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Nonlinear two-point boundary value problems: applications to a cholera epidemic model

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This paper is concerned primarily with constructive mathematical analysis of a general system of nonlinear two-point boundary value problem when an empirically constructed candidate for an approximate solution (*quasi-solution*) satisfies verifiable conditions. A local analysis in a neighbourhood of a *quasi-solution* assures the existence and uniqueness of solutions and, at the same time, provides error bounds for approximate solutions. Applying this method to a cholera epidemic model, we obtain an analytical approximation of the steady-state solution with rigorous error bounds that also displays dependence on a parameter. In connection with this epidemic model, we also analyse the basic reproduction number, an important threshold quantity in the epidemiology context. Through a complex analytic approach, we determine the principal eigenvalue to be real and positive in a range of parameter values.

1. Introduction

Nonlinear systems of two-point boundary value problem (BVP) are ubiquitous in applications. This paper primarily concerns a class of nonlinear two-point BVPs that arise as a steady state in a reaction–diffusion setting, though by no means limited to it. The approach relies on an empirically constructed analytical expression for a candidate approximate solution, henceforth referred to

as a *quasi-solution*. By inverting the linearized operator about this *quasi-solution*, and checking appropriate conditions, it is possible to prove existence and uniqueness of solution in a local neighbourhood. Indeed, this has been the basis of a large number of computer-assisted proofs in modern literature (see for instance [1–5]). There are however a few instances [6–9] we know of where the *quasi-solution* is simple enough to allow checks of estimates with minimal or no computer assist. Here, we seek a transparent approach for a class of nonlinear two-point BVP in one dimension with a changing parameter. Though simpler than [1–5], the complexity is comparable to a PDE in two variables. For the specific BVP where we applied this method, computer assist is limited to products of polynomials in two variables of low order (at most 12), conversion into Chebyshev representations polynomials, and explicit integrations. Furthermore, using rational numbers, we avoid round-off errors.

We formulate the main theoretical result for a general BVP in theorem 2.4 in terms of explicit conditions that can be checked for a *quasi-solution* chosen for a specific BVP arising in an epidemic modelling of cholera [10] and other reaction–diffusion models in biology (e.g. [11–15]).

A second mathematical problem analysed in this paper is specific to the the epidemiological model whose steady states have been analysed in the first part. This concerns the basic reproduction number $\mathcal{R}_0$, a most important threshold parameter for disease extinction in mathematical epidemiology [16–19], which measures the expected number of the secondary infection produced by one primary case in an otherwise susceptible population. Recent work [10] shows that the mathematical problem can be reduced to a relatively simple eigenvalue problem of a linear two-point BVP. Though, theoretical expressions are relatively simple, a significant open problem addressed here in theorem 5.1 is whether (i) all eigenvalues are in the right half complex plane, (ii) the eigenvalue with the smallest real part is on the real axis. The basic reproduction number $\mathcal{R}_0$ is the reciprocal of the smallest eigenvalue, and therefore, it is important that these properties are displayed for any epidemiological model to be relevant. Computer assist in this case is limited to calculating exponential of matrices with fixed entries and their determinants, each of size 4×4 , and calculation of roots of the low-order polynomials.

2. Constructive existence proof for a general system of nonlinear two-point boundary value problem

Consider the following two-point boundary value problem (BVP)

$$\begin{aligned} \mathcal{N}(U) &:= U''(\tau) + AU'(\tau) + f(U) = 0, & \tau \in (0, 1) \\ U'(0) + \mathcal{B}_0 U(0) &= 0, \quad U'(1) + \mathcal{B}_1 U(1) = 0, \end{aligned} \quad (2.1)$$

where $U: [0, 1] \rightarrow \mathbb{R}^n$, $A = \text{diag}[a_{11}, a_{22}, \dots, a_{nn}]$ is a constant matrix¹ of size n with $a_{jj} \geq 0$ for $(j = 1, 2, \dots, n)$, $\mathcal{B}_0$ and $\mathcal{B}_1$ are constant diagonal matrices of size n with non-negative entries, and $f \in C^2(\mathcal{D})$ for $\mathcal{D} \subset \mathbb{R}^n$, an open bounded set, where $D^2 f$ is assumed bounded. We choose a smooth *quasi-solution* $U^0 \in C^2[0, 1]$ so that $U^0(\tau) \in \mathcal{D}$ for each $\tau \in [0, 1]$ and $U^{0'}(0) + \mathcal{B}_0 U^0(0) = 0$, while errors ϵ_R and ϵ_B , defined in (2.6), are each small.² We allow $\mathcal{N}, \mathcal{B}_0, \mathcal{B}_1, A, U^0$ to depend on some parameter $\rho \in [-1, 1]$, though this dependence is suppressed, except when needed.

(a) Notations

- (i) $|U| := \sqrt{U_1^2 + U_2^2 + \dots + U_n^2}$ for any vector $U = (U_1, U_2, \dots, U_n) \in \mathbb{R}^n$.
- (ii) $|M| := \sqrt{\text{trace}(M^T M)}$ is the L^2 compatible matrix norm for an $n \times n$ real matrix M .
- (iii) $\mathcal{S}$ denotes the Banach space of continuous real-valued vector functions on $[0, 1]$ equipped with supremum norm $\|F\|_{\mathcal{S}} := \sup_{\tau \in [0, 1]} |F(\tau)|$, for any $F \in \mathcal{S}$.

¹The proof given works for non-constant positive definite matrix A ; it can be adapted to any non-constant bounded A at the cost of much larger bounds arising from Gronwall Lemma.

²Smallness criteria is set in theorem 2.4.

- (iv) $L_0 = Df(U^0)$ denotes the $n \times n$ matrix (Jacobian) function f at U^0 .
(v) For $E \in C^2([0, 1], \mathbb{R}^n)$, with $\|E\|_{\mathcal{S}}$ small enough to ensure $U^0(\tau) + E(\tau) \in \mathcal{D}$ for each $\tau \in [0, 1]$, we define operators $\mathcal{L}$ and $\mathcal{N}_1$ as follows

$$\mathcal{L}[E] := E'' + AE' + L_0E \quad \text{and} \quad \mathcal{N}_1[E] := f(U^0 + E) - f(U^0) - L_0E. \quad (2.2)$$

- (vi) Define functional $\mathcal{F}$ of any real-valued vector or matrix function $g \in \mathcal{S}$:

$$\mathcal{F}[g] := g(1) + \mathcal{B}_1 \int_0^1 g(s) \, ds. \quad (2.3)$$

Remark 2.1. It follows from above that

$$|\mathcal{F}[g]| \leq (1 + |\mathcal{B}_1|) \|g\|_{\mathcal{S}}. \quad (2.4)$$

- (vii) We define operator $\mathcal{G} : \mathcal{S} \rightarrow \mathcal{S}$ such that

$$\mathcal{G}[r](\tau) := \tilde{E}'(\tau), \quad (2.5)$$

where $\tilde{E}$ satisfies $\mathcal{L}\tilde{E} = r$ with $\tilde{E}(0) = \tilde{E}'(0) = 0$.

(b) Definitions

- (i) ϵ_R and ϵ_B denote upper-bounds as follows:

$$\|\mathcal{N}[U^0]\|_{\mathcal{S}} \leq \epsilon_R, \quad \left| U^{0'}(1) + \mathcal{B}_1 U^0(1) \right| \leq \epsilon_B. \quad (2.6)$$

- (ii) The $(2n) \times (2n)$ matrix $\Psi(\tau)$ associated with the linearized system of (2.1) at U^0 is defined to be the unique solution of the following system

$$\Psi' + \begin{pmatrix} 0 & -I_{n \times n} \\ L_0 & A \end{pmatrix} \Psi = 0, \quad \Psi(0) = I_{2n \times 2n}. \quad (2.7)$$

Remark 2.2. From the ordinary differential equation (ODE) theory, $\mathcal{G}[r](\tau)$ is a vector function consisting of the last n components of the $2n$ -component vector

$$\Psi(\tau) \int_0^\tau \Psi^{-1}(s) \begin{pmatrix} 0 \\ r(s) \end{pmatrix} \, ds \quad (2.8)$$

- (iii) For convenience, it is useful to define a fundamental matrix $n \times n$ matrix Φ related to Ψ as the unique solution to:

$$\mathcal{L}\Phi = 0, \quad \Phi'(0) + \mathcal{B}_0\Phi(0) = 0, \quad \Phi(0) = I_{n \times n}. \quad (2.9)$$

A *quasi-fundamental matrix* Φ^0 is chosen to satisfy $\|\mathcal{L}\Phi^0\| = \epsilon_\phi$ to be small, while satisfying the same initial conditions as Φ .

- (iv) Define $n \times n$ matrices $\mathcal{Q}, \mathcal{Q}^0$, depending on ρ alone, so that

$$\mathcal{Q} = \mathcal{F}[\Phi'] + \mathcal{B}_1, \quad \mathcal{Q}^0 = \mathcal{F}\left[(\Phi^0)'\right] + \mathcal{B}_1. \quad (2.10)$$

When $\mathcal{Q}$ is invertible, we define bound $M_{\mathcal{Q}}$ satisfying

$$|\mathcal{Q}^{-1}| \leq M_{\mathcal{Q}}. \quad (2.11)$$

- (v) We define constants M and $M_{\mathcal{G}}$ to be upper-bounds as follows

$$\|\Phi'\|_{\mathcal{S}} \leq M, \quad \|\mathcal{G}[r]\|_{\mathcal{S}} \leq M_{\mathcal{G}} \|r\|_{\mathcal{S}}. \quad (2.12)$$

- (vi) For $E, E_1, E_2 \in \mathcal{B}_{\rho_0} \subset \mathcal{S}$, a closed ball of some radius ρ_0 centred at the origin that ensures $U^0(\tau) + E(\tau) \in \mathcal{D} \subset \mathbb{R}^n$, for each $\tau \in [0, 1]$, we define C_N^3 so that

$$\|Df(U^0 + E) - Df(U^0)\|_{\mathcal{S}} \leq C_N \|E\|_{\mathcal{S}} \quad (2.13)$$

and

$$\|\mathcal{N}_1[E]\| \leq \frac{1}{2} C_N \|E\|_{\mathcal{S}}^2, \quad \|\mathcal{N}_1[E_1] - \mathcal{N}_1[E_2]\|_{\mathcal{S}} \leq \rho_0 C_N \|E_1 - E_2\|_{\mathcal{S}}. \quad (2.14)$$

Remark 2.3. From (2.10), (2.4), (2.5) and (2.12), and the definition of Φ^0 , it follows that $|\mathcal{Q} - \mathcal{Q}^0| \leq (1 + |\mathcal{B}_1|) M_{\mathcal{G}} \epsilon_{\phi}$, and hence when calculations show $\mathcal{Q}^0$ to be invertible, then $\mathcal{Q}$ will be invertible with

$$M_{\mathcal{Q}} \leq |\mathcal{Q}^{0^{-1}}| \left[1 - (1 + |\mathcal{B}_1|) M_{\mathcal{G}} \epsilon_{\phi} \right]^{-1}, \quad \text{when } (1 + |\mathcal{B}_1|) M_{\mathcal{G}} \epsilon_{\phi} < 1. \quad (2.15)$$

We will state the main theorem in this section and apply it to the epidemic model in §3.

