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Green's functions for unsymmetric composite laminates with inclusions

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It is known that the stretching and bending deformations will be coupled together for the unsymmetric composite laminates under in-plane force and/or out-of-plane bending moment. Although Green's functions for unsymmetric composite laminates with elliptical elastic inclusions have been obtained by using Stroh-like formalism around 10 years ago, due to the ignoring of inconsistent rigid body movements of matrix and inclusion, the existing solution may lead to displacement discontinuity across the interface between matrix and inclusion. Due to the multi-valued characteristics of complex logarithmic functions appeared in Green's functions, special attention should be made on the proper selection of branch cuts of mapped variables. To solve these problems, in this study, the existing Green's functions are corrected and a simple way to correctly evaluate the mapped complex variable logarithmic functions is suggested. Moreover, to apply the obtained solutions to boundary element method, we also derive the explicit closed-form solution for Green's function of deflection. Since the continuity conditions along the interface have been satisfied in Green's functions, no meshes are required along the interface, which will save a lot of computational time and the results are much more accurate than any other numerical methods.

1. Introduction

After the pioneering work of Eshelby [1,2], the stress analyses considering the elliptical, circular and line inclusions have aroused appreciable concern, such as the in-plane problem [3–14], the antiplane problem [9,10,12,15–19] and the out-of-plane bending problem [7,13,20–22]. Note that the term 'inclusion' here denotes a

material whose elastic properties are different from those of the matrix, which is different from the one defined as a subdomain of homogeneous media inside which the eigenstrain occurs [23]. In these studies, the material properties are usually assumed to be symmetric with respect to the mid-plane, therefore the in-plane stretching and the out-of-plane bending are decoupled and can be individually analysed. For unsymmetric laminates, however, the material properties possess no symmetry with respect to the mid-plane, and the stretching and bending deformations would be coupled together. Based upon the Kirchhoff's assumptions made in the classical lamination theory [24], some complex variable formalisms for stretching–bending coupling analysis have been proposed in the literature. Beom & Earmme [25] provided a complex variable method for the laminated plate composed of isotropic laminae. If the laminates are made up of multiple orthotropic layers or multiple anisotropic layers, several different complex variable formalisms have been developed by using different basic variables such as displacement-based [26–29], and mixed-based [29–31]. Among them, the Stroh-like formalism proposed by Hwu [29] is purposely arranged to possess the same mathematical form, both the general solution and its associated material eigen-relation, as the Stroh formalism for two-dimensional linear anisotropic elasticity [32,33]. With such advantageous form, almost all the mathematical techniques developed in Stroh formalism can be directly applied to the coupled stretching–bending analysis.

Through the complex variable formalisms proposed in the literature, some analytical solutions for the inclusion problems have been obtained for the unsymmetric laminated plates. The field solution of a thin laminated plate with an elliptical elastic inclusion subjected to uniform load at infinity was derived in [34]. For the extreme cases such as holes or cracks, some associated solutions can be found in [27,31,35–37]. However, due to the lack of consideration to single-valued deflection, the application of these solutions is restricted to some special cases such as the cross-ply laminates without shear or twisting load. To satisfy the single-valued deflection, Wu & Hsiao [38] revised the traction-free condition of stress functions, and derived the explicit solutions for anisotropic plates with elliptical holes under uniform load at infinity, which is for the problems of out-of-plane bending and is valid only for the symmetric laminated plates. For the behaviour of interface, Wang & Zhou [39] studied the three-phase inclusions for isotropic laminated plates under uniform load.

To consider Green's functions for unsymmetric composite laminates, the loads applied on the laminates are the concentrated forces and/or bending moments at the arbitrary location. This solution can provide more applicability by serving as the kernel functions of integral equations or the fundamental solutions in boundary element method (BEM). The basic Green's function for the infinite homogeneous laminates can be found in [40–46], and that for the laminates with an elliptical hole or crack was presented in [47–52]. Some other Green's functions related to the thin laminated plates are: a semi-infinite laminate [43,46,51], a rigid line inclusion [49], a multi-wedge composed of isotropic laminated plates [52], an elliptical rigid/elastic inclusion [51–53], etc. General cases of static Green's functions for anisotropic media, two dimensional or three dimensional, can be found in the book of Pan & Chen [54].

Green's function obtained in [53] was solved by using the Stroh-like formalism, in which all types of concentrated forces and bending moments are considered and the location of the loads can be outside or inside the elastic inclusion. By limiting the geometry and properties of the elastic inclusion, most of the special Green's functions such as those for holes, rigid inclusions or cracks can be reduced from this general one. This Green's function is well general and complete, and is suitable for the development of a special boundary element for the unsymmetric laminates with inclusions. However, recently when we tried to employ this function to BEM, we found some errors and deficiencies in this existing solution, such as displacement discontinuity across the matrix/inclusion interface, lack of closed-form explicit solution for Green's function of deflection, etc. Moreover, due to the multi-valued characteristics of complex logarithmic functions appeared in Green's function, special attention should be made on the proper selection of branch cuts of mapped variables as mentioned in [55]. In this paper, we will provide a simple way to evaluate the complex logarithmic functions and the solutions provided in [53] will be corrected and further simplified. To verify our correction and to illustrate its applicability to BEM, two representative

examples are presented and compared with the available analytical solutions and the solutions by commercial finite-element software ANSYS.

2. Stroh-like formalism for coupled stretching–bending analysis

According to Kirchhoff's assumptions, the kinematic relations, the constitutive laws and the equilibrium equations for the coupled stretching and bending analysis of thin laminated plates can be written in terms of tensor notation as [29]

$$\left. \begin{aligned} \varepsilon_{ij} &= \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \kappa_{ij} = \frac{1}{2}(\beta_{i,j} + \beta_{j,i}), \quad \beta_i = -w_{,i} \\ N_{ij} &= A_{ijkl}\varepsilon_{kl} + B_{ijkl}\kappa_{kl}, \quad M_{ij} = B_{ijkl}\varepsilon_{kl} + D_{ijkl}\kappa_{kl}, \\ N_{ij,j} &= 0, \quad M_{ij,ij} + q = 0, \quad Q_i = M_{ij,j}, \quad i, j, k, l = 1, 2, \end{aligned} \right\} \quad (2.1)$$

in which the repeated indices imply summation through 1 to 2, and a subscript comma stands for differentiation; u_i and β_i are the mid-plane displacements and negative slopes in x_i -directions, while w is the mid-plane deflection; ε_{ij} and κ_{ij} are the mid-plane strains and plate curvatures, respectively; N_{ij} , M_{ij} and Q_i denote the stress resultants, bending moments and transverse shear forces, respectively; A_{ijkl} , B_{ijkl} and D_{ijkl} are, respectively, the extensional, coupling and bending stiffness tensors; q is the lateral distributed load applied on the laminates.

In order to find a solution satisfying all the basic equations stated in (2.1), several different complex variable formalisms have been proposed in the literature [26–31]. No matter which kind of formalism is used, the solution is generally expressed in terms of four complex functions $f_\alpha(z_\alpha)$, $\alpha = 1, 2, 3, 4$ whose variables $z_\alpha = x_1 + \mu_\alpha x_2$. Here, x_1 and x_2 are two coordinate variables, and μ_α are complex numbers determined by the laminate properties. If the laminate is isotropic, it has been proved that μ_α are repeated and equal to the pure imaginary number i [28], i.e. $\mu_1 = \mu_2 = \mu_3 = \mu_4 = i$. In order to cover all the possible situations of symmetric and unsymmetric laminates, most of the complex variable formalisms consider the general cases of non-degenerate anisotropic plates whose material eigenvalues μ_α are distinct. For the degenerate cases such as the isotropic laminates, small perturbation of material properties can be used to suit for the formalism constructed by the assumption of distinct material eigenvalues [26–30]. With this understanding, in this study we employ the Stroh-like complex variable formalism for coupled stretching–bending analysis of general laminates, in which a general solution satisfying all the basic equations of (2.1) can be expressed as [33]

$$\mathbf{u}_d = 2\text{Re}\{\mathbf{A}\mathbf{f}(z)\} \quad \text{and} \quad \boldsymbol{\phi}_d = 2\text{Re}\{\mathbf{B}\mathbf{f}(z)\}, \quad (2.2a)$$

where

$$\left. \begin{aligned} \mathbf{u}_d &= \begin{Bmatrix} u_1 \\ u_2 \\ \beta_1 \\ \beta_2 \end{Bmatrix}, \quad \boldsymbol{\phi}_d = \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \psi_1 \\ \psi_2 \end{Bmatrix}, \quad \mathbf{f}(z) = \begin{Bmatrix} f_1(z_1) \\ f_2(z_2) \\ f_3(z_3) \\ f_4(z_4) \end{Bmatrix}, \\ \mathbf{A} &= [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4], \quad \mathbf{B} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{b}_4], \\ z_\alpha &= x_1 + \mu_\alpha x_2, \quad \alpha = 1, 2, 3, 4. \end{aligned} \right\} \quad (2.2b)$$

