} - \gamma_h \overline{H}_m + d_h \overline{H}_r = -\gamma_h N_h^* + \frac{\beta_m \gamma_h - \lambda_{R_0} d_h}{2\beta_m} N_h^* \geq 0.$$

Moreover,

$$\begin{aligned} \frac{\partial \underline{H}_r}{\partial t} - D_h \frac{\partial^2 \underline{H}_r}{\partial x^2} - \gamma_h \underline{H}_m + d_h \underline{H}_r &= \left[-\frac{\gamma_h(\beta_m + \gamma_h + 2d_h)}{2(\beta_m + d_h)} + \frac{\lambda_{R_0} \gamma_h}{2(\beta_m + d_h)} \right] \underline{H}_m < 0, \\ 0 = (h_0)' &\leq -\mu D_b \frac{\partial \underline{B}_i}{\partial x}(h_0, t), \quad t > 0, \\ 0 = (-h_0)' &\geq -\mu D_b \frac{\partial \underline{B}_i}{\partial x}(-h_0, t), \quad t > 0. \end{aligned}$$

Considering the initial values, if δ_m is sufficiently small, then we have

$$\begin{aligned} \delta_m \phi(x) &\leq H_{m0}(x) \leq \overline{H}_m(x, 0), \quad \text{for } -h_0 < x < h_0, \\ \frac{\delta_m \gamma_h}{2(\beta_m + d_h)} \phi(x) &\leq H_{r0}(x) \leq \overline{H}_r(x, 0), \quad \text{for } -h_0 < x < h_0. \end{aligned}$$

Combining the above results with Theorems 5.7 and 5.9, and applying Lemma 5.8 yield that

$$\begin{aligned} B_i &\leq \overline{B}_i = \varepsilon e^{-\delta t} \varphi(xh_0/\sigma(t)), \quad H_a \leq \overline{H}_a = \varepsilon_a e^{-\delta t} \varphi(xh_0/\sigma(t)), \\ H_m &\leq \overline{H}_m(x, t) = N_h^*, \quad H_r \leq \overline{H}_r = \frac{\beta_m \gamma_h - \lambda_{R_0} d_h}{2\beta_m d_h} N_h^*, \end{aligned}$$

for $(x, t) \in (g(t), h(t)) \times (0, \infty)$, and

$$\begin{aligned} B_i &\geq \underline{B}_i = 0, \quad H_r \geq \underline{H}_r = \frac{\gamma_h \delta_m}{2\beta_m + d_h} \phi, \quad \text{for } -h_0 < x < h_0, \quad t > 0, \\ H_a &\geq \underline{H}_a = 0, \quad H_m \geq \underline{H}_m = \delta_m \phi(x), \quad \text{for } -h_0 < x < h_0, \quad t > 0. \end{aligned}$$

Hence, we can get the conclusion that

$$\begin{aligned} \lim_{t \rightarrow +\infty} \|B_i(\cdot, t)\|_{C([g(t), h(t)])} &= 0, \quad \lim_{t \rightarrow +\infty} \|H_a(\cdot, t)\|_{C([g(t), h(t)])} = 0, \\ \liminf_{t \rightarrow +\infty} \|H_m(\cdot, t)\|_{C([g(t), h(t)])} &> 0, \quad \liminf_{t \rightarrow +\infty} \|H_r(\cdot, t)\|_{C([g(t), h(t)])} > 0. \end{aligned}$$

Therefore, $h_\infty - g_\infty < \infty$ by Theorem 5.7. $\square$

7 Spreading in the bird world

In this section, we are considering the case that the avian influenza spreads in the bird world.

Theorem 7.1. *If $r_0^F(0) := r_0^D((-h_0, h_0)) \geq 1$, then*

$$h_\infty - g_\infty = \infty \quad \text{and} \quad \liminf_{t \rightarrow +\infty} \|B_i(\cdot, t)\|_{C([g(t), h(t)])} > 0.$$

Proof. In the case that $r_0^D((-h_0, h_0)) > 1$, we can get $\lambda_{r_0} < 0$ by Lemma 4.2(i). Then we choose its corresponding eigenfunction $\varphi > 0$ such that

$$\begin{cases} -D_b \Delta \varphi = \beta_b \varphi - d_b \varphi + \lambda_{r_0} \varphi, & -h_0 < x < h_0, \\ \varphi(-h_0) = \varphi(h_0) = 0. \end{cases} \quad (7.1)$$

Now, we want to construct a lower solution to the bird system with the free boundary.