Theorem 2.4. *If for some choice of quasi-solution U^0 , $\mathcal{Q}^0$ is invertible and satisfies (2.15), and ϵ_R and ϵ_B are each small enough so that $\epsilon = M_{\mathcal{G}} \epsilon_R + M M_{\mathcal{Q}} [(1 + |\mathcal{B}_1|) M_{\mathcal{G}} \epsilon_R + \epsilon_B]$ satisfies*

$$\alpha := 2\epsilon C_N M_{\mathcal{G}} \left(1 + \frac{1}{M} \right)^2 (1 + M M_{\mathcal{Q}} (1 + |\mathcal{B}_1|)) < 1, \quad (2.16)$$

then, there is a unique solution U to the two-point BVP (2.1) in a neighbourhood of quasi-solution U^0 with

$$\|U'(\tau) - U^{0'}(\tau)\|_{\mathcal{S}} \leq 2\epsilon, \quad |U(0) - U^0(0)| \leq \frac{2\epsilon}{M}. \quad (2.17)$$

The proof of theorem 2.4 will have to await a few more lemmas.

It is clear from (2.2) that (2.1) is equivalent to the following two-point BVP for $E = U - U^0$:

$$\begin{aligned} \mathcal{L}E &= -\mathcal{N}[U^0] - \mathcal{N}_1[E], \quad 0 < \tau < 1 \\ E'(0) + \mathcal{B}_0 E(0) &= 0, \quad E'(1) + \mathcal{B}_1 E(1) = -U^{0'}(1) - \mathcal{B}_1 U^0(1). \end{aligned} \quad (2.18)$$

(c) Solution to boundary value problem (2.18)

Any small solution to (2.18) will necessarily be a solution to the initial value problem (IVP)

$$\mathcal{L}E = -\mathcal{N}[U^0] - \mathcal{N}_1[E], \quad E'(0) + \mathcal{B}_0 E(0) = 0, \quad E(0) = \delta, \quad (2.19)$$

for some small δ . From ODE theory, for τ sufficiently small, a unique solution E to (2.19) is guaranteed when quasi-solution U^0 is smooth. We want to show that this solution extends all the way to $\tau = 1$ and remains small in the $\|\cdot\|_{\mathcal{S}}$ norm, when $|\delta|$ is sufficiently small, and there is a unique δ in some ball of $\mathbb{R}^n$ that ensures that E satisfies the boundary condition of (2.18) at $\tau = 1$. In the following, we derive an integral reformulation of the two-point BVP (2.18) that will involve both $E' \in \mathcal{S}$ and $\delta \in \mathbb{R}^n$.

Let $\tilde{E} = E - \Phi\delta$. Clearly, $\mathcal{L}\tilde{E} = r$ with $\tilde{E}(0) = \tilde{E}'(0) = 0$. Hence, it follows from the ODE theory and definition of operator $\mathcal{G}$ in (2.5) that the initial value problem (IVP) (2.19) is equivalent to the nonlinear integral equation

$$E'(\tau) = \Phi'(\tau)\delta - \mathcal{G}[\mathcal{N}[U^0]](\tau) - \mathcal{G}[\mathcal{N}_1[E]](\tau), \quad (2.20)$$

where E is considered known in terms of E' through the relation

$$E(\tau) = \delta + \int_0^\tau E'(\tau') d\tau'. \quad (2.21)$$

Equation (2.21) implies

$$\|E\|_{\mathcal{S}} \leq |\delta| + \|E'\|_{\mathcal{S}}. \quad (2.22)$$

³Since C_N is only a property of f and the set $\mathcal{D} \subset \mathbb{R}^n$, it does not depend on τ .

Now, consider the boundary condition on the right end; i.e. $E'(1) + B_1 E(1) = -U^{0'}(1) - B_1 U^0(1)$, which from definition of functional $\mathcal{F}$ becomes

$$\mathcal{F}[E'] + B_1 \delta = -U^{0'}(1) - B_1 U^0(1). \quad (2.23)$$

Substituting (2.20) into (2.23) and using functional $\mathcal{F}$ and matrix $\mathcal{Q}$ in (2.3) and (2.10), (2.23) becomes

$$\mathcal{Q}\delta = \mathcal{F}[\mathcal{G}[\mathcal{N}[U^0] + \mathcal{N}_1[E]]] - U^{0'}(1) - B_1 U^0(1). \quad (2.24)$$

When $\mathcal{Q}$ is invertible, as is ensured when (2.15) holds, (2.24) is equivalent to

$$\delta = \delta^{(0)} + \mathcal{Q}^{-1} \mathcal{F}[\mathcal{G}\mathcal{N}_1[E]] =: \mathcal{M}_2[E', \delta], \quad (2.25)$$

where

$$\delta^{(0)} = \mathcal{Q}^{-1} \mathcal{F}[\mathcal{G}[\mathcal{N}[U^0]]] - \mathcal{Q}^{-1} [U^{0'}(1) + B_1 U^0(1)]. \quad (2.26)$$

as far as nonlinear On substituting the right-hand side of (2.25) in (2.20), we obtain

$$E' = E^{(0)'} - \mathcal{G}[\mathcal{N}_1[E]] + \Phi' \mathcal{Q}^{-1} \mathcal{F}[\mathcal{G}\mathcal{N}_1[E]] =: \mathcal{M}_1[E', \delta], \quad (2.27)$$

where

$$E^{(0)'}(\tau) = -\mathcal{G}[\mathcal{N}[U^0]](\tau) + \Phi'(\tau)\delta^{(0)}. \quad (2.28)$$

Equation (2.27) and (2.25) determine $(E', \delta) \in \mathcal{S} \times \mathbb{R}^n$ as shall be seen shortly. It is clear from the derivation that as long as $\mathcal{Q}$ is invertible, finding such solution is equivalent to finding a solution to the original two-point BVP. Indeed, the arguments above allow us to conclude the following proposition.

Proposition 2.5. *If $\mathcal{Q}^0$ is invertible and satisfies (2.15), then the BVP (2.18) is equivalent to $(E', E(0))$ satisfying (2.27) and (2.25), where $\delta \equiv E(0)$.*

Lemma 2.6. *$\delta^{(0)}$ defined in (2.26) satisfies the following bound*

$$|\delta^{(0)}| \leq M_{\mathcal{Q}} \left\{ (1 + |B_1|) M_{G \in R} + \epsilon_B \right\}. \quad (2.29)$$

Proof. The proof follows from the expression in (2.26) using bounds on $\mathcal{Q}^{-1}$, $\mathcal{F}$, $\mathcal{G}$, $\mathcal{N}[U^0]$ and $U^{0'}(1) + B_1 U^0(1)$ (see (2.6), (2.11), (2.4) and (2.12)). ■

Lemma 2.7. *For ϵ defined in theorem 2.4, $E^{(0)'}$ defined in (2.28) satisfies*

$$\|E^{(0)'}\|_{\mathcal{S}} \leq \epsilon. \quad (2.30)$$

Proof. From expression (2.28), the proof follows from bounds in $\delta^{(0)}$ in lemma (2.6), and the bounds on $\mathcal{N}[U^0]$ and Φ' in (2.6) and (2.11), respectively. ■

Definition 2.8. We define $\chi = (E', \delta) \in \mathcal{S} \times \mathbb{R}^n$ and define a norm in this space

$$\|\chi\| = \max \{ \|E'\|_{\mathcal{S}}, M|\delta| \}. \quad (2.31)$$

We define an operator $\mathcal{M} : \mathcal{S} \times \mathbb{R}^n \rightarrow \mathcal{S} \times \mathbb{R}^n$ so that

$$\mathcal{M}[\chi] = (\mathcal{M}_1[E', \delta], \mathcal{M}_2[E', \delta]), \quad (2.32)$$

where $\mathcal{M}_1 : \mathcal{S} \times \mathbb{R}^n \rightarrow \mathcal{S}$ and $\mathcal{M}_2 : \mathcal{S} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are defined by (2.27) and (2.25).

Then, by above definition, equations (2.27) and (2.25) for (E', δ) can be written abstractly as $\chi = \mathcal{M}[\chi]$.

Proof of theorem 2.4. It suffices to show that $\mathcal{M}: \mathcal{B}_{2\epsilon} \rightarrow \mathcal{B}_{2\epsilon}$ is contractive. For that purpose, applying bounds for Φ' , $\mathcal{G}$, $\mathcal{Q}^{-1}$, $\mathcal{F}$ and $\mathcal{N}_1$ (see (2.14)) and using (2.30) and (2.16), (2.27) lead to

$$\begin{aligned} \|\mathcal{M}_1[E', \delta]\|_S &\leq \epsilon + \frac{1}{2} M_G C_N [1 + M M_Q (1 + |\mathcal{B}_1|)] (\|E'\|_S + |\delta|)^2 \\ &\leq \epsilon + 2 M_G C_N [1 + M M_Q (1 + |\mathcal{B}_1|)] \epsilon^2 \left(1 + \frac{1}{M}\right)^2 \leq 2\epsilon, \end{aligned}$$

where the last inequality is obtained from assumption (2.16). Likewise, using definition (2.25), we find that

$$|\mathcal{M}_2[E', \delta]| \leq \frac{\epsilon}{M} + 2 M_Q (1 + |\mathcal{B}_1|) M_G C_N \epsilon^2 \left(1 + \frac{1}{M}\right)^2 \leq \frac{2\epsilon}{M}, \quad (2.33)$$

which immediately implies from definition of $\|\cdot\|$ that $\mathcal{M}: \mathcal{B}_{2\epsilon} \rightarrow \mathcal{B}_{2\epsilon}$. Now, consider two different $\chi_1 = (E'_1, \delta_1)$ and $\chi_2 = (E'_2, \delta_2)$ in $\mathcal{B}_{2\epsilon} \subset \mathcal{S} \times \mathbb{R}^n$. We first note that

$$\|E_i\|_S \leq \|E'_i\|_S + |\delta_i| \leq 2\epsilon \left(1 + \frac{1}{M}\right), \quad i = 1, 2,$$

and

$$\|E_1 - E_2\|_S \leq \|E'_1 - E'_2\|_S + |\delta_1 - \delta_2| \leq \left(1 + \frac{1}{M}\right) \|\chi_1 - \chi_2\|.$$

By (2.2) and (2.13),

$$\|\partial_{E_i} \mathcal{N}_1[E_i]\|_S = \|Df(U^0 + E_i) - Df(U^0)\|_S \leq C_N \|E_i\|_S, \quad i = 1, 2,$$

