Research

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Beacon-referenced pursuit for collective motions in three dimensions

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Motivated by real-world applications of unmanned aerial vehicles, this paper introduces a decentralized control mechanism to guide steering control of autonomous agents manoeuvring in the vicinity of multiple moving entities (e.g. other autonomous agents) and stationary entities (e.g. fixed beacons or points of reference) in a three-dimensional environment. The proposed control law, which can be perceived as a modification of the three-dimensional constant bearing (CB) pursuit law, provides a means to allocate simultaneous attention to multiple entities. We investigate the behaviour of the closed-loop dynamics for a system with one agent referencing two beacons, as well as a two-agent *mutual pursuit* system wherein each agent employs the beacon-referenced CB pursuit law with regards to the other agent and a stationary beacon. Under certain assumptions on the associated control parameters, we demonstrate that this problem admits *circling equilibria* with agents moving on circular orbits with a common radius, in planes perpendicular to a common axis passing through the beacons. As the common radius and distances from the beacon are determined by the choice of parameters in the pursuit law, this approach provides a means to engineer desired formations in a three-dimensional setting.

1. Introduction

As pursuit and collective motion play significant roles in various contexts of robotics and engineering, it seems appealing to seek inspiration from nature, which abounds with many such examples. Among the various

possible ways to pursue and intercept a moving target, evidence of a constant bearing (CB) pursuit strategy can be observed in a variety of animal species, such as flies [1,2], dogs [3], raptors [4] and humans [5]. While pursuing a target using CB pursuit strategy, the pursuer (i.e. the following agent) moves in such a way that the angle between its own velocity and the line-of-sight to the target (i.e. the baseline) remains constant. By prescribing a fixed offset between the baseline and the pursuer's velocity, this strategy provides a generalization of the classical pursuit strategy (wherein the pursuer's velocity is always aligned with the line-of-sight). While pursuit strategies are often employed in contexts with a single pursuer–target pair, previous work [6–8] has demonstrated that the CB pursuit strategy can be employed as a building block for designing coordinated manoeuvres in a collective of agents by implementing the strategy in a cyclic manner (i.e. agent i pursues agent $i + 1$, with the last agent pursuing the first). In the planar setting, Galloway *et al.* [9,10] demonstrated the existence and stability of a rich class of behaviours such as circular motion, rectilinear motion, shape-preserving spirals and periodic orbits. In [11,12], the same authors defined a CB pursuit strategy and associated control law for the three-dimensional setting and determined conditions for the existence of a similar class of motions. While this line of research has demonstrated the existence of circling equilibria in which agents moved on a common circular trajectory, both the location of the circumcentre of the formation (with respect to some inertial frame) and its size were determined by initial conditions rather than control parameters. To overcome this aspect and to broaden the scope from a design perspective, the work in [13–15] introduced a modified version of the CB pursuit law (in the planar setting), wherein the pursuer pays attention to its neighbour as well as to a stationary beacon. The beacon can represent an attractive food source in a biological setting, or some target of interest for an unmanned vehicle. (See also [16,17] for a related, but different, control formulation.)

In the current work, we extend the beacon-referenced approach to a three-dimensional setting and explore the possible equilibrium formations for two agents in beacon-referenced mutual pursuit. More specifically, we first state a modified version of the beacon-referenced three-dimensional pursuit law introduced in [18], which is based only on relative bearing measurements to two targets (typically a fixed beacon and a moving neighbour agent). This modified version of the control law is easier to implement since it does not require any estimate of optic flow or relative velocity, and it is more tractable for mathematical analysis. We then consider the corresponding dynamics for a mutual pursuit system in which two agents apply this pursuit law with respect to each other and to a (possibly distinct) fixed beacon. Earlier work [19–21] has demonstrated that mutual pursuit can lead to a variety of interesting motion patterns, while providing better tractability from a nonlinear analysis perspective, and it can be viewed as a building block towards understanding the more general cyclic pursuit framework. The main contribution of this work is to then derive existence conditions (and a mathematical characterization) for circling equilibria under the various possible combinations of one or two mobile agents interacting with one or two stationary beacons.

This paper is organized as follows. We begin by describing the self-steering particle model for agents moving in three dimensions. Then, in the latter part of §2, we introduce the three-dimensional beacon-referenced CB pursuit law. In the spirit of beginning with the configuration of least complexity, in §3 we consider the case in which a single mobile agent employs the beacon-referenced control law with respect to two fixed beacons, and we characterize the existence of circling equilibria. Before proceeding to configurations involving two mobile agents, in §4 we describe a reduction to an *effective shape space* which can be parameterized by corresponding scalar shape variables, and we state the closed-loop mutual pursuit dynamics in terms of these shape variables. Section 5 addresses the case where two agents reference the same beacon, which is similar in spirit to the presentation in [18] but incorporates the modified control law. Finally, in §6, we consider the most general case in which two agents employ the beacon-referenced pursuit law with regards to two distinct beacons. In the course of our analysis, we uncover the existence of a variety of coordinated three-dimensional circling motions (as depicted by the figures shown in §6) for which pertinent attributes such as circling radius, angular separation of the agents and vertical separation of the agents are all prescribed by control parameters rather

than initial conditions. Since these control laws require only bearing measurements relative to the agent's forward velocity vector, we hypothesize that they could provide effective methods for station-keeping in a variety of contexts such as space exploration.

2. Modelling agent dynamics and control

(a) Generative model: agents as self-steering particles

Similar to earlier works [11,22], we treat the agents as unit-mass self-steering particles moving along twice-differentiable paths in a three-dimensional environment. This allows us to describe the agents' motion, i.e. $\mathbf{r}_i \in \mathbb{R}^3$ (with respect to an inertial reference frame), in terms of its natural Frenet frame [23] defined by an orthonormal triad of vectors $[\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i]$. This formulation, in contrast to alternative modelling approaches based on curvature and torsion, does not preclude us from considering linear segments. In addition, by constraining the agents to move at a common but non-vanishing speed, we can assume, without loss of generality, that the agents are moving at unit speed, and subsequently describe the dynamics of a pair of agents as

$$\text{and} \quad \left. \begin{aligned} \dot{\mathbf{r}}_i &= \mathbf{x}_i \\ \dot{\mathbf{x}}_i &= u_i \mathbf{y}_i + v_i \mathbf{z}_i, \quad \dot{\mathbf{y}}_i = -u_i \mathbf{x}_i, \quad \dot{\mathbf{z}}_i = -v_i \mathbf{x}_i, \quad i = 1, 2, \end{aligned} \right\} \quad (2.1)$$

where the natural curvatures u_i and v_i can be perceived as gyroscopic steering controls influencing the agents' directions of motion. We also define two (possibly collocated) fixed beacons $\mathbf{r}_{b1}$ and $\mathbf{r}_{b2}$, which can be assumed (without loss of generality) to be located at $\mathbf{r}_{b1} = (0, 0, -b)$ and $\mathbf{r}_{b2} = (0, 0, b)$ for some $b \geq 0$. The vector from beacon 1 to beacon 2 will be denoted as $\hat{\mathbf{b}} = \mathbf{r}_{b2} - \mathbf{r}_{b1} = (0, 0, 2b)$. We also denote the vectors $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, $\mathbf{r}_{1b1} = \mathbf{r}_1 - \mathbf{r}_{b1}$ and $\mathbf{r}_{2b2} = \mathbf{r}_2 - \mathbf{r}_{b2}$ to express the relative configuration of the agents and the beacons, noting the resulting constraint

$$\hat{\mathbf{b}} + \mathbf{r}_{2b2} + \mathbf{r} - \mathbf{r}_{1b1} = \mathbf{0}. \quad (2.2)$$

These vector relationships are depicted in figure 1. (Note that only the $\mathbf{x}_i$ vectors are depicted rather than the entire frame.)

(b) Beacon-referenced constant bearing pursuit in three dimensions

Previous work in [11,12] introduced and analysed a control law for executing the CB pursuit strategy in three dimensions. Making use of the notation $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ and letting $a_i \in [-1, 1]$ be selected as the CB pursuit parameter for agent i , we say that agent i has attained the $\text{CB}(a_i)$ pursuit strategy with respect to agent j if $\mathbf{x}_i \cdot (\mathbf{r}_{ij}/|\mathbf{r}_{ij}|) = a_i$. The $\text{CB}(a_i)$ pursuit strategy can be thought of as prescribing a desired angle between the pursuer's velocity vector and the bearing vector to the desired target.¹ Then distance from this pursuit state can be described by the CB cost function $\Delta_i = \frac{1}{2}[(\mathbf{x}_i \cdot (\mathbf{r}_{ij}/|\mathbf{r}_{ij}|)) - a_i]^2 \in [0, 2]$, and a feedback law introduced in [11,12] was shown to enforce the property that $\dot{\Delta}_i \leq 0$ with $\dot{\Delta}_i = 0$ if and only if $\Delta_i = 0$ (i.e. the control law renders the CB pursuit state attractive and positively invariant). The control law from [11,12] is made up of one term referencing the bearing to the pursuee, and one term related to the relative velocity between the pursuer and pursuee. If we leave off the latter term, the resulting simplified CB pursuit law can be expressed as

$$u_i = -\mu \left[\left(\mathbf{x}_i \cdot \frac{\mathbf{r}_{ij}}{|\mathbf{r}_{ij}|} \right) - a_i \right] \left(\mathbf{y}_i \cdot \frac{\mathbf{r}_{ij}}{|\mathbf{r}_{ij}|} \right) \quad \text{and} \quad v_i = -\mu \left[\left(\mathbf{x}_i \cdot \frac{\mathbf{r}_{ij}}{|\mathbf{r}_{ij}|} \right) - a_i \right] \left(\mathbf{z}_i \cdot \frac{\mathbf{r}_{ij}}{|\mathbf{r}_{ij}|} \right), \quad (2.3)$$

where $\mu > 0$ is a control gain. Though this simplified version lacks the invariance property of the original CB pursuit law, it does satisfy $u_i = v_i = 0$ on the CB pursuit state and is particularly suited to the modification we introduce next.

¹We note that the bearing vector $\mathbf{r}_{ij}$ actually points *towards* the pursuer; therefore, if a_i represents the desired CB pursuit angle, the corresponding $\text{CB}(a_i)$ parameter is given by $a_i = -\cos(\alpha_i)$.

Figure 1. Two mobile agents $\mathbf{r}_1$ and $\mathbf{r}_2$ manoeuvring in the vicinity of fixed beacons $\mathbf{r}_{b1}$ and $\mathbf{r}_{b2}$. (Online version in colour.)

