How to deal with partially analyzable acts?

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Abstract In some situations, a decision is best represented by an incompletely analyzed act: conditionally on a given event A , the consequences of the decision on sub-events are perfectly known and uncertainty becomes probabilizable, whereas the plausibility of this event itself remains vague and the decision outcome on the complementary event $\bar{A}$ is imprecisely known. In this framework, we study an axiomatic decision model and prove a representation theorem. Resulting decision criteria aggregate partial evaluations consisting of (i) the conditional expected utility associated with the analyzed part of the decision, and (ii) the best and worst consequences of its non-analyzed part. The representation theorem is consistent with a wide variety of decision criteria, which allows for expressing various degrees of knowledge on $(A, \bar{A})$ and various types of attitude toward ambiguity and uncertainty. This diversity is taken into account by specific models already existing in the literature. We exploit this fact and propose some particular forms of our model incorporating these models as sub-models and moreover expressing various types of beliefs concerning the relative plausibility of the analyzed and the non-analyzed events ranging from probabilities to complete ignorance that include capacities.

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1 Introduction

Decision makers frequently encounter choice situations for which decision theory does not offer appropriate representations and choice criteria. In standard models, decisions are evaluated on the basis of the probabilities that they lead to such or such result. Now, whereas it is possible to estimate with precision some risks, such as automobile risks (thanks to actuarial data), the evaluation of others, such as natural disasters is more difficult (statistics on frequencies and magnitudes are scarce). Such an evaluation becomes even more problematic for environmental risks such as those resulting from global warming or from genetically modified organisms (GMO) and for some health risks such as epidemics. Although all these risks are perfectly identified, it seems difficult to assign probabilities to the relevant events as well as to evaluate their effects precisely.

On closer look, these last situations often display a common, and simple, pattern, which will be the pattern studied in this article. Either the corresponding risk does not happen and the situation evolves “normally”, i.e., available data allows one to associate precisely decision outcomes with specific events; or the risk happens, and such an association is no longer possible. We assume that in the latter case feasible outcomes can only be located within a range of results. In addition, the risk likelihood is generally not known and, moreover, may depend on the decision envisioned.

Take for instance a situation of epidemic risk such as that faced by the World Health Organization at the time of the outbreak of SARS in February 2003. Various strategies can be encompassed for controlling the propagation of a disease. The effectiveness of a given strategy depends on the values of several parameters which are initially unknown such as ways of spreading of the disease, the type of infection (viral/bacterial), etc. The introduction, for a given strategy, of the event associated with an effective control and of the complementary event can always be made, although generally it will not be possible to assess the relative likelihood of these two events (no quantitative information is attached to them). Then, conditionally on an effective control, a routine situation is recovered and standard utility/probability evaluation can be performed; whereas, conditionally on a poor control, predictions that can be made are vague (for instance, a number of casualties ranging from one thousand to one million) and, further, cannot be improved by recourse to a refined analysis.

As another example, consider the question of the use of GMO's in agriculture. Without GMO's, farmers' incomes depend basically on climatic and market variables. Available data allow one to estimate their probability distribution and their impact on income. With GMO's, expected income remains assessable conditionally on the absence of cross-fertilization and contamination of other plants. However, neither the plausibility of the contamination nor its consequences on the farmers' income can be evaluated precisely. The decision of investment in a country with unknown insolvency risk provides another example of such situation.

A suitable representation of these situations by a formal model must allow for ambiguity on the events and uncertainty on the consequences. The literature on decision making under uncertainty contains many theories with ambiguous events such as Choquet Expected Utility (Schmeidler 1989) and Cumulative Prospect Theory (Wakker and Tversky 1993) and few with uncertain consequences (Ghirardato 2001) but none combining the two in the required way. We hence propose a new axiomatic model that attempts to formalize such situations and to justify adapted decision criteria.

The crucial assumption of our model is that decisions can only be partially analyzed i.e., there exists an irreducible ignorance about outcomes, that the model expresses through the association, with each decision d , of an event/consequence range pair $(\bar{A}, [l, g])$ which should be understood as follows: (i) although non predictable, consequences of decision d on event $\bar{A}$ can be neither lower than l nor can be higher than g ; (ii) no sub-event of $\bar{A}$ with assessable weight can be associated with a narrower consequence range. The appropriateness of this assumption will be considered further (in the discussion section).

This article is organized as follows. Section 2 explains the distinction between the analyzed and the non-analyzed parts of a decision and expresses them formally. Sections 3 and 4 are devoted to conditional preferences on the analyzed part. Preferences on the non-analyzed part are modelled in Sect. 5. A representation theorem is achieved in Sect. 6 and an illustrative example is proposed in Sect. 7. Finally, the model is discussed in Sect. 8.

2 The model

2.1 Decisions

Consider: Ω , set of states of nature; $\mathcal{E}$, σ -algebra of events; $\mathcal{C}$, a set of consequences; $\mathcal{G}$, σ -algebra of subsets of $\mathcal{C}$ containing all singletons. A decision problem involves a particular set of decisions, $\mathcal{D}$, which are (measurable) acts in the sense of Savage,¹ i.e., mappings $(\Omega, \mathcal{E}) \rightarrow (\mathcal{C}, \mathcal{G})$. However, in the decision model below, these acts are not completely known by the decision maker. Specifically, the decisions are only partially analyzed, i.e., for any decision $a \in \mathcal{D}$ there is an event A such that the restriction of a to A —the analyzed part of a —denoted $a|_A$ is exactly known, but the only information about $a|_{\bar{A}}$ —the non-analyzed part of a —is its range $M_a = a(\bar{A})$. Preferences will depend on, and only on, pairs $(a|_A, M_a)$.

The reason why we assume that the relevant data concerning the non-analyzed part of a consists solely in the consequence range M_a is the following: the fact that a yields a given consequence c of M_a on sub-event e_c of $\bar{A}$, rather than on some other sub-event, should have no bearing on preferences if nothing is known about e_c beside the property defining it, namely “ $a|_{\bar{A}}$ yields c exactly on e_c ”, which clearly contributes no information by itself on the likelihood of e_c . This assumption may nonetheless appear as somewhat restrictive, since it requires, for instance, that the decision maker

¹ More precisely, we use Savage’s remark (§ 3.4., p. 42) that the results in his model remain valid with events, consequences, and acts defined in the present way.

Fig. 1 A partially analyzed act

be indifferent between two decisions a and b , analyzed on the same event A , even if in addition to $a|_A = b|_A$ and $M_a = M_b$, it is known that a weakly dominates b on $\bar{A}$. This last trait of the model is discussed and justified in Sect. 8.

A specific feature of the model is that $\mathcal{D}$ is not assumed to contain all conceivable pairs $(a|_A, M_a)$. The reason is that decision makers cannot be expected to meaningfully evaluate unrealistic decisions. Thus, the range M on an “unfavorable” event (such as a natural catastrophe) should not include any “happy” consequence. Similarly, in some situations, major ignorance about the effects of an event will necessarily imply much uncertainty about outcomes i.e., a wide consequence range M on this event.

Completely analyzed decisions, denoted by $(a|_\Omega, \cdot)$, can exist. In particular, for evaluation purposes, we shall assume the existence of completely analyzed $\mathcal{R}$ —measurable acts, where subalgebra $\mathcal{R}$ of $\mathcal{E}$ can be interpreted as events associated with sequences of heads and tails or with a roulette wheel (see Savage 1954, pp. 38–39 and de Finetti 1974, pp. 199–202).

