Pattern dynamics of an epidemic model with nonlinear incidence rate

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Abstract All species live in space, and the research of spatial diseases can be used to control infectious diseases. As a result, it is more realistic to study the spatial pattern of epidemic models with space and time. In this paper, spatial dynamics of an epidemic model with nonlinear incidence rate is investigated. We find that there are different types of stationary patterns by amplitude equations and numerical simulations. The obtained results may well explain the distribution of disease observed in the real world and provide some insights on disease control.

Keywords Nonlinear incidence · Spatial epidemic model · Amplitudes equation · Pattern selection

1 Introduction

The popularity and spread of infectious diseases have always been a huge threat to human survival [1–3]. The black death (bubonic plague) ravaged Europe four times in history. The first time was in 600 AD, and about half of Europe's people had been killed at that time; the second outbreak was in 1346–1350 AD, and it lead to reduce the Europe population by one-third; the third occurred from 1665 to 1666, and 1/6 of the population in London died; the last time was from 1720 to

1722, and it caused half of the population in Marseilles in France. After entering the 21st century, we still face the threat from infectious diseases. After Severe Acute Respiratory Syndrome (SARS) was found in Guang Dong province of China in November, 2002, it spreads in 32 countries and regions, more than eight thousand cases who got the disease and more than eight hundred people died in just a few months [4]. In 2009, H1N1 influenza virus caused a global outbreak and at least 11,516 deaths [5,6]. In February 2013, there appears a new type of avian influenza named H7N9 in China, and there has been 128 confirmed human cases reported by China's Ministry of Health, among 27 died [7,8]. Therefore, understanding the rule of infectious diseases transmission rule and providing control strategy is becoming world's significant problems which need to be solved urgently.

Traditional epidemic models are usually established using ordinary differential equation, difference equation, or delay differential equations which ignore spatial factors, to get the threshold of the spread of disease or not. However, all the species in the world are living in space, and thus the study of infectious diseases in space can well provide theoretical basis for the prevention and control of the infectious diseases.

Reaction–diffusion equations belong to time and space type, and they suppose that environment changes continuous and the individual migrates randomly or spreads in all directions with the same probability. The reaction terms indicate that changes or interaction process of individuals without diffusion; diffusion

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term describes space motion of the individual. Suppose $N(x, t)$ is individual density in t moment and x location, the corresponding reaction–diffusion equations can be written as [9–12]:

$$\frac{\partial N}{\partial t} = F(N) + D\nabla^2 N, \quad (1)$$

where $F(N)$ is reaction term, $D\nabla^2 N$ is diffusion term, and D is diffusion coefficient (diffusion rate).

For general reaction–diffusion models of infectious diseases, they usually can be written as:

$$\frac{\partial S}{\partial t} = F(S, I) + D_1 \nabla^2 S, \quad (2a)$$

$$\frac{\partial I}{\partial t} = G(S, I) + D_2 \nabla^2 I, \quad (2b)$$

where D_1 and D_2 are diffusion coefficients.

In the spread of disease, spatial pattern was first founded in the host-parasitoids model [13]. Ballegoijen and Boerlijst [14] founded that spiral wave pattern and target wave pattern in SIRS epidemic model and obtained the relationship between transmission frequency and wave velocity. Gubler [15] studied the transmission of DHF in 1930, 1970, and 2001 and founded that DHF presents typical pattern structures in space. Liu and Jin [16] established a SIR model and found that there are stable spotted and stripe coexistence pattern but no isolated spotted pattern. Sun et al. [17] investigated an SI epidemic model with nonlinear incidence rate and found isolated spotted pattern in two-dimensional space. Li et al. [18] studied an epidemic model with migration and obtained typical traveling pattern. Sun et al. [19] presented an epidemic model with cross diffusion, and anti-phase dynamics of different spatial points were found.

In this article, we will use the method mentioned and combine the standard multiple-scale analysis to study the pattern selection, where the control parameter (s) and the derivatives are expanded with respect to a small parameter ε , and the Fredholm solubility condition is used. Sun et al. [20] investigated spatial dynamics of a predator–prey system with Allee effect and found that predator mortality plays an important role in the pattern formation of populations. Sun et al. [21] modeled a vegetation model in an arid flat environment using reaction–diffusion form and presented the rich dynamics by means of amplitude equation.