In (2.2a), Re stands for the real part of a complex number. In (2.2b), ϕ_i , $i = 1, 2$, are the stress functions related to in-plane forces N_{ij} , and ψ_i , $i = 1, 2$, are the stress functions related to bending moments M_{ij} , transverse shear forces Q_i and effective transverse shear forces V_i . Their relations

are

$$\left. \begin{aligned} N_{11} &= -\phi_{1,2}, & N_{22} &= \phi_{2,1}, & N_{12} &= \phi_{1,1} = -\phi_{2,2} = N_{21}, \\ M_{11} &= -\psi_{1,2}, & M_{22} &= \psi_{2,1}, & M_{12} &= \frac{(\psi_{1,1} - \psi_{2,2})}{2} = M_{21}, \\ Q_1 &= -\eta_{,2}, & Q_2 &= \eta_{,1}, & \eta &= \frac{(\psi_{1,1} + \psi_{2,2})}{2}, \\ V_1 &= -\psi_{2,22}, & V_2 &= \psi_{1,11}. \end{aligned} \right\} \quad (2.3)$$

The material eigenvalues μ_α and their associated eigenvectors $\mathbf{a}_\alpha$ and $\mathbf{b}_\alpha$ can be determined from the following eigen-relation:

$$\mathbf{N}\boldsymbol{\xi} = \mu\boldsymbol{\xi}, \quad (2.4a)$$

where

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{bmatrix} \quad \text{and} \quad \boldsymbol{\xi} = \begin{Bmatrix} \mathbf{a} \\ \mathbf{b} \end{Bmatrix}, \quad (2.4b)$$

and the superscript T stands for the transpose of a matrix. $\mathbf{N}_1$, $\mathbf{N}_2$ and $\mathbf{N}_3$ are three 4×4 real matrices related to the extensional, coupling and bending stiffness. With the eigen-relation (2.4), the material eigenvalues μ_α have been proved to be four pairs of complex conjugates [28], and hence the ones with positive imaginary parts are purposely arranged to be the first four eigenvalues in the Stroh-like formalism. Their associated complex functions $f_\alpha(z_\alpha)$, $\alpha = 1, 2, 3, 4$, will then be determined through the satisfaction of boundary conditions.

From the relations given in (2.1) and (2.3), we know that the deflection w , mid-plane strains ε_{ij} , curvatures κ_{ij} , stress resultants N_{ij} , bending moments M_{ij} , transverse shear forces Q_i , and effective transverse shear forces V_i , are related to $\mathbf{u}_d$ and $\boldsymbol{\phi}_d$ by

$$w = - \int \beta_1 dx_1 = - \int \beta_2 dx_2, \quad (2.5a)$$

$$\boldsymbol{\varepsilon}_1 = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \kappa_{11} \\ \kappa_{12} \end{Bmatrix} = \begin{Bmatrix} u_{1,1} \\ u_{2,1} - \varepsilon_0 \\ \beta_{1,1} \\ \beta_{2,1} \end{Bmatrix}, \quad \boldsymbol{\varepsilon}_2 = \begin{Bmatrix} \varepsilon_{12} \\ \varepsilon_{22} \\ \kappa_{12} \\ \kappa_{22} \end{Bmatrix} = \begin{Bmatrix} u_{1,2} + \varepsilon_0 \\ u_{2,2} \\ \beta_{1,2} \\ \beta_{2,2} \end{Bmatrix}, \quad (2.5b)$$

$$\boldsymbol{\sigma}_1 = \begin{Bmatrix} N_{11} \\ N_{12} \\ M_{11} \\ M_{12} \end{Bmatrix} = - \begin{Bmatrix} \phi_{1,2} \\ \phi_{2,2} \\ \psi_{1,2} \\ \psi_{2,2} - \eta \end{Bmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{Bmatrix} N_{12} \\ N_{22} \\ M_{12} \\ M_{22} \end{Bmatrix} = \begin{Bmatrix} \phi_{1,1} \\ \phi_{2,1} \\ \psi_{1,1} - \eta \\ \psi_{2,1} \end{Bmatrix}, \quad (2.5c)$$

$$\text{and} \quad \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{Bmatrix} -\eta_{,2} \\ \eta_{,1} \end{Bmatrix}, \quad \begin{Bmatrix} V_1 \\ V_2 \end{Bmatrix} = \begin{Bmatrix} -\psi_{2,22} \\ \psi_{1,11} \end{Bmatrix}, \quad (2.5d)$$

where

$$\varepsilon_0 = \frac{1}{2}(u_{2,1} - u_{1,2}). \quad (2.5e)$$

Equations (2.5b,c) can also be written in matrix form as

$$\left. \begin{aligned} \boldsymbol{\varepsilon}_1 &= \mathbf{u}_{d,1} - \varepsilon_0 \mathbf{i}_2, & \boldsymbol{\varepsilon}_2 &= \mathbf{u}_{d,2} + \varepsilon_0 \mathbf{i}_1, \\ \boldsymbol{\sigma}_1 &= -\boldsymbol{\phi}_{d,2} + \eta \mathbf{i}_4, & \boldsymbol{\sigma}_2 &= \boldsymbol{\phi}_{d,1} - \eta \mathbf{i}_3, \end{aligned} \right\} \quad (2.6a)$$

where $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$, $\mathbf{i}_4$, are the base vectors defined by

$$\mathbf{i}_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \mathbf{i}_2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix}, \quad \mathbf{i}_3 = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \quad \text{and} \quad \mathbf{i}_4 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix}. \quad (2.6b)$$

From (2.5a) and (2.2a), we obtain

$$w = -2\mathbf{i}_3^T \text{Re}\{\mathbf{A}\tilde{\mathbf{f}}(z)\} = -2\mathbf{i}_4^T \text{Re}\{\mathbf{A}\langle\mu_\alpha^{-1}\rangle\tilde{\mathbf{f}}(z)\}, \quad \tilde{\mathbf{f}}(z) = \int \mathbf{f}(z)dz, \quad (2.7)$$

Figure 1. (a) A composite laminate with an elastic inclusion subjected to concentrated loads. (b) Cross section of the laminate.

in which the angular bracket $\langle \rangle$ stands for a 4×4 diagonal matrix with the components varied as the subscript α , e.g. $\langle \mu_\alpha^{-1} \rangle = \text{diag}[\mu_1^{-1}, \mu_2^{-1}, \mu_3^{-1}, \mu_4^{-1}]$. Note that $\tilde{f}(z)$, the integral of $f(z)$, can be evaluated numerically by the standard Gaussian quadrature rule as

$$\tilde{f}(z) = \begin{Bmatrix} \tilde{f}_1(z_1) \\ \tilde{f}_2(z_2) \\ \tilde{f}_3(z_3) \\ \tilde{f}_4(z_4) \end{Bmatrix} = \begin{Bmatrix} \int_0^{z_1} f_1(z_1) dz_1 \\ \int_0^{z_2} f_2(z_2) dz_2 \\ \int_0^{z_3} f_3(z_3) dz_3 \\ \int_0^{z_4} f_4(z_4) dz_4 \end{Bmatrix} \cong \begin{Bmatrix} \left(\frac{z_1}{2}\right) \sum_{i=1}^{n_G} w_i f_1\left(\frac{(t_i+1)z_1}{2}\right) \\ \left(\frac{z_2}{2}\right) \sum_{i=1}^{n_G} w_i f_2\left(\frac{(t_i+1)z_2}{2}\right) \\ \left(\frac{z_3}{2}\right) \sum_{i=1}^{n_G} w_i f_3\left(\frac{(t_i+1)z_3}{2}\right) \\ \left(\frac{z_4}{2}\right) \sum_{i=1}^{n_G} w_i f_4\left(\frac{(t_i+1)z_4}{2}\right) \end{Bmatrix}, \quad (2.8)$$

where n_G is the number of Gaussian points, t_i and w_i are the abscissae and weights of Gaussian quadrature rule, and the lower integral limit is arbitrarily chosen. Note that the second equality of (2.7)₁ can be proved by referring to the explicit expressions of **A** given in [56].