A decision a analyzed on an event A is called an A -act.² It generates a σ -algebra of subsets of A $\left(\left\{a|_A^{-1}(G), G \in \mathcal{G}\right\}\right)$, which we embed into a richer one, the σ -algebra $\mathcal{A}_a$ of subsets of A generated by the act a and the completely analyzed $\mathcal{R}$ —measurable acts previously defined, that is, by the sets $\left\{a|_A^{-1}(G) \cap R, G \in \mathcal{G}, R \in \mathcal{R}\right\}$ (Fig. 1).

To every act a we associate the set $\mathcal{F}_a$ of all acts which are $\mathcal{A}_a$ -measurable on A and have a consequence range M_a on $\bar{A}$. $\mathcal{F}_a$ is thus the set of all pairs $g = (g|_A, M_a)$ where $g|_A$ is any conceivable Savagean act $(A, \mathcal{A}_a) \rightarrow (\mathcal{C}, \mathcal{G})$, $g \in \mathcal{F}_a$ implies then $M_g = M_a$. There is a set $\mathcal{F}_a$ corresponding to each $a \in \mathcal{D}$ and the union of these sets is denoted by $\mathcal{F}$. We denote by $\mathcal{A}_{\mathcal{F}}$ the set of all events A such that $\mathcal{F}$ contains at least one A -act.

² So as to avoid ambiguities, whenever two acts are analyzed on different events, these events are given a different name.

Note that the fact that two acts a' and a'' are both A -acts, i.e., are analyzed on the same event A , does not imply the identity of $\mathcal{A}_{a'}$ and $\mathcal{A}_{a''}$, nor that of $\mathcal{F}_{a'}$ and $\mathcal{F}_{a''}$.

Example 1 Acts a, a', a'' characterize various oil field management strategies in the same country. The set $\mathcal{C}$ of consequences of these acts is a subset of $\mathbb{R}$ and corresponds to monetary amounts. Political risk (event $\bar{A}$) may imply a partial or complete loss of the investment. Act a' involves the same investment level I as a but concerns the exploitation of different oil fields, whereas act a'' corresponds to the more intensive exploitation of the same fields as a . Thus, it is likely that $M_{a'} = M_a = [0, -I]$ but $\mathcal{A}_{a'} \neq \mathcal{A}_a$ (oil yields depend on different events), whereas $M_{a''} = [0, -I''] \neq M_a$ and $\mathcal{A}_{a''} = \mathcal{A}_a$. Hence, although the three acts are analyzed on the same event A , $\mathcal{F}_a, \mathcal{F}_{a'}$ and $\mathcal{F}_{a''}$ all differ from one another.

2.2 Preferences

Preferences on $\mathcal{F}$ are expressed by a binary relation $\succsim$. We assume:

Axiom 1 $\succsim$ is a weak order on $\mathcal{F}$.

We want to endow $\succsim$ with standard properties and, moreover, to establish links between its restrictions $\succsim_a$ to the various $\mathcal{F}_a$.³ For this, we in particular need an appropriate version of Savage's Sure Thing Principle.

Because of the partial information on the decisions, the common part $Com(a, b)$ of two acts a and b analyzed on events A and B , respectively, is naturally defined as

$$Com(a, b) = \begin{cases} \{\omega \in A \cap B : a(\omega) = b(\omega)\} & \text{if } M_a \neq M_b \\ \{\omega \in A \cap B : a(\omega) = b(\omega)\} \cup (\bar{A} \cap \bar{B}) & \text{if } M_a = M_b \end{cases}$$

Thus, on any state of nature belonging to $Com(a, b)$, either the precise consequences of both events are known, in which case they must be the same, or they are not known, in which case the possible consequence ranges of the two decisions must be the same.

Axiom 2 (Sure Thing Principle for partially analyzed decisions)

Let $a, \hat{a}, b, \hat{b} \in \mathcal{F}$ where $\hat{a}$ results from a and $\hat{b}$ from b by a common modification in the sense that a and b are not modified outside $Com(a, b)$ and that $Com(\hat{a}, \hat{b}) = Com(a, b)$.

Then $a \succsim b \iff \hat{a} \succsim \hat{b}$.

Note that the feasible common modifications of a given pair of acts are strongly limited by the fact that the modified acts must still belong to $\mathcal{F}$ (which makes Axiom 2 a rather weak form of the Sure Thing Principle).

The axiom can nonetheless apply to quadruples $a, \hat{a}, b, \hat{b}$ with different $\mathcal{F}_a, \mathcal{F}_{\hat{a}}, \mathcal{F}_b, \mathcal{F}_{\hat{b}}$.

³ For simplicity, we denote preferences on domain $\mathcal{F}_a$ (the class of acts generated by element a) by $\succsim_a$ rather than by $\succsim_{\mathcal{F}_a}$.

Fig. 2 Two acts with common part $(A \cap B) \cup (\bar{A} \cap \bar{B})$

Fig. 3 A common modification of the previous acts

Example 2 Suppose there are three countries: $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$. Country $\mathbb{A}$ (resp. $\mathbb{B}$) may possibly face an economic crisis (event $\bar{A}$ (resp. $\bar{B}$)) which however is unlikely in country $\mathbb{C}$. A firm has to take a decision concerning a productive investment of amount I . The decision a of investing I in country $\mathbb{A}$ will generate sales revenues from countries $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ in proportions 45% in country $\mathbb{A}$, 5% in country $\mathbb{B}$ and 50% in country $\mathbb{C}$, unless an economic crisis (event $\bar{A}$) happens in $\mathbb{A}$ in which case I may be partially or completely lost, independently of a crisis occurring or not in country $\mathbb{B}$. See Fig. 2.

On the other hand, consider a' with the same amount of investment in $\mathbb{A}$ as a but generating a 70, 30 and 0% sales revenue from countries $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ respectively, if there is no economic crisis. With this investment decision, the firm may lose up to I if a crisis occurs only in $\mathbb{A}$, but is sure to lose the entire investment if the crisis takes place in $\mathbb{A}$ and $\mathbb{B}$ (event $\bar{A} \cap \bar{B}$) simultaneously.

Decisions b and b' have similar characteristics with the roles of countries $\mathbb{A}$ and $\mathbb{B}$ interchanged. We assume moreover that the countries are “similar”, in the sense that the return from sales is the same in $\mathbb{A}$ as in $\mathbb{B}$, that is $a|_A = c$ and $b|_B = c$ with $c \in \mathcal{C}$. See Fig. 3.