The simplified CB pursuit law given by (2.3) serves as our building block for developing a beacon-referenced CB pursuit law which uses only bearing measurements and references two targets rather than only one. For most of this work (except² §3), we will consider the case where one target is a fixed beacon and the second target is the other moving agent. Then, by letting $\lambda \in [0, 1]$ and $a_{bi} \in [-1, 1]$ represent the attention split between the targets and the ‘CB-like’ control parameter used by the i -th agent to reference beacon i , respectively, we can express the beacon-referenced control law as

$$u_i = -\mu(1 - \lambda) \left[\left(\mathbf{x}_i \cdot \frac{\mathbf{r}_{ij}}{|\mathbf{r}_{ij}|} \right) - a_i \right] \left(\mathbf{y}_i \cdot \frac{\mathbf{r}_{ij}}{|\mathbf{r}_{ij}|} \right) - \mu\lambda \left[\left(\mathbf{x}_i \cdot \frac{\mathbf{r}_{ibi}}{|\mathbf{r}_{ibi}|} \right) - a_{bi} \right] \left(\mathbf{y}_i \cdot \frac{\mathbf{r}_{ibi}}{|\mathbf{r}_{ibi}|} \right) \quad (2.4)$$

and

$$v_i = -\mu(1 - \lambda) \left[\left(\mathbf{x}_i \cdot \frac{\mathbf{r}_{ij}}{|\mathbf{r}_{ij}|} \right) - a_i \right] \left(\mathbf{z}_i \cdot \frac{\mathbf{r}_{ij}}{|\mathbf{r}_{ij}|} \right) - \mu\lambda \left[\left(\mathbf{x}_i \cdot \frac{\mathbf{r}_{ibi}}{|\mathbf{r}_{ibi}|} \right) - a_{bi} \right] \left(\mathbf{z}_i \cdot \frac{\mathbf{r}_{ibi}}{|\mathbf{r}_{ibi}|} \right).$$

Note that the first term in both of these equations corresponds to the i -th agent’s attention towards the j -th agent, whereas the second term captures its attention to beacon i . However, in general, the neighbour-tracking goal may conflict with the beacon-referencing goal, i.e. there are no guarantees that $(\mathbf{x}_i \cdot (\mathbf{r}_{ij}/|\mathbf{r}_{ij}|)) = a_i$ and $(\mathbf{x}_i \cdot (\mathbf{r}_{ibi}/|\mathbf{r}_{ibi}|)) = a_{bi}$. Also, since $\lambda = 0$ and $\lambda = 1$ correspond to extreme cases of either exclusive tracking of the other agent or exclusive tracking of the beacon, we will assume $\lambda \in (0, 1)$ in this work. It is worth noting that this control law (2.4) can easily be adapted³ to consider any arbitrary number of targets, with attentional weights that add up to 1.

Remark 2.1. Our previous work in [18] introduced a beacon-referenced CB law similar to (2.4) which used the full version of the CB pursuit law from [11,12] with an added beacon-referencing term. In this work, we have opted for the version given in (2.4) for the following reasons. First, the version given in (2.4) is simpler for an autonomous agent to implement in practice because it requires only bearing measurements. In particular, no range measurements or velocity estimations are required. Second, the version given in (2.4) leads to a more tractable mathematical analysis. Last, simulations suggest that steady-state behaviours under (2.4) and the control law given in [18] are qualitatively similar.

In what follows, we will consider the various combinations presented by systems of one or two mobile agents employing the beacon-referenced control law (2.4) with respect to one or two fixed beacons. For each case, we will analyse the closed-loop dynamics to determine

²Our analysis in §3 involves a single mobile agent which references two fixed beacons during its manoeuvre.

³Implementation of such an extended scheme would be limited only by sensor availability and computational power. An analysis of this generalized ‘target-referenced’ control law will be the subject of future research.

existence conditions and steady-state characterization of relative equilibria⁴ (which correspond to circling motions). Our approach is to start with the lowest level of complexity in terms of the dimensionality of the system dynamics, and then progress to increasing levels of complexity. More specifically, we will consider the following three configurations (in order):

- one mobile agent with two fixed beacons (§3);
- two mobile agents with one fixed beacon (§5);
- two mobile agents with two distinct fixed beacons (§6).

3. Configuration I: one agent with two fixed beacons

We start by considering the case of a single agent employing the beacon-referenced control law (2.4) with respect to two fixed beacons. To this end, we let $a_{bi} \in [-1, 1]$ represent the control parameter with reference to beacon i (for $i = 1, 2$), and denote⁵

$$\bar{x}_{1bi} \triangleq \mathbf{x}_1 \cdot \frac{\mathbf{r}_{1bi}}{|\mathbf{r}_{1bi}|}, \quad \bar{y}_{1bi} \triangleq \mathbf{y}_1 \cdot \frac{\mathbf{r}_{1bi}}{|\mathbf{r}_{1bi}|} \quad \text{and} \quad \bar{z}_{1bi} \triangleq \mathbf{z}_1 \cdot \frac{\mathbf{r}_{1bi}}{|\mathbf{r}_{1bi}|}, \quad i = 1, 2, \quad (3.1)$$

so that we can express a two-beacon version of (2.4) as

$$\left. \begin{aligned} u_1 &= -\mu(1 - \lambda)(\bar{x}_{1b2} - a_{b2})\bar{y}_{1b2} - \mu\lambda(\bar{x}_{1b1} - a_{b1})\bar{y}_{1b1} \\ v_1 &= -\mu(1 - \lambda)(\bar{x}_{1b2} - a_{b2})\bar{z}_{1b2} - \mu\lambda(\bar{x}_{1b1} - a_{b1})\bar{z}_{1b1}. \end{aligned} \right\} \quad (3.2)$$

and

Since $\mathbf{x}_1$, $\mathbf{y}_1$ and $\mathbf{z}_1$ make up an orthonormal frame, we have the vector decomposition $\mathbf{r}_{1bi}/|\mathbf{r}_{1bi}| = \bar{x}_{1bi}\mathbf{x}_1 + \bar{y}_{1bi}\mathbf{y}_1 + \bar{z}_{1bi}\mathbf{z}_1$, $i = 1, 2$, and therefore substituting (3.2) into the first two equations from (2.1) yields

$$\left. \begin{aligned} \dot{\mathbf{r}}_1 &= \mathbf{x}_1, \\ \text{and} \quad \dot{\mathbf{x}}_1 &= u_1\mathbf{y}_1 + v_1\mathbf{z}_1 \\ &= -(1 - \lambda)\mu(\bar{x}_{1b2} - a_{b2})(\bar{y}_{1b2}\mathbf{y}_1 + \bar{z}_{1b2}\mathbf{z}_1) - \lambda\mu(\bar{x}_{1b1} - a_{b1})(\bar{y}_{1b1}\mathbf{y}_1 + \bar{z}_{1b1}\mathbf{z}_1) \\ &= -(1 - \lambda)\mu(\bar{x}_{1b2} - a_{b2})\left(\frac{\mathbf{r}_{1b2}}{|\mathbf{r}_{1b2}|} - \bar{x}_{1b2}\mathbf{x}_1\right) - \lambda\mu(\bar{x}_{1b1} - a_{b1})\left(\frac{\mathbf{r}_{1b1}}{|\mathbf{r}_{1b1}|} - \bar{x}_{1b1}\mathbf{x}_1\right). \end{aligned} \right\} \quad (3.3)$$

Note that (3.3) is a self-contained set of dynamics and the complete $[\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1]$ frame can be reconstructed from the evolution of this reduced dynamics [22]. If we restrict our analysis away from states corresponding to collocation of the agent with the beacons, we can denote

$$\rho_{1b1} \triangleq |\mathbf{r}_{1b1}| > 0 \quad \text{and} \quad \rho_{1b2} \triangleq |\mathbf{r}_{1b2}| > 0. \quad (3.4)$$

This constraint is required to ensure that the control law (3.2) is well posed, and can generally be achieved by reasonable choices for initial conditions away from the collision state. Then, as is demonstrated in the electronic supplementary material, the corresponding shape dynamics (i.e.

⁴The model described at the beginning of §2 requires the agents to move at a fixed non-zero speed; therefore the corresponding dynamics do not admit any equilibria corresponding to stationary formations. However, the model does permit relative equilibria, for which the *relative* configurations of agents and beacons remains constant.

⁵See table 1 for a summary of definitions and physical interpretations of the shape variables used in this work.

the relative configuration of the agent with respect to the beacons) can be expressed as

$$\left. \begin{aligned} \dot{\tilde{x}}_{1b1} &= -(1-\lambda)\mu(\tilde{x}_{1b2} - a_{b2}) \left(\frac{\mathbf{r}_{1b1}}{|\mathbf{r}_{1b1}|} \cdot \frac{\mathbf{r}_{1b2}}{|\mathbf{r}_{1b2}|} - \tilde{x}_{1b1}\tilde{x}_{1b2} \right) \\ &\quad - \left(\lambda\mu(\tilde{x}_{1b1} - a_{b1}) - \frac{1}{\rho_{1b1}} \right) (1 - \tilde{x}_{1b1}^2), \\ \dot{\tilde{x}}_{1b2} &= -\lambda\mu(\tilde{x}_{1b1} - a_{b1}) \left(\frac{\mathbf{r}_{1b1}}{|\mathbf{r}_{1b1}|} \cdot \frac{\mathbf{r}_{1b2}}{|\mathbf{r}_{1b2}|} - \tilde{x}_{1b1}\tilde{x}_{1b2} \right) \\ &\quad - \left((1-\lambda)\mu(\tilde{x}_{1b2} - a_{b2}) - \frac{1}{\rho_{1b2}} \right) (1 - \tilde{x}_{1b2}^2), \\ \dot{\rho}_{1b1} &= \tilde{x}_{1b1} \\ \text{and} \quad \dot{\rho}_{1b2} &= \tilde{x}_{1b2}, \end{aligned} \right\} \quad (3.5)$$

where (2.2) can be used to derive the law of cosines relationship

$$\frac{\mathbf{r}_{1b1}}{|\mathbf{r}_{1b1}|} \cdot \frac{\mathbf{r}_{1b2}}{|\mathbf{r}_{1b2}|} = \frac{\rho_{1b1}^2 + \rho_{1b2}^2 - (2b)^2}{2\rho_{1b1}\rho_{1b2}}. \quad (3.6)$$

Since the dot product of unit vectors must take values in $[-1, 1]$, and the extreme values are not an option in this case because they correspond to configurations in which the agent and both beacons are collinear,⁶ (3.6) implies that

$$\rho_{1b1} + \rho_{1b2} > 2b \quad \text{and} \quad |\rho_{1b1} - \rho_{1b2}| < 2b. \quad (3.7)$$

Our goal is to determine the existence conditions and steady-state characterization for equilibria of the shape dynamics (3.5); therefore, we set the shape dynamics to zero to obtain the necessary conditions $\tilde{x}_{1b1} = 0 = \tilde{x}_{1b2}$ and

$$\left. \begin{aligned} (1-\lambda)\mu a_{b2} \left(\frac{\rho_{1b1}^2 + \rho_{1b2}^2 - (2b)^2}{2\rho_{1b1}\rho_{1b2}} \right) + \lambda\mu a_{b1} + \frac{1}{\rho_{1b1}} &= 0 \\ \text{and} \quad \lambda\mu a_{b1} \left(\frac{\rho_{1b1}^2 + \rho_{1b2}^2 - (2b)^2}{2\rho_{1b1}\rho_{1b2}} \right) + (1-\lambda)\mu a_{b2} + \frac{1}{\rho_{1b2}} &= 0. \end{aligned} \right\} \quad (3.8)$$

Noting that the parameters show up together in the same patterns, we will denote

$$\tilde{a}_{b1} \triangleq \lambda\mu a_{b1} \in \mathbb{R} \quad \text{and} \quad \tilde{a}_{b2} \triangleq (1-\lambda)\mu a_{b2} \in \mathbb{R}. \quad (3.9)$$

