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Author for correspondence:

Sharath Jose

e-mail: skjose@smail.iitm.ac.in

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Non-normal origin of modal instabilities in rotating plane shear flows

Sharath Jose¹ and Rama Govindarajan²

¹Department of Aerospace Engineering, Indian Institute of Technology Madras, Chennai, India

²International Centre for Theoretical Sciences, Tata Institute of Fundamental Research, Bengaluru, India

SJ, 0000-0002-9412-4105

Small variations introduced in shear flows are known to affect stability dramatically. Rotation of the flow system is one example, where the critical Reynolds number for exponential instabilities falls steeply with a small increase in rotation rate. We ask whether there is a fundamental reason for this sensitivity to rotation. We answer in the affirmative, showing that it is the non-normality of the stability operator in the absence of rotation which triggers this sensitivity. We treat the flow in the presence of rotation as a perturbation on the non-rotating case, and show that the rotating case is a special element of the pseudospectrum of the non-rotating case. Thus, while the non-rotating flow is always modally stable to streamwise-independent perturbations, rotating flows with the smallest rotation are unstable at zero streamwise wavenumber, with the spanwise wavenumbers close to that of disturbances with the highest transient growth in the non-rotating case. The instability critical rotation number scales inversely as the square of the Reynolds number, which we demonstrate is the same as the scaling obeyed by the minimum perturbation amplitude in non-rotating shear flow needed for the pseudospectrum to cross the neutral line. Plane Poiseuille flow and plane Couette flow are shown to behave similarly in this context.

1. Introduction

The effects of rotation are of practical importance in numerous fluid flow systems that are of engineering and geophysical interests [1,2]. Since the pioneering work of Taylor [3], the flow between two concentric cylinders rotating at different angular speeds (the

Taylor–Couette flow) has served as a canonical system wherein the processes involved in the emergence of instabilities, pattern formation, flow transition and fully turbulent states are examined (e.g. [4,5]). Plane shear flows are another class of system where the effect of rotation is significant on the flow characteristics. In particular, the rotating plane Poiseuille flow (RPPF) and the rotating plane Couette flow (RPCF) have been studied extensively, with emphasis given to instabilities, bifurcations and turbulence [6–13]. There is additional appeal in studying these systems as analogies may be drawn with other canonical flows such as the Rayleigh–Bénard flow and the Taylor–Couette flow [14–16].

For RPPF and RPCF, the laminar state is highly susceptible to exponentially growing instabilities and undergoes transition to yield a diverse range of dynamical states. Several of the recent investigations on RPPF and RPCF have been more focused on regimes where the flow is turbulent and chaotic [17–24]. An inviscid criterion for the conditions under which instabilities can set in has been proposed in terms of whether the base state vorticity is parallel or anti-parallel to the axis of the imposed rotation [25,26], and it provides a reasonably accurate prediction even when viscous effects are present. In both RPPF and RPCF, the modal critical Reynolds number, above which exponentially growing instabilities exist, is found to change drastically with the rotation rate. For RPPF, when compared to its non-rotating counterpart (PPF), the modal critical Reynolds number is as much as two orders of magnitude lower for a range of rotation strengths [6,14,15]. Similarly, in contrast to the non-rotating plane Couette flow (PCF) that is modally stable for any finite Reynolds number, the RPCF can support exponential instabilities at a Reynolds number as low as 20.7 [9,15].

Despite the numerous studies on RPPF and RPCF, the extreme sensitivity of the modal critical Reynolds number with rotation has, to our knowledge, not been previously addressed or explained. In this article, our approach is to treat rotation as a perturbation on the non-rotating case, and present evidence that the reason a rotating flow goes unstable at far lower Reynolds numbers for a small rate of rotation has its roots in the non-normal nature of the stability operator of the corresponding *non-rotating* flow. Even when all eigenmodes decay, the non-normality of the linear operator in shear flow can lead to transient algebraic growth of disturbances [27,28]. This growth can trigger nonlinearities, and lead to transition to turbulence, as has been shown for PPF and PCF [29,30]. In RPPF, Jose *et al.* [31] showed that even when the flow is linearly unstable, there exist regimes where perturbation energy can be enhanced significantly by transient algebraic growth. Besides this well-known property of non-normality, we show here that non-normality in a particular flow system can be the reason for a large decrease in the critical Reynolds number of a closely related flow.