3. The existing Green's functions for laminates with inclusions

Consider an infinite composite laminate containing an elliptical inclusion subjected to a concentrated force $\hat{\mathbf{f}} = (\hat{f}_1, \hat{f}_2, \hat{f}_3)$ and moment $\hat{\mathbf{m}} = (\hat{m}_1, \hat{m}_2, \hat{m}_3)$ at point $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2)$ as shown in figure 1. The inclusion and the matrix are assumed to be perfectly bonded along the interface. The contour of the inclusion boundary is represented by

$$x_1 = a \cos \varphi \quad \text{and} \quad x_2 = b \sin \varphi, \quad (3.1)$$

where $2a$ and $2b$ are the major and minor axes of the ellipse, respectively and φ is a real parameter. For the problems with elliptical boundary, the following transformed complex variables ζ_α are usually used

$$\zeta_\alpha = \frac{z_\alpha + \sqrt{z_\alpha^2 - a^2 - b^2 \mu_\alpha^2}}{a - ib \mu_\alpha}, \quad \alpha = 1, 2, 3, 4. \quad (3.2)$$

Through the relations (2.2b)₆ and (3.2), an ellipse with axes $2a$ and $2b$ in z -plane ($z = x_1 + ix_2$) will be mapped onto a unit circle in ζ_α -plane.

(a) Loads outside the inclusion

When the concentrated force $\hat{\mathbf{f}} = (\hat{f}_1, \hat{f}_2, \hat{f}_3)$ and moment $\hat{\mathbf{m}} = (\hat{m}_1, \hat{m}_2, \hat{m}_3)$ are outside the inclusion, the field solution can be expressed as

$$\left. \begin{aligned} \mathbf{u}_d^{(1)} &= 2\text{Re}\{\mathbf{A}_1[\mathbf{f}_0(\zeta) + \mathbf{f}_1(\zeta)]\}, & \phi_d^{(1)} &= 2\text{Re}\{\mathbf{B}_1[\mathbf{f}_0(\zeta) + \mathbf{f}_1(\zeta)]\}, \\ \mathbf{u}_d^{(2)} &= 2\text{Re}\{\mathbf{A}_2\mathbf{f}_2(\zeta^*)\}, & \phi_d^{(2)} &= 2\text{Re}\{\mathbf{B}_2\mathbf{f}_2(\zeta^*)\}, \end{aligned} \right\} \quad (3.3)$$

where the subscripts 1 and 2 or the superscripts (1) and (2) denote, respectively, the values related to the matrix and the inclusion. The variable with the superscript * denotes the value related to the properties of inclusion, e.g.

$$\zeta_\alpha^* = \frac{z_\alpha^* + \sqrt{(z_\alpha^*)^2 - a^2 - b^2(\mu_\alpha^*)^2}}{a - ib\mu_\alpha^*}, \quad z_\alpha^* = x_1 + \mu_\alpha^*x_2, \quad \alpha = 1, 2, 3, 4. \quad (3.4)$$

In (3.3), the complex function vectors $\mathbf{f}_0$, $\mathbf{f}_1$ and $\mathbf{f}_2$ have been solved as [53]

$$\left. \begin{aligned} \mathbf{f}_0(\zeta) &= \frac{1}{2\pi i} \left\{ \langle \ln(\zeta_\alpha - \hat{\zeta}_\alpha) \rangle \mathbf{A}_1^T \hat{\mathbf{p}} + \hat{f}_3 \langle g(\zeta_\alpha) \rangle \mathbf{A}_1^T \mathbf{i}_3 - \hat{m}_3 \left\langle \frac{c_{4\alpha} \hat{\zeta}_\alpha}{\zeta_\alpha - \hat{\zeta}_\alpha} \right\rangle \mathbf{A}_1^T \mathbf{i}_2 \right\}, \\ \mathbf{f}_1(\zeta) &= \sum_{k=1}^{\infty} \langle \zeta_\alpha^{-k} \rangle \mathbf{d}_k, \quad \mathbf{f}_2(\zeta^*) = \sum_{k=1}^{\infty} \left\langle (\zeta_\alpha^*)^k + \left(\frac{\gamma_\alpha^*}{\zeta_\alpha^*} \right)^k \right\rangle \mathbf{c}_k, \end{aligned} \right\} \quad (3.5a)$$

where

$$\left. \begin{aligned} g(\zeta_\alpha) &= (z_\alpha - \hat{z}_\alpha) \ln(\zeta_\alpha - \hat{\zeta}_\alpha) + c_{2\alpha}(\zeta_\alpha - \hat{\zeta}_\alpha) - c_{3\alpha}(\zeta_\alpha^{-1} - \hat{\zeta}_\alpha^{-1}), \\ c_{2\alpha} &= c_\alpha(\ln c_\alpha - 1), \quad c_{3\alpha} = c_\alpha \gamma_\alpha \ln(-\hat{\zeta}_\alpha), \quad c_{4\alpha} = \frac{1}{c_\alpha(\hat{\zeta}_\alpha - \gamma_\alpha/\hat{\zeta}_\alpha)}, \\ c_\alpha &= \frac{1}{2}(a - ib\mu_\alpha), \quad \gamma_\alpha = \frac{a + ib\mu_\alpha}{a - ib\mu_\alpha}, \end{aligned} \right\} \quad (3.5b)$$

and

$$\hat{\mathbf{p}} = [\hat{f}_1 \quad \hat{f}_2 \quad \hat{m}_2 \quad -\hat{m}_1]^T. \quad (3.5c)$$

Besides, $\mathbf{c}_k$ and $\mathbf{d}_k$ are constant vectors, whose definitions are given in appendix A.

(b) Loads inside the inclusion

When the concentrated force $\hat{\mathbf{f}} = (\hat{f}_1, \hat{f}_2, \hat{f}_3)$ and moment $\hat{\mathbf{m}} = (\hat{m}_1, \hat{m}_2, \hat{m}_3)$ are inside the inclusion, the field solution can be expressed as [53]

$$\left. \begin{aligned} \mathbf{u}_d^{(1)} &= 2\text{Re}\{\mathbf{A}_1[\mathbf{f}_0(\zeta) + \mathbf{f}_1(\zeta)]\}, & \phi_d^{(1)} &= 2\text{Re}\{\mathbf{B}_1[\mathbf{f}_0(\zeta) + \mathbf{f}_1(\zeta)]\}, \\ \mathbf{u}_d^{(2)} &= 2\text{Re}\{\mathbf{A}_2[\mathbf{f}_0^*(\zeta^*) + \mathbf{f}_2(\zeta^*)]\}, & \phi_d^{(2)} &= 2\text{Re}\{\mathbf{B}_2[\mathbf{f}_0^*(\zeta^*) + \mathbf{f}_2(\zeta^*)]\}, \end{aligned} \right\} \quad (3.6a)$$

where

$$\left. \begin{aligned} \mathbf{f}_0(\zeta) &= \frac{1}{2\pi i} \left\{ \langle \ln \zeta_\alpha \rangle \left[\langle \zeta_\alpha \rangle \mathbf{v}_1 + \langle \zeta_\alpha^{-1} \rangle \mathbf{v}_{-1} + \mathbf{v}_0 \right] + \langle \zeta_\alpha^{-1} \rangle \mathbf{k}_{-1} + \langle \zeta_\alpha \rangle \mathbf{k}_1 \right\}, \\ \mathbf{f}_0^*(\zeta^*) &= \frac{1}{2\pi i} \left\{ \langle \ln(z_\alpha^* - \hat{z}_\alpha^*) \rangle \mathbf{A}_2^T \hat{\mathbf{p}} + \hat{f}_3 \langle (z_\alpha^* - \hat{z}_\alpha^*) [\ln(z_\alpha^* - \hat{z}_\alpha^*) - 1] \rangle \mathbf{A}_2^T \mathbf{i}_3 - \hat{m}_3 \langle (z_\alpha^* - \hat{z}_\alpha^*)^{-1} \rangle \mathbf{A}_2^T \mathbf{i}_2 \right\} \\ \text{and } \mathbf{f}_1(\zeta) &= \sum_{k=1}^{\infty} \langle \zeta_\alpha^{-k} \rangle \mathbf{d}_k, \quad \mathbf{f}_2(\zeta^*) = \sum_{k=1}^{\infty} \left\langle (\zeta_\alpha^*)^k + \left(\frac{\gamma_\alpha^*}{\zeta_\alpha^*} \right)^k \right\rangle \mathbf{c}_k. \end{aligned} \right\} \quad (3.6b)$$

In the above, $\mathbf{v}_1, \mathbf{v}_{-1}, \mathbf{v}_0, \mathbf{k}_{-1}, \mathbf{k}_1, \mathbf{c}_k$ and $\mathbf{d}_k$ are all constant vectors whose details are given in appendix A.