Defining

$$\underline{B}_i(x, t) = \delta_1 \varphi(x), \quad \text{for } -h_0 \leq x \leq h_0, \quad t > 0,$$

we then have that

$$\begin{aligned} \frac{\partial \underline{B}_i}{\partial t} - D_b \frac{\partial^2 \underline{B}_i}{\partial x^2} - \frac{\beta_b(N_b^* - \underline{B}_i)\underline{B}_i}{N_b^*} + d_b \underline{B}_i \\ = -D_b \delta_1 \varphi'' - \frac{\beta_b(N_b^* - \underline{B}_i)\underline{B}_i}{N_b^*} + d_b \underline{B}_i \\ = \left(\lambda_{r_0} + \frac{\delta_1 \beta_b \varphi}{N_b^*} \right) \underline{B}_i \leq 0, \quad -h_0 \leq x \leq h_0, \quad t > 0, \end{aligned}$$

if δ_1 is sufficiently small.

Moreover,

$$\begin{aligned} \underline{B}_i(-h_0, t) &= \underline{B}_i(h_0, t) = 0, \\ 0 &= h'_0 \leq -\mu D_b \frac{\partial \underline{B}_i}{\partial x}(h_0, t), \quad \text{for } t > 0, \\ 0 &= -h'_0 \geq -\mu D_b \frac{\partial \underline{B}_i}{\partial x}(-h_0, t), \quad \text{for } t > 0. \end{aligned}$$

Noting that δ_1 is small such that $\underline{B}_i(x, 0) \leq B_{i0}$, we have $B_i(x, t) \geq \underline{B}_i(x, t)$ for $(x, t) \in [-h_0, h_0] \times [0, \infty)$ by using Lemma 5.6. Thus $\liminf_{t \rightarrow +\infty} \|B_i(\cdot, t)\|_{C([-h_0, h_0])} \geq \delta_1 \varphi(x) > 0$ and it follows from Corollary 5.3 that $h_\infty - g_\infty = \infty$.

The other case $r_0^F(0) = 1$ can follow from the above. In fact, for any $T_0 > 0$, we have $g(T_0) < -h_0$ and $h(T_0) > h_0$ by Lemma 3.2. From Lemma 4.3(iv), we get $r_0^F((g(T_0), h(T_0))) > r_0^F(0) = 1$. $\square$

Following the proof of Theorem 7.1, we have the following result.

Corollary 7.2. For any $t_0 \in (0, +\infty)$, if $r_0^F(t_0) \geq 1$, then

$$h_\infty - g_\infty = \infty \quad \text{and} \quad \liminf_{t \rightarrow +\infty} \|B_i(\cdot, t)\|_{C([g(t), h(t)])} > 0.$$

Theorem 7.3. Assume that $r_0^F(0) (= r_0^D((-h_0, h_0))) < 1 < r_0$. Then

$$h_\infty - g_\infty = \infty \quad \text{and} \quad \liminf_{t \rightarrow +\infty} \|B_i(\cdot, t)\|_{C([g(t), h(t)])} > 0,$$

if $\|B_{i0}(x)\|_{C([-h_0, h_0])}$ is sufficiently large.

Proof. Let λ be the principle eigenvalue of the problem

$$\begin{cases} -D_b \psi'' - \frac{1}{2} \psi' = \lambda \psi, & 0 < x < 1, \\ \psi'(0) = \psi(1) = 0, \end{cases} \quad (7.2)$$

its corresponding eigenfunction ψ can be chosen positive in $[0, 1)$ and $\|\psi\|_{L^\infty([0, 1])} = 1$. Then it satisfies that $\psi'(x) > 0$ in $[-1, 0)$ and $\psi'(x) < 0$ in $(0, 1]$. Now, if we extend ψ to be defined in $[-1, 1]$, then it satisfies

$$\begin{cases} -D_b \Psi'' - \frac{\text{sgn}(x)}{2} \Psi' = \lambda \Psi, & -1 < x < 1, \\ \Psi(-1) = \Psi(1) = 0. \end{cases} \quad (7.3)$$