2. Flow configurations and perturbation equations

(a) Base flows

The configurations we consider are shown in figure 1. The systems are both rotated about the spanwise axis z at a constant angular velocity Ω . The flow is confined between walls at $y = \pm h$, and h is our reference length. The nondimensional Navier–Stokes and continuity equations governing the fluid velocity $\mathbf{u} = (u, v, w)$ and pressure p in a coordinate frame rotating with the system are

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{u} - 2Ro \hat{\mathbf{z}} \times \mathbf{u} \quad (2.1)$$

and

$$\nabla \cdot \mathbf{u} = 0. \quad (2.2)$$

In the above, $\hat{\mathbf{z}}$ denotes the unit vector along the spanwise coordinate. The nondimensional parameters governing the system are the Reynolds number $Re \equiv U_0 h / \nu$ and the rotation number $Ro \equiv \Omega h / U_0$, which is set ≥ 0 throughout this study. The RPPF is pressure-driven, with a centreline velocity U_0 , while the walls in the RPCF move in opposite directions at U_0 . The

Figure 1. Base flow configurations: (a) rotating plane Poiseuille flow (RPPF); (b) rotating plane Couette flow (RPCF). In both, the rotation axis (z) is perpendicular to the plane (x – y) of the shear.

kinematic viscosity of the fluid is ν . The primary difference between these systems and their non-rotating counterparts is the non-zero base wall-normal pressure gradient set up due to the Coriolis force: $\partial_y P = -2Ro U$, where $U(y)$ is the base velocity profile, given by a parabolic profile ($U = 1 - y^2$) for RPPF and a linear profile ($U = y$) for RPCF. Tritton & Davies [32] offer a displaced particle argument for the instability mechanism on the basis of this wall-normal pressure gradient.

(b) Linearized perturbation equations

The disturbances about the base state are expressed in normal mode form $f = \tilde{f}(y)e^{i(\alpha x + \beta z - \omega t)}$, α and β are the streamwise and spanwise wavenumber respectively. In terms of the wall-normal velocity perturbation $\tilde{v}$ and wall-normal vorticity perturbation $\tilde{\eta}$, the linearized system of equations can be posed as an eigenvalue problem with the complex temporal frequency ω being the eigenvalue, as follows:

$$-i\omega \tilde{q} = L_{Ro} \tilde{q} \quad (2.3)$$

where

$$\tilde{q} = \begin{bmatrix} \tilde{v} \\ \tilde{\eta} \end{bmatrix}, L_{Ro} = \begin{bmatrix} D^2 - k^2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} L_{os} & -2iRo\beta \\ -i\beta U' + 2iRo\beta & L_{sq} \end{bmatrix}.$$

Here $D = d(\cdot)/dy$ and $k^2 = \alpha^2 + \beta^2$; the primes are used to denote the derivatives of the base flow velocity with respect to y . When there exists an eigenvalue whose imaginary part $\omega_i > 0$, the system is unstable to an exponentially growing mode. The mode with the greatest ω_i is the one that will survive at long times. The Orr–Sommerfeld and Squire operators are given respectively by

$$L_{os} = i\alpha U'' - i\alpha U(D^2 - k^2) + \frac{1}{Re}(D^2 - k^2)^2 \quad (2.4)$$

and

$$L_{sq} = -i\alpha U + \frac{1}{Re}(D^2 - k^2). \quad (2.5)$$

The boundary conditions that are imposed for the above system of equations are

$$\tilde{v}(\pm 1) = D\tilde{v}(\pm 1) = \tilde{\eta}(\pm 1) = 0. \quad (2.6)$$

When $Ro = 0$, the celebrated Orr–Sommerfeld–Squire system of equations for parallel shear flows [33] is recovered.