4. Simplification and correction of the existing Green's functions

Although Green's function has been solved around 10 years ago, due to its complexity and discontinuity problem till now no successful application has been presented. Since Green's function is the key function for the development of BEM, the simplification and correction of the existing Green's function are unavoidable in order to deal with the problems with unsymmetric composite laminates containing elastic inclusions.

(a) Simplification

From the solutions shown in (3.5a) and (3.6b), we see that their relations with the concentrated forces and moments have all been expressed explicitly. Among them, the relation for $\mathbf{f}_0(\zeta)$ of (3.6b) when the loads are applied inside the inclusion is relatively implicit and complicated than the others. In this study, with the aids of the following identities [33]:

$$\left. \begin{aligned} c_\alpha(1 + \gamma_\alpha) &= a, \quad c_\alpha(1 - \gamma_\alpha) = -ib\mu_\alpha, \quad c_\alpha(\zeta_\alpha + \gamma_\alpha/\zeta_\alpha) = z_\alpha, \\ \mathbf{H} &= 2i\mathbf{A}\mathbf{A}^T, \quad \mathbf{L} = -2i\mathbf{B}\mathbf{B}^T, \quad \mathbf{S} = i(2\mathbf{A}\mathbf{B}^T - \mathbf{I}), \\ 2\mathbf{A}\langle\mu_\alpha\rangle\mathbf{A}^T &= \mathbf{N}_2 - i(\mathbf{N}_1\mathbf{H} + \mathbf{N}_2\mathbf{S}^T), \\ 2\mathbf{B}\langle\mu_\alpha\rangle\mathbf{A}^T &= \mathbf{N}_1^T - i(\mathbf{N}_3\mathbf{H} + \mathbf{N}_1^T\mathbf{S}^T), \\ 2\mathbf{A}\langle z_\alpha\rangle\mathbf{A}^T &= -ix_1\mathbf{H} + x_2\{\mathbf{N}_2 - i(\mathbf{N}_1\mathbf{H} + \mathbf{N}_2\mathbf{S}^T)\}, \\ 2\mathbf{B}\langle z_\alpha\rangle\mathbf{A}^T &= x_1(\mathbf{I} - i\mathbf{S}^T) + x_2\{\mathbf{N}_1^T - i(\mathbf{N}_3\mathbf{H} + \mathbf{N}_1^T\mathbf{S}^T)\}, \\ \mathbf{A}^T\mathbf{i}_4 &= \langle\mu_\alpha\rangle\mathbf{A}^T\mathbf{i}_3, \quad \mathbf{N}_1^T\mathbf{i}_3 = \mathbf{i}_4, \quad \mathbf{N}_2\mathbf{i}_3 = \mathbf{0}, \end{aligned} \right\} \quad (4.1)$$

the solution for $\mathbf{f}_0(\zeta)$ of (3.6b) can be further simplified as

$$\mathbf{f}_0(\zeta) = \frac{1}{2\pi i} \left\{ \langle \ln \zeta_\alpha \rangle \mathbf{A}_1^T \hat{\mathbf{p}} + \hat{f}_3 \langle (z_\alpha - \hat{z}_\alpha) \ln \zeta_\alpha \rangle \mathbf{A}_1^T \mathbf{i}_3 + \langle \zeta_\alpha^{-1} \rangle \mathbf{k}_{-1} + \langle \zeta_\alpha \rangle \mathbf{k}_1 \right\}. \quad (4.2)$$

Note that $\mathbf{S}$, $\mathbf{H}$ and $\mathbf{L}$ of (4.1) are three 4×4 real matrices related to the extensional, coupling and bending stiffness, and are usually called *Barnett–Lothe tensors* in Stroh formalism.

(b) Correction

One of the requirements for Green's functions is the satisfaction of interface continuity, which can be expressed as [53]

$$\mathbf{u}_d^{(1)} = \mathbf{u}_d^{(2)}, \quad \boldsymbol{\phi}_d^{(1)} = \boldsymbol{\phi}_d^{(2)}, \quad w^{(1)} = w^{(2)}. \quad (4.3)$$

Usually for a solid with homogeneous materials, the constant terms appearing in the function vector $\mathbf{f}(z)$ of (2.2a) can be ignored since it represents rigid body motion. However, for the problems with heterogeneous materials, such as the one with elastic inclusion, the constant terms cannot be ignored due to the difference among the materials. Otherwise, the displacements would become discontinuous due to the misfit across the interface as shown in figures 2 and 3 by using the existing Green's functions presented in §3. With this consideration and after deliberately inspecting the derivation in [53], we found that the interface continuity conditions (4.3)_{1,2} were improperly enforced in [53] during the derivation of the existing Green's functions shown in (3.3)–(3.6) where some constant terms were missing. They passed their analytical verification through the reduction to two special cases: (1) without inclusion (made by the same matrix and inclusion) and (2) hole (treated as a very soft inclusion). Since only one material is considered in these two special cases, missing of the constant terms (usually represents ignoring of rigid body motion) will not influence the final results of stresses and deformations. However, it really affects the correctness of the general cases of elastic inclusions. Thus, we know that the violation of interface continuity for the existing Green's functions derived in [53] is totally due to the missing constant terms. In order to find the missing constant terms to correct these solutions such that the

Figure 2. Contour plot of displacement u_1 using the existing Green's function (3.3)–(3.5). (Online version in colour.)

satisfaction of continuity conditions (4.3)_{1,2} are guaranteed, we express the related logarithmic functions by Taylor's expansion such as

$$\left. \begin{aligned} \ln(\zeta_\alpha - \hat{\zeta}_\alpha) &= \ln(-\hat{\zeta}_\alpha) - \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{\zeta_\alpha}{\hat{\zeta}_\alpha} \right)^k, \\ \ln(z_\alpha^* - \hat{z}_\alpha^*) &= \ln \left\{ c_\alpha^* (\zeta_\alpha^* - \hat{\zeta}_\alpha^*) \left(1 - \frac{\gamma_\alpha^*}{\hat{\zeta}_\alpha^* \zeta_\alpha^*} \right) \right\} \\ &= \ln c_\alpha^* + \ln \zeta_\alpha^* - \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{\hat{\zeta}_\alpha^*}{\zeta_\alpha^*} \right)^k - \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{\gamma_\alpha^*}{\hat{\zeta}_\alpha^* \zeta_\alpha^*} \right)^k. \end{aligned} \right\} \quad (4.4)$$

With the series expressions of (4.4), we see that

$$\left. \begin{aligned} \ln(\zeta_\alpha - \hat{\zeta}_\alpha) &: \text{ holomorphic in } |\zeta_\alpha| \leq 1 \text{ if } |\hat{\zeta}_\alpha| > 1, \\ \ln(z_\alpha^* - \hat{z}_\alpha^*) - \ln \zeta_\alpha^* &: \text{ holomorphic in } |\zeta_\alpha^*| \geq 1 \text{ if } |\hat{\zeta}_\alpha^*| < 1. \end{aligned} \right\} \quad (4.5)$$

Using (4.4) and (4.5) with reference to the derivation of the original Green's functions presented in [53], we found that the continuity condition (4.3) can be satisfied by making the following correction for $\mathbf{f}_0(\zeta)$ of (3.5a) and $\mathbf{f}_0^*(\zeta^*)$ of (3.6b).

$$\left. \begin{aligned} \mathbf{f}_0^{\text{new}}(\zeta) &= \mathbf{f}_0(\zeta) - \frac{1}{2\pi i} \left\{ \langle \ln(-\hat{\zeta}_\alpha) \rangle \mathbf{A}_1^T \hat{\mathbf{p}} + \hat{f}_3 \langle c_\alpha \hat{\zeta}_\alpha [1 - \ln(-c_\alpha \hat{\zeta}_\alpha) - \gamma_\alpha \hat{\zeta}_\alpha^{-2}] \rangle \mathbf{A}_1^T \mathbf{i}_3 + \hat{m}_3 \langle c_{4\alpha} \rangle \mathbf{A}_1^T \mathbf{i}_2 \right\}, \\ \mathbf{f}_0^{*\text{new}}(\zeta^*) &= \mathbf{f}_0^*(\zeta^*) - \frac{1}{2\pi i} \left\{ \langle \ln c_\alpha^* \rangle \mathbf{A}_2^T \hat{\mathbf{p}} - \hat{f}_3 \langle \hat{\zeta}_\alpha^* \ln c_\alpha^* \rangle \mathbf{A}_2^T \mathbf{i}_3 \right\}. \end{aligned} \right\} \quad (4.6)$$

And all the other complex functions remain unchanged.