Then setting each equation in (3.8) over a common denominator, we have

$$\left. \begin{aligned} \tilde{a}_{b2} (\rho_{1b1}^2 + \rho_{1b2}^2 - 4b^2) + 2\tilde{a}_{b1}\rho_{1b1}\rho_{1b2} + 2\rho_{1b2} &= 0 \\ \text{and} \quad \tilde{a}_{b1} (\rho_{1b1}^2 + \rho_{1b2}^2 - 4b^2) + 2\tilde{a}_{b2}\rho_{1b1}\rho_{1b2} + 2\rho_{1b1} &= 0. \end{aligned} \right\} \quad (3.10)$$

Before proceeding, we **consider a special case for which $\tilde{a}_{b1} = \tilde{a}_{b2} \neq 0$** , i.e. $\lambda a_{b1} = (1-\lambda)a_{b2} \neq 0$. Substituting into (3.10) and taking the difference of the two equations yields $\rho_{1b1} = \rho_{1b2}$. Substituting this constraint back into the first equation in (3.10) yields

$$4\tilde{a}_{b1} \left[\rho_{1b1}^2 + \left(\frac{1}{2\tilde{a}_{b1}} \right) \rho_{1b1} - b^2 \right] = 0. \quad (3.11)$$

From our constraint (3.7), we have $\rho_{1b1} + \rho_{1b2} > 2b$, which in this case requires $\rho_{1b1} > b$; therefore, it follows from (3.11) that we must have $\tilde{a}_{b1} < 0$. Thus we can summarize this case by stating that

⁶Collinear configurations correspond to $\mathbf{r}_{1b1}/|\mathbf{r}_{1b1}| \cdot \mathbf{r}_{1b2}/|\mathbf{r}_{1b2}| = \pm 1$. Substitution of this condition into (3.5) leads to a violation of the positivity constraints in (3.4), i.e. these collinear configurations are not invariant under the dynamics (3.5).

if $\tilde{a}_{b1} = \tilde{a}_{b2} < 0$, then an equilibrium exists with

$$\rho_{1b1} = \rho_{1b2} = \left(\frac{1}{2}\right) \left(-\frac{1}{2\tilde{a}_{b1}} - \sqrt{\frac{1}{4\tilde{a}_{b1}^2} + 4}\right) = \left(\frac{1}{-4\tilde{a}_{b1}}\right) \left(1 + \sqrt{1 + (4\tilde{a}_{b1}b)^2}\right). \quad (3.12)$$

If $\tilde{a}_{b1} \neq \tilde{a}_{b2}$, then the two equations in (3.10) represent the intersection of two conic sections. While simple substitution leads to a fourth-order equation and difficult mathematical analysis, one can use methods such as conic pencils to determine the intersection points. However, in the present case, a straightforward change of variables greatly simplifies our analysis. In particular, if we let $\rho_{1b+} = \rho_{1b2} + \rho_{1b1}$ and $\rho_{1b-} = \rho_{1b2} - \rho_{1b1}$, i.e. $\rho_{1b1} = (\rho_{1b+} - \rho_{1b-})/2$ and $\rho_{1b2} = (\rho_{1b+} + \rho_{1b-})/2$, then we can express (3.10) (after some algebraic manipulation) as

$$\left. \begin{aligned} &\tilde{a}_{b2} \left(\frac{\rho_{1b+}^2 + \rho_{1b-}^2}{2}\right) + \tilde{a}_{b1} \left(\frac{\rho_{1b+}^2 - \rho_{1b-}^2}{2}\right) + (\rho_{1b+} + \rho_{1b-}) - 4\tilde{a}_{b2}b^2 = 0 \\ \text{and} \quad &\tilde{a}_{b1} \left(\frac{\rho_{1b+}^2 + \rho_{1b-}^2}{2}\right) + \tilde{a}_{b2} \left(\frac{\rho_{1b+}^2 - \rho_{1b-}^2}{2}\right) + (\rho_{1b+} - \rho_{1b-}) - 4\tilde{a}_{b1}b^2 = 0. \end{aligned} \right\} \quad (3.13)$$

Then taking the sum and difference of the two equations in (3.13), we have

$$\left. \begin{aligned} &(\tilde{a}_{b2} + \tilde{a}_{b1}) \rho_{1b+}^2 + 2\rho_{1b+} - 4(\tilde{a}_{b2} + \tilde{a}_{b1})b^2 = 0 \\ \text{and} \quad &(\tilde{a}_{b2} - \tilde{a}_{b1}) \rho_{1b-}^2 + 2\rho_{1b-} - 4(\tilde{a}_{b2} - \tilde{a}_{b1})b^2 = 0. \end{aligned} \right\} \quad (3.14)$$

We note that if $\tilde{a}_{b2} + \tilde{a}_{b1} = 0$ then the first equation would require $\rho_{1b+} = 0$, which is not possible; thus, we assume $\tilde{a}_{b2} + \tilde{a}_{b1} \neq 0$. Since we have already assumed $\tilde{a}_{b2} - \tilde{a}_{b1} \neq 0$, we can express (3.14) as

$$\rho_{1b+}^2 + \left(\frac{2}{\tilde{a}_{b2} + \tilde{a}_{b1}}\right) \rho_{1b+} - (2b)^2 = 0 \quad \text{and} \quad \rho_{1b-}^2 + \left(\frac{2}{\tilde{a}_{b2} - \tilde{a}_{b1}}\right) \rho_{1b-} - (2b)^2 = 0. \quad (3.15)$$

Since the first constraint in (3.7) requires $\rho_{1b+} > 2b > 0$, i.e. $\rho_{1b+}^2 > (2b)^2$, it follows from the first equation in (3.15) that we must have $\tilde{a}_{b2} + \tilde{a}_{b1} < 0$ with the equilibrium value

$$\rho_{1b+} = \left(\frac{1}{-(\tilde{a}_{b2} + \tilde{a}_{b1})}\right) \left(1 + \sqrt{1 + (2b(\tilde{a}_{b2} + \tilde{a}_{b1}))^2}\right). \quad (3.16)$$

Since the second constraint in (3.7) requires $|\rho_{1b-}| < 2b$, i.e. $\rho_{1b-}^2 < (2b)^2$, it follows from the second equation in (3.15) that we must have $(\tilde{a}_{b2} - \tilde{a}_{b1})\rho_{1b-} > 0$. Solving for the roots of the second equation in (3.15) yields

$$\rho_{1b-} = \left(\frac{1}{\tilde{a}_{b2} - \tilde{a}_{b1}}\right) \left(-1 \pm \sqrt{1 + (2b(\tilde{a}_{b2} - \tilde{a}_{b1}))^2}\right), \quad (3.17)$$

and application of the constraint $(\tilde{a}_{b2} - \tilde{a}_{b1})\rho_{1b-} > 0$ requires that the second term in the (3.17) product is positive. The following proposition summarizes these configuration I results.

Proposition 3.1. *Consider a system in which a single agent employs the beacon-referenced CB pursuit law (2.4) with respect to two fixed beacons, according to the shape dynamics (3.5) parameterized by μ , λ , and the CB parameters a_{b1} and a_{b2} . Then a circling equilibrium exists if and only if one of the following cases holds. (Note that, in each case, at equilibrium we have $\tilde{x}_{1b1} = 0 = \tilde{x}_{1b2}$.)*

(a) If $\lambda a_{b1} = (1 - \lambda)a_{b2} < 0$, then a circling equilibrium exists with

$$\rho_{1b1} = \rho_{1b2} = \left(\frac{1}{-4\mu\lambda a_{b1}}\right) \left(1 + \sqrt{1 + (4\mu\lambda a_{b1}b)^2}\right). \quad (3.18)$$

Figure 2. Circling equilibria for configuration I (i.e. one agent referencing two fixed beacons) with beacon positioning (a) $b = 10$ and control parameters $a_0 = -0.156$, $a = -0.187$ and $\lambda = 0.45$ and (b) $a_0 = -0.156$, $a = -0.156$ and $\lambda = 0.5$. (Online version in colour.)

(b) If $\lambda a_{b1} + (1 - \lambda)a_{b2} < 0$, then a circling equilibrium exists with

$$\rho_{1b1} = \frac{(\rho_{1b+} - \rho_{1b-})}{2} \quad \text{and} \quad \rho_{1b2} = \frac{(\rho_{1b+} + \rho_{1b-})}{2}, \quad (3.19)$$

where

$$\rho_{1b+} = \frac{1 + \sqrt{1 + [2b\mu(\lambda a_{b1} + (1 - \lambda)a_{b2})]^2}}{-\mu(\lambda a_{b1} + (1 - \lambda)a_{b2})}$$

and

$$\rho_{1b-} = \frac{-1 + \sqrt{1 + [2b\mu(\lambda a_{b1} - (1 - \lambda)a_{b2})]^2}}{-\mu(\lambda a_{b1} - (1 - \lambda)a_{b2})}.$$

Proof. Follows from the discussion above. ■

Remark 3.2. The position of the agent with respect to the two beacons forms a triangle, geometry reveals that the vertical positioning of the agent (i.e. the third component of $\mathbf{r}_1$, denoted as r_{13}) is given by $r_{13} = (\rho_{1b+}\rho_{1b-})/(4b) + b$ and the radius of the circle is given by $\text{radius} = \sqrt{\rho_{1b2}^2 - r_{13}^2}$. Representative trajectories are shown in figure 2. Note also that if one of the CB parameters is zero while the other is negative (e.g. if $a_{b2} = 0$ and $a_{b1} < 0$), then the conditions of case (b) are satisfied, and simplifying (3.19) yields $\rho_{1b1} = 1/\lambda\mu(-a_{b1})$ with $\rho_{1b2} = \sqrt{\rho_{1b1}^2 + (2b)^2}$.

4. Shape variable representation for the two-agent configurations

In the remainder of this work, every configuration will involve at least one beacon and two mobile agents (engaged in a mutual pursuit). Moreover, to simplify the analysis, we introduce the following assumptions:⁷

- (A1) The bearing offset parameters with respect to the beacon are common for both agents, i.e. $a_{b1} = a_{b2} = a_0$.
- (A2) The bearing offset parameters with respect to the other agent are the same for both agents, i.e. $a_1 = a_2 = a$.

⁷For future work, we intend to relax some of these assumptions to explore the broader space of possible system behaviours.