This article mainly aims at the pattern dynamics generated by means of an SI model. In Sect. 2, we describe

an SI model. In Sect. 3, we analyze the bifurcation and get the Turing bifurcation under the critical condition. Furthermore, we apply nonlinear multi-scale analysis to gain amplitude equation and obtain different types of Turing pattern. Finally, some conclusions are given.

2 Main model

First, we give two assumptions:

- Pathogens are alive in the population, and include two subgroups: the healthy individuals who are susceptible (S) to infection and the already infected individuals (I) who can transmit the disease to the healthy ones.
- The infection term is $\beta S^p I^q$, where β is infectious rate. In this paper, we consider the case: $p = 1$, $q = 2$.

The spatial epidemic model is as follows:

$$\frac{\partial S}{\partial t} = A - \beta S^p I^q - dS + d_1 \nabla^2 S, \quad (3a)$$

$$\frac{\partial I}{\partial t} = \beta S^p I^q - (d + \mu)I + d_2 \nabla^2 I, \quad (3b)$$

where A is the rate of population increase, d is natural mortality rate of population, and μ is the disease-induced death rate of infected. $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ presents Laplace operator in the two-dimensional space, d_1 and d_2 are diffusion coefficient of susceptible individuals and infected individuals, respectively. We assume that all parameters in this article are positive. The initial condition of model is $S(0) > 0$ and $I(0) > 0$. The boundary condition is $\frac{\partial S}{\partial n}|_{(x,y)} = \frac{\partial I}{\partial n}|_{(x,y)} = 0$. Here n is spatial vector $(x, y) \in \partial\Omega$, and Ω is the space domain.

3 Pattern dynamics

3.1 Local dynamics

In order to get the Turing instability in reaction–diffusion systems, considering the local dynamics of the system is very important. The corresponding model is:

$$\frac{dS}{dt} = A - \beta S^p I^q - dS = f(S, I), \quad (4a)$$

$$\frac{dI}{dt} = \beta S^p I^q - (d + \mu)I = g(S, I). \quad (4b)$$

In the region $S \geq 0$ and $I \geq 0$, we let $f(S, I) = 0$ and $g(S, I) = 0$ and obtain equilibrium points:

- (i) $E_0(S_0, I_0) = (\frac{A}{d}, 0)$, corresponding to disease free equilibrium;
- (ii) $E_1(S_1, I_1) = \left(\frac{A\beta + \sqrt{A^2\beta^2 - 4d^3\beta - 8d^2\beta\mu - 4d\beta\mu^2}}{2d\beta}, \frac{2d(d+\mu)}{A\beta + \sqrt{A^2\beta^2 - 4d^3\beta - 8d^2\beta\mu - 4d\beta\mu^2}} \right)$ corresponding to the coexistence of the susceptible and infective;
- (iii) $E^*(S^*, I^*) = \left(\frac{A\beta - \sqrt{A^2\beta^2 - 4d^3\beta - 8d^2\beta\mu - 4d\beta\mu^2}}{2d\beta}, \frac{2d(d+\mu)}{A\beta - \sqrt{A^2\beta^2 - 4d^3\beta - 8d^2\beta\mu - 4d\beta\mu^2}} \right)$ corresponding to the coexistence of the susceptible and infective.

It is known that E_1 is unstable by means of calculation, and we only need to study the dynamical behavior of E^* . The Jacobian matrix of the equilibrium is as follows:

$$\mathbf{J} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where

$$a_{11} = -d - \beta(I^*)^2, \quad (5a)$$

$$a_{12} = -2\beta S^* I^*, \quad (5b)$$

$$a_{21} = \beta(I^*)^2, \quad (5c)$$

$$a_{22} = 2\beta S^* I^* - (d + \mu). \quad (5d)$$

We make the following nonuniform perturbations:

$$\begin{pmatrix} S \\ I \end{pmatrix} = \begin{pmatrix} S^* \\ I^* \end{pmatrix} + \varepsilon \begin{pmatrix} S_\kappa \\ I_\kappa \end{pmatrix} e^{\lambda t + i\kappa \mathbf{r}} + c.c. + O(\varepsilon^2), \quad (6)$$

where λ is the growth rate of perturbations in time t , i is imaginary unit, κ presents wavelength, and $\mathbf{r} = (X, Y)$ is two-dimensional spatial factor which stands for complex conjugate plane. The linear stability of the equilibrium can be deduced from the dispersion relations. After substituting the above Eq. (6) into equation (3), we can get the determinant of $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} a_{11} - d_1\kappa^2 - \lambda & a_{12} \\ a_{21} & a_{22} - d_2\kappa^2 - \lambda \end{pmatrix}.$$