It then follows from (2.14) that

$$\|\mathcal{N}_1[E_1] - \mathcal{N}_1[E_2]\|_S \leq 2 C_N \epsilon \left(1 + \frac{1}{M}\right)^2 \|\chi_1 - \chi_2\|.$$

Therefore, from (2.27), we have

$$\|\mathcal{M}_1[E'_1, \delta_1] - \mathcal{M}_1[E'_2, \delta_2]\|_S \leq 2 C_N \epsilon M_G \left(1 + \frac{1}{M}\right)^2 (1 + M M_Q [1 + |\mathcal{B}_1|]) \|\chi_1 - \chi_2\|,$$

whereas from (2.25), we have

$$|\mathcal{M}_2[E'_1, \delta_1] - \mathcal{M}_2[E'_2, \delta_2]| \leq 2 C_N \epsilon M_G \left(1 + \frac{1}{M}\right)^2 M_Q [1 + |\mathcal{B}_1|] \|\chi_1 - \chi_2\|. \quad (2.34)$$

It follows immediately from definition of norm $\|\cdot\|$ in $\mathcal{S} \times \mathbb{R}^n$ that

$$\|\mathcal{M}[\chi_1] - \mathcal{M}[\chi_2]\| \leq \alpha \|\chi_1 - \chi_2\| \quad (2.35)$$

and hence $\mathcal{M}: \mathcal{B}_{2\epsilon} \rightarrow \mathcal{B}_{2\epsilon}$ is contractive. Theorem 2.4 follows from the Banach Contraction Theorem. ■

3. Determining quasi-solution, ϵ_R , ϵ_B , M , M_G and M_Q

Determining quasi-solution based on numerical solutions is fairly straight forward, and one can use a number of different basis. In the simplest case, we can consider a discrete set $\rho = \rho_m \in [-1, 1]$, and for each such $\rho = \rho_m$, numerically determine the solution to the two-point BVP at a set of discrete points $\tau_l \in [0, 1]$. We employ a polynomial or Chebyshev fit procedure involving

$T_j(\rho)T_k(2\tau - 1)$ basis, using numerical values of U, U' to determine $U'' = -AU' - f(U)$, which leads to

$$U^{0''}(\tau; \rho) = \sum_{j=0}^{N_\tau} \sum_{k=0}^{N_\rho} u_{j,k} \tau^j \rho^k, \quad (3.1)$$

where $u_{j,k} \in \mathbb{R}^n$ are constants. We may choose each entry of $u_{j,k}$ to be a rational number to avoid round-off errors. Independently, numerical solution at $\tau = 0$ at a sequence of ρ values as above, is used to determine $\hat{u}_k$ in the representation

$$U^0(0; \rho) = \sum_{k=0}^{N_B} \hat{u}_k \rho^k. \quad (3.2)$$

Using boundary conditions (3.2) and $U^{0'}(0) = -\mathcal{B}_0 U^0(0)$, we determine expression U^0 that satisfies the left boundary condition exactly. Expression for *quasi-fundamental matrix* $\Phi^0(\tau; \rho)$ is determined in a similar manner, and each column can be generated through an ODE solver for IVP and use of polynomial fit, as for (3.1).

Without much loss of generality,⁴ we assume that $f(U)$ is either a polynomial or rational function. To determine residual bounds ϵ_R rigorously, we multiply the i -th scalar component of (2.1), call it e_i , by a non-negative⁵ polynomial $P_i(U_1, U_2, \dots, U_n)$ so that $P_i e_i$ is a polynomial in $U_1, U_2, \dots, U_n$, where $U = (U_1, \dots, U_n)$. Therefore, substituting $U = U^0$, obtained from integrating (3.1), will necessarily result in a polynomial in τ, ρ whose Chebyshev representation is

$$P_i(U^0) e_i(U^0) = \sum_{j=0}^{\tilde{N}} \sum_{k=0}^{\tilde{M}} r_{j,k}^i T_j(2\tau - 1) T_k(\rho)$$

from which it follows that

$$\|P_i(U^0) e_i(U^0)\|_S \leq \sum_{j=0}^{\tilde{N}} \sum_{k=0}^{\tilde{M}} |r_{j,k}^i|.$$

We then determine lower bounds on P_i once again by using usual minimization in calculus on a low order polynomial approximation in τ and ρ , if necessary by breaking up into smaller subintervals, and using Chebyshev representation to bound the rest. This process leads to determination of bounds on $\epsilon_R := \|\mathcal{N}[U^0]\|$. The same process may be followed to find bound for $\epsilon_\phi := \|\mathcal{L}\Phi^0\|$. To find ϵ_B , we substitute U^0 and its derivative at $\tau = 1$, use Chebyshev representation in ρ , and add absolute value of all coefficients.

It is clear that finding explicit fundamental solution Φ for a non-constant system of arbitrary order ODEs is not possible. The purpose of this section is to show that this is not needed. An 'energy type' argument combined with introduction of a 'cumulative norm' that has been used earlier in a scalar context [20] may be modified to find bounds upper M and $M_{\mathcal{G}}$ on Φ' and operator $\mathcal{G} := \frac{d}{d\tau} \mathcal{L}^{-1}$ respectively. This is crucial in checking that the conditions of theorem 2.4 apply.

Here we seek to determine bounds for Φ' and operator $\mathcal{G}$. For that purpose, we pose the IVP:

$$\mathcal{L}E := E'' + AE' + L_0 E = r, \quad \text{with } E(0) = \delta, \quad E'(0) = -\mathcal{B}_0 \delta. \quad (3.3)$$

with initial value Taking inner-product of (3.3) with E' , we obtain

$$\frac{1}{2} \frac{d}{d\tau} \langle E', E' \rangle + \langle E', AE' \rangle + \langle E', L_0 E \rangle = \langle E', r \rangle. \quad (3.4)$$

⁴Every smooth function in bounded domain can approximate by a rational function.

⁵Since by assumption f is smooth in $\mathcal{D}$, it is clear that when it is rational in U , we can choose non-zero denominators, which without loss of generality will be positive.

Write $E' = (\epsilon_1, \epsilon_2, \dots, \epsilon_n)^T$. Then (3.4) can be written as

$$\frac{1}{2} \frac{d}{d\tau} \left(\sum_{i=1}^n \epsilon_i^2 \right) + \sum_{i=1}^n a_{ii} \epsilon_i^2 + \langle E', L_0 E \rangle = \langle E', r \rangle. \quad (3.5)$$

Integration of (3.4) results in

$$\begin{aligned} \frac{1}{2} \sum_{i=1}^n \epsilon_i^2 &= \frac{1}{2} \delta^T \mathcal{B}_0^T \mathcal{B}_0 \delta - \sum_{i=1}^n a_{ii} \int_0^\tau |\epsilon_i(s)|^2 ds - \int_0^\tau \langle E'(s), L_0(E(s) - E(0)) \rangle ds \\ &\quad - \int_0^\tau \langle E'(s), L_0 \delta \rangle + \int_0^\tau \langle E'(s), r(s) \rangle ds. \end{aligned} \quad (3.6)$$

Since $a_{ii} \geq 0$ for all $1 \leq i \leq n$, we have

$$\begin{aligned} \frac{1}{2} |E'(\tau)|^2 &\leq \frac{1}{2} |\mathcal{B}_0 \delta|^2 + \int_0^\tau |L_0(s)| |E'(s)| |E(s) - E(0)| ds \\ &\quad + |\delta| \int_0^\tau |L_0(s)| |E'(s)| ds + \int_0^\tau |E'(s)| |r(s)| ds, \end{aligned} \quad (3.7)$$

where the last inequality is obtained from Cauchy–Schwartz inequality. Note we can determine E in terms of E' through integration

$$E(\tau) = E(0) + \int_0^\tau E'(s) ds. \quad (3.8)$$

In what follows, we will determine constants $M_{\mathcal{G}}$ and M , which are an upper bound for the operators $\mathcal{G}$ and Φ' defined in (2.9) and (2.5), respectively.

(a) Finding bound $M_{\mathcal{G}}$

In this case, we have zero boundary conditions on the left and so (3.7) implies

$$\frac{1}{2} |E'(\tau)|^2 \leq \int_0^\tau |L_0(s)| |E'(s)| |E(s)| ds + \int_0^\tau |E'(s)| |r(s)| ds. \quad (3.9)$$

Define

$$\mathcal{D}(\tau) = \sup_{s \in [0, \tau]} |E'(s)|. \quad (3.10)$$

We notice

$$|E(\tau) - E(0)| \leq \tau \mathcal{D}(\tau), \quad |E'(\tau)| \leq \mathcal{D}(\tau), \quad \forall \tau \geq 0. \quad (3.11)$$

Then, using (3.8) in (3.9), we obtain

$$\frac{1}{2} |E'(\tau)|^2 \leq \int_0^\tau |L_0(s)| s \mathcal{D}^2(s) ds + \int_0^\tau \mathcal{D}(s) |r(s)| ds \leq \int_0^{\tau'} \left(s |L_0(s)| + \frac{1}{2} \right) \mathcal{D}^2(s) ds + \frac{1}{2} \int_0^{\tau'} r^2(s) ds, \quad (3.12)$$

for any $0 \leq \tau \leq \tau' \leq 1$. Since the right-hand side of (3.12) is independent of τ , by taking the supremum over all possible $0 \leq \tau \leq \tau'$ of the left-hand side of (3.12), we immediately obtain

$$\frac{1}{2} \mathcal{D}^2(\tau') \leq \int_0^{\tau'} \left(s |L_0(s)| + \frac{1}{2} \right) \mathcal{D}^2(s) ds + \frac{1}{2} \int_0^{\tau'} r^2(s) ds \quad (3.13)$$

Gronwall's Lemma immediately implies that

$$\mathcal{D}^2(\tau) \leq \int_0^\tau \exp \left[\int_s^\tau (1 + 2s' |L_0(s')|) ds' \right] r^2(s) ds, \quad (3.14)$$

implying

$$\mathcal{D}(1) \leq \exp \left[\int_0^1 s |L_0(s)| ds \right] \sqrt{(e-1)} \|r\|_S. \quad (3.15)$$

Therefore, it follows from (2.12) that

$$M_{\mathcal{G}} \leq \sqrt{(e-1)} \exp \left[\int_0^1 s |L_0(s)| \, ds \right]. \quad (3.16)$$