Since $r_0 > 1$ by assumption, it follows from Lemma 4.4 that $\lim_{T \rightarrow +\infty} r_0^D((-\sqrt{T}, \sqrt{T})) = r_0 > 1$ and there exists a large T^* such that $r_0^D((-\sqrt{T^*}, \sqrt{T^*})) > 1$. Now, let any $T > T^*$ and we want to construct a suitable lower solution $(\underline{B}_i; \underline{g}, \underline{h})$ to (2.6) for $0 \leq t \leq T$.

Define

$$\begin{aligned} \underline{g}(t) &= -\sqrt{t + \sigma}, \quad \underline{h}(t) = \sqrt{t + \sigma}, \quad \text{for } t > 0, \\ \underline{B}_i(x, t) &= \frac{M}{(t + \sigma)^k} \Psi\left(\frac{x}{\sqrt{t + \sigma}}\right), \quad -\sqrt{t + \sigma} \leq x \leq \sqrt{t + \sigma}, \quad t \geq 0. \end{aligned}$$

Let $0 < \sigma \leq \min\{1, h_0^2\}$ and $k > \lambda + d_b(T + 1)$. Direct computation yields

$$\begin{aligned} & \frac{\partial \underline{B}_i}{\partial t} - D_b \frac{\partial^2 \underline{B}_i}{\partial x^2} - \frac{\beta_b(N_b^* - \underline{B}_i)\underline{B}_i}{N_b^*} + d_b \underline{B}_i \\ &= -M(t + \sigma)^{-k-1} \left\{ D_b \Psi'' + \frac{x}{2\sqrt{t + \sigma}} \Psi' + k \Psi + \left[\frac{\beta_b(N_b^* - \underline{B}_i)}{N_b^*} - d_b \right] (t + \sigma) \Psi \right\} \\ &\leq -M(t + \sigma)^{-k-1} \left(D_b \Psi'' + \frac{\text{sgn}(x)}{2} \Psi' + \lambda \Psi \right) = 0, \end{aligned}$$

for all $\underline{g} \leq x \leq \underline{h}$ and $0 < t \leq T$. We now choose $M > -\frac{(T+1)^k}{2\mu D_b \Psi'(1)}$ sufficiently large such that

$$\begin{aligned} \underline{h}'(t) + \mu D_b(\underline{B}_i)_x(\sqrt{t + \sigma}, t) &= \frac{1}{2\sqrt{t + \sigma}} + \frac{\mu D_b M}{(t + \sigma)^{k+1/2}} \Psi'(1) < 0, \\ \underline{g}'(t) + \mu D_b(\underline{B}_i)_x(-\sqrt{t + \sigma}, t) &= -\frac{1}{2\sqrt{t + \sigma}} + \frac{\mu D_b M}{(t + \sigma)^{k+1/2}} \Psi'(-1) > 0. \end{aligned}$$

Then we obtain

$$\left\{ \begin{array}{ll} \frac{\partial \underline{B}_i}{\partial t} - D_b \frac{\partial^2 \underline{B}_i}{\partial x^2} \leq \frac{\beta_b(N_b^* - \underline{B}_i)\underline{B}_i}{N_b^*} - d_b \underline{B}_i, & |x| < \sqrt{t + \sigma}, \quad 0 < t \leq T, \\ \underline{B}_i(g(t), t) = \underline{B}_i(h(t), t) = 0, & 0 < t \leq T, \\ \underline{g}(0) = -\sqrt{\sigma} \geq -h_0, \quad \underline{g}'(t) \geq -\mu D_b \frac{\partial \underline{B}_i}{\partial x}(\underline{g}, t), & 0 < t \leq T, \\ \underline{h}(0) = \sqrt{\sigma} \leq h_0, \quad \underline{h}'(t) \leq -\mu D_b \frac{\partial \underline{B}_i}{\partial x}(\underline{h}, t), & 0 < t \leq T. \end{array} \right.$$