Eigenvalue equation is as follows:

$$\lambda^2 + a_\kappa \lambda + \eta_\kappa = 0, \quad (7)$$

where

$$a_\kappa = (d_1 + d_2)\kappa^2 - (a_{11} + a_{22}), \quad (8a)$$

$$\eta_\kappa = d_1 d_2 \kappa^4 - (d_2 a_{11} + d_1 a_{22}) \kappa^2 + a_{11} a_{22} - a_{12} a_{21}. \quad (8b)$$

Then we can obtain the eigenvalues λ_κ as follows:

$$\lambda_\kappa = \frac{-a_\kappa \pm \sqrt{(a_\kappa)^2 - 4\eta_\kappa}}{2}. \quad (9)$$

Hopf bifurcation occurs when $Im(\lambda_\kappa) \neq 0$, $Re(\lambda_\kappa) = 0$ with $\kappa = 0$, i.e. $a_{11} + a_{22} = 0$. Hopf bifurcation parameter β can be deduced by means of calculation:

$$\beta_H = \frac{d^4 + 4d^3\mu + 6d^2\mu^2 + 4d\mu^3 + \mu^4}{\mu A^2}. \quad (10)$$

Turing bifurcation occurs when $Im(\lambda_\kappa) = 0$, $Re(\lambda_\kappa) = 0$ with $\kappa = \kappa_T \neq 0$, and

$$\kappa_T^4 = \frac{a_{11}a_{22} - a_{12}a_{21}}{d_1 d_2}. \quad (11)$$

By calculation, we can get the Turing threshold of this bifurcation parameter β :

$$\begin{aligned} \beta_T = & d_1(d^3 + 3d^2\mu + 3d\mu^2 + \mu^3)[d^2d_1^2 + 3d^2d_1d_2 \\ & + 8d^2d_2^2 + 2dd_1(dd_1 + \mu d_1 - \sqrt{J}) \\ & + 2d\mu d_1 + 4dd_2(dd_1 + \mu d_1 - \sqrt{J}) + 3d\mu d_1d_2 \\ & + 2d_1(dd_1 + \mu d_1 - \mu\sqrt{J} + \mu^2d_1^2)] \\ & / [A^2d_2(d^2d_1^2 + 2d^2d_1d_2 + d^2d_2^2 + 2d\mu d_1^2 \\ & + 2d\mu d_1d_2 + \mu^2d_1^2)], \end{aligned}$$

where

$$J = 2d^2d_1^2 + 4d\mu d_1^2 + 2\mu^2d_1^2 - 2d^2d_1d_2 - 2d\mu d_1d_2.$$

In Fig. 1, we show the bifurcations in the parameter space spanned by the parameters β and μ . In the space marked by T , stationary inhomogeneous patterns can be observed. And we find that as β increases, the real part of the character value decreases.

3.2 Amplitude equations

The well-known amplitude equations can be deduced by the standard multiple-scale analysis. Close to the onset $\beta = \beta_T$, the eigenvalues associated to the critical modes are close to zero, and they are slowly varying modes which implies that we only need to consider perturbations κ around κ_T .

To deduce the amplitude equation, we first write out the linearized form of the model (3) at the equilibrium point E^* as follows:

Fig. 1 Bifurcation diagram for the system (3). Turing space is marked as T. Parameters values: $A = 1$, $d = 1$, $d_1 = 6$, and $d_2 = 1$

$$\frac{\partial x}{\partial t} = a_{11}x + a_{12}y - 2\beta I^*xy - \beta S^*y^2 - \beta xy^2 + d_1\Delta^2x, \quad (12a)$$

$$\frac{\partial y}{\partial t} = a_{21}x + a_{22}y + 2\beta I^*xy + \beta S^*y^2 + \beta xy^2 + d_2\Delta^2y. \quad (12b)$$

Close to onset $\beta = \beta_T$, the solutions of the above model can be expanded as the following form:

$$U = U_S + \sum_{j=1}^3 U_0[A_j \exp(i\kappa_j \cdot r) + \bar{A}_j \exp(-i\kappa_j \cdot r)]. \quad (13)$$