Thus, a and b are an A -act and a B -act respectively with

$$Com(a, b) = (A \cap B) \cup (\bar{A} \cap \bar{B})$$

and $M_a = M_b = [0, -I]$.

a' and b' are $(A \cap B) \cup (\bar{A} \cap \bar{B})$ -acts resulting from a and b by a modification of their common part. More precisely,

$$\begin{aligned} a|_{A \cap B} &= b|_{A \cap B} = a'|_{A \cap B} = b'|_{A \cap B}, \\ M_{a'} &= M_{b'} = [0, -I] \text{ and } a'|_{\bar{A} \cap \bar{B}} = b'|_{\bar{A} \cap \bar{B}} = -I. \end{aligned}$$

3 Preferences on analyzed events and SEU

From preferences $\succsim_a$ on $\mathcal{F}_a$, we can now derive, “à la Savage”, for any event $E \in \mathcal{A}_a$, conditional preferences given E , denoted by $\succsim_a^E$ as follows:

$$g \succsim_a^E h \Leftrightarrow g' \succsim_a h' \text{ where } g'|_E = g|_E, h'|_E = h|_E \text{ and } g'|_{A \setminus E} = h'|_{A \setminus E}.$$

Axiom 2 ensures that the ordering of g' and h' is independent from their common values on $A \setminus E$.

Note that $\succsim_a^A$ is the same as $\succsim_a$.

More generally, given an A -act $a \in \mathcal{D}$, and a B -act $b \in \mathcal{D}$, where $B \in \mathcal{A}_a$ (hence $B \subset A$), orderings $\succsim_a^B$ on $\mathcal{F}_a$ and $\succsim_b$ on $\mathcal{F}_b$ are related as shown by the following lemma.

Lemma 1 Let $a' = (a'|_A, M_a)$, $a'' = (a''|_A, M_a)$ with $a', a'' \in \mathcal{F}_a$. Suppose that, for some $B \in \mathcal{A}_a$, b is a B -act and $b' = (a'|_B, M_b)$, $b'' = (a''|_B, M_b) \in \mathcal{F}_b$. Then

$$a' \succsim_a^B a'' \Leftrightarrow b' \succsim_b b''.$$

Proof See Appendix

As a direct consequence of Lemma 1, conditional preferences given E are intrinsic in the sense that they do not depend on which $\mathcal{A}_a$ containing E (hence on which a in $\mathcal{D}$) is considered, and can be defined by

$$g \succsim_a^B h \Leftrightarrow \text{there is } a \in \mathcal{D} \text{ such that } g \succsim_a^E h.$$

We also need slightly modified versions of Savage’s other definitions and axioms.

A constant A -act f_a^c in $\mathcal{F}_a$ is defined by $f_a^c(A) = \{c\}$ with $c \in \mathcal{C}$ and $f_a^c(\bar{A}) = M_a$.

Savage’s P3 becomes:

Axiom 3 For every $a \in \mathcal{D}$ and associated event A (i.e., a is an A -act) and every $c', c'' \in \mathcal{C}$, let $f_a^{c'}, f_a^{c''}$ be constant A -acts in $\mathcal{F}_a$, and similarly let $f_b^{c'}, f_b^{c''}$ be constant B -acts in $\mathcal{F}_b$, then

$$f_a^{c'} \succsim f_a^{c''} \iff f_b^{c'} \succsim f_b^{c''}.$$

Preferences among consequences can now be defined by

$c' \succeq_C c'' \iff$ there exist $a \in \mathcal{D}$ and constant A -acts $f_a^{c'}$, $f_a^{c''}$ in $\mathcal{F}_a$ such that $f_a^{c'} \succsim f_a^{c''}$.

Since $\mathcal{C}$ can always be replaced by its quotient w.r.t. the symmetric part of $\succeq_C$, we henceforth assume w.l.o.g. that $\succeq_C$ is an order (i.e., is antisymmetric so that no ties are allowed) which justifies the use of symbol $\succeq_C$.

We now require Savage's P4 (irrelevance of the values of the prizes on the events) in every $\mathcal{F}_e$, where $e \in \mathcal{D}$ is an E -act.

Axiom 4 Let $A, B \in \mathcal{A}_e$ where $e \in \mathcal{D}$ is an E -act; let $c_1, c'_1, c_2, c'_2 \in \mathcal{C}$ be such that $c_1 \succ_C c'_1$ and $c_2 \succ_C c'_2$. Define acts $f, f', g, g' \in \mathcal{F}_e$ by:

- (i) $f(\bar{E}) = f'(\bar{E}) = g(\bar{E}) = g'(\bar{E}) = M_e$;
- (ii) $f(\omega) = c_1, f'(\omega) = c'_1, \text{ for } \omega \in A$;
 $f(\omega) = c_2, f'(\omega) = c'_2, \text{ for } \omega \in E \setminus A$;
- (iii) $g(\omega) = c_1, g'(\omega) = c'_1, \text{ for } \omega \in B$;
 $g(\omega) = c_2, g'(\omega) = c'_2, \text{ for } \omega \in E \setminus B$;
 then $f \succsim g \iff f' \succsim g'$.

Whenever $f \succsim g$ holds for f, g defined as in Axiom 4, we can write $A \succsim_e^E B$. However, this ordering does not depend on e since if $A, B \in \mathcal{A}_{e^*}$ for some other $e^* \in \mathcal{D}$ which is also an E -act, then by Axiom 2 $A \succsim_e^E B \iff A \succsim_{e^*}^E B$: the common modification of f and g consists in replacing $f(\bar{E}) = f'(\bar{E}) = g(\bar{E}) = g'(\bar{E}) = M_e$ by $f(\bar{E}) = f'(\bar{E}) = g(\bar{E}) = g'(\bar{E}) = M_{e^*}$.

We can therefore drop the subscript e and simply write $A \succsim^E B$ and read "event A is qualitatively more probable than B conditionally on event E ".

The next axiom assumes both that $\succeq_C$ is not trivial and that worst and best consequences exist.

Axiom 5 There exists a pair $\bar{c}, \underline{c} \in \mathcal{C}$ such that $\bar{c} \succ_C \underline{c}$ and $\bar{c} \succeq_C c \succeq_C \underline{c}$ for all $c \in \mathcal{C}$.

We also introduce a version of Savage's P6. It shows that one of the roles of the coin-toss related subalgebra of events $\mathcal{R}$ is to make all $(\mathcal{A}_a, \succsim_a)$ atomless.

Axiom 6 For every $a \in \mathcal{D}$ and associated event A (i.e., a is an A -act) and every $f, g \in \mathcal{F}_a$, with $f \succ g$, and $c \in \mathcal{C}$, there exists a partition of A , consisting of events $R \cap A, R \in \mathcal{R}$, such that if f (resp. g) is modified on any element of the partition and given constant outcome c on this element, then the modified act f' (resp. g') also satisfies $f' \succ g$ (resp. $f \succ g'$).

We also need Savage's P7 for each $\succsim_a$.

Axiom 7 For every $a \in \mathcal{D}$ and associated event A (i.e., a is an A -act), every $f, g \in \mathcal{F}_a$ and $E \in \mathcal{A}_a$, if $f \succsim_a^E$ (resp. $\prec_a^E$) $g(\omega)$ for all $\omega \in E$, then $f \succsim_a^E$ (resp. $\prec_a^E$) g .

Axioms 1–7 imply the validity of Savage's P1–7 in every $\mathcal{F}_a$, where his main result consequently holds: preferences in $\mathcal{F}_a$ can be represented by a subjective expected utility (SEU) criterion with respect to an atomless probability on $\mathcal{A}_a$.