5. Green's functions of deflection

As stated in (2.5)–(2.7), after getting Green's functions of $\mathbf{u}_d$ and ϕ_d , most of the physical quantities should be evaluated by knowing their derivatives and integrals. If the explicit expressions of $\mathbf{u}_d$ and ϕ_d have been obtained, such as those presented in the previous two sections, usually the

Figure 3. Contour plot of slope β_1 using the existing Green's function (3.6). (Online version in colour.)

derivatives can be derived directly from their expressions. However, the explicit expressions for the integrals cannot be guaranteed to be obtained analytically. This is also why we introduce the numerical integration for the general situation in (2.8). Although the standard Gaussian quadrature rule provides a systematic way to calculate the deflection, the analytical integration, if available, is overwhelming in terms of accuracy and computational efficiency.

To find the deflection w analytically, we use the first equality of (2.7)₁ in which the integrals of $\mathbf{f}$ with respect to the argument z_α are required. Since the solutions shown in §§3 and 4 are in terms of ζ_α , we need to transform the integral of (2.7) into the form of ζ_α through the relation of $z_\alpha = c_\alpha(\zeta_\alpha + \gamma_\alpha/\zeta_\alpha)$. With this relation, the integral of (2.7)₂ can be rewritten as

$$\tilde{\mathbf{f}}(\zeta) = \int \mathbf{f}(\zeta) \frac{\partial z}{\partial \zeta} d\zeta = \langle c_\alpha \rangle \left[\int \mathbf{f}(\zeta) d\zeta - \int \left(\frac{\gamma_\alpha}{\zeta_\alpha^2} \right) \mathbf{f}(\zeta) d\zeta \right]. \quad (5.1)$$

To obtain Green's functions of deflection analytically and to avoid the violation of the interface continuity (4.3)₃ by the integration constants, the following procedure is made in our derivation:

- (1) expand $\mathbf{f}(\zeta)$ into Taylor's series,
- (2) substitute the solution of series form into (5.1),
- (3) evaluate the integrals of each term analytically,
- (4) check the continuity condition (4.3)₃ by adjusting the integration constants,
- (5) merge the series form to the proper logarithmic and fractional functions.

Substituting the solutions obtained in the previous two sections into (5.1), and following the procedure stated above, we obtain Green's functions of deflection as follows.

(a) Loads outside the inclusion

$$\left. \begin{aligned} w^{(1)} &= -2\mathbf{i}_3^T \text{Re}\{\mathbf{A}_1[\tilde{\mathbf{f}}_0(\zeta) + \tilde{\mathbf{f}}_1(\zeta)]\}, \\ w^{(2)} &= -2\mathbf{i}_3^T \text{Re}\{\mathbf{A}_2\tilde{\mathbf{f}}_2(\zeta^*)\}, \end{aligned} \right\} \quad (5.2a)$$

where

$$\begin{aligned}\tilde{f}_0(\zeta) = & \frac{1}{2\pi i} \left\{ \left\langle c_\alpha \left[\frac{\gamma_\alpha}{\hat{\zeta}_\alpha} (\ln \zeta_\alpha + 1) - \zeta_\alpha \right] + (z_\alpha - \hat{z}_\alpha) \ln \left(1 - \frac{\zeta_\alpha}{\hat{\zeta}_\alpha} \right) \right\rangle \mathbf{A}_1^T \hat{\mathbf{p}} \right. \\ & \left. + \hat{f}_3 \left(\tilde{g}(\zeta_\alpha) \right) \mathbf{A}_1^T \mathbf{i}_3 - \hat{m}_3 \left\langle \ln \left(1 - \frac{\zeta_\alpha}{\hat{\zeta}_\alpha} \right) + c_\alpha c_{4\alpha} \left(\zeta_\alpha + \frac{\gamma_\alpha}{\hat{\zeta}_\alpha} \ln \zeta_\alpha \right) \right\rangle \mathbf{A}_1^T \mathbf{i}_2 \right\},\end{aligned}\quad (5.2b)$$

$$\tilde{f}_1(\zeta) = \langle c_\alpha \rangle \left\{ \left\langle \ln \zeta_\alpha + \frac{\gamma_\alpha}{2\zeta_\alpha^2} \right\rangle \mathbf{d}_1 + \sum_{k=2}^{\infty} \left\langle \zeta_\alpha^{-k} \left[\frac{\gamma_\alpha/\zeta_\alpha}{k+1} - \frac{\zeta_\alpha}{k-1} \right] \right\rangle \mathbf{d}_k \right\}, \quad (5.2c)$$

$$\begin{aligned}\tilde{f}_2(\zeta^*) = & \langle c_\alpha^* \rangle \left\{ \frac{1}{2} \left\langle (\zeta_\alpha^*)^2 + \left(\frac{\gamma_\alpha^*}{\zeta_\alpha^*} \right)^2 \right\rangle \mathbf{c}_1 \right. \\ & \left. + \sum_{k=2}^{\infty} \left\langle (\zeta_\alpha^*)^k \left[\frac{\zeta_\alpha^*}{k+1} - \frac{\gamma_\alpha^*/\zeta_\alpha^*}{k-1} \right] + \left(\frac{\gamma_\alpha^*}{\zeta_\alpha^*} \right)^k \left[\frac{\gamma_\alpha^*/\zeta_\alpha^*}{k+1} - \frac{\zeta_\alpha^*}{k-1} \right] \right\rangle \mathbf{c}_k \right\}.\end{aligned}\quad (5.2d)$$

In (5.2b), $\tilde{g}(\zeta_\alpha)$ in $\tilde{f}_0(\zeta)$ is

$$\begin{aligned}\tilde{g}(\zeta_\alpha) = & \frac{1}{2} (z_\alpha - \hat{z}_\alpha)^2 \ln \left(1 - \frac{\zeta_\alpha}{\hat{\zeta}_\alpha} \right) - \frac{1}{2} \left(\frac{c_\alpha \gamma_\alpha}{\hat{\zeta}_\alpha} \right)^2 \ln \zeta_\alpha \\ & + c_\alpha^2 \left[\gamma_\alpha \left(\frac{2\zeta_\alpha}{\hat{\zeta}_\alpha} - 1 \right) - \left(\zeta_\alpha \hat{\zeta}_\alpha + \frac{\gamma_\alpha^2}{\zeta_\alpha \hat{\zeta}_\alpha} \right) \left(\frac{3\zeta_\alpha}{4\hat{\zeta}_\alpha} - \frac{1}{2} \right) + \left(\frac{\zeta_\alpha^2}{2} - \gamma_\alpha \ln \zeta_\alpha \right) \ln(-c_\alpha \hat{\zeta}_\alpha) \right].\end{aligned}\quad (5.3)$$