Then by substituting (2.4) into (2.1), we can express the closed-loop dynamics as

$$\left. \begin{aligned} \dot{\mathbf{r}}_1 &= \mathbf{x}_1, \\ \dot{\mathbf{r}}_2 &= \mathbf{x}_2, \\ \dot{\mathbf{x}}_1 &= -(1-\lambda)\mu(\bar{\mathbf{x}}_1 - a) \left(\frac{\mathbf{r}}{|\mathbf{r}|} - \bar{\mathbf{x}}_1 \mathbf{x}_1 \right) - \lambda\mu(\bar{\mathbf{x}}_{1b1} - a_0) \left(\frac{\mathbf{r}_{1b1}}{|\mathbf{r}_{1b1}|} - \bar{\mathbf{x}}_{1b1} \mathbf{x}_1 \right) \\ \text{and} \quad \dot{\mathbf{x}}_2 &= -(1-\lambda)\mu(\bar{\mathbf{x}}_2 - a) \left(-\frac{\mathbf{r}}{|\mathbf{r}|} - \bar{\mathbf{x}}_2 \mathbf{x}_2 \right) - \lambda\mu(\bar{\mathbf{x}}_{2b2} - a_0) \left(\frac{\mathbf{r}_{2b2}}{|\mathbf{r}_{2b2}|} - \bar{\mathbf{x}}_{2b2} \mathbf{x}_2 \right), \end{aligned} \right\} \quad (4.1)$$

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and

$$\bar{\mathbf{x}}_1 = \mathbf{x}_1 \cdot \frac{\mathbf{r}_{12}}{|\mathbf{r}_{12}|} = \mathbf{x}_1 \cdot \frac{\mathbf{r}}{|\mathbf{r}|} \quad \text{and} \quad \bar{\mathbf{x}}_2 = \mathbf{x}_2 \cdot \frac{\mathbf{r}_{21}}{|\mathbf{r}_{21}|} = \mathbf{x}_2 \cdot \left(-\frac{\mathbf{r}}{|\mathbf{r}|} \right). \quad (4.2)$$

This closed-loop dynamics evolves on a 10-dimensional *reduced space* (similar to [18]). As the complete frame $[\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i]$ can be reconstructed from the evolution of this reduced dynamics, we restrict our attention to (4.1) in the remainder of this work. As a step towards developing a scalar parameterization of the shape dynamics (i.e. the dynamics of the relative configuration of the agents and beacons), we also define

$$\tilde{\mathbf{x}} = \mathbf{x}_1 \cdot \mathbf{x}_2 \quad \text{and} \quad \rho = |\mathbf{r}| > 0. \quad (4.3)$$

This allows us to express the dynamics of these variables as:⁸

$$\left. \begin{aligned} \dot{\tilde{\mathbf{x}}} &= -(1-\lambda)\mu(\bar{\mathbf{x}}_1 - a) \left(1 - \tilde{\mathbf{x}}_1^2 \right) - \lambda\mu(\bar{\mathbf{x}}_{1b1} - a_0) \left(\frac{\mathbf{r}_{1b1}}{|\mathbf{r}_{1b1}|} \cdot \frac{\mathbf{r}}{|\mathbf{r}|} - \bar{\mathbf{x}}_{1b1} \tilde{\mathbf{x}}_1 \right) \\ &\quad + \frac{1}{\rho} \left(1 - \tilde{\mathbf{x}} - \tilde{\mathbf{x}}_1^2 - \bar{\mathbf{x}}_1 \tilde{\mathbf{x}}_2 \right), \\ \dot{\tilde{\mathbf{x}}}_2 &= -(1-\lambda)\mu(\bar{\mathbf{x}}_2 - a) \left(1 - \tilde{\mathbf{x}}_2^2 \right) - \lambda\mu(\bar{\mathbf{x}}_{2b2} - a_0) \left(-\frac{\mathbf{r}_{2b2}}{|\mathbf{r}_{2b2}|} \cdot \frac{\mathbf{r}}{|\mathbf{r}|} - \bar{\mathbf{x}}_{2b2} \tilde{\mathbf{x}}_2 \right) \\ &\quad + \frac{1}{\rho} \left(1 - \tilde{\mathbf{x}} - \tilde{\mathbf{x}}_2^2 - \bar{\mathbf{x}}_1 \tilde{\mathbf{x}}_2 \right), \\ \dot{\tilde{\mathbf{x}}}_{1b1} &= -(1-\lambda)(\mu(\bar{\mathbf{x}}_1 - a)) \left(\frac{\mathbf{r}_{1b1}}{|\mathbf{r}_{1b1}|} \cdot \frac{\mathbf{r}}{|\mathbf{r}|} - \bar{\mathbf{x}}_{1b1} \tilde{\mathbf{x}}_1 \right) \\ &\quad - \left(\lambda\mu(\bar{\mathbf{x}}_{1b1} - a_0) - \frac{1}{\rho_{1b1}} \right) \left(1 - \tilde{\mathbf{x}}_{1b1}^2 \right), \\ \dot{\tilde{\mathbf{x}}}_{2b2} &= -(1-\lambda)(\mu(\bar{\mathbf{x}}_2 - a)) \left(-\frac{\mathbf{r}_{2b2}}{|\mathbf{r}_{2b2}|} \cdot \frac{\mathbf{r}}{|\mathbf{r}|} - \bar{\mathbf{x}}_{2b2} \tilde{\mathbf{x}}_2 \right) \\ &\quad - \left(\lambda\mu(\bar{\mathbf{x}}_{2b2} - a_0) - \frac{1}{\rho_{2b2}} \right) \left(1 - \tilde{\mathbf{x}}_{2b2}^2 \right), \\ \dot{\tilde{\mathbf{x}}} &= (1-\lambda)\mu(\bar{\mathbf{x}}_1 - a) (\bar{\mathbf{x}}_2 + \tilde{\mathbf{x}} \bar{\mathbf{x}}_1) + (1-\lambda)\mu(\bar{\mathbf{x}}_2 - a) (\bar{\mathbf{x}}_1 + \tilde{\mathbf{x}} \bar{\mathbf{x}}_2) \\ &\quad - \lambda\mu(\bar{\mathbf{x}}_{1b1} - a_0) \left(\mathbf{x}_2 \cdot \frac{\mathbf{r}_{1b1}}{|\mathbf{r}_{1b1}|} - \bar{\mathbf{x}}_{1b1} \tilde{\mathbf{x}} \right) - \lambda\mu(\bar{\mathbf{x}}_{2b2} - a_0) \left(\mathbf{x}_1 \cdot \frac{\mathbf{r}_{2b2}}{|\mathbf{r}_{2b2}|} - \bar{\mathbf{x}}_{2b2} \tilde{\mathbf{x}} \right), \\ \dot{\rho} &= \bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2, \\ \dot{\rho}_{1b1} &= \bar{\mathbf{x}}_{1b1} \\ \text{and} \quad \dot{\rho}_{2b2} &= \bar{\mathbf{x}}_{2b2}, \end{aligned} \right\} \quad (4.4)$$

where we note that the explicit dot product terms still need to be represented in terms of the shape variables to make this completely self-contained. We can derive expressions for these dot product

⁸Please refer to the electronic supplementary material for a detailed derivation of (4.4).

terms by making use of our vector closure constraint (2.2). For example, (2.2) implies that

$$\mathbf{x}_2 \cdot \mathbf{r}_{1b1} = \mathbf{x}_2 \cdot (\hat{\mathbf{b}} + \mathbf{r}_{2b2} + \mathbf{r}) = \mathbf{x}_2 \cdot \hat{\mathbf{b}} + \rho_{2b2} \bar{x}_{2b2} - \rho \bar{x}_2, \quad (4.5)$$

from which it follows that

$$\mathbf{x}_2 \cdot \frac{\mathbf{r}_{1b1}}{|\mathbf{r}_{1b1}|} = \frac{1}{\rho_{1b1}} (\mathbf{x}_2 \cdot \hat{\mathbf{b}} + \rho_{2b2} \bar{x}_{2b2} - \rho \bar{x}_2) \in [-1, 1]. \quad (4.6)$$

Similar calculations lead to

$$\mathbf{x}_1 \cdot \frac{\mathbf{r}_{2b2}}{|\mathbf{r}_{2b2}|} = \frac{1}{\rho_{2b2}} (-\mathbf{x}_1 \cdot \hat{\mathbf{b}} + \rho_{1b1} \bar{x}_{1b1} - \rho \bar{x}_1) \in [-1, 1], \quad (4.7)$$

$$\frac{\mathbf{r}_{1b1}}{|\mathbf{r}_{1b1}|} \cdot \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{1}{2\rho\rho_{1b1}} (\rho_{1b1}^2 + \rho^2 - \rho_{2b2}^2 - 2(\mathbf{r}_2 \cdot \hat{\mathbf{b}})) \in [-1, 1] \quad (4.8)$$

and

$$\frac{\mathbf{r}_{2b2}}{|\mathbf{r}_{2b2}|} \cdot \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{1}{2\rho\rho_{2b2}} (\rho_{1b1}^2 - \rho^2 - \rho_{2b2}^2 - 2(\mathbf{r}_1 \cdot \hat{\mathbf{b}})) \in [-1, 1]. \quad (4.9)$$

From these calculations, it is clear that we can make the shape dynamics (4.4) self-contained if we augment them with the variables

$$\hat{r}_1 = \mathbf{r}_1 \cdot \hat{\mathbf{b}}, \quad \hat{r}_2 = \mathbf{r}_2 \cdot \hat{\mathbf{b}}, \quad \hat{x}_1 = \mathbf{x}_1 \cdot \hat{\mathbf{b}} \quad \text{and} \quad \hat{x}_2 = \mathbf{x}_2 \cdot \hat{\mathbf{b}}, \quad (4.10)$$

with dynamics given by

$$\left. \begin{aligned} \dot{\hat{r}}_1 &= \hat{x}_1, \quad \dot{\hat{r}}_2 = \hat{x}_2, \\ \dot{\hat{x}}_1 &= -(1-\lambda)\mu(\bar{x}_1 - a) \left(\frac{\mathbf{r}}{|\mathbf{r}|} \cdot \hat{\mathbf{b}} - \bar{x}_1 \mathbf{x}_1 \cdot \hat{\mathbf{b}} \right) - \lambda\mu(\bar{x}_{1b1} - a_0) \left(\frac{\mathbf{r}_{1b1}}{|\mathbf{r}_{1b1}|} \cdot \hat{\mathbf{b}} - \bar{x}_{1b1} \mathbf{x}_1 \cdot \hat{\mathbf{b}} \right) \\ &= -\frac{(1-\lambda)\mu(\bar{x}_1 - a)}{\rho} (\mathbf{r} \cdot \hat{\mathbf{b}}) - \frac{\lambda\mu(\bar{x}_{1b1} - a_0)}{\rho_{1b1}} (\mathbf{r}_{1b1} \cdot \hat{\mathbf{b}}) \\ &\quad + \mu \left((1-\lambda)\bar{x}_1(\bar{x}_1 - a) + \lambda\bar{x}_{1b1}(\bar{x}_{1b1} - a_0) \right) (\mathbf{x}_1 \cdot \hat{\mathbf{b}}) \\ &= -\mu \left(\frac{(1-\lambda)(\bar{x}_1 - a)}{\rho} + \frac{\lambda(\bar{x}_{1b1} - a_0)}{\rho_{1b1}} \right) \hat{r}_1 + \left(\frac{(1-\lambda)\mu(\bar{x}_1 - a)}{\rho} \right) \hat{r}_2 \\ &\quad - \frac{2\lambda\mu(\bar{x}_{1b1} - a_0)b^2}{\rho_{1b1}} + \mu \left((1-\lambda)\bar{x}_1(\bar{x}_1 - a) + \lambda\bar{x}_{1b1}(\bar{x}_{1b1} - a_0) \right) \hat{x}_1, \\ \dot{\hat{x}}_2 &= -(1-\lambda)\mu(\bar{x}_2 - a) \left(-\frac{\mathbf{r}}{|\mathbf{r}|} \cdot \hat{\mathbf{b}} - \bar{x}_2 \mathbf{x}_2 \cdot \hat{\mathbf{b}} \right) - \lambda\mu(\bar{x}_{2b2} - a_0) \left(\frac{\mathbf{r}_{2b2}}{|\mathbf{r}_{2b2}|} \cdot \hat{\mathbf{b}} - \bar{x}_{2b2} \mathbf{x}_2 \cdot \hat{\mathbf{b}} \right) \\ &= -\frac{(1-\lambda)\mu(\bar{x}_2 - a)}{\rho} (-\mathbf{r} \cdot \hat{\mathbf{b}}) - \frac{\lambda\mu(\bar{x}_{2b2} - a_0)}{\rho_{2b2}} (\mathbf{r}_{2b2} \cdot \hat{\mathbf{b}}) \\ &\quad + \mu \left((1-\lambda)\bar{x}_2(\bar{x}_2 - a) + \lambda\bar{x}_{2b2}(\bar{x}_{2b2} - a_0) \right) (\mathbf{x}_2 \cdot \hat{\mathbf{b}}) \\ &= -\mu \left(\frac{(1-\lambda)(\bar{x}_2 - a)}{\rho} + \frac{\lambda(\bar{x}_{2b2} - a_0)}{\rho_{2b2}} \right) \hat{r}_2 + \left(\frac{(1-\lambda)\mu(\bar{x}_2 - a)}{\rho} \right) \hat{r}_1 \\ &\quad + \frac{2\lambda\mu(\bar{x}_{2b2} - a_0)b^2}{\rho_{2b2}} + \mu \left((1-\lambda)\bar{x}_2(\bar{x}_2 - a) + \lambda\bar{x}_{2b2}(\bar{x}_{2b2} - a_0) \right) \hat{x}_2. \end{aligned} \right\} \quad (4.11)$$