(c) Numerical method

For discretization along the wall-normal coordinate y , we use a Chebyshev collocation method with 81 Chebyshev polynomials [34,35]. Higher resolutions did not change the answer to at least 12 decimal places. The code has been validated by recovering the eigenspectra for the non-rotating plane Poiseuille and plane Couette flows (Appendix A.7 in [27]). With the effect of rotation included, we recover the marginal stability curves for RPPF and RPCF [15].

Figure 2. (a) The marginal stability curves in the Re - Ro phase plane for RPPF and RPCF. The inset shows the decreasing Re_{mar} at larger values of Ro . (b) The spanwise disturbance wavenumber at marginal stability.

3. Marginal stability characteristics

We briefly revisit the modal stability characteristics of RPPF and RPCF in this section to highlight some key features. In PPF, exponential instabilities can be excited only when $Re > 5772.2$ [36]. The PCF on the other hand is modally stable for all values of Re [37]. The energy critical Reynolds number Re_{crE} , below which all perturbations decay monotonically, is 49.6 and 20.7 for PPF [38] and PCF [39] respectively. Even transient amplification of perturbation energy in these non-rotating flows may only be observed above these thresholds, and can be attributed to the non-normal nature of the linearized operator of the flow. Our first check to see if exponential growth of disturbances in rotating flows has its origin in the non-normality of the underlying operator is to verify if the critical *modal* Reynolds numbers in RPPF and RPCF are greater than 49.6 and 20.7 respectively. Figure 2a shows the marginal stability curves for RPPF and RPCF; the flow is unstable to exponential perturbations for configurations that lie above each curve. The critical modal Reynolds number drops steeply with increase in rotation rate in both cases. The smallest values of the modal critical Reynolds number are 66.4 (at $Ro = 0.167$) for RPPF [8] and around 20.7 (at $Ro = 0.25$) for RPCF [15], and these values are greater than and close to the corresponding Re_{crE} for the non-rotating cases.

It is well known that non-rotating plane shear flows are modally stable at any value of Reynolds number to Squire ($\alpha = 0$) modes [33]. In PPF, which can support exponential instabilities when $Re > 5772.2$, the first mode to become unstable is the two-dimensional Tollmien-Schlichting mode with $\beta = 0$. However for RPPF and RPCF, the modes that first destabilize the flow are standing modes that are streamwise independent with $\alpha = 0$ [14,15]. Thus the marginally stable states are also streamwise independent. In figure 2b, we plot the spanwise wavenumber β_{mar} of the marginally stable mode for RPPF and RPCF. While there is a departure from $\beta_{\text{mar}} = 2$ at higher rotation rates in RPPF, $\beta_{\text{mar}} (= 1.56)$ is insensitive to Ro in RPCF. To relate this to the non-rotating counterpart flows, it is to be noted that the disturbances that first allow for transient growth there are streamwise independent and the values of the wavenumber are close to those displaying exponential growth of instabilities in the rotating flows. This is the second indication that the mechanism causing transient algebraic growth in the non-rotating flows is the same as that giving rise to exponential growth in the rotating flows. The next section makes a direct connection between the two.

4. Analysis using pseudospectra

To understand the sensitivity of the Reynolds number of marginal stability Re_{mar} to the rotation number Ro , we now turn to the pseudospectra of the non-rotating flow [40]. In this vein, it is

instructive to rewrite the eigenvalue problem given in equation (2.3) as follows:

$$-i\omega\tilde{q} = L\tilde{q} + RoP\tilde{q} \quad (4.1)$$

where

$$L = \begin{bmatrix} D^2 - k^2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} L_{os} & 0 \\ -i\beta U' & L_{sq} \end{bmatrix}$$

and

$$P = \begin{bmatrix} D^2 - k^2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -2i\beta \\ 2i\beta & 0 \end{bmatrix}.$$