It can also be expanded as:

$$U^0 = \sum_{j=1}^3 U_0[A_j \exp(i\kappa_j \cdot r) + \bar{A}_j \exp(-i\kappa_j \cdot r)], \quad (14)$$

where U_S represents the uniform steady state, and

$$U_0 = \left(\frac{a_{11}^*d_2 + a_{22}^*d_1}{2a_{21}^*d_1}, 1 \right)^T \quad (15)$$

is eigenvector of linear operator. A_j and the conjugate $\bar{A}_j$ are the amplitudes associated with the modes κ_j and $-\kappa_j$, respectively. Through the standard multiple-scale analysis, the amplitude of the evolution of time and space can be described by the following equations:

$$\tau_0 \frac{\partial A_1}{\partial t} = \xi A_1 + h A_2 A_3 - (g_1 |A_1|^2 + g_2 (|A_2|^2 + |A_3|^2)) A_1, \quad (16a)$$

$$\tau_0 \frac{\partial A_2}{\partial t} = \xi A_2 + h A_1 A_3 - (g_1 |A_2|^2 + g_2 (|A_1|^2 + |A_3|^2)) A_2, \quad (16b)$$

$$\tau_0 \frac{\partial A_3}{\partial t} = \xi A_3 + h A_1 A_2 - (g_1 |A_3|^2 + g_2 (|A_1|^2 + |A_2|^2)) A_3, \quad (16c)$$

where $\xi = (\beta_T - \beta)/\beta_T$ is a normalized distance, and τ_0 is a typical relaxation time. In the following, we will give the exact expressions of the coefficients τ_0 , h , g_1 and g_2 .

Setting $X = (x, y)^T$, $N = (N_1, N_2)$, system (12) can be converted to the following system:

$$\frac{\partial X}{\partial t} = LX + N, \quad (17)$$

where

$$L = \begin{pmatrix} a_{11} + d_1 \nabla^2 & a_{12} \\ a_{21} & a_{22} + d_2 \nabla^2 \end{pmatrix},$$

$$N = \begin{pmatrix} -2\beta I^*xy - \beta S^*y^2 - \beta xy^2 \\ 2\beta I^*xy + \beta S^*y^2 + \beta xy^2 \end{pmatrix}.$$

We just analyze the dynamics when $\beta = \beta_T$. Then we can expand β to the following form:

$$\beta_T - \beta = \varepsilon \beta_1 + \varepsilon^2 \beta_2 + \varepsilon^3 \beta_3 + O(\varepsilon^4), \quad (18)$$

where ε is a small parameter. Expanding the variable X and the nonlinear term N to the series form about ε :

$$X = \begin{pmatrix} x \\ y \end{pmatrix} = \varepsilon \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \varepsilon^3 \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} + O(\varepsilon^4), \quad (19)$$

$$N = \varepsilon^2 h^2 + \varepsilon^3 h^3 + O(\varepsilon^4), \quad (20)$$

where h^2 and h^3 are corresponding to the second and the third order of ε in the expansion, respectively, of the nonlinear term N . The linear operator L can be expanded as follows:

$$L = L_T + (\beta_T - \beta)M, \quad (21)$$

where

$$L_T = \begin{pmatrix} a_{11}^* + d_1 \nabla^2 & a_{12}^* \\ a_{21}^* & a_{22}^* + d_1 \nabla^2 \end{pmatrix}, M = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

The core of the standard multiple-scale analysis is separating the dynamical behavior of the system according to different time scales and spatial scale. We need to

separate the time scale for model (17) (i.e. $T_0 = t$, $T_1 = \varepsilon t$ and $T_2 = \varepsilon^2 t$). Each time scale T_i can be considered as an independent variable. The derivative of T_i with respect to time can turn to the following form:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} + \varepsilon^2 \frac{\partial}{\partial T_2} + O(\varepsilon^3). \quad (22)$$

Since the variation of the amplitude A changes slowly, the derivative with respect to time $\frac{\partial}{\partial T_0}$ almost does not have an effect on the amplitude A . We have the following result:

$$\frac{\partial A}{\partial t} = \varepsilon \frac{\partial A}{\partial T_1} + \varepsilon^2 \frac{\partial A}{\partial T_2} + O(\varepsilon^3). \quad (23)$$

Substituting the Eqs. (19), (20), (21), and (22) into (17), we can obtain three equations as follows.