Moreover, due to the explicit introduction of the σ -algebra $\mathcal{R}(A) = \{A \cap R, R \in \mathcal{R}\}$ in Axiom 6, it is clear that this result still holds if $\mathcal{F}_a$ is replaced by its restriction to $\mathcal{R}(A)$ -measurable acts. We can thus state:

Proposition 1 Under Axioms 1–7, for every $a \in \mathcal{D}$, there exist a bounded utility u_a and an additive probability P_a such that

$$f \succsim g \Leftrightarrow \int_A u_a \circ f \, dP_a \geq \int_A u_a \circ g \, dP_a, \quad \forall f, g \in \mathcal{F}_a$$

where

u_a is unique up to an affine transformation;

P_a is unique and for every $\rho \in [0, 1]$ there exists $B \in \mathcal{A}_a$ such that $P_a(B) = \rho$.

Moreover, these existence and uniqueness statements are also valid when $\mathcal{F}_a$ is replaced by its restriction to $\mathcal{R}(A)$ -measurable acts and thus $\mathcal{A}_a$ is replaced by $\mathcal{R}(A)$.

NB: In the rest of this article, we shall simply write “probability” for “additive probability”.

4 Intrinsic utility and probability consistency

It is well known that Savage’s axioms do not imply the existence of certainty equivalents for the acts. However, this existence is easily acceptable for sufficiently rich consequence sets (for instance when $\mathcal{C}$ is a real interval) and, although not necessary (see the discussion in Sect. 7), will make later statements simpler and more easily interpretable. So, we assume:

Axiom 8 For any $a \in \mathcal{D}$ there exist $c \in \mathcal{C}$ such that the constant A -act f_a^c satisfies $f_a^c \sim_a a$.

The next assumption and the lemma that follows assert that coin-toss related events are “qualitatively” independent and thus “quantitatively” independent from events in $\mathcal{E}$.

Axiom 9 For every $A, B \in \mathcal{A}_{\mathcal{F}}$ conditional preferences on events $\succsim^A$ and $\succsim^B$ satisfy, for all $R', R'' \in \mathcal{R}$:

$$A \cap R' \succsim^A A \cap R'' \iff B \cap R' \succsim^B B \cap R''.$$

Lemma 2 Let $a, b \in \mathcal{D}$. For every $R \in \mathcal{R}$, $P_a(A \cap R) = P_b(B \cap R)$.

Proof See Appendix

Whenever $A \cap R' \succsim^A A \cap R''$ holds for $R', R'' \in \mathcal{R}$ and some $A \in \mathcal{A}_{\mathcal{F}}$, we shall simply write $R' \succsim^{\mathcal{R}} R''$ and read “event R' is qualitatively more probable than R'' ”. Qualitative probability $\succsim^{\mathcal{R}}$ is uniquely represented by probability $P_{\mathcal{R}}$ defined by $P_{\mathcal{R}}(R) = P_a(A \cap R)$ for some A .

Thus, Axiom 9 ensures the existence of an intrinsic probability $P_{\mathcal{R}}$ on $\mathcal{R}$.

We shall use this result to derive properties of utilities. Thus far, all we know about the $u_a, a \in \mathcal{D}$ is that they represent the same ordering $\succeq_{\mathcal{C}}$ and are therefore increasing

transforms from one another. We would like the functions u_a to be identical (after proper rescaling).

According to Proposition 1, for every triple $c' \succeq_C c \succeq_C c''$, with $c' \succ_C c''$, there is an event $R \in \mathcal{R}$ such that act $g \in \mathcal{F}_a$ with $g(\omega) = c'$, for $\omega \in A \cap R$, and $g(\omega) = c''$ for $\omega \in A \cap R^c$ is indifferent to the constant A -act f_a^c in $\mathcal{F}_a$. In other words, there is $R \in \mathcal{R}$ such that $P_a(A \cap R)$ satisfies

$$u_a(c) = P_a(A \cap R)u_a(c') + (1 - P_a(A \cap R))u_a(c''). \quad (1)$$

Hence, according to the definition that follows Lemma 2

$$u_a(c) = P_{\mathcal{R}}(R)u_a(c') + (1 - P_{\mathcal{R}}(R))u_a(c'').$$

Thus, all we need is an axiom ensuring that event R in (1) only depends on c .

Axiom 10 For every triple $c' \succeq_C c \succeq_C c''$, with $c' \succ_C c''$, there exists an event $R \in \mathcal{R}$ such that for every $a \in \mathcal{D}$, act $g \in \mathcal{F}_a$, with $g(\omega) = c'$, for $\omega \in A \cap R$, and $g(\omega) = c''$, for $\omega \in A \cap R^c$, is indifferent to the constant A -act f_a^c in $\mathcal{F}_a$.

It follows immediately that

$$\frac{u_a(c) - u_a(c'')}{u_a(c') - u_a(c'')} = \frac{u_b(c) - u_b(c'')}{u_b(c') - u_b(c'')} \quad \forall a, b \in \mathcal{D}.$$

Hence, since this holds for all $c', c'' \in \mathcal{C}$ with $c' \succ_C c''$, we get Proposition 2.

Proposition 2 If Axioms 1–10 hold, utilities u_a ($a \in \mathcal{D}$) are affine transforms of one another.

Thus, after rescaling u_a 's are identical, and we will from now on write u instead of u_a . Note that u is a utility function representing $\succeq_C$.

The next proposition guarantees the existence of intrinsic conditional probabilities in the sense that they are independent from the context in which they are evaluated.

Proposition 3 Let $a, b \in \mathcal{D}$ be analyzed on A and B , respectively, with $B \in \mathcal{A}_a$ (such that $P_a(B) > 0$), and let, moreover, $E \in \mathcal{A}_b$ (hence $E \subset B \subset A$). If Axioms 1–10 hold, then

$$P_a(E/B) = \frac{P_a(E)}{P_a(B)} = \frac{P_b(E)}{P_b(B)} = P_b(E).$$

Proof See Appendix

Thus, as for conditional preferences, intrinsic conditional probabilities can be defined by $P(E/B) = P_a(E/B)$ where $E, B \in \mathcal{A}_a$ and $E \subset B$.

5 Preferences on non-analyzed events

Let us now turn to the non-analyzed part of the decisions.

Denote by $\mathcal{M}$ the union of all ranges corresponding to all decisions:

$$\mathcal{M} = \{M_a, \exists a \in \mathcal{D} \text{ such that } a = (a|_A, M_a)\}.$$

We define a partial preference relation over $\mathcal{M}$. For this, two axioms are needed: Axiom 11 ensures the existence of the relation and Axiom 12 its transitivity.

Axiom 11 For every $a', a'' \in \mathcal{D}$ with the same associated event A (i.e., a', a'' are A -acts) such that $a' = (a'|_A, M_{a'})$, $a'' = (a''|_A, M_{a''})$ with $a'|_A = a''|_A$ and let $b', b'' \in \mathcal{D}$ with the same associated event B such that $b' = (b'|_B, M_{b'})$, $b'' = (b''|_B, M_{b''})$ with $b'|_B = b''|_B$, $M_{b'} = M_{a'}$ and $M_{b''} = M_{a''}$, then

$$a' \succsim a'' \Leftrightarrow b' \succsim b''.$$

The *preferences among ranges* can now be defined by the transitive closure $\succsim_{\mathcal{M}}$ of the relation $\succsim_{\mathcal{M}}^0$ given by:

$M' \succsim_{\mathcal{M}}^0 M'' \iff$ there exist A -acts $a', a'' \in \mathcal{D}$ such that $M_{a'} = M'$, $M_{a''} = M''$, $a'|_A = a''|_A$ and $a' \succsim a''$.
 $\succsim_{\mathcal{M}}$ is automatically a partial order if

Axiom 12 $\succsim_{\mathcal{M}}^0$ is acyclic i.e., there is no sequence M^i , $i = 1 \dots n$ in $\mathcal{M}$ such that $M^i \succsim_{\mathcal{M}}^0 M^{i+1}$, $i = 1 \dots n - 1$ and $M^n \succ_{\mathcal{M}}^0 M^1$.