(b) Loads inside the inclusion

$$\begin{aligned}w^{(1)} = & -2\mathbf{i}_3^T \text{Re}\{\mathbf{A}_1[\tilde{f}_0(\zeta) + \tilde{f}_1(\zeta)]\}, \\ w^{(2)} = & -2\mathbf{i}_3^T \text{Re}\{\mathbf{A}_2[\tilde{f}_0^*(\zeta^*) + \tilde{f}_2(\zeta^*)]\},\end{aligned}\quad (5.4a)$$

where

$$\begin{aligned}\tilde{f}_0(\zeta) = & \frac{1}{2\pi i} \left\{ \tilde{\mathbf{k}}(\zeta_\alpha) + \left\langle z_\alpha \ln \zeta_\alpha - c_\alpha \left(\zeta_\alpha - \frac{\gamma_\alpha}{\hat{\zeta}_\alpha} \right) \right\rangle \mathbf{A}_1^T \hat{\mathbf{p}} \right. \\ & \left. + \hat{f}_3 \left[\left\langle z_\alpha \left(\frac{z_\alpha}{2} - \hat{z}_\alpha \right) - c_\alpha^2 \gamma_\alpha \right\rangle \ln \zeta_\alpha - c_\alpha \left(\frac{z_\alpha}{4} - \hat{z}_\alpha \right) \left(\zeta_\alpha - \frac{\gamma_\alpha}{\hat{\zeta}_\alpha} \right) \right] \mathbf{A}_1^T \mathbf{i}_3 \right\},\end{aligned}\quad (5.4b)$$

$$\begin{aligned}\tilde{f}_0^*(\zeta^*) = & \frac{1}{2\pi i} \left\{ (z_\alpha^* - \hat{z}_\alpha^*) [\ln(z_\alpha^* - \hat{z}_\alpha^*) - \ln c_\alpha^* - 1] \mathbf{A}_2^T \hat{\mathbf{p}} \right. \\ & \left. + \hat{f}_3 \langle p(\zeta_\alpha^*) \rangle \mathbf{A}_2^T \mathbf{i}_3 - \hat{m}_3 \langle \ln(z_\alpha^* - \hat{z}_\alpha^*) - \ln c_\alpha^* \rangle \mathbf{A}_2^T \mathbf{i}_2 \right\}\end{aligned}\quad (5.4c)$$

and

$$\begin{aligned}\tilde{\mathbf{k}}(\zeta_\alpha) = & \left\langle c_\alpha \left(\frac{\zeta_\alpha^2}{2} - \gamma_\alpha \ln \zeta_\alpha \right) \right\rangle \mathbf{k}_1 + \left\langle c_\alpha \left(\frac{\gamma_\alpha}{2\zeta_\alpha^2} + \ln \zeta_\alpha \right) \right\rangle \mathbf{k}_{-1}, \\ p(\zeta_\alpha^*) = & \frac{1}{2} (z_\alpha^* - \hat{z}_\alpha^*)^2 \left[\ln(z_\alpha^* - \hat{z}_\alpha^*) - \ln c_\alpha^* - \frac{3}{2} \right] + \frac{1}{2} (z_\alpha^*)^2 \ln c_\alpha^* - (c_\alpha^*)^2 \gamma_\alpha^* (\ln c_\alpha^* - 1).\end{aligned}\quad (5.4d)$$

The integrals of $\mathbf{f}_1$ and $\mathbf{f}_2$ have the same form as those shown in (5.2c) and (5.2d), respectively.

6. Evaluation of complex logarithmic functions

It is well known that the complex logarithmic function is a multi-valued function, and a single function value can be obtained by choosing a region with proper branch cut, which may induce the problem of discontinuity. For example, $\ln z = \ln r + i\theta$ is single-valued if a proper branch $(-\pi, \pi]$ is selected for θ , and the discontinuity occurs at the point $z = re^{\pm i\pi} = -r$ that across the cut. For a continuous body this discontinuity is not allowed to occur for the physical quantities such

Figure 4. Branch cuts in (a) z -plane, (b) z_α -plane and (c) ζ_α -plane. ($a = 0.2 \text{ m}$, $b/a = 0.75$, $\hat{x}_1/a = 2$, $\hat{x}_2/a = 1$, $\mu_1 = -0.7227 + 1.974i$, $\mu_2 = 0.7145 + 1.062i$, $\mu_3 = -0.7393 + 0.6735i$, $\mu_4 = 9.377 \times 10^{-4} + 0.6016i$). (Online version in colour.)

as displacements. Thus, during the derivation for Green's functions, the requirement of single-valued displacement should be satisfied. As stated in [55] this satisfaction requires simultaneous jumping across all the branch cuts for the subscript $\alpha = 1, 2, 3, 4$. Consider two adjacent points $\mathbf{x}^+$ and $\mathbf{x}^-$ upper and below the branch cut, if $\mathbf{x}^+ - \hat{\mathbf{x}} = (-R, \varepsilon)$ and $\mathbf{x}^- - \hat{\mathbf{x}} = (-R, -\varepsilon)$, where $R, \varepsilon > 0$, and the material eigenvalues $\mu_\alpha = \mu_\alpha^R + i\mu_\alpha^I$ where $\mu_\alpha^I > 0$, we can prove that (also figure 4b)

$$(\ln(z_\alpha^+ - \hat{z}_\alpha) - \ln(z_\alpha^- - \hat{z}_\alpha)) = 2\pi i \mathbf{I}, \text{ when } \varepsilon \rightarrow 0. \quad (6.1)$$

in which $\mathbf{I}$ denotes the identity matrix.

When the logarithmic function is expressed in terms of the mapped complex variables such as ζ_α , $\alpha = 1, 2, 3, 4$, defined in (3.2), one straight branch cut will lead to four curvilinear branch-cuts (figure 4) and the requirement of simultaneous jumping (6.1) cannot be satisfied by direct evaluation of $\ln(\zeta_\alpha^+ - \hat{\zeta}_\alpha) - \ln(\zeta_\alpha^- - \hat{\zeta}_\alpha)$. To deal with this problem, a corrective evaluation was suggested in [55]. Here, an alternative way that can be achieved more instinctively than before is proposed by considering the first equality of (4.4)₂. With this consideration, the complex

logarithmic function with mapped variables can simply be evaluated by

$$\ln(\zeta_\alpha - \hat{\zeta}_\alpha) = \ln(z_\alpha - \hat{z}_\alpha) - \ln c_\alpha - \ln \left(1 - \frac{\gamma_\alpha}{\hat{\zeta}_\alpha \zeta_\alpha} \right). \quad (6.2)$$

Knowing that $|\gamma_\alpha| < |\hat{\zeta}_\alpha \zeta_\alpha|$, which can be proved from the mapped domain shown in [33], the real part of $1 - \gamma_\alpha/(\hat{\zeta}_\alpha \zeta_\alpha)$ will always be positive. Thus, $\ln(1 - \gamma_\alpha/\hat{\zeta}_\alpha \zeta_\alpha)$ will never cross the region of discontinuity, which locates at the portion with negative real part if the branch for the principal value of logarithmic function is selected to be $(-\pi, \pi]$. Therefore, relation (6.2) would synchronize the discontinuity of $\ln(\zeta_\alpha - \hat{\zeta}_\alpha)$ with $\ln(z_\alpha - \hat{z}_\alpha)$. This synchronization would then lead to

$$\langle \ln(\zeta_\alpha^+ - \hat{\zeta}_\alpha) - \ln(\zeta_\alpha^- - \hat{\zeta}_\alpha) \rangle = 2\pi i \mathbf{I}, \text{ when } \varepsilon \rightarrow 0. \quad (6.3)$$

Similar to $\ln(\zeta_\alpha - \hat{\zeta}_\alpha)$, all the other complex logarithmic functions in Green's function can be evaluated as follows:

$$\left. \begin{aligned} \ln \zeta_\alpha &= \ln z_\alpha - \ln c_\alpha - \ln \left(1 + \frac{\gamma_\alpha}{\hat{\zeta}_\alpha^2} \right) \\ \ln(-\hat{\zeta}_\alpha) &= \ln(-\hat{z}_\alpha) - \ln c_\alpha - \ln \left(1 + \frac{\gamma_\alpha}{\hat{\zeta}_\alpha^2} \right) \end{aligned} \right\} \quad (6.4)$$

and

7. Application to boundary element method

To effectively treat the coupled stretching–bending deformation, a boundary element [57] using Green's function for the infinite composite laminates was developed based upon the following boundary integral equations [58]:

$$\begin{aligned} c_{ip}(\hat{\mathbf{x}})u_p(\hat{\mathbf{x}}) &+ \int_{\Gamma}^* t_{ij}^*(\hat{\mathbf{x}}, \mathbf{x}) \hat{u}_j(\mathbf{x}) d\Gamma(\mathbf{x}) + \sum_{k=1}^{N_c^*} t_{ic}^*(\hat{\mathbf{x}}, \mathbf{x}_k) \hat{u}_3(\mathbf{x}_k) \\ &= \int_{\Gamma} u_{ij}^*(\hat{\mathbf{x}}, \mathbf{x}) t_j(\mathbf{x}) d\Gamma(\mathbf{x}) + \int_{\Omega} u_{ij}^*(\hat{\mathbf{x}}, \mathbf{x}) q_j(\mathbf{x}) d\Omega(\mathbf{x}) \\ &+ \sum_{k=1}^{N_c^*} u_3^*(\hat{\mathbf{x}}, \mathbf{x}_k) t_c(\mathbf{x}_k), \quad i, j = 1, 2, 3, 4, \quad p = 1, 2, 3, 4, 5. \end{aligned} \quad (7.1)$$