Table 1. Definitions of shape variables (from the perspective of agent 1).

variable	definition	physical interpretation
$\bar{x}_1$	$\mathbf{x}_1 \cdot \frac{\mathbf{r}}{ \mathbf{r} }$	direction cosine between agent 1 heading and bearing towards agent 2
$\bar{x}_{1bi}$	$\mathbf{x}_1 \cdot \frac{\mathbf{r}_{1bi}}{ \mathbf{r}_{1bi} }$	direction cosine between agent 1 heading and bearing towards beacon i
$\tilde{x}$	$\mathbf{x}_1 \cdot \mathbf{x}_2$	dot product of agent 1 heading and agent 2 heading
ρ	$ \mathbf{r} $	range between agents
ρ_{1bi}	$ \mathbf{r}_{1bi} $	range from agent 1 to beacon i
$\hat{r}_1$	$\mathbf{r}_1 \cdot \hat{\mathbf{b}}$	dot product of agent 1 position with beacon axis
$\hat{x}_1$	$\mathbf{x}_1 \cdot \hat{\mathbf{b}}$	dot product of agent 1 heading with beacon axis

To assist the reader in following the meaning of these many variables, we have provided table 1 to summarize the definitions and physical interpretation of each variable. In the next sections, we will characterize equilibria for the shape dynamics (4.4) and (4.11) and determine conditions under which these equilibria exist.

5. Configuration II: two agents referencing a single beacon

We first consider the case in which there is a single beacon (at the origin) referenced by two agents in mutual pursuit.⁹ In this case, $\hat{\mathbf{b}}$ is the zero vector, i.e. $\hat{r}_1 = 0 = \hat{r}_2$ and $\hat{x}_1 = 0 = \hat{x}_2$; therefore, our shape dynamics simplify to (4.4) with constraints given by (4.6), (4.7), (4.8) and (4.9) (with $\hat{\mathbf{b}} = 0$). Then setting $\dot{\rho} = 0$ and $\dot{\rho}_{1b1} = 0 = \dot{\rho}_{2b2}$ requires $\bar{x}_2 = -\bar{x}_1$ and $\bar{x}_{1b} = 0 = \bar{x}_{2b}$, and substituting these constraints into (4.4) yields

$$\left. \begin{aligned} \dot{\bar{x}}_1 &= -(1-\lambda)\mu(\bar{x}_1 - a) \left(1 - \bar{x}_1^2\right) + \lambda\mu a_0 \left(\frac{\rho_{1b1}^2 + \rho^2 - \rho_{2b2}^2}{2\rho\rho_{1b1}}\right) + \frac{1}{\rho}(1 - \tilde{x}), \\ \dot{\bar{x}}_2 &= -(1-\lambda)\mu(-\bar{x}_1 - a) \left(1 - \bar{x}_1^2\right) + \lambda\mu a_0 \left(\frac{-\rho_{1b1}^2 + \rho^2 + \rho_{2b2}^2}{2\rho\rho_{2b2}}\right) + \frac{1}{\rho}(1 - \tilde{x}), \\ \dot{\bar{x}}_{1b1} &= -(1-\lambda)\mu(\bar{x}_1 - a) \left(\frac{\rho_{1b1}^2 + \rho^2 - \rho_{2b2}^2}{2\rho\rho_{1b1}}\right) + \frac{1}{\rho_{1b1}} + \lambda\mu a_0, \\ \dot{\bar{x}}_{2b2} &= -(1-\lambda)\mu(-\bar{x}_1 - a) \left(\frac{-\rho_{1b1}^2 + \rho^2 + \rho_{2b2}^2}{2\rho\rho_{2b2}}\right) + \frac{1}{\rho_{2b2}} + \lambda\mu a_0 \end{aligned} \right\} \quad (5.1)$$

and

$$\dot{\tilde{x}} = \mu\bar{x}_1 \left(\lambda a_0 \rho \left(\frac{\rho_{2b2} - \rho_{1b1}}{\rho_{1b1}\rho_{2b2}} \right) - 2(1-\lambda)\bar{x}_1(1 - \tilde{x}) \right).$$

(a) Special case for configuration II: $a_0 = 0$

If $a_0 = 0$, then the first two equations in (5.1) can both be zero if and only if $\bar{x}_1 = \pm 1$ or $\bar{x}_1 = 0$. Since the latter option is the only one that results in $\dot{\tilde{x}} = 0$, we conclude that $\bar{x}_1 = 0$ at equilibrium. From this, we can state the following proposition.

Proposition 5.1. *Consider a beacon-referenced mutual CB pursuit system with shape dynamics (5.1) parameterized by μ , λ , and the CB parameters a , a_0 , with $a_0 = 0$. Then, a circling equilibrium exists if and*

⁹This is the case that was considered exclusively in [18], but with a different version of the beacon-referenced control law. As such, many of the results in this section are similar in spirit to those presented in [18].

only if $a < 0$, and the corresponding equilibrium values satisfy

$$\text{and} \quad \left. \begin{aligned} & \bar{x}_1 = \bar{x}_2 = 0, \quad \bar{x}_{1b1} = \bar{x}_{2b2} = 0, \quad \bar{x} = -1 \\ & \rho = \frac{2}{(1-\lambda)\mu(-a)}, \quad \rho_{1b1} = \rho_{2b2}. \end{aligned} \right\} \quad (5.2)$$

Proof. Substituting $a_0 = 0$ and $\bar{x}_1 = 0$ into (5.1) and setting to zero leads to

$$(1-\lambda)\mu a + \frac{1}{\rho}(1-\bar{x}) = 0, \quad (5.3a)$$

$$(1-\lambda)\mu a \left(\frac{\rho_{1b1}^2 + \rho^2 - \rho_{2b2}^2}{2\rho\rho_{1b1}} \right) = \frac{-1}{\rho_{1b1}} \quad (5.3b)$$

$$\text{and} \quad (1-\lambda)\mu a \left(\frac{\rho_{2b2}^2 + \rho^2 - \rho_{1b1}^2}{2\rho\rho_{2b2}} \right) = \frac{-1}{\rho_{2b2}}. \quad (5.3c)$$

Now, if $\bar{x} = 1$, the first condition (5.3a) holds true if and only if $a = 0$. But with these choices for $\bar{x}$ and a , the last two conditions (5.3b) and (5.3c) lead to $1/\rho_{1b} = 1/\rho_{2b} = 0$, which cannot be true since both ρ_{1b} and ρ_{2b} are finite. Therefore, we must have $\bar{x} \neq 1$ at an equilibrium, and then the first condition (5.3a) yields the equilibrium value of ρ as

$$\rho = \frac{(1-\bar{x})}{(1-\lambda)\mu(-a)}. \quad (5.4)$$

As ρ must be positive and finite, (5.4) yields a meaningful solution if and only if $a < 0$. Substituting this solution for ρ into (5.3b) and (5.3c), we have

$$\frac{1}{2\rho_{1b1}} \left[-(1-\bar{x}) \left(\frac{\rho_{1b1}^2 + \rho^2 - \rho_{2b2}^2}{\rho^2} \right) + 2 \right] = 0 \quad (5.5a)$$

and

$$\frac{1}{2\rho_{2b2}} \left[-(1-\bar{x}) \left(\frac{-\rho_{1b1}^2 + \rho^2 + \rho_{2b2}^2}{\rho^2} \right) + 2 \right] = 0, \quad (5.5b)$$

which holds true if and only if $\rho_{1b1} = \rho_{2b2}$ and $\bar{x} = -1$. ■

Figure 3 illustrates the type of circling equilibrium which is described in proposition 5.1. Note that the distance between individual agents and the beacon is determined by initial conditions, but the inter-agent separation (i.e. ρ) is determined by the control parameters according to (5.2).

(b) General case for configuration II: $a_0 \neq 0$

We now shift our attention to the case where $a_0 \neq 0$, and show that circling equilibria exist in this scenario as well. Note that the following proposition provides sufficient (but not necessary) conditions for the existence of equilibria, and therefore should not be regarded as an exhaustive categorization of equilibria for configuration II. We also note that some choices of control parameters will satisfy the conditions for both cases (a) and (b) in proposition 5.2. In this situation, initial conditions determine which (if any) of the equilibria the agents will converge to.

Proposition 5.2. *Consider a beacon-referenced mutual CB pursuit system with shape dynamics (5.1) parameterized by μ , λ , and the CB parameters a , a_0 , with $a_0 \neq 0$. The following statements provide sufficient conditions for the existence of circling equilibria.*

Figure 3. Circling equilibria for configuration II with a single beacon located at the origin. (a) A representative of proposition 5.1, with control parameters $\lambda = 0.5$, $a = -0.7071$, $a_0 = 0$. (b) A representative of proposition 5.2 case (b), with control parameters $\lambda = 0.5$, $a = 0.3090$, $a_0 = -0.7071$, that appears to have a very small region of attraction. (Online version in colour.)