(b) Calculation of M

In this case, we need to set $r=0$ and allow for inhomogeneous boundary condition at $\tau=0$. Returning to (3.7), we now obtain

$$\begin{aligned} |E'(\tau)|^2 &\leq |B_0|^2 |\delta|^2 + \int_0^\tau 2s |L_0(s)| \mathcal{D}^2(s) \, ds + 2|\delta| \int_0^\tau |L_0(s)| \mathcal{D}(s) \, ds \\ &\leq |B_0|^2 |\delta|^2 + \int_0^{\tau'} (2s+1) |L_0(s)| \mathcal{D}^2(s) \, ds + |\delta|^2 \int_0^{\tau'} |L_0(s)| \, ds \end{aligned} \quad (3.17)$$

for $0 \leq \tau \leq \tau'$. It follows that

$$\mathcal{D}^2(\tau') \leq \alpha(\tau') + \int_0^{\tau'} \beta(s) \mathcal{D}^2(s) \, ds, \quad (3.18)$$

where

$$\beta(\tau) = (2\tau+1) |L_0(\tau)|, \quad \alpha(\tau) = |\delta|^2 \left(\int_0^\tau |L_0(s)| \, ds + |B_0|^2 \right). \quad (3.19)$$

From Gronwall's leads to

$$\mathcal{D}^2(\tau) \leq \alpha(\tau) + \int_0^\tau \alpha(s) \beta(s) \exp \left[\int_s^\tau \beta(s') \, ds' \right]. \quad (3.20)$$

On integration by parts, and using definition of α, β we obtain

$$\mathcal{D}^2(\tau) \leq |\delta|^2 \left\{ \exp \left[\int_0^\tau (2s'+1) |L_0(s')| \, ds' \right] (|B_0|^2 + 1) - 1 \right\}. \quad (3.21)$$

It follows that

$$M \leq \sqrt{\exp \left[\int_0^1 (2s+1) |L_0(s)| \, ds \right] (|B_0|^2 + 1) - 1}. \quad (3.22)$$

Remark 3.1. In estimating $M_{\mathcal{G}}, M$ in (3.16) and (3.22), if $L_0 = p(s)/q(s)$ for polynomial p, q with scalar $q > 0$, we note

$$\int_0^1 2s |L_0(s)| \, ds \leq \frac{1}{q_m} \sqrt{\int_0^1 s |p(s)|^2 \, ds}, \quad \int_0^1 |L_0(s)| \, ds \leq \frac{1}{q_m} \sqrt{\int_0^1 |p_0(s)|^2 \, ds}, \quad (3.23)$$

where $q_m = \min_{s \in [0,1]} q(s)$. Integrals of Euclidean norm $|p(s)|^2$ is explicit even when dependence on ρ is a polynomial. When needed, breaking up the integrals into subintervals affords sharper bounds.

(c) Calculation of $M_{\mathcal{Q}}$

Recall expression of $M_{\mathcal{Q}}$ in (2.15). However, to apply this we need bounds on $|\mathcal{Q}^{0-1}|$. Since *quasi-solution* procedure leaves us with $\mathcal{Q}^0$ a polynomial in ρ , directly determining uniform bounds on $|\mathcal{Q}^{0-1}|$ becomes more difficult. Instead, using numerical inverse $\mathcal{Q}^{0-1}$ at a finite set of ρ values in $[-1, 1]$ and using polynomial fit, we determine empirically a matrix polynomial $\mathcal{Q}^I$ for which

multiplication confirms that $\mathcal{Q}_{E,I} := \mathcal{Q}^I \mathcal{Q}^0 - I$ is small through a Chebyshev representation of each entry and addition of absolute value of coefficients. Then, it follows that

$$\mathcal{Q}^{0^{-1}} = (I + \mathcal{Q}_{E,I})^{-1} \mathcal{Q}^I, \quad \text{implying } \|\mathcal{Q}^{0^{-1}}\| \leq \frac{\|\mathcal{Q}^I\|}{1 - \|\mathcal{Q}_{E,I}\|}. \quad (3.24)$$

Using (3.24) in (2.15), we determine invertibility of $\mathcal{Q}$ and bound $M_{\mathcal{Q}}$. In principle, this procedure will work for any $\rho \in [-1, 1]$, except at bifurcation point(s) where $\mathcal{Q}$ is necessarily singular. This is because *quasi-fundamental matrix* Φ^0 may be constructed to approximate Φ to any desired accuracy, though accuracy demands are unrealistic when too close to a bifurcation point.

4. Cholera model and steady-state solution

Let's consider a cholera partial differential equation (PDE) model along a theoretical river that was proposed in [21] and studied in [10,22,23].

$$\left. \begin{aligned} \frac{\partial S}{\partial t} &= \Lambda - \beta_1 SI - \beta_2 S \frac{B}{B+K} - dS + \sigma R + D_1 \frac{\partial^2 S}{\partial x^2}, & 0 < x < L, t > 0, \\ \frac{\partial I}{\partial t} &= \beta_1 SI + \beta_2 S \frac{B}{B+K} - (d + \gamma)I + D_2 \frac{\partial^2 I}{\partial x^2}, & 0 < x < L, t > 0, \\ \frac{\partial R}{\partial t} &= \gamma I - (d + \sigma)R + D_3 \frac{\partial^2 R}{\partial x^2}, & 0 < x < L, t > 0 \\ \text{and} \quad \frac{\partial B}{\partial t} &= \xi I + gB \left(1 - \frac{B}{K_B}\right) - \delta B - v \frac{\partial B}{\partial x} + D_4 \frac{\partial^2 B}{\partial x^2}, & 0 < x < L, t > 0, \end{aligned} \right\} \quad (4.1)$$

with no flux boundary condition

$$\frac{\partial S}{\partial x}(x, t) = \frac{\partial I}{\partial x}(x, t) = \frac{\partial R}{\partial x}(x, t) = D_4 \frac{\partial B}{\partial x}(x, t) - vB(x, t) = 0, \quad x = 0, L, \quad t > 0,$$

and the initial condition $S(x, 0), I(x, 0), R(x, 0), B(x, 0) \in C([0, L], \mathbb{R}^+)$.

Here $x \in [0, L]$ is the position variable and $t \geq 0$ is the time variable. $x = 0$ and L represent two ends of the river. $S = S(x, t)$, $I = I(x, t)$, and $R = R(x, t)$ measure the density of susceptible, infectious, and recovered human hosts at location x and time t , respectively. $B = B(x, t)$ denotes the concentration of cholera in the water environment. Λ is the influx of susceptible humans. β_1 and β_2 represent direct (i.e. human-to-human) and indirect (i.e. environment-to-human) transmission rates, respectively. D_i ($1 \leq i \leq 4$) is the diffusion coefficient of S , I , R and B , respectively. v is the convection coefficient depicting the drift for vibrio's transport. d is the natural death rate of human hosts. σ is the rate at which recovered individuals lose immunity, γ represents the recovery rate of infectious hosts, and ξ denotes the time-dependent shedding rate of bacteria by infectious human hosts, and δ is the natural death rate of the bacteria. K is the half saturation rate that describes the infectious dose in water sufficient to produce disease in 50% of those exposed. K_B is the maximal capacity of pathogen in water environment. The base values of these parameters are taken from [21], and $D_2 = D_4 = 1$ and $v = 0.1$ are assumed.

In this section, we apply the theory developed in §2–3 to find accurate analytic approximations to the steady-state solution of cholera model (4.1). Consider the steady-state solution $(\tilde{S}, \tilde{I}, \tilde{R}, \tilde{B})$ of model (4.1). With $\tau = 1 - \frac{x}{L}$ and scaling as follows:

$$\tilde{S}(x) = \frac{\Lambda}{d} (1 + S(\tau)), \quad \tilde{I}(x) = \frac{\beta_2 \Lambda L^2}{d D_2} I(\tau), \quad \tilde{R}(x) = \frac{\beta_2 \Lambda L^2 \gamma}{d D_2 (d + \sigma)} R(\tau), \quad \tilde{B}(x) = K B(\tau) \quad (4.2)$$

the steady-state equations reduce to

$$\left. \begin{aligned} -c_1 S(\tau) - n_2 I(\tau)(1 + S(\tau)) - \frac{[n_2 B(\tau)(1 + S(\tau))]}{[1 + B(\tau)]} + c_2 R(\tau) + S''(\tau) &= 0, \\ c_3 I(\tau) + n_3 S(\tau)I(\tau) + \frac{[n_2 B(\tau)(1 + S(\tau))]}{[1 + B(\tau)]} + I''(\tau) &= 0, \\ -c_4 R(\tau) + c_4 I(\tau) + R''(\tau) &= 0 \end{aligned} \right\} \quad (4.3)$$

and $-n_4 (B(\tau))^2 + c_5 B(\tau) + c_6 I(\tau) + c_7 B'(\tau) + B''(\tau) = 0,$

with the two-point boundary conditions $S'(\tau) = I'(\tau) = R'(\tau) = B'(\tau) + c_7 B(\tau) = 0$ for $\tau = 0, 1$, where

$$\left. \begin{aligned} c_1 &= \frac{dL^2}{D_1}, \quad n_1 = \frac{\Lambda L^4 \beta_1 \beta_2}{D_1 D_2 d}, \quad n_2 = \frac{\beta_2 L^2}{D_1}, \quad c_2 = \frac{\sigma \gamma \beta_2 L^4}{D_1 D_2 (\sigma + d)}, \\ c_3 &= \frac{\beta_1 \Lambda L^2}{d D_2} - \frac{(d + \gamma) L^2}{D_2}, \quad n_3 = \frac{n_1}{n_2}, \quad c_4 = \frac{(d + \sigma) L^2}{D_3}, \\ n_4 &= \frac{K g L^2}{D_4 K_b}, \quad c_5 = \frac{(g - \delta) L^2}{D_4}, \quad c_6 = \frac{\xi L^4 \beta_2 \Lambda}{K D_4 D_2 d}, \quad c_7 = \frac{\nu L}{D_4}. \end{aligned} \right\} \quad (4.4)$$

We have now reduced the problem to 10 independent non-dimensional parameters $c_1, \dots, c_7, n_1, n_2$ and n_4 (note $n_3 = n_1/n_2$ is not independent). It is clear that if we define vector

$$U = (S, I, R, B)^T, \quad (4.5)$$

then equation (4.3) takes the form (2.1), where $n = 4$, and $a_{jj} = 0$ for $j = 1, 2, 3$, but $a_{44} = c_7$. In our case the matrices B_0 and B_1 are also the same as A . The vector function $f(U)$ in this case is given by

$$f(U) = \begin{pmatrix} -c_1 U_1 - n_1 U_2 (1 + U_1) - \frac{n_2 U_4 (1 + U_1)}{1 + U_4} + c_2 U_3 \\ c_3 U_2 + n_3 U_1 U_2 + \frac{U_4 (1 + U_1)}{(U_4 + 1)} \\ -c_4 U_3 + c_4 U_2 \\ -n_4 U_4^2 + c_5 U_4 + c_6 U_2 \end{pmatrix}. \quad (4.6)$$