In the above, $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2)$ and $\mathbf{x} = (x_1, x_2)$ represent, respectively, the source point and field point of the boundary integral equations. Γ and Ω are the boundary and domain of mid-plane in the elastic solid, respectively. The symbol $\int_{\Gamma}^*$ denotes an integral taken in the sense of *Cauchy principal value*. $c_{ip}(\hat{\mathbf{x}})$ are the free term coefficients dependent on the location of the source point $\hat{\mathbf{x}}$, which equal to $\delta_{ip}/2$ for a smooth boundary and $c_{ip} = \delta_{ip}$ for an internal point. The symbol δ_{ip} is the Kronecker delta. In practical applications, $c_{ip}(\hat{\mathbf{x}})$ can be computed by considering rigid body motion. $u_j(\mathbf{x})$ and $t_j(\mathbf{x})$ are the generalized displacements and surface tractions at the field point $\mathbf{x}$. $u_{ij}^*(\hat{\mathbf{x}}, \mathbf{x})$ and $t_{ij}^*(\hat{\mathbf{x}}, \mathbf{x})$ are, respectively, the fundamental solution of generalized displacements and tractions.

The differences between equation (7.1) and the conventional one for two- or three-dimensional analysis are: (1) the appearance of the symbol $\hat{\bullet}$, (2) the physical meaning of u_j and t_j , and the range of the sub-indices, (3) the surface integral related to the loads q_i on the plate surface, (4) the summation terms related to the corner force t_c and deflection u_3 . These are explained as follows:

$$\hat{u}_i(\mathbf{x}) = u_i(\mathbf{x}), \quad i = 1, 2, 4, \quad \hat{u}_3(\mathbf{x}) = u_3(\mathbf{x}) - u_3(\hat{\mathbf{x}}) \quad (7.2a)$$

and

$$\left. \begin{aligned} u_1 &= u_0, u_2 = v_0, u_3 = w, u_4 = \beta_n, u_5 = \beta_s, \\ t_1 &= T_x, t_2 = T_y, t_3 = V_n, t_4 = M_n, t_c = M_{ns}^+ - M_{ns}^-, \\ q_1 &= q_x, q_2 = q_y, q_3 = q, q_4 = m_n, \end{aligned} \right\} \quad (7.2b)$$

Figure 5. Illustration of the integral path for numerical integration. (Online version in colour.)

where u_0 and v_0 are the mid-plane displacements in x - and y -directions, respectively; β_n and β_s are the negative slopes of deflection in normal and tangential directions, respectively; T_x and T_y are the x - and y -components of surface traction resultant, respectively; V_n and M_n are the effective transverse shear force and bending moment on the surface with normal direction $\mathbf{n}$, respectively; q_x, q_y, q and m_n represent the distributed loads in x, y, z directions and the moment, respectively; t_c is the corner force related to the twisting moments M_{ns} ahead (+) and behind (−) of the corner. $u_{ij}^*(\hat{\mathbf{x}}, \mathbf{x})$, $t_{ij}^*(\hat{\mathbf{x}}, \mathbf{x})$ and $t_{ic}^*(\hat{\mathbf{x}}, \mathbf{x})$, $i = 1, 2, 3$, $j = 1, 2, 3, 4$, are the fundamental solutions which represent, respectively, u_j , t_j and t_c at point $\mathbf{x}$ corresponding to a unit point force acting in the x_i direction applied at point $\hat{\mathbf{x}}$, whereas $u_{4j}^*(\hat{\mathbf{x}}, \mathbf{x})$, $t_{4j}^*(\hat{\mathbf{x}}, \mathbf{x})$ and $t_{4c}^*(\hat{\mathbf{x}}, \mathbf{x})$, $j = 1, 2, 3, 4$, are the fundamental solutions which represent u_j , t_j and t_c at point $\mathbf{x}$ corresponding to a unit point moment acting on the surface with normal $\mathbf{n}$ applied at point $\hat{\mathbf{x}}$. N_c^* of the summation terms is related to the number of corners N_c by

$$N_c^* = \begin{cases} N_c, & \text{if the source point } \hat{\mathbf{x}} \text{ is not a corner point;} \\ N_c - 1, & \text{if the source point } \hat{\mathbf{x}} \text{ is a corner point.} \end{cases} \quad (7.3)$$

In other words, when the source point $\hat{\mathbf{x}}$ is a corner, the location $\mathbf{x}_k$ of the corner of the summation terms in (7.1) does not include the source point itself. Thus, no singularity occurs in the terms of summation.

Following the procedure developed in [57], noticing all the problems discussed in [59–61], and using Green's functions obtained in the previous sections, we can now develop a special boundary element for the coupled stretching–bending analysis of composite laminates with inclusion, and call it SCBEM.

8. Numerical examples

To verify all the newly derived solutions and evaluation methods, such as the simplification and correction made in §4, the analytical solution of deflection, the proposed corrective evaluation of complex logarithmic functions, and the application to BEM, two representative numerical examples were implemented. In these examples, we consider a four-layered graphite/epoxy fibre-reinforced composite laminate with an isotropic steel inclusion. The thickness of each lamina is 1 mm, and the axes of the inclusion are $a = 0.2$ m, $b = 0.15$ m. The material properties of each

Table 1. Stress resultants, bending moments and deflection along the interface for Green’s function with $\hat{f}_3$ outside the inclusion. ($N_{ss}^{(1)} = aN_{ss}^{(1)}/\hat{f}_3$, $N_{ss}^{(2)} = aN_{ss}^{(2)}/\hat{f}_3$, $M_{ss}^{(1)} = M_{ss}^{(1)}/\hat{f}_3$, $M_{ss}^{(2)} = M_{ss}^{(2)}/\hat{f}_3$, $w_A^{(1)} = w_A^{(1)} \times 10^6/a\hat{f}_3$, $w_N^{(1)} = w_N^{(1)} \times 10^6/a\hat{f}_3$).

φ	$\tilde{N}_{ss}^{(1)}$	$\tilde{N}_{ss}^{(2)}$	$\tilde{M}_{ss}^{(1)}$	$\tilde{M}_{ss}^{(2)}$	$\tilde{w}_A^{(1)}$	$\tilde{w}_N^{(1)}$	$\tilde{w}_A^{(1)} - \tilde{w}_N^{(1)}$
symmetric laminate [45/0] _s							
0°	0	0	0.103	0.114	−4.062	0	−4.062
45°	0	0	0.025	0.332	12.06	16.12	−4.062
90°	0	0	0.111	0.145	3.887	7.949	−4.062
135°	0	0	0.101	0.038	−10.12	−6.061	−4.062
180°	0	0	0.045	0.050	−6.788	−2.726	−4.062
225°	0	0	0.015	0.196	8.079	12.14	−4.062
270°	0	0	0.041	0.185	6.490	10.55	−4.062
315°	0	0	0.131	0.039	−9.570	−5.508	−4.062
unsymmetric laminate [45/0/45/−45]							
0°	13.09	−5.393	0.115	0.079	−5.887	0	−5.887
45°	6.580	−7.593	0.074	0.289	8.959	14.85	−5.887
90°	−0.330	−3.577	0.073	0.159	6.059	11.95	−5.887
135°	−10.25	−1.301	0.041	0.054	−7.404	−1.517	−5.887
180°	6.168	−0.162	0.053	0.025	−8.166	−2.279	−5.887
225°	3.799	−0.862	0.045	0.162	5.247	11.13	−5.887
270°	−1.283	−1.062	0.027	0.181	7.557	13.44	−5.887
315°	−14.70	−1.047	0.054	0.069	−6.366	−0.479	−5.887

lamina are

$$E_L = 138 \text{ GPa}, E_T = 9 \text{ GPa}, G_{LT} = 6.9 \text{ GPa}, \nu_{LT} = 0.3,$$

whereas the properties of steel are

$$E = 210 \text{ GPa}, \nu = 0.3.$$

Here, E , G and ν are, respectively, Young’s modulus, shear modulus and Poisson’s ratio, and the subscripts L and T represent the longitudinal and transverse directions, respectively.

For numerical calculation, the infinite series contained in Green’s functions must be truncated into finite terms. After some convergence tests, we consider only the first 20 terms for the following examples, which would lead to the results with the acceptable error less than 10^{-3} .