- (a) Whenever $(1 - \lambda)a + \lambda a_0 < 0$, a circling equilibrium exists with agents on opposite sides of the circling orbit (i.e. $\tilde{x} = -1$), and corresponding equilibrium values are given by

$$\left. \begin{aligned} & \bullet \quad \bar{x}_1 = \bar{x}_2 = 0, \quad \bar{x}_{1b1} = \bar{x}_{2b2} = 0, \quad \tilde{x} = -1, \\ & \bullet \quad \rho_{1b1} = \rho_{2b2} = \frac{1}{-\mu((1 - \lambda)a + \lambda a_0)} \\ & \bullet \quad \rho = 2\rho_{1b1} = \frac{2}{-\mu((1 - \lambda)a + \lambda a_0)}. \end{aligned} \right\} \quad (5.6)$$

and

- (b) In addition, if the condition from (a) holds **and** $a_0 < 0$ with $a > 0$, a second type of circling equilibrium exists for which the agents orbit on circling trajectories in parallel planes (as in figure 3b). The corresponding equilibrium values are given by

$$\left. \begin{aligned} & \bullet \quad \bar{x}_1 = \bar{x}_2 = 0, \quad \bar{x}_{1b1} = \bar{x}_{2b2} = 0, \quad \tilde{x} = 1, \\ & \bullet \quad \rho_{1b1} = \rho_{2b2} = \frac{\lambda a_0}{\mu((1 - \lambda)^2 a^2 - \lambda^2 a_0^2)} \\ & \bullet \quad \rho = \frac{-2(1 - \lambda)a}{\mu((1 - \lambda)^2 a^2 - \lambda^2 a_0^2)}. \end{aligned} \right\} \quad (5.7)$$

and

Proof. See the electronic supplementary material. ■

6. Configuration III: two agents referencing two distinct beacons

We now consider the most general case, for which two agents in mutual pursuit each reference a different beacon. For this case, our shape dynamics are given by (4.4) and (4.11) with constraints given by (4.6), (4.7), (4.8) and (4.9). As in previous cases, setting $\dot{\rho} = 0$ and $\dot{\rho}_{1b1} = 0 = \dot{\rho}_{2b2}$ requires $\bar{x}_2 = -\bar{x}_1$ and $\bar{x}_{1b} = 0 = \bar{x}_{2b}$. If we also set $\hat{r}_1 = 0 = \hat{r}_2$, i.e. $\hat{x}_1 = 0 = \hat{x}_2$, then our shape dynamics

(4.4)–(4.11) along these nullclines are given by

$$\left. \begin{aligned} \dot{\bar{x}}_1 &= -(1-\lambda)\mu(\bar{x}_1 - a) \left(1 - \bar{x}_1^2\right) + \lambda\mu a_0 \left(\frac{\rho_{1b1}^2 + \rho^2 - \rho_{2b2}^2 - 2\hat{r}_2}{2\rho\rho_{1b1}}\right) + \frac{1}{\rho}(1 - \tilde{x}), \\ \dot{\bar{x}}_2 &= -(1-\lambda)\mu(-\bar{x}_1 - a) \left(1 - \bar{x}_1^2\right) + \lambda\mu a_0 \left(\frac{-\rho_{1b1}^2 + \rho^2 + \rho_{2b2}^2 + 2\hat{r}_1}{2\rho\rho_{2b2}}\right) + \frac{1}{\rho}(1 - \tilde{x}), \\ \dot{x}_{1b1} &= -(1-\lambda)\left(\mu(\bar{x}_1 - a)\right) \left(\frac{\rho_{1b1}^2 + \rho^2 - \rho_{2b2}^2 - 2\hat{r}_2}{2\rho\rho_{1b1}}\right) + \frac{1}{\rho_{1b1}} + \lambda\mu a_0, \\ \dot{x}_{2b2} &= -(1-\lambda)\left(\mu(-\bar{x}_1 - a)\right) \left(\frac{-\rho_{1b1}^2 + \rho^2 + \rho_{2b2}^2 + 2\hat{r}_1}{2\rho\rho_{2b2}}\right) + \frac{1}{\rho_{2b2}} + \lambda\mu a_0, \\ \dot{\tilde{x}} &= \mu\bar{x}_1 \left(\lambda a_0 \rho \left(\frac{\rho_{2b2} - \rho_{1b1}}{\rho_{1b1}\rho_{2b2}}\right) - 2(1-\lambda)\bar{x}_1(1 - \tilde{x})\right), \\ \dot{\hat{x}}_1 &= -\mu \left(\frac{(1-\lambda)(\bar{x}_1 - a)}{\rho} - \frac{\lambda a_0}{\rho_{1b1}}\right) \hat{r}_1 + \left(\frac{(1-\lambda)\mu(\bar{x}_1 - a)}{\rho}\right) \hat{r}_2 + \frac{2\lambda\mu a_0 b^2}{\rho_{1b1}} \\ \text{and } \dot{\hat{x}}_2 &= -\mu \left(\frac{(1-\lambda)(-\bar{x}_1 - a)}{\rho} - \frac{\lambda a_0}{\rho_{2b2}}\right) \hat{r}_2 + \left(\frac{(1-\lambda)\mu(-\bar{x}_1 - a)}{\rho}\right) \hat{r}_1 - \frac{2\lambda\mu a_0 b^2}{\rho_{2b2}}. \end{aligned} \right\} \quad (6.1)$$

Before proceeding, we note the following helpful result.

Lemma 6.1. Let $\bar{x}_{1b} = 0 = \bar{x}_{2b}$ and $\hat{x}_1 = 0 = \hat{x}_2$, with $\hat{\mathbf{b}} \neq \mathbf{0}$ and $\hat{\mathbf{b}}$ not parallel to either $\mathbf{r}_{1b1}$ or $\mathbf{r}_{2b2}$. Then if $\bar{x}_1 = 0 = \bar{x}_2$, $\tilde{x}$ must be either 1 or -1 .

Proof. See the electronic supplementary material. ■

As before, we will first consider a special case.

(a) Case 1 for configuration III: $a_0 = 0$

If $a_0 = 0$, then (6.1) simplifies to

$$\left. \begin{aligned} \dot{\bar{x}}_1 &= -(1-\lambda)\mu(\bar{x}_1 - a) \left(1 - \bar{x}_1^2\right) + \frac{1}{\rho}(1 - \tilde{x}), \\ \dot{\bar{x}}_2 &= -(1-\lambda)\mu(-\bar{x}_1 - a) \left(1 - \bar{x}_1^2\right) + \frac{1}{\rho}(1 - \tilde{x}), \\ \dot{x}_{1b1} &= -(1-\lambda)\mu(\bar{x}_1 - a) \left(\frac{\rho_{1b1}^2 + \rho^2 - \rho_{2b2}^2 - 2\hat{r}_2}{2\rho\rho_{1b1}}\right) + \frac{1}{\rho_{1b1}}, \\ \dot{x}_{2b2} &= -(1-\lambda)\mu(-\bar{x}_1 - a) \left(\frac{-\rho_{1b1}^2 + \rho^2 + \rho_{2b2}^2 + 2\hat{r}_1}{2\rho\rho_{2b2}}\right) + \frac{1}{\rho_{2b2}}, \\ \dot{\tilde{x}} &= -2\mu(1-\lambda)\bar{x}_1^2(1 - \tilde{x}), \\ \dot{\hat{x}}_1 &= -\mu \left(\frac{(1-\lambda)(\bar{x}_1 - a)}{\rho}\right) \hat{r}_1 + \left(\frac{(1-\lambda)\mu(\bar{x}_1 - a)}{\rho}\right) \hat{r}_2 \\ \text{and } \dot{\hat{x}}_2 &= -\mu \left(\frac{(1-\lambda)(-\bar{x}_1 - a)}{\rho}\right) \hat{r}_2 + \left(\frac{(1-\lambda)\mu(-\bar{x}_1 - a)}{\rho}\right) \hat{r}_1. \end{aligned} \right\} \quad (6.2)$$

Figure 4. (a,b) Circling equilibria for configuration III with beacon positioning $b = 10$ and control parameters $a_0 = 0$, $a = -0.771$, $\lambda = 0.57$ and $\mu = 1$, for two different sets of initial conditions. (Online version in colour.)

As in the same special case for configuration II, the only way that we can have $\dot{\tilde{x}}_1$, $\dot{\tilde{x}}_2$ and $\dot{\tilde{x}}$ all equal to zero is if $\tilde{x}_1 = 0$. Substituting this constraint into (6.2) and setting equal to zero yields

$$(1 - \lambda)\mu a + \frac{1}{\rho}(1 - \tilde{x}) = 0, \quad (6.3a)$$

$$(1 - \lambda)\mu a \left(\frac{\rho_{1b1}^2 + \rho^2 - \rho_{2b2}^2 - 2\hat{r}_2}{2\rho\rho_{1b1}} \right) = \frac{-1}{\rho_{1b1}}, \quad (6.3b)$$

$$(1 - \lambda)\mu a \left(\frac{\rho_{2b2}^2 + \rho^2 - \rho_{1b1}^2 + 2\hat{r}_1}{2\rho\rho_{2b2}} \right) = \frac{-1}{\rho_{2b2}} \quad (6.3c)$$

and
$$\left(\frac{(1 - \lambda)\mu a}{\rho} \right) (\hat{r}_1 - \hat{r}_2) = 0, \quad (6.3d)$$

and, since $a = 0$ is not valid (based on the fact that ρ_{1b1} and ρ_{2b2} are nonzero and finite), we must have $\hat{r}_1 = \hat{r}_2$, $\tilde{x} = -1$ (based on (6.3a) and lemma 6.1) and $\rho = \frac{2}{(1 - \lambda)\mu(-a)}$. Then substituting these values into (6.3b) and (6.3c) yields the following result.

Proposition 6.2. Consider a beacon-referenced mutual CB pursuit system with beacon positioning parameter b and shape dynamics (4.4), (4.11) parameterized by μ , λ , and CB parameters a and $a_0 = 0$. Then, a circling equilibrium exists if and only if $a < 0$, and the corresponding equilibrium values satisfy

$$\begin{aligned} & \bullet \quad \tilde{x}_1 = \tilde{x}_2 = 0, \quad \tilde{x}_{1b1} = \tilde{x}_{2b2} = 0, \quad \hat{x}_1 = \hat{x}_2 = 0, \quad \tilde{x} = -1 \\ & \bullet \quad \rho = \frac{2}{(1 - \lambda)\mu(-a)}, \quad \hat{r}_1 = \hat{r}_2, \quad \rho_{1b1}^2 - \rho_{2b2}^2 - 2\hat{r}_1 = 0. \end{aligned} \quad (6.4)$$

Proof. Follows from previous discussion. ■

We note that if we express r_1 component-wise as (r_{11}, r_{12}, r_{13}) , then it follows that $r_{13} = \hat{r}_1/(2b)$. Therefore, the circling equilibria described by proposition 6.2 involve both agents manoeuvring in a common plane which is orthogonal to $\hat{b}$, with circling radius given by $\rho/2 = 1/(1 - \lambda)\mu(-a)$ and $r_{13} = r_{23}$ determined by initial conditions. Typical circling equilibria corresponding to proposition 6.2 for two different sets of initial conditions are depicted in figure 4.