Let us now turn to the representation of the preference relation $\succsim_{\mathcal{M}}$.

Proposition 4 below states that intermediate values of the range M_a can have no influence on preferences. This is straightforward if there is only one intermediate value: consider a decision a generating A -act $(a|_A, M_a)$ with range $M_a = \{l, m, g\}$ where $g \succ_C m \succ_C l$, and let $\{e_g, e_m, e_l\}$ be the partition of $\bar{A}$ such that d takes value g on e_g , m on e_m , and l on e_l . By weak (pointwise) dominance, decision a^+ satisfying $a^+|_A = a|_A$ and taking value g on $e_g \cup e_m$ and l on e_l should be preferred to a ; but, by weak dominance again, a should be preferred to a^- satisfying $a^-|_A = a|_A$ and taking value g on e_g and l on $e_m \cup e_l$; since a^+ and a^- generate the same A -act $(a|_A, \{l, g\})$, necessarily $a^+ \sim a \sim a^-$; hence, $\{l, m, g\} \sim_{\mathcal{M}} \{l, g\}$.

Barbera et al. (1984) show that a finite number of intermediate values cannot influence preferences either. The following requirement will allow us to extend this result to the case of infinite ranges.

Axiom 13 (i) For all $M \in \mathcal{M}$ and $c \in \mathcal{C}$, there exist an event A and two A -acts a', a'' such that $M_{a'} = M$ and $M_{a''} = M \cup \{c\}$;
 (ii) Let $c_1, c_2 \in \mathcal{C}$ be such that $c_1 \succ_C c_2$. Then, for any $M_0 \in \mathcal{M} \cup \{\emptyset\}$ such that $c_1, c_2 \notin M_0$,

$$\{c_1\} \cup M_0 \succsim_{\mathcal{M}} \{c_1, c_2\} \cup M_0 \succsim_{\mathcal{M}} \{c_2\} \cup M_0.$$

Moreover, if $c \succ_C c_2$ for all $c \in M_0$, then:

$$\{c_1\} \cup M_0 \succ_{\mathcal{M}} \{c_1, c_2\} \cup M_0$$

and if $c_1 \succ_C c$ for all $c \in M_0$, then

$$\{c_1, c_2\} \cup M_0 \succ_{\mathcal{M}} \{c_2\} \cup M_0.$$

Axiom 13 makes both existence and comparability requirements. In particular, it implies that $M \in \mathcal{M} \Rightarrow M \cup \{c_1, \dots, c_k\} \in \mathcal{M}$. Moreover, for $M_0 = \emptyset$, we get $\{c_1\} \succ_{\mathcal{M}} \{c_1, c_2\} \succ_{\mathcal{M}} \{c_2\}$.

Our representation theorem involves the $\succeq_C$ -greatest and the $\succeq_C$ -lowest consequences of M , respectively, denoted $g(M)$ and $l(M)$. They clearly exist if M is finite. For infinite M , we shall require their existence. This assumption could be dispensed with, but the price to pay in technicalities is too high for our purposes (see Cohen and Jaffray 1983). Anyhow, in the main applications that we have in mind (environmental and health risk management), greatest and lowest consequences are likely to exist.

Lemma 3 (i) For all finite $M \in \mathcal{M}$ such that $g(M) \succ_C l(M)$, $M \sim_{\mathcal{M}} \{g(M), l(M)\}$.
(ii) For finite $M', M'' \in \mathcal{M}$:

$$\left. \begin{array}{l} g(M') \succeq_C g(M'') \\ l(M') \succeq_C l(M'') \end{array} \right\} \Rightarrow M' \precsim_{\mathcal{M}} M''. \quad (2)$$

Moreover,

$$\left. \begin{array}{l} g(M') \succ_C g(M'') \\ l(M') \succ_C l(M'') \end{array} \right\} \Rightarrow M' \succ_{\mathcal{M}} M''. \quad (3)$$

Proof See Appendix

Lemma 3 directly implies that, for a finite sequence $(M_i)_{i=1}^n$ of finite M_i with $g(M_i)$ and $l(M_i)$ independent of i , $\cup_{j=1}^n M_j \sim_{\mathcal{M}} M_i$, $i = 1 \dots n$.

The following axiom requires both the existence of $g(M)$ and $l(M)$ for any (infinite) $M \in \mathcal{M}$ and extends the last proposition to infinite unions.

Axiom 14 (i) Every $M \in \mathcal{M}$ has a $\succeq_C$ -greatest and a $\succeq_C$ -lowest consequence;
(ii) For any family $(M_i)_{i \in I}$, of finite $M_i \in \mathcal{M}$ such that for all $i, j \in I$, $g(M_i) = g(M_j)$ and $l(M_i) = l(M_j)$, $\cup_{j \in I} M_j \sim_{\mathcal{M}} M_i$, $i \in I$.

We can then prove the following proposition:

Proposition 4 If Axioms 11–14 hold, then for all $M \in \mathcal{M}$ such that $g(M) \succ_C l(M)$, $M \sim_{\mathcal{M}} \{g(M), l(M)\}$.

Proof It is sufficient to note that any $M \in \mathcal{M}$ is the infinite union of finite subsets of it also in $\mathcal{M}$ and with the same greatest and lowest consequences. $\square$

Proposition 5 *If Axioms 11–14 hold, then there exists a mapping $v : \mathcal{M} \rightarrow \mathbb{R}$ such that*

$$\begin{aligned} M' \succ_{\mathcal{M}} M'' &\Rightarrow v(M') > v(M'') \\ M' \sim_{\mathcal{M}} M'' &\Rightarrow v(M') = v(M'') \end{aligned}$$

with $M \mapsto v(M) = \varphi(g(M), l(M))$ for some $\varphi : \mathcal{C} \times \mathcal{C} \rightarrow \mathbb{R}$ and that

$$\left. \begin{array}{l} g(M') \succ_{\mathcal{C}} g(M''), l(M') \succeq_{\mathcal{C}} l(M'') \\ \text{or} \\ g(M') \succeq_{\mathcal{C}} g(M''), l(M') \succ_{\mathcal{C}} l(M'') \end{array} \right\} \Rightarrow v(M') > v(M'').$$

Proof See Appendix

Note that v is not unique and, in general, not even unique up to a strictly increasing transformation.

6 Preference representation

6.1 Representation theorem

We now want to construct a utility representation of preferences $\succsim$ in $\mathcal{F}$ that incorporates the results obtained so far concerning its restrictions $\succsim_a$ to the various $\mathcal{F}_a$ as well as those concerning $\succsim_{\mathcal{M}}$.

Note that, since completely analyzed acts exist, Axiom 8 implies that for any act $a \in \mathcal{D}$, there exists $k \in \mathcal{C}$ such that $f^k \sim a$ where f^k denotes the constant act $f^k(\Omega) = \{k\}$.