Thus $L_{Ro} = L + RoP$, where L is the linearized stability operator for the non-rotating flow. Rotation thus perturbs the stability operator L by $E_{Ro} = RoP$. This provides us a framework for examining the pseudospectrum of the non-rotating operator L . The perturbation operator E_{Ro} has a Frobenius norm Q , which can be readily verified to be equal to $2\beta Ro$ for both RPPF and RPCF. Pseudospectrum is typically defined in terms of a measure of random operator perturbations, where the measure is chosen in an arbitrary manner. In Q , we have a measure derived from physical considerations that can be used to characterize the pseudospectrum of L . The values of Ro are chosen such that the focus is on regions in the $Re-Ro$ phase space that are close to the marginal stability curve.

The resolvent norm $\mathcal{R}(z) = \|(z - L)^{-1}\|$ is evaluated at different points in the complex plane. The region bounded by the contour $\mathcal{R}(z) = Q^{-1}$ contains points that belong to the Q -pseudospectrum (σ_Q) of L . If we then generate random perturbation matrices $E \in \mathbb{C}^{n \times n}$ having a Frobenius norm less than or equal to Q , we would find elements of this pseudospectrum which will all lie within the contour. For all the results to follow, we generate 5000 random matrices that have a Frobenius norm equal to Q . The entries in E are picked from a uniform distribution with a zero mean, and renormalized to have the requisite norm Q . We have verified that our findings do not change qualitatively when E is generated with inputs from a normal distribution. One of our aims is to examine where the spectrum of the rotation operator L_{Ro} lies with respect to this pseudospectrum.

(a) Rotating plane Poiseuille flow

We set $Re = 1000$, $\alpha = 0$ and $\beta = 2$, with the wavenumbers chosen based on the marginal stability conditions for RPPF. For these parameters, the rotation number for marginal stability Ro_{mar} is 2.9206×10^{-4} . We set Ro to be 3×10^{-4} such that the configuration allows for an exponentially unstable mode; the corresponding value of Q is 1.2×10^{-3} . We have verified that the results do not qualitatively change for other values of Ro .

In figure 3a we plot the contour where the resolvent norm of operator L is equal to Q^{-1} . The region bounded by this curve is seen to extend well into the unstable half of the complex plane, highlighting the non-normal nature of L . Also shown in the figure are the eigenspectra of L_{Ro} (red dots) and L (green crosses), and those of 5000 perturbed matrices (small black dots). It is apparent that the unstable eigenmode that arises due to rotation belongs to the Q -pseudospectrum σ_Q . This is clearer in figure 3b, where we show only the least stable eigenvalues of the various perturbed matrices, along with the spectra of L and L_{Ro} .

In figure 3b, the complex plane is densely populated near the origin, well away from the Q^{-1} contour. This feature will also be seen for the RPCF configuration in the following subsection. This selective populating of the region bounded by the Q^{-1} contour is worth commenting on, since if all possible random perturbations with $\|E\| < Q$ were considered, the superposition of the various spectra should fill the region bounded by the Q^{-1} contour. We report however that in spite of increasing the number of random matrices E up to a million to generate the pseudospectra, no significant change in the range of perturbed eigenvalues is observed. This hints at specific forms of the perturbation operator E being required to populate the regions away from the origin.

Figure 3. RPPF: $Ro = 3 \times 10^{-4}$, $Re = 1000$, $\alpha = 0$ and $\beta = 2$. (a) Solid black curve—the Q^{-1} contour; small black dots—elements of Q -pseudospectrum σ_Q ; (larger) red dots—RPPF eigenvalues; green crosses—PPF eigenvalues. (b) Zoom in, with only the least stable eigenvalues of each perturbed matrix. (Online version in colour.)