The first order of ε :

$$L_T \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0;$$

The second order of ε :

$$L_T \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \frac{\partial}{\partial T_1} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \beta_1 M \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} -2\beta I^* x_1 y_1 - \beta S^* y_1^2 \\ 2\beta I^* x_1 y_1 + \beta S^* y_1^2 \end{pmatrix};$$

The third order of ε :

$$L_T \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \frac{\partial}{\partial T_1} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \frac{\partial}{\partial T_2} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \beta_1 M \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} - \beta_2 M \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - P, \quad (24)$$

Where

$$P = \begin{pmatrix} -2\beta I^*(x_1 y_2 + y_1 x_2) - 2\beta S^* y_1 y_2 - \beta x_1 y_1^2 \\ 2\beta I^*(x_1 y_2 + y_1 x_2) + 2\beta S^* y_1 y_2 + \beta x_1 y_1^2 \end{pmatrix}.$$

First the first order about ε , we have that:

$$L_T \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0. \quad (25)$$

As L_T is the linear operator of the system close to initial point, $(x_1, y_1)^T$ is the linear combination of the eigenvectors that corresponds to the eigenvalue 0. Solving the first order of ε , we have:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \frac{a_{11}^* d_2 - a_{22}^* d_1}{2a_{21}^* d_1} \\ 1 \end{pmatrix} (W_1 \exp(i\kappa_1 \mathbf{r}) + W_2 \exp(i\kappa_2 \mathbf{r}) + W_3 \exp(i\kappa_3 \mathbf{r})) + c.c., \quad (26)$$

where $|\kappa_j| = \kappa_T^*$, W_j is the modulus of $\exp(i\kappa_j \mathbf{r})$ when the system is under the first order perturbation. The

form is determined by means of the higher order perturbation terms.

The second-order differential equation of ε , we let

$$L_T \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \frac{\partial}{\partial T_1} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \beta_1 M \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} -2\beta I^* x_1 y_1 - \beta S^* y_1^2 \\ 2\beta I^* x_1 y_1 + \beta S^* y_1^2 \end{pmatrix} = \begin{pmatrix} F_x \\ F_y \end{pmatrix}. \quad (27)$$

According to the Fredhold solubility condition, in order to ensure the nontrivial solution of equation, the vector function of the right hand of equation (27) must be orthogonal with the zero eigenvectors of operator L_c^+ . L_c^+ is the adjoint operator of L_c . In this system, the zero eigenvectors of operator L_c^+ are

$$\left(-\frac{1}{\frac{a_{11}^* d_2 - a_{22}^* d_1}{2a_{21}^* d_2}} \right) \exp(-i\kappa_j \mathbf{r}) + c.c. \quad (j = 1, 2, 3). \quad (28)$$

We can get the following from the orthogonality

$$\left(1, -\frac{a_{11}^* d_2 - a_{22}^* d_1}{2a_{21}^* d_2} \right) \begin{pmatrix} F_x^i \\ F_y^i \end{pmatrix} = 0, \quad (29)$$

where F_x^i and F_y^i represent the coefficients corresponding to $\exp(i\kappa_j \mathbf{r})$ in F_x and F_y . Taking $\exp(i\kappa_1 \mathbf{r})$ for example, we have

$$\begin{aligned} \left(1 - \frac{d_1}{d_2} \right) \frac{\partial W_1}{\partial T_1} &= \beta_1 (lb_{11} + b_{12} \\ &- \frac{\partial d_1}{\partial d_2} l(b_{21} + b_{22})) W_1 \\ &- \left(1 + \frac{d_1}{d_2} l \right) (4l\beta_T I^* + 2l\beta_T S^*) \bar{W}_2 \bar{W}_3. \end{aligned} \quad (30)$$