Theorem 1 *Under Axioms 1–14, the weak order $\succsim$ on $\mathcal{F}$ is representable by a utility function V such that:*

- For an A -act a with $A \neq \Omega$ and with $\succeq_{\mathcal{C}}$ -greatest and $\succeq_{\mathcal{C}}$ -lowest consequences in M_a denoted $g(M_a), l(M_a)$, respectively,

$$a = (a|_A, M_a) \mapsto V(a) = \Phi \left(A, \int_A u \circ a \, dP_a, g(M_a), l(M_a) \right)$$

where P_a is a subjective additive probability on the σ -algebra $\mathcal{A}_a$, and Φ is weakly increasing with $\int_A u \circ a \, dP_a, g(M_a), l(M_a)$.

- Otherwise, for $A = \Omega$,

$$a = (a|_{\Omega}, \cdot) \mapsto V(a) = \Psi \left(\int_{\Omega} u \circ a \, dP_a \right)$$

with Ψ weakly increasing with its argument.

- Moreover, probabilities $\{P_a, a \in \mathcal{D}\}$ are unique and possess the following consistency property:
for $a, b \in \mathcal{D}$ with $B \in \mathcal{A}_a$ and $E \in \mathcal{A}_b$: $P_a(E/B) = P_b(E)$.

Proof See Appendix

Moreover, the certainty equivalent k depends on $a|_A$ only through $\int_A u \circ a \, dP_a$ (by Proposition 1) and on M_a only through $(g(M_a), l(M_a))$ (by Proposition 4). $\square$

6.2 Some particular criteria

The representation theorem is consistent with a wide variety of decision criteria which allows for expressing various degrees of knowledge about $(A, \bar{A})$ and various types of attitude toward ambiguity and uncertainty. This diversity is taken into account by specific models already existing in the literature. We can take advantage of that fact and propose some particular forms of our model incorporating these models as sub-models and, moreover, expressing various types of beliefs concerning the relative plausibility of the analyzed and the non-analyzed events ranging from probabilities ($P(A) + P(\bar{A}) = 1$) to complete ignorance that include capacities ($v(A) + v(\bar{A}) \neq 1$).

- Case of $P(A)$ precisely known:

$$V(a) = P(A) \int_A u \circ a \, dP_a + (1 - P(A)) \varphi(g(M_a), l(M_a)) \quad (4)$$

where u and φ express attitudes toward risk and ambiguity, respectively. The particular form of φ due to Hurwicz (1951) only involves a pessimism index α which leads to:

$$V(a) = P(A) \int_A u \circ a \, dP_a + (1 - P(A)) [\alpha u(l(M_a)) + (1 - \alpha) u(g(M_a))]. \quad (5)$$

- Case of $P(A)$ imprecisely known. The representation of the imprecision on $(A, \bar{A})$ by a capacity v together with a rank-dependent criterion (Schmeidler 1989) leads to the following form for V :

$$- \int_A u \circ a \, dP_a \leq \varphi(g(M_a), l(M_a)),$$

$$V(a) = (1 - v(\bar{A})) \int_A u \circ a \, dP_a + v(\bar{A}) \varphi(g(M_a), l(M_a)) \quad (6)$$

$$- \int_A u \circ a \, dP_a \geq \varphi(g(M_a), l(M_a)),$$

$$V(a) = v(A) \int_A u \circ a \, dP_a + (1 - v(A)) \varphi(g(M_a), l(M_a)). \quad (7)$$

- *Case of complete ignorance on $(A, \bar{A})$.* According to models proposed by [Arrow and Hurwicz \(1972\)](#), [Cohen and Jaffray \(1980\)](#), and [Barbera et al. \(1984\)](#) only best and worst evaluations on A or $\bar{A}$ are relevant which leads to:

$$V(a) = \varphi(\max\{u(g(M_a)), \int_A u \circ a \, dP_a\}, \min\{u(l(M_a)), \int_A u \circ a \, dP_a\}). \quad (8)$$

Despite its generality, our model has normative implications. It rules out some well-established criteria and suggests alternative ones, as in the following example.

Example 3 A common practice in international borrowing consists in classifying countries into various groups according to their degree of insolvency risk. The rating is generally based on a check-list of economic indicators through a multiple criteria decision model; probability evaluations are rarely involved ([Saini and Bates 1984](#)). A given country is then allowed to borrow money at an interest rate equal to the LIBOR, i , plus a risk spread Δi , which depends on its group. Thus, the net expected present value of a one period investment I with expected return ER is

$$EV = -I + \frac{ER}{(1+i+\Delta i)} \quad (9)$$

which, by neglecting a second-order term in $i \times \Delta i$, yields equivalence

$$EV \approx -I + \frac{ER}{(1+i)} - \frac{\Delta i \times ER}{(1+i+\Delta i)}.$$

Thus, whenever two projects, characterized respectively by (I_1, R_1) and (I_2, R_2) are such that

$$I_1 > I_2 \text{ and } ER_1 > ER_2 \text{ with } -I_1 + \frac{ER_1}{(1+i)} = -I_2 + \frac{ER_2}{(1+i)},$$

then $EV_1 < EV_2$, which means that the criterion favors low investment/low return projects over high investment/high return ones in high-risk countries, which preference has no apparent economic justification.

Now, our model (see Proposition 6), applies to investment projects with $\bar{A}$ as the insolvency event, $\int_A u \circ a \, dP_a = ER$, $g(M_a) = 0$, and $l(M_a) = -L \geq -I$; thus, a particular, additive, instance of this model would evaluate a project according to the formula:

$$V^* = -I + \frac{ER}{(1+i)} - k(A) \times L \quad (10)$$

i.e., would require the risk premium to be proportional to the maximal possible loss, here L (and independent from ER), which appears as reasonable. Of course, the weight $k(A)$ will only depend on the investment country through its insolvency risk group.

Note that preference for low investment/low return projects is not excluded but now only prevails among projects for which the whole investment can be lost ($L = I$).

7 An illustrative example: influenza pandemic prevention plans

The influenza A viruses, better known as avian viruses, which can infect humans as well as birds, can undergo two types of mutation:

- step-by-step changes in their genetic make-up (called *antigenic drift*) at the origin of common seasonal flu epidemics. The regularity of these epidemics makes them highly predictable in terms of numbers of casualties as well as in terms of prevention and treatment costs (for instance, Sentiweb predicts for 2006–2007 in France a mean of 2.2 (million) cases with an interdecile range of [1.3, 3]).
- a drastic change (called *antigenic shift*) at the origin of a new hybrid virus, combination of human and avian viruses. If this virus causes severe disease and allows easy and sustainable human-to-human transmission, it will ignite a pandemic. According to the WHO, this mutation cannot be predicted, making it difficult if not impossible to know if or when a virus such as H5N1 might acquire the properties needed to spread easily and sustainably among humans.

Moreover, the mortality and morbidity rates during a pandemic are difficult to evaluate. Historically, the number of deaths during a pandemic has varied greatly. Death rates are largely determined by four factors: the number of people who become infected, the virulence of the virus, the underlying characteristics, and vulnerability of affected populations and the effectiveness of preventive measures. Current estimates of mortality and morbidity rates in case of pandemics use the data of past pandemics and build scenarios by considering different “plausible” values for the different factors. This provides intervals for mortality and morbidity rates (see for instance [Doyle et al. 2006](#)). We assume below that for any given values of these rates, it is possible to associate a generalized cost (incorporating human as well as material costs) and thus, to any combination of intervals of rates corresponds a single interval of costs.