We notice that if we write $U = U^0 + E$, the linearization of $f(U)$ at $U = U^0$ is given by $L_0 E$, where L_0 is in general an $n \times n$ matrix coming from calculation of Jacobian of $f(U)$ at $U = U^0$. In our case with

$$U^0 = (U_1^0, U_2^0, U_3^0, U_4^0) = (s_0, i_0, r_0, b_0),$$

and we have

$$L_0 = \begin{pmatrix} -c_1 - n_1 U_2^0 - \frac{n_2 U_4^0}{1 + U_4^0} & -n_1 (1 + U_1^0) & c_2 & -\frac{n_2 (1 + U_1^0)}{(1 + U_4^0)^2} \\ n_3 U_2^0 + \frac{U_4^0}{(1 + U_4^0)} & c_3 + n_3 U_1^0 & 0 & \frac{(1 + U_1^0)}{(1 + U_4^0)^2} \\ 0 & c_4 & -c_4 & 0 \\ 0 & c_6 & 0 & c_5 - 2n_4 U_4^0 \end{pmatrix}. \quad (4.7)$$

We also have for any U , the following expression that is useful in estimating C_N in (2.13)–(2.14)

$$\sum_{j,k,m=1}^4 |\partial_{U_k U_m}^2 f_j(U)|^2 = 4n_4^2 + \frac{4(1 + U_1)^2}{(1 + U_4)^6} + \frac{2}{(1 + U_4)^4} + 2n_3^2 + \frac{4n_2^2(1 + U_1)^2}{(1 + U_4)^6} + \frac{2n_2^2}{(1 + U_4)^4} + 2n_1^2. \quad (4.8)$$

In epidemic dynamics of cholera, the dependence of steady state on ν , and hence on non-dimensional c_7 . We choose to explore the range:

$$c_7 = \frac{11 + 9\rho}{40}, \quad \text{for } \rho \in [-1, 1], \quad (4.9)$$

all other parameters $c_1, c_2, c_3, c_4, c_5, c_6, n_1, n_2, n_3, n_4$ are constants in table 1.

Following the procedure outlined in the last section, we determined a suitable *quasi-solution* in the form

$$\left. \begin{aligned} S_0(\tau, \rho) &= \sum_{k=0}^6 s_k^0 \rho^k + \sum_{j=0}^7 \sum_{k=0}^6 \frac{s_{j,k}}{(j+1)(j+2)} \rho^k \tau^{j+2}, \\ I_0(\tau, \rho) &= \sum_{k=0}^6 i_k^0 \rho^k + \sum_{j=0}^7 \sum_{k=0}^6 \frac{i_{j,k}}{(j+1)(j+2)} \rho^k \tau^{j+2}, \\ R_0(\tau, \rho) &= \sum_{k=0}^6 r_k^0 \rho^k + \sum_{j=0}^7 \sum_{k=0}^6 \frac{r_{j,k}}{(j+1)(j+2)} \rho^k \tau^{j+2} \end{aligned} \right\} \quad (4.10)$$

and

$$B_0(\tau, \rho) = \left(\sum_{k=0}^{10} b_k^0 \rho^k \right) (1 - c_7(\rho)\tau) + \sum_{j=0}^{10} \sum_{k=0}^{10} \frac{b_{j,k}}{(j+1)(j+2)} \rho^k \tau^{j+2}.$$

where coefficients of (4.10) are provided in tables 3 and 4. In a similar manner, the *quasi-fundamental matrix* Φ^0 listed in section 1 of the electronic supplementary material.

(a) Checking conditions of theorem 2.4

First, we determined C_N in (2.13) satisfies $C_N \leq 2.114$, computed from expression (4.8) for given values of parameter n_1, n_2, n_3, n_4 in table 1, when $U_0, U^0 + E \in \mathcal{D}^6$ chosen to be

$$\mathcal{D} := \left\{ U \in \mathbb{R}^4 : 0 < 1 + U_1 < \frac{1}{5}, U_4 > \frac{1}{3} \right\}.$$

Demonstrating $U^0 \in \mathcal{D}$ was based on extremizing a low-order polynomial approximation of U^0 in τ and ρ and noting small bounds for the remaining terms through Chebyshev representation. The 2ϵ bound quoted below on $\|E\|$ was more than sufficient to ensure $U^0(\tau) + E(\tau) \in \mathcal{D}$ for each $\tau \in [0, 1]$.

Using procedures described in the previous section, also we determined $\epsilon_R \leq 1.29 \times 10^{-11}$, $\epsilon_B \leq 5.35 \times 10^{-13}$, $\epsilon_\phi \leq 1.001 \times 10^{-6}$ and $\epsilon \leq 1.324 \times 10^{-6}$. We calculated the bounds M and M_G based in (3.22) and (3.16) as a function of ρ and using Chebyshev representation in ρ of exponents (see remark 3.23 for integration) able to conclude $M_G \leq 2.53$, and $M \leq 3.18$. Using *quasi-fundamental solution* Φ^0 given in section 1 of the electronic supplementary material, through a process described in the last section, we determined for any $\rho \in [-1, 1]$, $M_Q \leq 6178$, which gave rise to $\alpha = 0.529$, enough for contractivity condition in theorem 2.4, with quasi-solution having 1.94×10^{-6} accuracy. More details appear in the electronic supplementary material.

5. Basic reproduction number

The basic reproduction number $\mathcal{R}_0$ is an important threshold quantity for disease extinction and persistence. By [10], $\mathcal{R}_0$ of our model can be characterized as the principal eigenvalue of the

⁶ $\mathcal{D}$ is tailored to contain U^0 through verifiable bounds with enough margin to contain $U^0 + E$ for any E in theorem 2.4.

Figure 1. The basic reproduction number $\mathcal{R}_0$ as a function of the bacterial intrinsic growth rate g with $g = p_7$. Let Δp_7 denote the step size of the subsequent p_j . In our case, $\Delta p_7 = 1/100$ for $p_7 \in [0.01, 0.15]$, $\Delta p_7 = 1/200$ for $p_7 \in [0.15, 0.25]$ and $\Delta p_7 = 1/500$ for $p_7 \in [0.25, 0.3]$. (a) $0.01 \leq g \leq 0.3$. (b) $0.01 \leq g \leq 0.16$. (Online version in colour.)

following problem.

$$\left. \begin{aligned} -D_2\psi_I''(x) + (d + \gamma)\psi_I &= \lambda N(\beta_1\psi_I + \beta_2/K\psi_B), & 0 < x < L, \\ -D_4\psi_B''(x) + v\psi_B'(x) + \delta\psi_B &= \lambda(\xi\psi_I + g\psi_B), & 0 < x < L \\ \psi_I'(x) = 0, & x = 0, L, \\ (D_4\psi_B'(x) - v\psi_B)(x) &= 0, & x = 0, L, \end{aligned} \right\} \quad (5.1)$$

where $N = \Lambda/d$.

We rescale (5.1) by introducing non-dimensional parameters as follows: $p_1 = (d + \gamma)/(D_2/L^2)$; $p_2 = N\beta_1/(D_2/L^2)$; $p_3 = N\beta_2/(D_2K/L^2)$; $p_4 = v/(D_4/L^2)$; $p_5 = \delta/(D_4/L^2)$; $p_6 = \xi/(D_4/L^2)$; $p_7 = g/(D_4/L^2)$; $\tau = 1 - x/L$. Then equation (5.1) becomes

$$\left. \begin{aligned} \psi_I''(\tau) - p_1\psi_I(\tau) &= \lambda(-p_2\psi_I(\tau) - p_3\psi_B(\tau)), & 0 < \tau < 1, \\ \psi_B''(\tau) + p_4\psi_B'(\tau) - p_5\psi_B(\tau) &= \lambda(-p_6\psi_I(\tau) - p_7\psi_B(\tau)), & 0 < \tau < 1, \\ \psi_I'(\tau) = 0, & \tau = 0, 1, \\ \psi_B'(\tau) + p_4\psi_B(\tau) &= 0, & \tau = 0, 1. \end{aligned} \right\} \quad (5.2)$$

Theorem 5.1. For parameter $p_7 \in I_7 = [0.01, 0.3]$ and other parameter values listed in table 2, the following statements hold: (1) All eigenvalues $\lambda \in \mathbb{H}^+$, the right half complex plane. (2) The eigenvalue with the smallest real part $\lambda_0 \in \mathbb{R}^+$. (3) Furthermore, for $p_7 \in \{p_{7,j}\}_{j=1}^{60} \in I_7$, the corresponding basic reproduction number $\mathcal{R}_0 = 1/\lambda_0$ is within 10^{-8} of the computed values $\{\mathcal{R}_{0,j}\}_{j=1}^{60}$, where $\{(p_{7,j}, \mathcal{R}_{0,j})\}_{j=1}^{60}$ is listed at the end of section 5 of the electronic supplementary material (graphically represented in red stars in figure 1). Furthermore, there is a smooth curve $\mathcal{R}_0(p_7)$ that interpolates between these point values.

Remark 5.2. The proof of theorem 5.1 will have to await some preliminary results, and is given at the end of this section.

(a) *A priori* bounds on eigenvalues

In (5.2), we shall assume that $D = p_2p_7 - p_3p_6 > 0$, a necessary condition for the associated time evolution problem to be well-posed. This condition holds for parameter values listed in table 2.

Lemma 5.3. For $D = p_2p_7 - p_3p_6 > 0$, and for any choice of positive α, ϵ_j for $j = 1, \dots, 6$ that ensures the following conditions,

$$4 \left(p_7 - \frac{\epsilon_1}{2} \alpha p_3 p_4 \right) \left(p_2 - \frac{\epsilon_2}{2} p_2 p_4 \right) - 4 p_3 p_6 - \left(\alpha p_3 - \frac{p_6}{\alpha} \right)^2 > 0,$$

and then with any $\gamma > 0$ chosen small enough to assure

$$4 \left(p_7 - \frac{1}{2} [\epsilon_1 + \epsilon_6 \gamma] \alpha p_3 p_4 \right) \left(p_2 - \frac{1}{2} [\epsilon_2 + \epsilon_5 \gamma] p_2 p_4 \right) > \left(\alpha p_3 + \frac{p_6}{\alpha} + \gamma \left| \alpha p_3 - \frac{p_6}{\alpha} \right| \right)^2,$$

any eigenvalue $\lambda \in \mathbb{C}$ is restricted to the region:

$$D\Re\lambda \geq \min \left\{ p_2 p_5 - \frac{\alpha}{2\epsilon_1} p_3 p_4 - \frac{1}{2\epsilon_2} p_2 p_4 - \frac{\epsilon_3}{2} \left(\alpha p_5 p_3 + \frac{p_1 p_6}{\alpha} \right), p_7 p_1 - \frac{1}{2\epsilon_3} \left(\alpha p_3 p_5 + \frac{p_6 p_1}{\alpha} \right) \right\},$$

$$D\gamma |\Im\lambda| \leq D\Re\lambda + \mu, \quad (5.3)$$

where

$$\mu = \max \left\{ -p_2 p_5 + \frac{\alpha}{2\epsilon_1} p_3 p_4 + \frac{1}{2\epsilon_2} p_2 p_4 + \frac{\epsilon_3}{2} \left(\alpha p_5 p_3 + \frac{p_1 p_6}{\alpha} \right) + \frac{\gamma}{2\epsilon_6} \alpha p_3 p_4 + \frac{\gamma}{2\epsilon_5} p_2 p_4 \right. \\ \left. + \frac{\epsilon_7}{2} \gamma \left(\left| \alpha p_5 p_3 - \frac{p_6 p_1}{\alpha} \right| \right), -p_7 p_1 + \frac{1}{2\epsilon_3} \left(\alpha p_3 p_5 + \frac{p_6 p_1}{\alpha} \right) + \frac{1}{2\epsilon_7} \gamma \left(\left| \alpha p_5 p_3 - \frac{p_6 p_1}{\alpha} \right| \right) \right\}.$$