At the same time, we let:

$$\begin{aligned} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} &= \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} + \sum_{j=1}^3 \begin{pmatrix} X_j \\ Y_j \end{pmatrix} \exp(i\kappa_j \cdot \mathbf{r}) \\ &+ \sum_{j=1}^3 \begin{pmatrix} X_{jj} \\ Y_{jj} \end{pmatrix} \exp(i2\kappa_j \cdot \mathbf{r}) \\ &+ \begin{pmatrix} X_{12} \\ Y_{12} \end{pmatrix} \exp(i(\kappa_1 - \kappa_2) \cdot \mathbf{r}) + H + c.c., \end{aligned} \quad (31)$$

where

$$\begin{aligned} H &= \begin{pmatrix} X_{23} \\ Y_{23} \end{pmatrix} \exp(i(\kappa_2 - \kappa_3) \cdot \mathbf{r}) \\ &+ \begin{pmatrix} X_{31} \\ Y_{31} \end{pmatrix} \exp(i(\kappa_3 - \kappa_1) \cdot \mathbf{r}). \end{aligned}$$

The coefficients of Eq. (30) are obtained by solving the sets of the linear equations about $\exp(0)$, $\exp(i\kappa_j \mathbf{r})$, $\exp(i2\kappa_j \mathbf{r})$, and $\exp(i(\kappa_j - \kappa_k) \mathbf{r})$. We can get

$$\begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} (|W_1|^2 + |W_2|^2 + |W_3|^2), \quad X_i = lY_i, \\ \begin{pmatrix} X_{jj} \\ Y_{jj} \end{pmatrix} = \begin{pmatrix} x_{11} \\ y_{11} \end{pmatrix} W_j^2, \quad \begin{pmatrix} X_{jk} \\ Y_{jk} \end{pmatrix} = \begin{pmatrix} x^* \\ y^* \end{pmatrix} W_j \bar{W}_k.$$

We solve the third-order differential equation of ε and get

$$L_T \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \frac{\partial}{\partial T_1} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \frac{\partial}{\partial T_2} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \beta_1 M \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} - \beta_2 M \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - O, \quad (32)$$

where

$$O = \begin{pmatrix} -2\beta I^*(x_1 y_2 + y_1 x_2) - 2\beta S^* y_1 y_2 - \beta x_1 y_1^2 \\ 2\beta I^*(x_1 y_2 + y_1 x_2) + 2\beta S^* y_1 y_2 + \beta x_1 y_1^2 \end{pmatrix}.$$

We can get the following using the Fredholm solubility condition

$$\begin{aligned} & \frac{d_2 - d_1}{d_2} l \frac{\partial W_1}{\partial T_1} + \frac{d_2 - d_1}{d_2} l \frac{\partial Y_1}{\partial T_2} \\ &= \beta_2 \left(lb_{11} + b_{12} - \frac{d_1}{d_2} l (lb_{21} + b_{22}) \right) W_1 \\ &+ \beta_1 \left(lb_{11} + b_{12} - \frac{d_1}{d_2} l (lb_{21} + b_{22}) \right) Y_1 \\ &+ \left(1 + \frac{d_1}{d_2} l \right) \left(4\beta_T I^* + 2\beta_T S^* \right) (\bar{W}_3 \bar{Y}_2 + \bar{W}_2 \bar{Y}_3) \\ &- \left(G_1 |W_1|^2 + G_2 (|W_2|^2 + |W_3|^2) \right) W_1. \end{aligned} \quad (33)$$

In a similar way, the other two equations can be obtained, and the amplitude A_i can be expanded as

$$A_i = \varepsilon W_i + \varepsilon^2 V_i + O(\varepsilon^3). \quad (34)$$

We use Eqs. (30), (33) multiply by ε^2 and ε^3 , we can obtain the amplitude equation corresponding to A_1 by combining variables of Eqs. (23), (34) as follows

$$\begin{aligned} \tau_0 \frac{\partial A_1}{\partial t} &= \xi A_1 + h \bar{A}_2 \bar{A}_3 - (g_1 |A_1|^2 \\ &+ g_2 (|A_2|^2 + |A_3|^2)) A_1, \end{aligned} \quad (35)$$

The other two equations of (16) can be obtained through the transformation of the subscript of A . Exact expressions of coefficient l , G_1, G_2 , τ_0 , h , g_1 , and g_2 are shown in Appendix.