Noting that $\|B_{i0}(x)\|_{C([-h_0, h_0])}$ is sufficiently large, we get $\underline{B}_i(x, 0) = \frac{M}{\sigma^k} \Psi(\frac{x}{\sqrt{\sigma}}) \leq B_{i0}(x)$ in $[-\sqrt{\sigma}, \sqrt{\sigma}]$. By using Lemma 5.6, we get $h(t) \geq \underline{h}(t)$ and $g(t) \leq \underline{g}(t)$ in $[0, T]$. Then we have that $h(T) \geq \underline{h}(T) = \sqrt{T + \sigma} \geq \sqrt{T}$, and $g(T) \leq \underline{g}(T) = -\sqrt{T + \sigma} \leq -\sqrt{T}$. Therefore,

$$r_0^F(T^*) = r_0^D((-g(T^*), h(T^*))) \geq r_0^D((-\sqrt{T^*}, \sqrt{T^*})) > 1,$$

which together with Corollary 7.2 gives that $h_\infty - g_\infty = \infty$ and $\liminf_{t \rightarrow +\infty} \|B_i(\cdot, t)\|_{C([g(t), h(t)])} > 0$. $\square$

8 Discussion

Most prior work related to the disease transmitting has focused on the understanding of the disease dynamic in the fixed domain. There is almost no work which has investigated the characteristics of disease spreading in a moving area.

In this paper, we introduce two moving boundaries, which are called free boundaries, to describe the avian influenza virus transmitting in the habitat. The dynamical behaviors of the solution have been discussed. Two basic reproduction numbers $r_0^F(t)$ and $R_0^F(t)$ are given for the free boundary problem (2.4) in Section 4. The infected domain $(g(t), h(t))$ changes with t , so the basic reproduction numbers depend on time t . It has been shown that the two reproduction numbers $r_0^F(t)$ and $R_0^F(t)$ are monotonically increasing with respect to t , and $r_0^F(t)$ and $R_0^F(t)$ approach to r_0 and R_0 as $t \rightarrow \infty$, respectively, where r_0 and R_0 are the responding basic reproduction numbers of avian influenza in the homogeneous bird world and mutant avian influenza in the homogeneous world, respectively.

Recently, much work with the basic reproduction number has been done to study the diseases dynamic properties. The basic reproduction numbers are usually defined by constants. But our definition of the basic reproduction numbers are time-dependent, and thus $r_0^F(t)$ and $R_0^F(t)$ can be utilized to predict spreading or vanishing of the avian influenza virus.

The sufficient conditions are given to determine the transmission of the avian influenza virus. Our results show that spreading or vanishing of the avian influenza is not only related to the basic reproduction numbers $r_0^F(t)$ and $R_0^F(t)$, but the initial infection. If $r_0^F(0) < 1$, $R_0^F(0) < 1$ and the initial infection value is small, the avian influenza disease vanishes, i.e., $h_\infty - g_\infty < \infty$,

$$\lim_{t \rightarrow +\infty} (\|B_i(\cdot, t)\|_C, \|H_a(\cdot, t)\|_C, \|H_m(\cdot, t)\|_C, \|H_r(\cdot, t)\|_C) = (0, 0, 0, 0)$$

uniformly for $x \in [g(t), h(t)]$ (see Theorem 5.9). It is shown that the mutant avian influenza spreads in the human world if $r_0^F(0) < 1$ and

$$R_0^F(0) \geq 1 + \frac{M}{D_h(\frac{\pi}{2h_0})^2 + (d_h + \gamma_h)}$$

(see Theorem 6.1). If $r_0^F(t_0) \geq 1$ for any $t_0 \geq 0$, then the avian influenza transmits in the bird world (see Corollary 7.2).

Nowadays, epidemic models have attracted much attention [14], especially, avian influenza model. The study of our work tries to describe the spreading process of the disease, and encourages people to take good strategies to prevent the avian influenza virus from transmitting to human world. There are some work to do, for example, the spreading speed when spreading happens.

Acknowledgements The work was supported by National Natural Science Foundation of China (Grant No. 11071209), Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (Grant No. 2010-0025700), and Natural Science Foundation of the Higher Education Institutions of Jiangsu Province (Grant No. 12KJD110008).

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