Proof. First, we derive from (5.2) an equivalent set of two equations where λ appears exclusively as $\lambda D\psi_I$ in one equation and $\lambda D\psi_B$ in the other. A routine application of ‘energy method’ on a rescaled variables $(\alpha\psi_I, \psi_B)$ results in the above inequalities. Details appear in section 3 of the electronic supplementary material. ■

(b) Direct calculation of eigenvalues using matrix exponential

With $y = (\psi_I, \psi_I', \psi_B, \psi_B')$, equation (5.2) can be written as a system of first-order ODE

$$y' = My, \quad \mathcal{B}y[0] = 0 = \mathcal{B}y[1], \quad (5.4)$$

where

$$M = M(\lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ p_1 - \lambda p_2 & 0 & -\lambda p_3 & 0 \\ 0 & 0 & 0 & 1 \\ -\lambda p_6 & 0 & p_5 - \lambda p_7 & -p_4 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & p_4 & 1 \end{pmatrix}.$$

Since the general solution to $y' = My$ is $y(\tau) = e^{\tau M} y_0$, by enforcing the boundary conditions in (5.4), it follows eigenvalue λ is determined by

$$0 = \det G = \det \begin{pmatrix} \mathcal{B} \\ \mathcal{B}e^M \end{pmatrix} = p_4^2 (E_{2,1}E_{3,4} - E_{2,4}E_{3,1}) \\ + p_4 (E_{2,1}E_{4,4} - E_{2,4}E_{4,1} + E_{2,3}E_{3,1} - E_{2,1}E_{3,3}) + E_{2,3}E_{4,1} - E_{2,1}E_{4,3}, \quad (5.5)$$

where $E_{j,k}$ are the (j,k) -th entry of e^M .

Lemma 5.4 (A priori bound on $\det G$). For any $\lambda_c \in \mathbb{C}$ and $r > 0$, if $|\lambda - \lambda_c| \leq r$ and $p_7 \in I := [p_{7,m}, p_{7,M}]$, we have the following a priori bound

$$|\det G| \leq p_4^2 (E_{2,1}^0 E_{3,4}^0 + E_{2,4}^0 E_{3,1}^0) \\ + p_4 (E_{2,1}^0 E_{4,4}^0 + E_{2,4}^0 E_{4,1}^0 + E_{2,3}^0 E_{3,1}^0 + E_{2,1}^0 E_{3,3}^0) + E_{2,3}^0 E_{4,1}^0 + E_{2,1}^0 E_{4,3}^0 =: D_{\lambda_c, r},$$

where E_{ij}^0 are the $(i, j)^{th}$ entry of the matrix e^{M_0} with

$$M_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ |p_1 - \lambda_c p_2| + r p_2 & 0 & p_3 |\lambda_c| + r p_3 & 0 \\ 0 & 0 & 0 & 1 \\ r p_6 + |\lambda_c| p_6 & 0 & M_{0,4,3} & p_4 \end{pmatrix} \quad \text{with} \quad M_{0,4,3} = \sup_{p_7 \in I_7} \sup_{|\lambda - \lambda_c| = r} |p_5 - \lambda p_7|.$$

Proof. Since the $\det G(\lambda)$ is an analytic function of λ , it is enough to check that the bounds hold on $|\lambda - \lambda_c| = r$, where on inspection. We notice that the (i, j) entry of M is bounded above by the (i, j) -th entry of matrix M_0 . Since matrix multiplication involves addition and multiplication operation, using triangular inequality, it follows that for all $k \geq 0$, $|[M^k]_{ij}| \leq [M_0^k]_{ij}$ and therefore,

$$|E_{ij}| \leq E_{ij}^0 := [e^{M_0}]_{ij}. \quad (5.6)$$

Using this bound in (5.5), lemma 5.4 follows. $\blacksquare$

Lemma 5.5 (A priori bound on $\frac{\partial}{\partial p_7} \det G$). For any $\lambda_c \in \mathbb{C}$ and $r > 0$, if $|\lambda - \lambda_c| \leq r$ and $p_7 \in I_7 := [p_{7,m}, p_{7,M}]$, we have the following a priori bound

$$\begin{aligned} \left| \frac{d}{dp_7} \det G \right| \leq \alpha_7 = & (|\lambda_c| + r) p_4^2 \left[(B_{2,1} E_{3,4}^0 + E_{2,1}^0 B_{3,4} + B_{2,4} E_{3,1}^0 + E_{2,4}^0 B_{3,1}) + p_4 (B_{2,1} E_{4,4}^0 \right. \\ & + E_{2,1}^0 B_{4,4} + B_{2,4} E_{4,1}^0 + E_{2,4}^0 B_{4,1} + B_{2,3} E_{3,1}^0 + E_{2,3}^0 B_{3,1} + B_{2,1} E_{3,3}^0 + E_{2,1}^0 B_{3,3}) \\ & \left. + B_{2,3} E_{4,1}^0 + E_{2,3}^0 B_{4,1} + B_{2,1} E_{4,3}^0 + E_{2,1}^0 B_{4,3} \right], \end{aligned} \quad (5.7)$$

where $B_{ij} = E_{i,4}^0 E_{3,j}^0$.

Proof. We first note that

$$\partial_{p_7} M = -\lambda S,$$

where $S = (S_{ij})_{4 \times 4}$ with $S_{ij} = 1$ for $(i, j) = (4, 3)$ and zero otherwise.

It follows that

$$\partial_{p_7} M^k = -\lambda \sum_{j=0}^{k-1} M^{k-j-1} S M^j, \quad (5.8)$$

and hence

$$\partial_{p_7} e^M = -\lambda \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{n=0}^{k-1} M^{k-n-1} S M^n = -\lambda \sum_{n=0}^{\infty} \left(\sum_{k=n+1}^{\infty} \frac{M^{k-n-1}}{k!} \right) S M^n. \quad (5.9)$$

Since $[M_0]_{ij} \geq |M_{ij}|$ and $\frac{n!l!}{(n+l)!} \leq 1$, it follows that

$$\left| \partial_{p_7} E_{ij} \right| \leq |\lambda| \left[\sum_{n=0}^{\infty} \left(\sum_{l=0}^{\infty} \frac{M_0^l}{l!} \right) S \frac{M_0^n}{n!} \right]_{ij} = \left[|\lambda| e^{M_0} S e^{M_0} \right]_{ij} =: |\lambda| B_{ij}, \quad (5.10)$$

where $B_{ij} = E_{i,4}^0 E_{3,j}^0$. Therefore, our assertion follows by using (5.10) in $\partial_{p_7} \det G$ obtained from (5.5). $\blacksquare$

Corollary 5.6. If $|\lambda - \lambda_c| \leq qr$ for $0 < q < 1$, and $p_7 \in I_7$, then for any integer $k \geq 0$,

$$\frac{1}{k!} \left| \frac{d^k}{d\lambda^k} \det G(\lambda) \right| \leq \frac{1}{(1-q)^k r^k} D_{\lambda_c, r}. \quad (5.11)$$

In particular

$$\frac{1}{k!} \left| \frac{d^k}{d\lambda^k} \det G(\lambda_c) \right| \leq \frac{1}{r^k} D_{\lambda_c, r}. \quad (5.12)$$

Proof. The first result follows from Cauchy integral formula for derivatives of analytic function $\det G$. Taking the limit of $q \rightarrow 0^+$ in (5.11), we get the second result. ■

Corollary 5.7. *If $|\lambda - \lambda_c| \leq qr$ for $0 < q < 1$, and $p_7 \in I_7$, then for any integer $k \geq 0$,*

$$\frac{1}{k!} \left| \frac{\partial^{k+1}}{\partial \lambda^k \partial p_7} \det G(\lambda) \right| \leq \frac{1}{r^k (1-q)^k} \alpha_7. \quad (5.13)$$

In particular

$$\frac{1}{k!} \left| \frac{\partial^k}{\partial \lambda^k \partial p_7} \det G(\lambda_c) \right| \leq \frac{1}{r^k} \alpha_7. \quad (5.14)$$

Proof. Recognizing the analyticity of $(\partial/\partial p_7) \det G(\lambda; p_7)$ in λ , in $|\lambda - \lambda_c| \leq r$ with bound α_7 in lemma 5.5 on the outer boundary, the proof parallels the one in corollary 5.6. ■

The following lemma gives error bounds on a polynomial approximation of $\det G(\lambda)$ based on the convergent Taylor expansion

$$\det G(\lambda) = \sum_{m=0}^{\infty} A_m (\lambda - \lambda_c)^m, \quad A_m = \frac{1}{m!} \frac{d^m}{d\lambda^m} \det G(\lambda_c). \quad (5.15)$$

The following result is obtained from direct calculation. The detailed proof for lemma 5.8 is provided in section 4 of the electronic supplementary material.

Lemma 5.8. *Let $N \in \mathbb{Z}^+$ and $0 < q < 1$. Define $z_j = \lambda_c + qre^{i2\pi j/N}$.*

$$\hat{A}_m := \frac{1}{N} \sum_{j=0}^{N-1} \frac{\det G(z_j)}{(z_j - \lambda_c)^m}.$$

Define polynomial

$$P_N(\lambda) = \sum_{m=0}^{N-1} \hat{A}_m (\lambda - \lambda_c)^m. \quad (5.16)$$

Then, for $|\lambda - \lambda_c| \leq qr$,

$$\begin{aligned} |\det G(\lambda) - P_N(\lambda)| &\leq D_{\lambda_c, r} \left\{ \frac{q^N}{(1-q)(1+q^N)} + \frac{q^N}{1-q} \right\}, \\ \left| \frac{d}{d\lambda} [\det G(\lambda) - P_N(\lambda)] \right| &\leq \frac{1}{r} D_{\lambda_c, r} \left\{ \frac{q^N}{(1-q^N)} \left(\sum_{m=1}^{N-1} m q^{m-1} \right) + \sum_{m=N}^{\infty} m q^{m-1} \right\}. \end{aligned}$$

Proof. Using (5.15) in (5.4), calculation shows that $\hat{A}_m - A_m = \sum_{l=1}^{\infty} A_{m+lN}$. The rest follows from using expressions for $\det G$ and p_N in (5.15) and (5.16) and using bounds on A_m available in (5.12). ■

Remark 5.9. If for fixed $p_7 \in I_7$ and some $q \in [0, 1)$ on $|\lambda - \lambda_c| = qr$, $|(d/d\lambda)P_N(\lambda)| \geq m_P > 0$, and the error bounds in lemma 5.8 show $|(d/d\lambda) \det G - (d/d\lambda)P_N| \geq (m_P/2)$, then triangular inequality implies $|(d/d\lambda) \det G| \geq (m_P/2)$.