By means of substitutions, we have:

$$\begin{aligned} \tau_0 \frac{\partial \varphi}{\partial t} &= -h \frac{\rho_1^2 \rho_2^2 + \rho_1^2 \rho_3^2 + \rho_2^2 \rho_3^2}{\rho_1 \rho_2 \rho_3} \sin \varphi, \\ \tau_0 \frac{\partial \rho_1}{\partial t} &= \xi \rho_1 + h \rho_2 \rho_3 \cos \varphi - g_1 \rho_1^3 - g_2 (\rho_2^2 \rho_3^2) \rho_1, \\ \tau_0 \frac{\partial \rho_2}{\partial t} &= \xi \rho_2 + h \rho_1 \rho_3 \cos \varphi - g_1 \rho_2^3 - g_2 (\rho_1^2 \rho_3^2) \rho_2, \\ \tau_0 \frac{\partial \rho_3}{\partial t} &= \xi \rho_3 + h \rho_1 \rho_2 \cos \varphi - g_1 \rho_3^3 - g_2 (\rho_1^2 \rho_2^2) \rho_3. \end{aligned} \quad (36)$$

where, $\varphi = \varphi_1 + \varphi_2 + \varphi_3$.

The dynamical system (36) possesses five kinds of solutions [20–22].

(1) The stationary state (O), given by

$$\rho_1 = \rho_2 = \rho_3 = 0, \quad (37)$$

is stable for $\xi < \xi_2 = 0$, and unstable for $\xi > \xi_2$.

(2) Stripe patterns (S), given by

$$\rho_1 = \sqrt{\frac{\xi}{g_1}} \neq 0, \quad \rho_2 = \rho_3 = 0, \quad (38)$$

are stable for $\xi > \xi_3 = \frac{h^2 g_1}{(g_2 - g_1)^2}$, and unstable for $\xi < \xi_3$.

(3) Hexagon patterns (H_0, H_π) are given by

$$\rho_1 = \rho_2 = \rho_3 = \frac{|h| \pm \sqrt{h^2 + 4(g_1 + 2g_2\xi)}}{2(g_1 + 2g_2)}, \quad (39)$$

with $\varphi = 0$ or π , and exist when

$$\xi > \xi_1 = \frac{-h^2}{4(g_1 + 2g_2)}. \quad (40)$$

The solution $\rho^+ = \frac{|h| + \sqrt{h^2 + 4(g_1 + 2g_2\xi)}}{2(g_1 + 2g_2)}$ is stable only for

$$\xi < \xi_4 = \frac{2g_1 + g_2}{(g_2 - g_1)^2} h^2, \quad (41)$$

and $\rho^- = \frac{|h| - \sqrt{h^2 + 4(g_1 + 2g_2\xi)}}{2(g_1 + 2g_2)}$ is always unstable.

(4) The mixed states are given by

$$\rho_1 = \frac{|h|}{g_2 - g_1}, \quad \rho_2 = \rho_3 = \sqrt{\frac{\xi - g_1 \rho_1^2}{g_1 + g_2}}, \quad (42)$$

with $g_2 > g_1$. They exist when $\xi > \xi_3$ and are always unstable.

Fig. 2 (Color online) Snapshots of contour pictures of the time evolution of the infected at different instants with $A = 1$, $\mu = 1$, $d = 1$, $d_1 = 6$ and $d_2 = 1$, and $\beta = 16$. **a** 0 iteration; **b** 10,000 iterations; **c** 50,000 iterations; and **d** 1,00,000 iterations

4 Numerical results

In this section, we do lots of numerical simulations in the two-dimensional space to display pattern dynamics of spatial epidemic model (3). All numerical simulations are studied in a system size of 100×100 space units. We keep $A = 1$, $\mu = 1$, $d = 1$, $d_1 = 6$, $d_2 = 1$, and set β as a varied parameter. The numerical simulations will reach a stationary state or stop until they show a behavior that does not seem to change its characteristics anymore. In this paper, we want to know the distribution of the infected population (I), so we only analyze the form of the pattern I .

Figure 2 shows spatial pattern of infected population at 0, 10,000, 50,000, and 1,00,000 iterations in Turing space, and initial conditions are (S^*, I^*) with tiny disturbance. In that case, we obtain that $\xi_3 < \xi < \xi_4$ with $\beta = 16$, which means coexistence of spotted and stripe patterns will appear in the two-dimensional space.

By choosing $A = 1$, $d = 1$, $\mu = 1$, $d_1 = 6$, $d_2 = 1$, and $\beta = 17.5$, we obtain that $\xi > \xi_4$. In Fig. 3, we show the spatial pattern of infected population at 0, 10,000, 50,000, and 1,00,000 iterations. At the initial time, the infected population shows patched distribu-

tion. As time is large enough, stripe pattern appears, and the dynamics does not change anymore.