Figure 4. RPPF: $Re = 1000$, $Ro = 3 \times 10^{-4}$, $\alpha = 0$ and $\beta = 2$. (a) Unstable eigenmode. (b) Eigenmode with closest perturbed eigenvalue ω_d . In (a) and (b), the streamwise vorticity component is plotted, where the eigenmode is normalized such that its perturbation energy is unity. (c) Fraction of perturbation kinetic energy above the channel centreline $KE_{>}$ of unstable eigenfunctions obtained from perturbed matrices. The horizontal dashed line denotes $KE_{>}$ for the unstable mode in RPPF, and highlights the strong asymmetry of the mode about the channel centreline.

For example, by shifting the perturbations to support a non-zero mean (while maintaining the amplitude), we were able to shift the pseudospectrum elements away from the origin.

We proceed to compare the eigenfunction of the most unstable rotational eigenmode ω_{Ro} to that of its nearest counterpart ω_{cl} from the perturbed spectra of the non-rotating operator, shown respectively in figure 4a,b. A striking feature common to both eigenfunctions is the strong localization of the eigenfunction amplitude in one half of the channel. The wall-normal pressure gradient in a rotating channel gives rise to such an eigenfunction. But it is remarkable that the eigenfunction of a member of σ_Q breaks this symmetry in such a strong manner, considering that the eigenfunctions of L are symmetric about the channel centreline, and that we have perturbed all elements of the L matrix randomly with zero mean.

It would be interesting to see if this feature of localization is seen for other eigenfunctions of the pseudospectrum as well. For a quantitative measure of localization, we will take a closer look at the energy content of the perturbation eigenmodes of various eigenvalues populating the pseudospectrum. We obtain the perturbation kinetic energy, and integrate it separately in the upper ($0 \leq y \leq 1$) and lower ($-1 \leq y \leq 0$) halves of the channel. We then express each of these integrals as fractions of their sum. We define $KE_{>}$ as the greater of the energy fractions in the two halves of the channel. The further the value of $KE_{>}$ from 0.5, the stronger the localization in one half of the channel.

In figure 4c, a scatter plot of $KE_{>}$ is given with the dashed horizontal line representing the energy fraction of the rotational unstable mode. The range of $KE_{>}$ is evenly sampled, with several highly localized eigenfunction structures. The localization in the corresponding rotating case is shown by the horizontal dashed line, and we see that there are several members of the pseudospectrum whose eigenfunctions are similarly localized on one half of the channel. Thus we find that despite the symmetry of the base flow, of the linear operator of PPF, and despite the perturbations being of zero mean and evenly distributed across the channel width, random perturbations of the operator need not preserve this symmetry in their unstable eigenmodes, and most often do not. This is another consequence of the non-normal nature of the operator L as defined by the appropriate form of $U(y)$. This is consistent with how, while the rotation operator P in equation (4.1) contains no asymmetry about the channel centreline, the physically expected result of a disturbance mode that is concentrated on one side due to Coriolis effects, appears. We will see an interesting contrast to this localization in the following subsection.

(b) Rotating plane Couette flow

Next we examine the RPCF using a similar line of approach to that followed in the previous subsection. We select $Re = 600$, and for the wavenumbers, we choose the marginal stability values of $\alpha = 0$ and $\beta = 1.56$. The linear operator L is now evaluated with the definitions of base state velocity and its derivatives appropriate for PCF. For the given parameters, the marginal value Ro_{mar} is 1.4829×10^{-4} . For the results to follow, we set $Ro = 3 \times 10^{-4}$; the corresponding norm Q of E is 9.36×10^{-4} . As before, we report that the results do not qualitatively change while using other values of Ro .

As before, we plot the Q^{-1} contour along with the eigen spectra of L , L_{Ro} and 5000 perturbed matrices in figure 5a. The non-normal nature of L is once again evident as the Q^{-1} contour extends significantly into the unstable half of the complex plane, and the unstable mode due to rotation is shown to belong in the Q -pseudospectrum σ_Q . Figure 5b shows the least stable modes obtained from the randomly perturbed matrices. While we obtain eigenvalues belonging to the unstable half of the complex plane, their growth rates are found to be smaller than that of the rotational unstable eigenvalue ω_{Ro} . We have made several checks (not shown) which confirm that the less random we make our perturbation, and the closer to the actual rotation matrix, the closer the perturbed (least stable) eigenvalue is to ω_{Ro} .