Lemma 5.10. *If on the circle $|\lambda - \lambda_c| \leq qr$ for some $0 < q < 1$, at some $p_{7,0} \in I_7$,*

$$\left| \frac{\partial}{\partial \lambda} \det G(\lambda, p_{7,0}) \right| \geq 2m_D > 0.$$

Then, for $p_7 \in I_7$ satisfying $|p_7 - p_{7,0}| \leq \frac{m_D r(1-q)}{\alpha_7}$,

$$\left| \frac{\partial}{\partial \lambda} \det G(\lambda, p_7) \right| \geq m_D. \quad (5.17)$$

Proof. By using corollary 5.7, and applying condition, from the mean value inequality,

$$\left| \frac{\partial}{\partial \lambda} \det G(\lambda, p_7) - \frac{\partial}{\partial \lambda} \det G(\lambda, p_{7,0}) \right| \leq \frac{\alpha_7}{r(1-q)} |p_7 - p_{7,0}| \leq m_D. \quad (5.18)$$

The lemma readily follows from triangular inequality. ■

Proof of theorem 5.1. Here we outline the proof; details including values of bounds, choices of points and circles are provided in the electronic supplementary material. Through calculation of $\det G$ on the real axis, we identified smallest positive real eigenvalues $\{\lambda_{0,j}\}_{j=1}^I$ to within 10^{-8} accuracy for $p_7 \in \{p_{7,j}\}_{j=1}^I$, uniformly spaced out points in the interval $I = [1100, \frac{3}{10}]$, with other parameter values shown in table 2. The eigenvalues and their accuracy are justified through intermediate value theorem through evaluation of $\det G$, accounting also for any floating point errors in matrix exponential and determinants. The corresponding reproduction number $R_{0,j} = 1/\lambda_{0,j}$ at $p_{7,j}$ is identified as red points in figure 1. Define $I_j = [p_{7,j} - \Delta_L, p_{7,j} + \Delta_R]$ where $\Delta_L = \Delta_R = \Delta := \frac{1}{2}|p_{7,j} - p_{7,j\pm 1}|$, except for left- and right-most sub-interval where $\Delta_L = 0$ and $\Delta_R = 0$, respectively. Combining this result with *a priori* bounds in lemma 5.3 for $p_7 \in I$, we conclude that the eigenvalue with the smallest algebraic real part is necessarily restricted to

$$\mathcal{T} = \begin{cases} \left\{ \lambda \in \mathbb{C} : -1.065 \leq \Re \lambda \leq 1.1, |\Im \lambda| \leq \frac{\Re \lambda}{0.578} + 2.504 \right\}, & \text{for } p_7 \in \left[\frac{1}{100}, \frac{1}{10} \right], \\ \left\{ \lambda \in \mathbb{C} : 0.3 \leq \Re \lambda \leq 1, |\Im \lambda| \leq \frac{\Re \lambda}{0.578} + 0.062 \right\}, & \text{for } p_7 \in \left[\frac{1}{10}, \frac{3}{10} \right]. \end{cases} \quad (5.19)$$

We choose a sequence of circles $K_{\lambda_{c,m},r_0} = \{\lambda \in \mathbb{C} : |\lambda - \lambda_{c,m}| = r_m\}$ centred at $\{\lambda_{c,m}\}_{m=1}^M$ so that its union contains $\mathcal{T}$, with $\text{dist}(\lambda_{0,j}, K_{\lambda_{c,m},r_m}) \geq 0.1$. We also ensured that there is one and only one circle with centre $\lambda_{c,m} \in \mathbb{R}$ contained computed $\lambda_{0,j}$. For each I_j , *a priori* bounds on $\det G$ on $K_{\lambda_{c,m},3r_m}$ gives bounds on $(d/d\lambda) \det G$ on $K_{\lambda_{c,m},2r_0}$ from (5.11); together with evaluation of $\det G$ on uniformly spaced out points on this circle, we get bounds on $\det G$. We use this in lemma 5.8 to identify low-order (at most 10) polynomials $P_N(\lambda; p_{7,j})$ that approximates $\det G$ at $p_{7,j}$ on or inside $K_{\lambda_{c,m},r_0}$. Bounds on $\partial_{p_7} \det G$ on this circle for each I_j was also calculated from lemma 5.5. We formed polynomial $\tilde{P}_N = a \prod_{l=1}^{N-1} (\lambda - \lambda_{c,m} - y_l)$ by using numerical roots of P_N ; but expanding $\tilde{P}_N$ and finding l^1 bounds of all coefficients $P_N - \tilde{P}_N$ helped determine rigorous bounds on $\tilde{P}_N - \det G$ on $K_{\lambda_{c,m},r_0}$ for each $p_{7,j}$. For $\lambda \in K_{\lambda_{c,m},r_m}$, we checked $|\tilde{P}_N(\lambda; p_{7,j}) - \det G(\lambda, p_{7,j})| + \Delta |\partial_{p_7} \det G| < \inf_{\lambda \in K_{\lambda_{c,m},r_m}} |\tilde{P}_N(\lambda; p_{7,j})|$ to conclude from Rouché's theorem that $\det G(\lambda; p_7)$ and $\tilde{P}_N(\lambda; p_{7,j})$ have the same number of zeros for any $p_7 \in I_j$, where the roots and lower bounds of $\tilde{P}_N$ is manifest in the product form. The number of zeros was found to be exactly one when $\lambda_{0,j}$ was inside the particular $K_{\lambda_{c,m},r_0}$ with real $\lambda_{c,m}$, and zero otherwise. By repeating this process for each I_j , we concluded statements (1) and (2) of theorem 5.1. Since there is only one zero of $\det G(\lambda, p_7)$ when the circle $K_{\lambda_{c,m},r_m}$ centred on real axis contained $\lambda_{0,j}$ and none otherwise, it follows that this root is real. This is because complex roots of $\det G$ come in conjugate pairs, and this root is not repeated either as Rouché's theorem counts multiplicity. Therefore, $(d/d\lambda) \det G(\lambda, p_7) \neq 0$ at the root inside $K_{\lambda_{c,m},r_m}$ for any $p_7 \in I_j$ inside that circle. Implicit function theorem applies and the real root $\lambda = \lambda(p_7)$ is smooth for $p_7 \in I_j$, and this holds for any j . ■

Applying theorem 5.1 and taking the reciprocal of the obtained principal eigenvalue give us the corresponding basic reproduction number. The result of the basic reproduction number as a function of p_7 is shown in figure 1. Computational details are available in the electronic supplementary material and data file below.

Data accessibility. The main numerical codes used are available at <http://www.math.wsu.edu/faculty/xueying/publications/public/>.

Authors' contributions. S.T. and X.W. devised the project, and wrote the paper. A.C. conducted numerical work.

Competing interests. We declare we have no competing interests.

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Appendix A. Definition of model parameters

Please see tables 1 and 2.

Table 1. The base values of dimensionless parameters in model (4.3), with $c_7 \in [\frac{1}{20}, \frac{1}{2}]$.

c_1	c_2	c_3	c_4	c_5	c_6	n_1	n_2	n_3	n_4
$\frac{14}{31755}$	$\frac{609}{6200}$	$-\frac{2572534}{60858309}$	$\frac{217}{31755}$	$\frac{1}{15}$	$\frac{259287}{4000000}$	$\frac{407451}{4000000}$	$\frac{3}{40}$	$\frac{135817}{100000}$	$\frac{3}{20}$

Table 2. The base values of dimensionless parameters in (5.2), with $p_7 \in [\frac{1}{100}, \frac{3}{10}]$.

p_1	p_2	p_3	p_4	p_5	p_6
$\frac{44471}{31755}$	$\frac{27579}{20306}$	$\frac{1177}{1271024}$	$\frac{1}{10}$	$\frac{7}{30}$	7

Appendix B. Coefficients of analytical representation of the *Quasi-solution* (4.10)

$$\begin{aligned} \left\{ s_k^0 \right\}_{k=0}^6 &= \left[-\frac{1898047}{2329137'} - \frac{30841}{29156532'} - \frac{16072}{36496045'} - \frac{4149}{1212586487'} - \frac{1007}{1462729593'} \right. \\ &\quad \left. - \frac{32}{5969990527'} - \frac{25}{33025052188'} \right] \\ \left\{ t_k^0 \right\}_{k=0}^6 &= \left[\frac{287913}{5412755'} - \frac{40969}{151261861'} - \frac{11117}{404602566'} - \frac{145}{592337941'} - \frac{259}{7215368604'} \right. \\ &\quad \left. - \frac{37}{14158554642'} - \frac{4}{140243595661'} \right] \\ \left\{ r_k^0 \right\}_{k=0}^6 &= \left[\frac{601053}{11388691'} - \frac{6032}{85970461'} - \frac{541}{18964957'} - \frac{467}{2108455476'} - \frac{477}{10687996313'} - \frac{32}{95212284693'} \right. \\ &\quad \left. - \frac{18}{364327699753'} \right] \\ \left\{ b_k^0 \right\}_{k=0}^{10} &= \left[\frac{402245}{721054'} - \frac{298216}{5488743'} - \frac{43547}{73914987'} - \frac{56585}{288992416'} - \frac{1993}{359445976'} - \frac{139}{160452001'} - \frac{65}{1741506847'} \right. \\ &\quad \left. - \frac{39}{10459524989'} - \frac{11}{49705391730'} - \frac{4}{265859699649'} - \frac{1}{840315061408'} \right]. \end{aligned}$$

Please see tables 3 and 4.