Figure 4 shows the evolution of the spatial pattern of infected population at 0, 5,000, 20,000, and 1,00,000 iterations, with small random perturbation of the stationary solution of the spatially homogeneous systems (3). In the parameter set, $A = 1$, $d = 1$, $\mu = 1$, $d_1 = 6$, $d_2 = 1$, and $\beta = 19.8$, we find that $\xi > \xi_1$, which implies that spotted pattern will emerge. Under this situation, one can see that random initial distribution of the model leads to a highly irregular and very short pattern in the region. After the irregular pattern formation, spatial patterns go by a slow change, finally form regular spotted patterns, and fill up the whole space. Our theoretical results are confirmed by means of the numerical results.

5 Discussion and conclusion

Based on the epidemic model with nonlinear incidence rate, we study the corresponding pattern dynamics. Through the analysis and the numerical simulation, we obtain two main results. First, we use linear analysis and standard multiple-scale analysis, and gain the exact expressions of amplitude equation. Second, we reveal

Fig. 3 (Color online) Snapshots of contour pictures of the time evolution of the infected at different instants with $A = 1$, $\mu = 1$, $d = 1$, $d_1 = 6$ and $d_2 = 1$, and $\beta = 17.5$. **a** 0 iteration; **b** 10,000 iterations; **c** 50,000 iterations; and **d** 1,00,000 iterations

Fig. 4 (Color online) Snapshots of contour pictures of the time evolution of the infected at different instants with $A = 1$, $\mu = 1$, $d = 1$, $d_1 = 6$ and $d_2 = 1$, and $\beta = 19.8$. **a** 0 iteration; **b** 5,000 iterations; **c** 20,000 iterations; and **d** 1,00,000 iterations

that an epidemic model with spatial diffusion has rich dynamics by means of numerical simulation in parameters space. The results confirm that spatial motion of individuals can form high density of infectious diseases.

In this study, we only let one parameter β change, and other remaining parameters are fixed. The parameter β has big effects on the spatial patterns. In other words, the increase of β related to pattern selection, whether it is a stripe pattern or spotted pattern. However, we ignore many factors in the model (3). For example, we do not take into account migration of individuals, the recovery of the infected populations and so on [23–25]. We need to investigate the pattern dynamics of epidemic models with these factors in the future work. It should be noted that we just investigated Turing instability of system (3). Other instability (such as Benjamin-Feir instability) may be found in this system. Moreover, we can extend our results in more complex spatial epidemic models like SIR, SIRS, or SEIRS models.

From a practical standpoint, the results obtained in this paper indicate that large infection rate can induce stationary patterns which implies that it can form high density of disease. As a result, we need to take measures to decrease infection rate to control the spread of disease.

Appendix

$$l = \frac{a_{11}^* d_2 - a_{22}^* d_1}{2a_{21}^* d_2},$$

$$G_1 = \left(1 + \frac{d_1}{d_2}\right) [(-2\beta_T I^* l - 2\beta_T S^*)(y_0 + y_{11}) - 2\beta_T I^*(x_0 + x_{11}) - 3l\beta_T],$$

$$G_2 = \left(1 + \frac{d_1}{d_2}\right) [(-2\beta_T I^* l - 2\beta_T S^*)(y_0 + y_{11}) - 2\beta_T I^*(x_0 + x_{11}) - 6l\beta_T],$$

$$\tau_0 = \frac{d_2 - d_1}{\beta_T d_2 [lb_{11} + b_{12} - \frac{d_1}{d_2} l(lb_{21} + b_{12})]},$$

$$h = \frac{(1 + \frac{d_1}{d_2} l)(4\beta_T I^* + 2\beta_T S^*)}{\beta_T d_2 [lb_{11} + b_{12} - \frac{d_1}{d_2} l(lb_{21} + b_{12})]},$$

$$g_1 = \frac{G_1}{\beta_T d_2 [lb_{11} + b_{12} - \frac{d_1}{d_2} l(lb_{21} + b_{12})]},$$

$$g_2 = \frac{G_2}{\beta_T d_2 [lb_{11} + b_{12} - \frac{d_1}{d_2} l(lb_{21} + b_{12})]}.$$

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