Annual costs generated by flu epidemics are, therefore, likely to vary greatly depending on the type of mutation that has taken place that year. The precision of the available data in case of a simple drift (which includes the no-change case), as opposed to the vagueness of the information in case of a shift, justifies the identification of events “drift” and “shift” with, respectively, analyzed and non-analyzed events, A and $\bar{A}$, in our model.

Faced with the threat of pandemic, public health authorities can consider various courses of actions to fulfill one or the other of the following two objectives, defined by the WHO:

- averting a pandemic and controlling the outbreak in humans. Measures consist then in the immediate culling of infected and exposed birds, the quarantine and disinfection of farms, the geographic restriction of animals, etc.
- mitigating the impact of the pandemic (once it has broken out). The corresponding measures include vaccine development and antiviral drug production and storage.

For the sake of simplicity, we only consider four courses of action (decisions).

- d_0 : status quo (no action) resulting in a known cost probability distribution (P) with a range $[k^-, k^+]$ conditionally on A and an unknown cost in a range $[K^-, K^+]$

Fig. 4 The four partially analyzed decisions encompassed

estimated by experts taking into account the variable intensity of the pandemics if $\bar{A}$ obtains.

- d_1 : preventive measures, costing c , which may or may not succeed in preventing the outbreak of a pandemic in case of antigenic shift (event $\bar{A}$). Hence, if $\bar{A}$ occurs, an additional cost varying from the lowest feasible cost of the no pandemic case, k^- , to the highest feasible cost of the pandemic case, K^+ and thus a global cost range $[k^- + c, K^+ + c]$. If on the contrary, A occurs, probability distribution P above is just shifted by c .
- d_2 : standard mitigation measures (vaccine development), an alternative to prevention with the same cost c , which, if $\bar{A}$ occurs, limits the extension of the pandemics and the global cost range becomes $[K_2^- + c, K_2^+ + c]$ where $K_2^- + c > K^- (> K_2^-)$ and $K_2^+ + c < K^+$. If on the contrary, A occurs, as with d_1 , probability distribution P is shifted by c .
- d_3 : intensive mitigation measures, costing $C > c$, and guaranteeing drastic circumscribing of the pandemics at cost K_3 (hence, a global cost $K_3 + C$) if $\bar{A}$ occurs. If on the contrary, A occurs, cost is again random with probability distribution P shifted by C .

The partially analyzed acts corresponding to these four decisions are represented in Fig. 4.

Now, data concerning the relative likelihood of events A and $\bar{A}$ only consists in the observation that pandemics occurred three times in the past century. Consequently, these events can at best be endowed with probability intervals, for instance, $P(\bar{A}) \in [0.01, 0.10]$ and thus $P(A) \in [0.90, 0.99]$.

Thus, the formalization of the flu pandemic prevention problem involves both ambiguity of events and uncertainty on consequences. Our model, unlike other models in the literature, is able to express both kinds of imprecision. For instance, the particular criterion of formulas (6, 7) could be applied with $\nu(A) = 0.90$ and $\nu(\bar{A}) = 0.01$.

8 Discussion and conclusion

The dichotomy analyzed part/ non-analyzed part is a fundamental feature of our model which deserves additional comments:

- (i) Since the relevant events for two distinct decisions are in general not the same (see example 2), we must allow pairs $(A, \bar{A})$ to depend on the decisions. However, (A, A) can be the same for particular groups of decisions, for instance decisions associated with various investments in the same geographical/political context.
- (ii) We only introduce a single non-analyzed event for a given decision. The reason is that specifying more precise outcome intervals on subsets of $\bar{A}$ would be, in our opinion, deceptive. The fact that an event such as “the number of deaths resulting from poor control of SARS is between 100 000 and 500 000” can be precisely described in words does not endow it with any assessable weight (such as a conditional probability or capacity). Thus, in accordance with general principles of rational decision making under complete ignorance (Arrow and Hurwicz 1972; Cohen and Jaffray 1980; Barbera et al. 1984), decisions should depend only on the best and the worst outcomes on all subsets which is to say on the best and worst ones on the whole of $\bar{A}$ itself.

By the way, Cohen and Jaffray (1980, 1983) show that, under complete ignorance, consistency with the weak dominance principle rules out a weak ordering of the decisions although both properties are “approximately compatible”. For this reason, the satisfaction of weak dominance on the non-analyzed part of decisions is not required but the model still guarantees the absence of awkward implications.

The model is consistent with various generalizations of SEU. For instance partially analyzed acts can be considered as a special case of multi-valued acts; once restricted to this special class, the criteria of Ghirardato (2001) model, become a subfamily of ours. More generally, the family of criteria described by the representation theorem is rather wide and various behavioral assumptions could be added and lead to more specific criteria such as those exhibited above.

On the other hand, there remains a major difference between the multi-valued act approach (and Savage’s theory as well) and ours: we avoid introducing acts which are unrealistic or inconceivable, such as acts assigning precise consequences on non-analyzable events or happy consequences on catastrophic events—an important advantage from a prescriptive point of view.

A more technical point deserves discussion. The existence of certainty equivalents for acts (Axiom 8) has been directly assumed mainly for simplicity reasons. In fact, the introduction of subalgebra $\mathcal{R}$ of events associated with a roulette wheel insures the existence of a “bet-equivalent” for any A -act of the form c^+ on $R \cap A$, c^- on $\bar{R} \cap A$ (and M_a on $\bar{A}$). These bet-equivalents could replace the certainty equivalents in the

proof of our representation theorem. Thus, Axiom 8 could be dispensed with at some intricacy cost.

Finally, it should be noted that the building blocks of the model, SEU for the analyzed part and “(max, min)” for the non-analyzed one could easily be replaced by other theories: for instance, the analyzed part would still be probabilizable but Rank Dependent Utility (Quiggin 1982) would replace EU, or information on the non-analyzed part of the acts would not be quantified in terms of consequence sets but according to symbolic categories.

In conclusion, the model proposed is intended to be particularly suited for decision making in situations involving potential risks and will, hopefully, prove to be so. At any rate, should alternative models be considered, the technical solutions developed here (insuring the consistency of conditional preferences and beliefs as well as the merging of submodels) can be adapted for the axiomatic justification of these models.

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Appendix

A Proof of Lemma 1

Consider g' and g'' resulting from b' and b'' by the common modification consisting in giving them a constant common consequence $g'(\omega) = g''(\omega) = c$ for $\omega \in A \setminus B$ and the same range M_a on $\bar{A}$. By Axiom 2, $g' \succsim g'' \Leftrightarrow b' \succsim_b b''$. Moreover, g' and g'' also belong to $\mathcal{F}_a$ and can be obtained by modifying a' and a'' on $A \setminus B$ and giving them a constant value c on this event. By definition, $a' \succsim_a^B a'' \Leftrightarrow g' \succsim g''$. Hence, $a' \succsim_a^B a'' \Leftrightarrow b' \succsim_b b''$.