Table 3. Definition of $\{[s_{j,k=0}^6]_{j=0}^7, [l_{j,k=0}^6]_{j=0}^7$ and $\{[r_{j,k=0}^6]_{j=0}^7$ from top to bottom.

j	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
0	$\frac{6480}{15055597}$	$\frac{2001}{5599192}$	$\frac{435}{101437472}$	$\frac{8}{254538401}$	$\frac{76}{5801458263}$	$\frac{11}{4468723411}$	$\frac{3}{82783725253}$
1	$\frac{21247}{24213805}$	$\frac{1663}{2225631}$	$\frac{1661}{71742008}$	$\frac{126}{25636231}$	$\frac{71}{549632889}$	$\frac{11}{5257507556}$	$\frac{12}{15376631831}$
2	$\frac{673}{66720834}$	$\frac{1765}{55336366}$	$\frac{2397}{152791822}$	$\frac{121}{41223297}$	$\frac{73}{1081239925}$	$\frac{30}{1924459519}$	$\frac{7}{4293364217}$
3	$\frac{1308}{58803917}$	$\frac{1373}{55729611}$	$\frac{1120}{122525273}$	$\frac{352}{100984759}$	$\frac{97}{420330720}$	$\frac{38}{1761067265}$	$\frac{11}{10231437285}$
4	$\frac{193}{39868023}$	$\frac{478}{106208325}$	$\frac{311}{248161066}$	$\frac{275}{335024797}$	$\frac{52}{502910865}$	$\frac{30}{1412787787}$	$\frac{7}{10585297781}$
5	$\frac{741}{301514930}$	$\frac{245}{108648507}$	$\frac{284}{1331376621}$	$\frac{28}{817130373}$	$\frac{81}{1320887039}$	$\frac{18}{787003223}$	$\frac{19}{7647574836}$
6	$\frac{157}{575130785}$	$\frac{159}{516265709}$	$\frac{104}{1237189983}$	$\frac{85}{26371155054}$	$\frac{31}{1997993660}$	$\frac{52}{6012585717}$	$\frac{5}{3130767477}$
7	$\frac{80}{1413286849}$	$\frac{59}{127867661}$	$\frac{7}{2389204477}$	$\frac{6}{5743492525}$	$\frac{5}{24246954817}$	$\frac{2}{4395534015}$	$\frac{4}{19528401267}$
0	$\frac{23203}{4499026}$	$\frac{15695}{3660593}$	$\frac{7791}{139853866}$	$\frac{38}{98334913}$	$\frac{50}{267528253}$	$\frac{124}{4184564263}$	$\frac{5}{8441949442}$
1	$\frac{13027}{1113451}$	$\frac{31476}{3159379}$	$\frac{4627}{14988725}$	$\frac{1867}{284870865}$	$\frac{235}{136441011}$	$\frac{34}{1218568913}$	$\frac{167}{16042987164}$
2	$\frac{18877}{5043773}$	$\frac{15431}{4505495}$	$\frac{13109}{52824000}$	$\frac{11101}{285629523}$	$\frac{361}{350171567}$	$\frac{55}{294157576}$	$\frac{8}{360023835}$
3	$\frac{20209}{8305229}$	$\frac{4447}{2227866}$	$\frac{2151}{43146626}$	$\frac{1805}{37606878}$	$\frac{425}{158798273}$	$\frac{217}{775241818}$	$\frac{41}{238393737}$

(Continued.)

Table 3. (Continued.)

j	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
4	$\frac{3270}{-8745511}$	$\frac{8693}{-25491760}$	$\frac{291}{-23747360}$	$\frac{1889}{122303421}$	$\frac{355}{280915597}$	$\frac{153}{-510775156}$	$\frac{63}{-13175203483}$
5	$\frac{4183}{30333908}$	$\frac{5593}{50922252}$	$\frac{1390}{-219708103}$	$\frac{830}{-293222439}$	$\frac{336}{543605957}$	$\frac{145}{464134737}$	$\frac{17}{581449539}$
6	$\frac{695}{-53267656}$	$\frac{1037}{-93853349}$	$\frac{311}{564860216}$	$\frac{97}{143529570}$	$\frac{38}{-287216463}$	$\frac{121}{-1059973818}$	$\frac{119}{-6377349999}$
7	$\frac{617}{196229366}$	$\frac{256}{108096283}$	$\frac{190}{-789649851}$	$\frac{50}{-986995157}$	$\frac{143}{11263869712}$	$\frac{43}{6609567596}$	$\frac{2}{982963883}$
0	$\frac{454}{-159980813}$	$\frac{1058}{-454013349}$	$\frac{136}{-18954923841}$	$\frac{7}{-43957310998}$	$\frac{2}{33510087965}$	$\frac{1}{64263302316}$	$\frac{1}{-1877939294039}$
1	$\frac{1}{730521044594}$	$\frac{1}{616437133028}$	$\frac{1}{8702598626226}$	$\frac{1}{-4519807359420}$	$\frac{1}{35882683912148}$	$\frac{1}{4125345520601}$	$\frac{1}{49568983705593}$
2	$\frac{910}{51669939}$	$\frac{6458}{441070215}$	$\frac{122}{641039253}$	$\frac{6}{4528939049}$	$\frac{34}{53285633781}$	$\frac{3}{-29441195530}$	$\frac{1}{-425240271688}$
3	$\frac{1395}{-104691983}$	$\frac{1141}{-100559516}$	$\frac{151}{-429502724}$	$\frac{30}{4041168263}$	$\frac{61}{-31196754109}$	$\frac{2}{52587224603}$	$\frac{1}{71544845170}$
4	$\frac{1623}{758361851}$	$\frac{305}{155816778}$	$\frac{182}{1287359053}$	$\frac{41}{-1866299411}$	$\frac{13}{23176292442}$	$\frac{6}{7487173779}$	$\frac{1}{-50766539429}$
5	$\frac{249}{-298904096}$	$\frac{195}{-285558983}$	$\frac{19}{111776856}$	$\frac{1}{62318184}$	$\frac{10}{10267286459}$	$\frac{2}{-54962003035}$	$\frac{1}{52683733253}$
6	$\frac{127}{1559208089}$	$\frac{724}{9786938087}$	$\frac{16}{5748335457}$	$\frac{33}{-10683132116}$	$\frac{8}{-21978973755}$	$\frac{1}{-176626857217}$	$\frac{1}{-7777274922}$
7	$\frac{35}{-1839047047}$	$\frac{125}{-8496084667}$	$\frac{7}{7115397586}$	$\frac{4}{22311021271}$	$\frac{1}{-20598383276}$	$\frac{1}{466722408362}$	$\frac{1}{294184814471}$

Table 4. Definition of $[[b_{j,k}]^5_{j=0}]^{10}_{k=0}$ and $[[b_{j,k}]^{10}_{k=0}]^{10}_{j=0}$ from top to bottom.

j	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
0	$\frac{34474}{714781}$	$\frac{116646}{1484105}$	$\frac{105458}{2986985}$	$\frac{28789}{10931742}$	$\frac{1795}{31928008}$	$\frac{2509}{280385992}$
1	$\frac{132730}{4623083}$	$\frac{262264}{5340407}$	$\frac{32973}{1068113}$	$\frac{31455}{3569546}$	$\frac{14345}{25632822}$	$\frac{1747}{108806108}$
2	$\frac{28951}{2874062}$	$\frac{52931}{2528249}$	$\frac{63471}{4042859}$	$\frac{38849}{6941959}$	$\frac{14070}{12866123}$	$\frac{3867}{68752358}$
3	$\frac{1069}{404704}$	$\frac{20755}{3188578}$	$\frac{22465}{3612359}$	$\frac{45131}{15748573}$	$\frac{9971}{14937808}$	$\frac{3745}{41715003}$
4	$\frac{14443}{24323355}$	$\frac{9341}{5604937}$	$\frac{4294}{2244163}$	$\frac{12875}{11235506}$	$\frac{6973}{18851845}$	$\frac{8210}{134879851}$
5	$\frac{3968}{31221985}$	$\frac{8287}{21191866}$	$\frac{7218}{14085809}$	$\frac{1144}{3111213}$	$\frac{2391}{15484849}$	$\frac{689}{18415117}$
6	$\frac{2422}{100088187}$	$\frac{2901}{34499270}$	$\frac{7313}{58150583}$	$\frac{2517}{24046903}$	$\frac{2012}{38124267}$	$\frac{494}{29846553}$
7	$\frac{1556}{344641613}$	$\frac{335}{19727713}$	$\frac{4901}{173112439}$	$\frac{1433}{53276668}$	$\frac{389}{24463210}$	$\frac{213}{35466491}$
8	$\frac{53}{69688504}$	$\frac{1558}{494999797}$	$\frac{656}{113885809}$	$\frac{673}{110879180}$	$\frac{450}{109699709}$	$\frac{461}{255145706}$
9	$\frac{181}{1632275646}$	$\frac{190}{391254741}$	$\frac{6191}{6519066043}$	$\frac{365}{33830182}$	$\frac{338}{419721451}$	$\frac{285}{720009629}$
10	$\frac{28}{2911419669}$	$\frac{43}{965216757}$	$\frac{87}{954073159}$	$\frac{91}{837401944}$	$\frac{72}{831681775}$	$\frac{39}{856118722}$

(Continued.)

Table 4. (Continued.)

j	$k = 6$	$k = 7$	$k = 8$	$k = 9$	$k = 10$
0	$\frac{660}{1649132761}$	$\frac{31}{826869197}$	$\frac{11}{4561516678}$	$\frac{6}{40203534031}$	$\frac{1}{77645125759}$
1	$\frac{457}{255700684}$	$\frac{87}{823020842}$	$\frac{29}{4099507521}$	$\frac{2}{3367289039}$	$\frac{1}{20826281786}$
2	$\frac{2063}{930739040}$	$\frac{44}{278640283}$	$\frac{43}{2884697509}$	$\frac{7}{10103649175}$	$\frac{7}{12671640160}$
3	$\frac{393}{119845775}$	$\frac{142}{707032585}$	$\frac{105}{23940913448}$	$\frac{47}{17260006895}$	$\frac{19}{4459858327}$
4	$\frac{1290}{240017611}$	$\frac{87}{1267101596}$	$\frac{25}{67403386}$	$\frac{83}{10450506542}$	$\frac{41}{2018944114}$
5	$\frac{373}{83972651}$	$\frac{113}{588590866}$	$\frac{29}{213792624}$	$\frac{13}{602049401}$	$\frac{74}{1285272475}$
6	$\frac{463}{160749895}$	$\frac{511}{2672073064}$	$\frac{427}{1696687751}$	$\frac{39}{1012201445}$	$\frac{44}{437743297}$
7	$\frac{41}{34705764}$	$\frac{29}{287739839}$	$\frac{37}{133130481}$	$\frac{33}{767068937}$	$\frac{43}{397614913}$
8	$\frac{40}{116381447}$	$\frac{38}{1952075069}$	$\frac{121}{677704859}$	$\frac{13}{454890973}$	$\frac{116}{1686786955}$
9	$\frac{67}{987385112}$	$\frac{5}{35907566712}$	$\frac{89}{1471564260}$	$\frac{51}{5045041289}$	$\frac{55}{2384001572}$
10	$\frac{9}{118710618}$	$\frac{5}{25271372316}$	$\frac{14}{1742233887}$	$\frac{6}{4227709909}$	$\frac{13}{4364459194}$

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