Unlike the RPPF, where the mean shear changes sign across the centreline and hence the Coriolis effects lead to asymmetry, in the RPCF the mean shear is constant and we do not expect such asymmetry. Indeed, for a given Ro , the instability that sets in for RPCF manifests itself throughout the channel and is equally distributed in both halves (figure 6a). As before we identify the pseudospectrum eigenvalue ω_{cl} closest to ω_{Ro} , and plot its eigenfunction in figure 6b. It is readily seen that these two modes are similar in structure. In figure 6c, we show $KE_{>}$ of the different unstable eigenfunctions obtained from perturbing the linear operator L . In contrast to the PPF configuration (figure 4c), the range of values $KE_{>}$ takes is much smaller, and strong localizations are not observed. It is reassuring to find that this difference in the physics of rotation between the Couette and Poiseuille flows is encapsulated by the difference in behaviour of their entire pseudospectra.

Figure 5. RPCF: $Ro = 3 \times 10^{-4}$, $Re = 600$, $\alpha = 0$ and $\beta = 1.56$. (a) Solid black curve—the Q^{-1} contour; small black dots—elements of the Q -pseudospectrum σ_Q ; red dots—RPCF eigenvalues; green crosses—PCF eigenvalues. (b) Zoom in: only the least stable eigenvalues of the perturbed matrices. (Online version in colour.)

Figure 6. RPCF: $Re = 600$, $Ro = 3 \times 10^{-4}$, $\alpha = 0$ and $\beta = 1.56$. (a) Unstable eigenmode of RPCF. (b) Eigenmode with closest perturbed eigenvalue ω_d . In (a) and (b), the streamwise vorticity component is plotted, where the eigenmode is normalized such that its perturbation energy is unity. (c) $KE_{>}$ of unstable eigenfunctions obtained from perturbed matrices. The horizontal dashed line ($KE_{>} = 0.5$) shows the unstable mode in RPCF to be symmetric about $y = 0$.

(c) Scaling with the Reynolds number

We have now shown that modal instabilities due to rotation in RPPF and RPCF are a result of the non-normal nature of the hydrodynamic operator L . In particular, we have shown that, by this mechanism, low Reynolds number modal instabilities can be excited, where none existed in the non-rotating case, even for very small rotation. We define, for a given Reynolds number, $E_{\min}$ as the smallest operator perturbation such that the least stable eigenvalue belonging to the pseudospectrum lies on the neutral line of the complex plane ($\omega_i = 0$). In this subsection, we examine how E_{Ro} compares with $E_{\min}$ from the viewpoint of yielding unstable eigenvalues. To this end, we revisit the marginal stable states of RPPF and RPCF; we denote the marginal rotation number as Ro_{mar} and the corresponding spanwise wavenumber as β_{mar} . At conditions of marginal stability, the norm of E_{Ro} is $Q_{\text{mar}} = 2\beta_{\text{mar}}Ro_{\text{mar}}$.

Figure 7. $\varepsilon_{\min}$ (red squares) and Q_{mar} (black dots) as a function of Re for (a) PPF and (b) PCF configurations. In (a), the results are fitted by power-law forms where $P_{\varepsilon} = 93$, $P_Q \approx 12.66P_{\varepsilon}$. In (b), the fitting constants are $C_{\varepsilon} = 24.74$, $C_Q \approx 7.14C_{\varepsilon}$. (Online version in colour.)