B Proof of Lemma 2

For any $R', R'' \in \mathcal{R}$, $P_a(A \cap R') \geq P_a(A \cap R'') \Leftrightarrow A \cap R' \succsim^A A \cap R'' \Leftrightarrow B \cap R' \succsim^B B \cap R'' \Leftrightarrow P_b(B \cap R') \geq P_b(B \cap R'')$. Thus, the mapping $\mathcal{R}(A) \mapsto [0, 1]$ defined by $A \cap R \mapsto P_b(B \cap R)$ is a probability measure representing $\succsim^A$ which is uniquely represented by P_a . Therefore $P_a(A \cap R) = P_b(B \cap R)$ for every $R \in \mathcal{R}$.

C Proof of Proposition 3

By Proposition 2, there exists $R \in \mathcal{R}$ such that $R \cap B \sim_b E$, and thus, by Lemma 1, $R \cap B \sim_a^B E$, implying

$$P_b(R \cap B) = P_b(E) \text{ and } P_a(R \cap B/B) = P_a(E/B). \quad (11)$$

Moreover, by applying Lemma 1 to acts offering prizes on events $R' \cap B$ and $R'' \cap B$, where $R', R'' \in \mathcal{R}$, we get $R' \cap B \succsim_b R'' \cap B \Leftrightarrow R' \cap B \succsim_a^B R'' \cap B$. Thus, the same

ordering (say $\succsim_b$) on set of events $\{R \cap B, R \in \mathcal{R}\}$ is representable by (restrictions of) probabilities P_b and $P_a(\cdot/B)$; by unicity of such a representation (see Proposition 2), $P_b(R \cap B) = P_a(R \cap B/B)$, for all $R \in \mathcal{R}$. Then according to (11) $P_b(E) = P_a(E/B)$.

D Proof of Lemma 3

- (i) For $c \in M \setminus \{g(M), l(M)\}$, by Axiom 13, $g(M) \succ_C c$ implies $M \setminus \{c\} \succsim_{\mathcal{M}} M$ (take $M_0 = M \setminus \{g(M), c\}$) and symmetrically $c \succ_C l(M)$ implies $M \succsim_{\mathcal{M}} M \setminus \{c\}$; hence $M \sim_{\mathcal{M}} M \setminus \{c\}$.

Let $M = \{g(M), c_1, c_2, \dots, c_n, l(M)\}$ where $g(M) \succ_C c_1 \succ_C c_2 \succ_C \dots \succ_C c_n \succ_C l(M)$. Then, by repeated application of last relation:

$$\begin{aligned} M &\sim_{\mathcal{M}} M \setminus \{c_1\} \sim_{\mathcal{M}} M \setminus \{c_1, c_2\} \sim_{\mathcal{M}} \dots \\ &\sim_{\mathcal{M}} M \setminus \{c_1, c_2, \dots, c_n\} = \{g(M), l(M)\}. \end{aligned}$$

- (ii) From (i) above, we have $M' \sim_{\mathcal{M}} \{g(M'), l(M')\}$ and $M'' \sim_{\mathcal{M}} \{g(M''), l(M'')\}$. $\succsim_{\mathcal{M}}$ being transitive (Axiom 12), we just need to prove that $\{g(M'), l(M')\} \succsim_{\mathcal{M}} \{g(M''), l(M'')\}$. Assume that in the left side of (2) there is at least one strict preference, for instance $g(M') \succ_C g(M'')$ (if it is not the case, the result is straightforward). By Axiom 13 (point (ii)) with $M_0 = \{l(M')\}$, we have $\{g(M'), l(M')\} \succsim_{\mathcal{M}} \{g(M''), l(M')\}$. If $l(M') \succ_C l(M'')$, by the same axiom with $M_0 = \{g(M'')\}$, $\{g(M''), l(M')\} \succsim_{\mathcal{M}} \{g(M''), l(M'')\}$; else (case of $l(M') \sim_C l(M'')$), it follows from the proof of (i) that:

$$\{g(M''), l(M'), l(M'')\} \sim_{\mathcal{M}} \{g(M''), l(M')\} \sim_{\mathcal{M}} \{g(M''), l(M'')\}.$$

The proof of the second part of (ii) is similar and uses the second part of point (ii) in Axiom 13 (strict inequalities).

E Proof of Proposition 5

Since the linear order $\succeq_C$ is representable by a utility function, it is perfectly separable, i.e., there exist a countable subset $\Gamma = \{\gamma_n, n \in \mathbb{N}\}$ of $\mathcal{C}$ such that:

$$c \succ_C c' \Rightarrow \exists \gamma, \gamma' \in \Gamma \text{ s.t. } c \succeq_C \gamma \succ_C \gamma' \succeq_C c'.$$

Let the mapping $\varphi : \{(c, c') \in \mathcal{C} \times \mathcal{C} : c \succeq_C c'\} \rightarrow \mathbb{R}$ be defined by:

for

$$c \succ_C c', \text{ by } \varphi(c, c') = \sum_{(n,m) \in E(c,c')} \frac{1}{2^{n+m}}$$

where $E(c, c') = \{(n, m) \text{ s.t. } \gamma_n \succ_C \gamma_m \text{ and } \{c, c'\} \succ_{\mathcal{M}} \{\gamma_n, \gamma_m\}\}$;

$$\text{and for } c = c', \text{ by } \varphi(c, c') = \sum_{(n,m) \in F(c)} \frac{1}{2^{n+m}}$$

where $F(c) = \{(n, m) \text{ s.t. } \gamma_n \succ_C \gamma_m \text{ and } \{c\} \succ_{\mathcal{M}} \{\gamma_n, \gamma_m\}\}$.

Then, v defined by $v(M) = \varphi(g(M), l(M))$ has the required properties since, if $g(M) \succ_{\mathcal{M}} l(M)$, then $M \sim_{\mathcal{M}} \{c_i, c_j\}$ for some $c_i = g(M)$ and $c_j = l(M)$ and, if $g(M) = l(M)$, $M \sim_{\mathcal{M}} \{c_i\}$ for $c_i = g(M)$.

F Proof of Theorem 1

Any a in $\mathcal{F}$ has a certainty equivalent k in $\mathcal{C}$ and $\succeq_{\mathcal{C}}$ is representable by utility function u . A priori consequence k and, hence, number $u(k)$, depend on all the elements characterizing a namely A , $\mathcal{A}_a$, $a|_A$ and M_a .

By Axiom 8, there exists c in $\mathcal{C}$ such that $a \sim_a f_a^c$ which implies that the ranking of a only depends on quadruple $(A, \mathcal{A}_a, f_a^c|_A, M_a)$.

The constant A -act f_a^c being measurable with respect to any σ -algebra $\mathcal{A}_a$ of subsets of A , we have, for any A -acts a' , a'' such that $M_{a'} = M_{a''}$ and $f_{a'}^c = f_{a''}^c$, $a' \sim a''$. Thus, the preference between a' and a'' does not explicitly depend on $\mathcal{A}_{a'}$ and $\mathcal{A}_{a''}$ and the ranking of a in fact only depends on triple $(A, f_a^c|_A, M_a)$.

Moreover, the certainty equivalent k depends on $a|_A$ only through $\int_A u \circ a \, dP_a$ (by Proposition 1) and on M_a only through $(g(M_a), l(M_a))$ (by Proposition 4).

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