(b) Case 2 for configuration III: stacked circling equilibria

We now return to the dynamics given by (6.1) and assume $a_0 \neq 0$. From the form of the $\dot{\tilde{x}}$ expression we note that we can set it equal to zero by choosing $\tilde{x}_1 = 0$. Then (6.1) simplifies to

$$\left. \begin{aligned} \dot{\tilde{x}}_1 &= (1 - \lambda)\mu a + \lambda\mu a_0 \left(\frac{\rho_{1b1}^2 + \rho^2 - \rho_{2b2}^2 - 2\hat{r}_2}{2\rho\rho_{1b1}} \right) + \frac{1}{\rho}(1 - \tilde{x}), \\ \dot{\tilde{x}}_2 &= (1 - \lambda)\mu a + \lambda\mu a_0 \left(\frac{-\rho_{1b1}^2 + \rho^2 + \rho_{2b2}^2 + 2\hat{r}_1}{2\rho\rho_{2b2}} \right) + \frac{1}{\rho}(1 - \tilde{x}), \\ \dot{\tilde{x}}_{1b1} &= (1 - \lambda)\mu a \left(\frac{\rho_{1b1}^2 + \rho^2 - \rho_{2b2}^2 - 2\hat{r}_2}{2\rho\rho_{1b1}} \right) + \frac{1}{\rho_{1b1}} + \lambda\mu a_0, \\ \dot{\tilde{x}}_{2b2} &= (1 - \lambda)\mu a \left(\frac{-\rho_{1b1}^2 + \rho^2 + \rho_{2b2}^2 + 2\hat{r}_1}{2\rho\rho_{2b2}} \right) + \frac{1}{\rho_{2b2}} + \lambda\mu a_0, \\ \dot{\hat{x}}_1 &= \mu \left(\frac{(1 - \lambda)a}{\rho} + \frac{\lambda a_0}{\rho_{1b1}} \right) \hat{r}_1 - \left(\frac{(1 - \lambda)\mu a}{\rho} \right) \hat{r}_2 + \frac{2\lambda\mu a_0 b^2}{\rho_{1b1}}, \\ \dot{\hat{x}}_2 &= \mu \left(\frac{(1 - \lambda)a}{\rho} + \frac{\lambda a_0}{\rho_{2b2}} \right) \hat{r}_2 - \left(\frac{(1 - \lambda)\mu a}{\rho} \right) \hat{r}_1 - \frac{2\lambda\mu a_0 b^2}{\rho_{2b2}}. \end{aligned} \right\} \quad (6.5)$$

and

We note that taking the difference of $\dot{\tilde{x}}_1 - \dot{\tilde{x}}_2$ and the difference of $\dot{\tilde{x}}_{1b1} - \dot{\tilde{x}}_{2b2}$ yields

$$\dot{\tilde{x}}_1 - \dot{\tilde{x}}_2 = \frac{\lambda\mu a_0}{2\rho} \left(\frac{\rho_{1b1}^2 + \rho^2 - \rho_{2b2}^2 - 2\hat{r}_2}{\rho_{1b1}} + \frac{\rho_{1b1}^2 - \rho^2 - \rho_{2b2}^2 - 2\hat{r}_1}{\rho_{2b2}} \right) \quad (6.6)$$

and

$$\dot{\tilde{x}}_{1b1} - \dot{\tilde{x}}_{2b2} = \frac{(1 - \lambda)\mu a}{2\rho} \left(\frac{\rho_{1b1}^2 + \rho^2 - \rho_{2b2}^2 - 2\hat{r}_2}{\rho_{1b1}} + \frac{\rho_{1b1}^2 - \rho^2 - \rho_{2b2}^2 - 2\hat{r}_1}{\rho_{2b2}} \right) + \left(\frac{1}{\rho_{1b1}} - \frac{1}{\rho_{2b2}} \right) \quad (6.7)$$

and, since $a_0 \neq 0$, setting both equations equal to zero yields the requirement $\rho_{1b1} = \rho_{2b2}$. Substituting this equivalence back into (6.6) and setting equal to zero then requires $\hat{r}_2 = -\hat{r}_1$. Under these constraints, (6.5) becomes

$$\left. \begin{aligned} \dot{\tilde{x}}_1 &= (1 - \lambda)\mu a + \lambda\mu a_0 \left(\frac{\rho^2 + 2\hat{r}_1}{2\rho\rho_{1b1}} \right) + \frac{1}{\rho}(1 - \tilde{x}), \\ \dot{\tilde{x}}_{1b1} &= (1 - \lambda)\mu a \left(\frac{\rho^2 + 2\hat{r}_1}{2\rho\rho_{1b1}} \right) + \frac{1}{\rho_{1b1}} + \lambda\mu a_0 \\ \dot{\hat{x}}_1 &= \mu \left(\frac{2(1 - \lambda)a}{\rho} + \frac{\lambda a_0}{\rho_{1b1}} \right) \hat{r}_1 + \frac{2\lambda\mu a_0 b^2}{\rho_{1b1}}. \end{aligned} \right\} \quad (6.8)$$

and

Based on lemma 6.1, $\tilde{x}$ must be either 1 or -1 .

- **Consider the case where $\tilde{x} = 1$.** Then setting the first equation in (6.8) to zero yields

$$\left(\frac{\rho^2 + 2\hat{r}_1}{2\rho\rho_{1b1}} \right) = - \left(\frac{1 - \lambda}{\lambda} \right) \left(\frac{a}{a_0} \right), \quad (6.9)$$

from which substitution into the second equation in (6.8) and setting equal to zero yields

$$\frac{1}{\rho_{1b1}} = \mu \left(\frac{(1 - \lambda)^2 a^2}{\lambda a_0} \right) - \lambda\mu a_0 = \mu \left(\frac{(1 - \lambda)^2 a^2 - \lambda^2 a_0^2}{\lambda a_0} \right), \quad (6.10)$$

i.e.

$$\rho_{1b1} = \frac{\lambda a_0}{\mu \left((1-\lambda)^2 a^2 - \lambda^2 a_0^2 \right)}. \quad (6.11)$$

Note that this requires $\frac{a_0}{(1-\lambda)^2 a^2 - \lambda^2 a_0^2} > 0$. Denoting

$$\Phi = \frac{(1-\lambda)a}{\mu \left((1-\lambda)^2 a^2 - \lambda^2 a_0^2 \right)}, \quad (6.12)$$

so that $\rho_{1b1} = \frac{\lambda a_0}{(1-\lambda)a} \Phi$, we can substitute back into the first and third equations in (6.8) and set them to zero to obtain

$$\left. \begin{aligned} 0 &= 2\rho\Phi + \rho^2 + 2\hat{r}_1 \\ 0 &= 2\hat{r}_1\Phi + \rho(\hat{r}_1 + 2b^2). \end{aligned} \right\} \quad (6.13)$$

and

From the first equation, we have $\hat{r}_1 = -\rho(\rho/2 + \Phi)$, and substitution into the second equation results in

$$\begin{aligned} 0 &= -2\rho \left(\frac{\rho}{2} + \Phi \right) \Phi + \rho \left(-\rho \left(\frac{\rho}{2} + \Phi \right) + 2b^2 \right) \\ &= -\frac{\rho}{2} (\rho^2 + 4\Phi\rho + 4\Phi^2 - 4b^2). \end{aligned} \quad (6.14)$$

Solving the quadratic equation inside the parentheses then results in

$$\rho = 2(-\Phi \pm b), \quad (6.15)$$

which implies that $\rho/2 + \Phi = \pm b$; therefore, we also have

$$\hat{r}_1 = \mp b\rho = 2b(-b \pm \Phi). \quad (6.16)$$

(Note that the choice of $+$ or $-$ in the $\hat{r}_1$ equation must match the choice from the ρ equation, i.e. we have only two options rather than four combinations.) To characterize the existence conditions, we must ensure that (6.15) satisfies $\rho > 0$ and that the constraint (4.9) (equivalent to (4.8) in the present case) is satisfied. Substituting $\rho_{1b1} = \rho_{2b2} = \frac{\lambda a_0}{(1-\lambda)a} \Phi$ into (4.9) results in the requirement

$$\frac{-(1-\lambda)a}{2\rho\lambda a_0\Phi} (\rho^2 + 2\hat{r}_1) \in [-1, 1], \quad (6.17)$$

and, since $\rho^2 + 2\hat{r}_1 = -2\rho\Phi$, we obtain the parameter constraint

$$\left| \frac{(1-\lambda)a}{\lambda a_0} \right| < 1. \quad (6.18)$$

Note that this implies that the denominator of Φ is negative, and therefore $\text{sgn}(\Phi) = -\text{sgn}(a)$. Since the denominator of ρ_{1b1} is the same as the denominator of Φ and we require $\rho_{1b1} = \rho_{2b2} > 0$, we must have $a_0 < 0$. Then in order for $\rho > 0$, we must have either $\Phi < 0$ (i.e. $\text{sgn}(a) > 0$) or $\Phi > 0$ (i.e. $\text{sgn}(a) < 0$) with $b > \Phi$. Representative trajectories for this case are depicted in figure 5.

We now return to our earlier choice for $\tilde{x}$ and consider the other option.

• **Consider the case where $\tilde{x} = -1$.** As we have done in a previous context, we will denote

$$\tilde{a}_0 \triangleq \lambda \mu a_0 \in \mathbb{R} \quad \text{and} \quad \tilde{a} \triangleq (1-\lambda)\mu a \in \mathbb{R}. \quad (6.19)$$

Then substituting $\tilde{x} = -1$ into (6.8) and setting all three equations to zero (and eliminating denominators) yields

$$0 = 2\tilde{a}\rho\rho_{1b1} + \tilde{a}_0 (\rho^2 + 2\hat{r}_1) + 4\rho_{1b1}, \quad (6.20)$$

$$0 = 2\tilde{a}_0\rho\rho_{1b1} + \tilde{a} (\rho^2 + 2\hat{r}_1) + 2\rho \quad (6.21)$$

and

$$0 = 2\tilde{a}\rho_{1b1}\hat{r}_1 + \tilde{a}_0\rho\hat{r}_1 + 2\tilde{a}_0\rho b^2. \quad (6.22)$$

Figure 5. (a,b) Circling equilibria (corresponding to case (b) of proposition 6.3) for configuration III with beacon positioning $b = 2$ and control parameters $a_0 = -0.707$, $a = 0.707$, $\lambda = 0.6$ and $\mu = 0.9$, for two different sets of initial conditions. Note that, since $\Phi \approx -5.66$ is negative, we have two possible steady-state configurations for this set of parameters. If $\Phi > 0$, then there would be only one possible steady-state configuration. (Online version in colour.)

As in the previous case, we have $\rho_{1b1} = \rho_{2b2}$ and $\hat{r}_2 = -\hat{r}_1$. Taking the sum and difference of (6.20) and (6.21) results in

$$0 = (\tilde{a}_0 + \tilde{a}) \left(2\rho\rho_{1b1} + \rho^2 + 2\hat{r}_1 \right) + 2(2\rho_{1b1} + \rho) \quad (6.23)$$

and

$$0 = (\tilde{a}_0 - \tilde{a}) \left(-2\rho\rho_{1b1} + \rho^2 + 2\hat{r}_1 \right) + 2(2\rho_{1b1} - \rho), \quad (6.24)$$

which suggests the change of variables $\rho_+ = \frac{\rho}{2} + \rho_{1b1}$ and $\rho_- = \frac{\rho}{2} - \rho_{1b1}$, i.e. $\rho = \rho_+ + \rho_-$ and $\rho_{1b1} = (\rho_+ - \rho_-)/2$. In terms of this notation, we have $2\rho\rho_{1b1} = \rho_+^2 - \rho_-^2$ and $\rho^2 = (\rho_+ + \rho_-)^2$, and therefore we can express (6.23), (6.24) and (6.22) as

$$0 = (\tilde{a}_0 + \tilde{a}) \left(\rho_+^2 + \rho_+\rho_- + \hat{r}_1 \right) + 2\rho_+, \quad (6.25)$$

$$0 = (\tilde{a}_0 - \tilde{a}) \left(\rho_-^2 + \rho_+\rho_- + \hat{r}_1 \right) - 2\rho_-, \quad (6.26)$$

and

$$0 = (\tilde{a}_0 + \tilde{a})\hat{r}_1\rho_+ + (\tilde{a}_0 - \tilde{a})\hat{r}_1\rho_- + 2\tilde{a}_0b^2(\rho_+ + \rho_-). \quad (6.27)$$