As $E_{\min}$ is not known beforehand, we work with the values of the resolvent norm $\mathcal{R}(z)$ evaluated for the non-rotating systems. We identify the point $z_{\min}$ on the line $\omega_i = 0$ that is closest to the spectrum of L (or the least stable eigenvalue of L). If we set $\mathcal{R}(z_{\min}) = \varepsilon_{\min}^{-1}$, then we have Frobenius norm of $E_{\min}$ to be equal to $\varepsilon_{\min}$. In other words, $\varepsilon_{\min}$ gives the norm of the smallest perturbation $E_{\min}$ such that the least stable eigenvalue of the pseudospectrum is on the neutral boundary. The phase speed of the eigenmodes of L is zero when $\alpha = 0$, so the pseudospectrum crosses the horizontal axis at the origin. Since the smallest perturbation which takes the spectrum to the neutral boundary need not be the rotation operator, we do not expect the two to be identical. In other words, rotation of the system gives rise to one special perturbation matrix, whereas $\varepsilon_{\min}$ provides the minimum threshold for instability. So we must have $Q_{\text{mar}} \geq \varepsilon_{\min}$. Figure 7 shows Q_{mar} for the rotating flows and $\varepsilon_{\min}$ for their non-rotating counterparts as functions of Re , where this condition is seen to be satisfied. Extending this argument, for any $Ro > Ro_{\text{mar}}$, we have E_{Ro} belonging to a more general family of operator perturbations that extends the pseudospectra of L to regions that lie within the unstable half of the complex plane.

We also see in figure 7 that both $\varepsilon_{\min}$ and Q_{mar} scale excellently as Re^{-2} . The former is related to the well-known scaling of the maximum transient growth as Re^2 for streamwise independent disturbances [27], connecting the optimal operator perturbation to the crossing of the threshold by the resolvent norm. The Q_{mar} scaling is because rotation is closely related to the ‘best’ perturbation. Thus Q_{mar} is related to $\varepsilon_{\min}$ by a constant factor. A long time ago, Lezius & Johnston [15] noticed that the rotation number for marginal stability scales as Re^{-2} . Our work has shown that non-normality dictates the scaling, and preordains the dramatic drop with Reynolds number. The fact that the exponent is -2 , and is the same for RPPF and RPCF, is thus no longer a surprise.

5. Conclusion

In this study, we have shown why rotation has such a profound effect on the modal stability behaviour of plane shear flows. For rotating plane Poiseuille flow, at a moderate Reynolds number of $O(1000)$, we can get linear exponential instability for rotation number as low as $O(10^{-4})$, whereas when there is zero rotation, the first instability appears at 5772.2. For the plane Couette flow, a system where exponential instabilities do not exist for all Reynolds numbers, exponentially growing modes may be excited when a small rate of rotation is introduced.

Hitherto, to our knowledge, the non-normality of the stability operator in shear flows has been associated primarily with transient algebraic growth, which could lead to subcritical transition. Here, we have shown that non-normal effects are more powerful, in that they can also lead to exponentially unstable modes when suitable perturbations are made to the hydrodynamic

operator. We have taken the cases of two rotating flows, and shown that the sharp drop in critical Reynolds number for extremely small rotation is inevitably ordained by the stability operator of the corresponding non-rotating flows. Not just the critical Reynolds number, but also the shape and localization of the least stable eigenfunction are embedded in the operator of the non-rotating flow. Inverting the problem, in the small rotation limit, to study the instability-critical Rotation number as a function of the Reynolds number, we show that in both RPPF and RPCF the minimum rotation required to obtain exponential growth is related by a constant to the strength of the minimum random perturbation of the non-rotating operator required to cross the neutral boundary. Thus, both fall as Re^{-2} .

Given that almost any shear flow has a non-normal linear stability operator, it is worth examining the instances of sensitive dependence of instability threshold on small variations in flow geometry, viscosity and density stratification, and body forces to see if non-normality of the unperturbed operator is the driver there as well [40–45]. In other words, this facet of non-normality could possibly provide the explanation for the dramatic changes in instability critical Reynolds number in situations like very weakly diverging pipe and channel flows, and in boundary layers in adverse and favourable pressure gradients, which remain unexplained.

Data accessibility. The source code used to generate the results is provided as electronic supplementary material.

Authors' contributions. Both authors contributed to theoretical aspects of the paper, and the first author performed all the computations.

Competing interests. The authors state there are no competing interests.

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