Before proceeding, we note that ρ_+ must be positive (since $\rho > 0$ and $\rho_{1b1} > 0$), and therefore (6.25) implies that $\rho_+^2 + \rho_+\rho_- + \hat{r}_1$ must be non-zero. Additionally, if $\rho_-^2 + \rho_+\rho_- + \hat{r}_1 = 0$, then (6.26) requires $\rho_- = 0$, and the combination of the two constraints results in $\hat{r}_1 = 0$. But substituting both of these values into (6.27) results in $2\tilde{a}_0b^2\rho_+ = 0$, which is not possible since $\tilde{a}_0$, b and ρ_+ are all non-zero by assumption. Therefore, it is valid to rearrange (6.25) and (6.26) to obtain

$$\left. \begin{aligned} \tilde{a}_0 + \tilde{a} &= \frac{-2\rho_+}{\rho_+^2 + \rho_+\rho_- + \hat{r}_1} \\ \tilde{a}_0 - \tilde{a} &= \frac{2\rho_-}{\rho_-^2 + \rho_+\rho_- + \hat{r}_1} \end{aligned} \right\} \quad (6.28)$$

and

Noting that summing the two equations in (6.28) yields $2\tilde{a}_0$; we can substitute these expressions into (6.27) to obtain

$$\begin{aligned} 0 &= 2\hat{r}_1 \left(\frac{-\rho_+^2}{\rho_+^2 + \rho_+\rho_- + \hat{r}_1} + \frac{\rho_-^2}{\rho_-^2 + \rho_+\rho_- + \hat{r}_1} \right) \\ &\quad + 2b^2(\rho_+ + \rho_-) \left(\frac{\rho_-}{\rho_-^2 + \rho_+\rho_- + \hat{r}_1} - \frac{\rho_+}{\rho_+^2 + \rho_+\rho_- + \hat{r}_1} \right), \end{aligned} \quad (6.29)$$

which implies that

$$\begin{aligned}
 0 &= \hat{r}_1 \left(-\rho_+^2 \left(\rho_-^2 + \rho_+ \rho_- + \hat{r}_1 \right) + \rho_-^2 \left(\rho_+^2 + \rho_+ \rho_- + \hat{r}_1 \right) \right) \\
 &\quad + b^2 (\rho_+ + \rho_-) \left(\rho_- \left(\rho_+^2 + \rho_+ \rho_- + \hat{r}_1 \right) - \rho_+ \left(\rho_-^2 + \rho_+ \rho_- + \hat{r}_1 \right) \right) \\
 &= \hat{r}_1 (\rho_+ \rho_- + \hat{r}_1) (\rho_-^2 - \rho_+^2) + b^2 (\rho_+ + \rho_-) (\rho_- - \rho_+) \hat{r}_1 \\
 &= \hat{r}_1 (\rho_-^2 - \rho_+^2) (\rho_+ \rho_- + \hat{r}_1 + b^2). \tag{6.30}
 \end{aligned}$$

Since $\rho_-^2 - \rho_+^2 = -2\rho\phi_{1b1} \neq 0$ and it is straightforward to show that $\hat{r}_1 = 0$ leads to a contradiction, we must have $\hat{r}_1 = -\rho_+ \rho_- - b^2$. Substitution into (6.25) and (6.26) then results in

$$0 = (\tilde{a}_0 + \tilde{a}) (\rho_+^2 - b^2) + 2\rho_+ \tag{6.31}$$

and

$$0 = (\tilde{a}_0 - \tilde{a}) (\rho_-^2 - b^2) - 2\rho_- \tag{6.32}$$

If we represent our constraint (4.9) in terms of the ρ_+ and ρ_- variables, it is straightforward to obtain the requirement $\rho_+^2 > b^2 > \rho_-^2$. Thus the form of (6.31) imposes the requirement $\tilde{a}_0 + \tilde{a} < 0$, and solving the quadratic equation in ρ_+ (and selecting the only option that corresponds to $\rho_+ > 0$) leads to

$$\rho_+ = \frac{1}{-(\tilde{a}_0 + \tilde{a})} \left(1 + \sqrt{1 + ((\tilde{a}_0 + \tilde{a})b)^2} \right). \tag{6.33}$$

If $\tilde{a}_0 - \tilde{a} = 0$, then (6.32) requires $\rho_- = 0$. If $\tilde{a}_0 - \tilde{a} \neq 0$, then (6.32) can be arranged in the form

$$\rho_-^2 - \left(\frac{2}{\tilde{a}_0 - \tilde{a}} \right) \rho_- - b^2 = 0, \tag{6.34}$$

with constraint (4.9) requiring $(\tilde{a}_0 - \tilde{a})\rho_- < 0$ to ensure $b^2 > \rho_-^2$. This leads to the result

$$\rho_- = \frac{1}{\tilde{a}_0 - \tilde{a}} \left(1 - \sqrt{1 + ((\tilde{a}_0 - \tilde{a})b)^2} \right). \tag{6.35}$$

The following proposition summarizes the results of this entire subsection by presenting several sets of sufficient conditions for the existence of circling equilibria. (See remark 6.6 for further discussion of the necessity of conditions.)

Proposition 6.3. *Consider a beacon-referenced mutual CB pursuit system with beacon positioning parameter b and shape dynamics (4.4)–(4.11) parameterized by μ , λ , and CB parameters a and $a_0 \neq 0$, and define*

$$\Phi = \frac{(1 - \lambda)a}{\mu \left((1 - \lambda)^2 a^2 - \lambda^2 a_0^2 \right)}. \tag{6.36}$$

Circling equilibria exist under the following conditions, with equilibrium values in each case given by

$$\bullet \quad \bar{x}_1 = \bar{x}_2 = 0 \quad \text{and} \quad \bar{x}_{1b} = \bar{x}_{2b} = 0. \tag{6.37}$$

(a) *If $a_0 < 0$, $a < 0$, $(1 - \lambda)^2 a^2 - \lambda^2 a_0^2 < 0$ and $\Phi < b$, a circling equilibrium exists with corresponding equilibrium values given by*

$$\left. \begin{aligned}
 &\bullet \quad \tilde{x} = 1, \quad \hat{r}_1 = -\hat{r}_2 = 2b(-b + \Phi) \\
 &\bullet \quad \rho = 2(-\Phi + b), \quad \rho_{1b1} = \rho_{2b2} = \left(\frac{\lambda a_0}{(1 - \lambda)a} \right) \Phi.
 \end{aligned} \right\} \tag{6.38}$$

and

- (b) If $a_0 < 0$, $a > 0$ and $(1 - \lambda)^2 a^2 - \lambda^2 a_0^2 < 0$, a pair of circling equilibria exist with corresponding equilibrium values given by

$$\left. \begin{aligned} & \bullet \tilde{x} = 1, \quad \hat{r}_1 = -\hat{r}_2 = 2b(-b \pm \Phi) \\ & \bullet \rho = 2(-\Phi \pm b), \quad \rho_{1b1} = \rho_{2b2} = \left(\frac{\lambda a_0}{(1 - \lambda)a} \right) \Phi. \end{aligned} \right\} \quad (6.39)$$

and

- (c) If $a_0 < 0$ and $(1 - \lambda)a = \lambda a_0$, a circling equilibrium exists with corresponding equilibrium values given by

$$\left. \begin{aligned} & \bullet \tilde{x} = -1, \quad \hat{r}_1 = -\hat{r}_2 = -b^2 \\ & \bullet \rho = \frac{1}{-(2\lambda\mu a_0)} \left(1 + \sqrt{1 + (2\lambda\mu a_0 b)^2} \right), \quad \rho_{1b1} = \rho_{2b2} = \frac{\rho}{2}. \end{aligned} \right\} \quad (6.40)$$

and

- (d) If $(1 - \lambda)a + \lambda a_0 < 0$ and $(1 - \lambda)a - \lambda a_0 \neq 0$, a circling equilibrium exists with corresponding equilibrium values given by

$$\left. \begin{aligned} & \bullet \tilde{x} = -1, \quad \hat{r}_1 = -\hat{r}_2 = -\rho_+ \rho_- - b^2, \\ & \bullet \rho_+ = \frac{1}{-\mu((1 - \lambda)a + \lambda a_0)} \left(1 + \sqrt{1 + (\mu((1 - \lambda)a + \lambda a_0)b)^2} \right), \\ & \bullet \rho_- = \frac{1}{\mu((1 - \lambda)a - \lambda a_0)} \left(1 - \sqrt{1 + (\mu((1 - \lambda)a - \lambda a_0)b)^2} \right) \\ & \bullet \rho = \rho_+ + \rho_-, \quad \rho_{1b1} = \rho_{2b2} = \frac{(\rho_+ - \rho_-)}{2}. \end{aligned} \right\} \quad (6.41)$$

and

Proof. Follows from previous discussion. ■

Remark 6.4. Note that cases (a), (b) and (c) are all mutually exclusive. Cases (c) and (d) are also mutually exclusive, but the conditions of case (d) may hold true alongside either (a) or (b).

Remark 6.5. In each of the cases above, the third component of r_1 is given by $\hat{r}_1/(2b)$, and therefore the $\hat{r}_1$ values indicate the ‘vertical’ displacement of the planes of orbit. Radii for the circling orbits can be determined by projecting the vector $\mathbf{r}_{1b1}$ onto the ‘x–y’ plane (i.e. the plane normal to $\hat{\mathbf{b}}$ which passes through the origin), which yields the expression

$$\text{radius} = \sqrt{\rho_{1b1}^2 - \left(\frac{\hat{r}_1 + 2b^2}{2b} \right)^2}. \quad (6.42)$$

Representative trajectories for the equilibria described in the last two bullets of proposition 6.3 are depicted in figure 6.

Remark 6.6. It is important to note that proposition 6.3 does not provide an exhaustive characterization of circling equilibria for configuration III, in that it provides sufficient (but not necessary) conditions for existence. Numerical simulations suggest an additional type of circling equilibrium for this configuration, for which $\tilde{x}_1 \neq 0$, and the midpoint between the agents moves on a separate circling trajectory around the beacon axis (see electronic supplementary material, figure S7). Characterization of this type of circling equilibrium will be the focus of future work.

7. Conclusion

In this work, we have proposed a new control law which references multiple targets and relies only on bearing measurements. Closed-loop shape dynamics were formulated for the multiple configuration possibilities for one or two agents referencing one or two fixed beacons, and these shape dynamics were analysed to determine existence conditions and steady-state characterization for circling relative equilibria. In each case, we demonstrated that the radius of

Figure 6. Circling equilibria corresponding to the last two bullets of proposition 6.3 with beacon positioning $b = 10$. (a) The trajectories were generated using control parameters $a_0 = -0.707$, $a = -0.707$, $\lambda = 0.5$ and $\mu = 1$, and correspond to bullet (c) of proposition 6.3. (b) The trajectories were generated using control parameters $a_0 = 0.707$, $a = -0.588$, $\lambda = 0.35$, and $\mu = 1$, and correspond to bullet (d) of the proposition. (Online version in colour.)

the circling trajectory and the vertical separation of the mobile agents could be prescribed through the choice of control parameters. This decentralized control method could be used to coordinate the motions of autonomous vehicles in a circular stationing pattern with minimal required sensing (e.g. underwater vehicles sensing the relative bearing to sound sources serving as beacons).

There exist several clear paths for future research endeavours related to this control formulation. First, it will be important to explore the stability characteristics of these steady-state behaviours to determine parameter requirements to ensure that the circling equilibria are attractive. Numerical simulations suggest that for most of the circling equilibria described in this paper there exist a range of parameter values for which the circling trajectories are attractive, and future work will prove this through mathematical analysis. Additionally, the ideas in this paper can be extended in new directions by considering systems with more than two mobile agents, each referencing two (or more) targets.

Data accessibility. This article has no additional data.

Authors' contributions. Both authors were involved with the development of the framework and analysis of the pertinent equations.

Competing interests. We declare we have no competing interest.

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