To verify the analytical solutions for Green’s function of deflection presented in §5, the numerical integration introduced in (2.8) provides the convincing reference. Note in (2.8), the lower limit of integrals is arbitrarily selected to be zero. For inclusion problem, to avoid the misuse of the function vectors in wrong region, we suggest that the lower limit to be a point on the interface such as $\mathbf{x}_0 = (a, 0)$, instead of zero. To calculate the deflection numerically for the field point $\mathbf{x}$ located in the matrix region, the integral path is suggested to be split into two parts (figure 5):

- (1) elliptical curve from the lower limit to the intersection of radial line and ellipse, which requires the integral with respect to the parameter φ in (3.1);
- (2) straight segment from the intersection to the field point.

Figure 6. Contour plot of displacement u_1 using the corrected Green's function (4.6). (Online version in colour.)

Figure 7. Contour plot of deflection w using the newly derived Green's function (5.2)–(5.3). ($w, w_1, w_2 \neq 0$ at $\hat{\mathbf{x}}$). (Online version in colour.)

With this selection, the integral path would be purely in the matrix region and only the solution of $\mathbf{f}_0$ and $\mathbf{f}_1$ will be used during integration. Note that after the convergence test for numerical integration, in all the following examples 16 Gaussian points are selected for the standard Gaussian quadrature rule.

(a) Example 1: an infinite laminate subjected to a concentrated force or moment

Consider an infinite composite laminate with an elastic inclusion subjected to a concentrated transverse force $\hat{f}_3$ located at the point outside the inclusion $\hat{\mathbf{x}} = (2a, a)$. To study the coupling effects, both symmetric and unsymmetric laminates, represented by $[45/0]_s$ and $[45/0/45/-45]$, are considered. Table 1 shows the results of stress resultants, bending moments and deflection

Figure 8. Contour plot of deflection w using the newly derived Green's function (5.2)–(5.3). ($w = w_1 = w_2 = 0$ at $\hat{\mathbf{x}}$). (Online version in colour.)

Figure 9. Contour plot of slope β_1 using the corrected Green's function (4.6). (Online version in colour.)

along the interface. Since no comparison data can be obtained from the other sources, in this Table only parts of stress resultants ($N_{ss}^{(1)}, N_{ss}^{(2)}$) and bending moments ($M_{ss}^{(1)}, M_{ss}^{(2)}$) are presented to see the difference between symmetric and unsymmetric laminates. Here, the subscript s denotes the tangential direction of the inclusion interface, and the superscripts (1) and (2) denote, respectively, the values on the sides of matrix and inclusion. On the other hand, for the purpose of comparison, two sets of deflection are presented. One is $w_A^{(1)}$, and the other is $w_N^{(1)}$. Here, the subscripts A and N denote, respectively, the deflection obtained by analytical solutions shown in §5 and by numerical method introduced in (2.7) and (2.8). From the results shown in table 1, we see that the deflections calculated by these two methods are different only by a constant value, which is induced by the difference of the lower limit of the integrals and can be treated as a rigid

Figure 10. Contour plot of deflection w using the newly derived Green’s function (5.4). ($w, w_1, w_2 \neq 0$ at $\hat{\mathbf{x}}$). (Online version in colour.)

Table 2. Stress resultants, bending moments and deflection along the interface for Green’s function with $\hat{m}_1 = \hat{m}_2 = \hat{m}$ inside the inclusion. ($\tilde{N}_{ss}^{(1)} = a^2 N_{ss}^{(1)} / \hat{m}$, $\tilde{N}_{ss}^{(2)} = a^2 N_{ss}^{(2)} / \hat{m}$, $\tilde{M}_{ss}^{(1)} = a M_{ss}^{(1)} / \hat{m}$, $\tilde{M}_{ss}^{(2)} = a M_{ss}^{(2)} / \hat{m}$, $\tilde{w}_A^{(1)} = w_A^{(1)} \times 10^6 / a \hat{m}$, $\tilde{w}_N^{(1)} = w_N^{(1)} \times 10^6 / a \hat{m}$).

φ	$\tilde{N}_{ss}^{(1)}$	$\tilde{N}_{ss}^{(2)}$	$\tilde{M}_{ss}^{(1)}$	$\tilde{M}_{ss}^{(2)}$	$\tilde{w}_A^{(1)}$	$\tilde{w}_N^{(1)}$	$\tilde{w}_A^{(1)} - \tilde{w}_N^{(1)}$
symmetric laminate [45/0] _s							
0°	0	0	−0.161	−0.412	−84.83	0	−84.83
45°	0	0	0.005	0.137	17.10	101.9	−84.83
90°	0	0	0.198	0.322	83.76	168.6	−84.83
135°	0	0	0.141	0.064	6.432	91.26	−84.83
180°	0	0	0.055	0.044	−17.40	67.43	−84.83
225°	0	0	0.014	0.133	22.55	107.4	−84.80
270°	0	0	−0.053	0.071	19.42	104.3	−84.91
315°	0	0	−0.190	−0.121	−47.27	37.37	−84.64
unsymmetric laminate [45/0/45/−45]							
0°	−12.88	4.464	−0.149	−0.402	−83.40	0	−83.40
45°	5.459	4.681	0.074	0.122	14.55	97.95	−83.40
90°	0.175	−3.084	0.151	0.317	82.55	165.9	−83.40
135°	−14.63	−1.252	0.058	0.066	8.694	92.10	−83.40
180°	7.947	−0.938	0.062	0.031	−19.01	64.39	−83.41
225°	2.404	−0.940	0.026	0.122	20.51	103.9	−83.38
270°	−3.163	0.159	−0.053	0.077	21.87	105.3	−83.47
315°	17.97	0.301	−0.088	−0.125	−46.10	37.15	−83.25

Figure 11. An unsymmetric laminate with an elliptical elastic inclusion subjected to a uniform bending moment.

Figure 12. Stress resultants and bending moments along the interface of figure 10. ($\tilde{N}_{ss}^{(i)} = hN_{ss}^{(i)}/M^{\infty}$, $\tilde{M}_{ss}^{(i)} = M_{ss}^{(i)}/M^{\infty}$, $\tilde{M}_{sn}^{(i)} = M_{sn}^{(i)}/M^{\infty}$, $i = 1, 2$). (Online version in colour.)

body translation. Thus, we can conclude that the analytical solutions obtained in §5 well agree with the numerical solutions, and hence are highly useful for the practical application such as the BEM introduced in §7. Figures 6 and 7 show the contour plot of u_1 and w for $[45/0/45/-45]$ laminate in the region $-3 \leq x/a \leq 3$ and $-2 \leq y/a \leq -2$ by using the newly derived solution and the corrective evaluation introduced in the previous sections. From figure 6, we see that the discontinuity occurred in the existing solution shown in figure 2 has been corrected. Moreover, as shown in figure 7 Green's functions of deflection obtained in (5.2) and (5.3) really satisfy the continuity condition across the interface. At first glance, it looks odd that the contour plot shown in figure 7 does not encircle the concentrated loading point $\hat{x}$. This is totally due to the fact that the deflection and the rotation angle at $\hat{x}$ of figure 7 are not zero. Since Green's function is obtained by ignoring rigid body motion (including translation and rotation), to have an intuitively rational

Table 3. Stress resultants and bending moments along the interface for an unsymmetric composite laminate with an elastic inclusion subjected to a uniform bending moment. ($\tilde{N}_{nn}^{(i)} = hN_{nn}^{(i)}/M^\infty$, $\tilde{N}_{sn}^{(i)} = hN_{sn}^{(i)}/M^\infty$, $\tilde{M}_{nn}^{(i)} = M_{nn}^{(i)}/M^\infty$, $i = 1, 2$)

φ	$\tilde{N}_{nn}^{(1)}$	$\tilde{N}_{nn}^{(2)}$	$\tilde{N}_{sn}^{(1)}$	$\tilde{N}_{sn}^{(2)}$	$\tilde{M}_{nn}^{(1)}$	$\tilde{M}_{nn}^{(2)}$
numerical results of analytical solution						
0°	−0.300	−0.300	−0.353	−0.353	−2.334	−2.334
30°	0.276	0.276	−0.397	−0.397	−0.760	−0.760
60°	0.491	0.491	0.010	0.010	1.840	1.840
90°	0.333	0.333	0.353	0.353	2.987	2.987
120°	−0.024	−0.024	0.472	0.472	2.453	2.453
150°	−0.406	−0.406	0.216	0.216	0.052	0.052
180°	−0.300	−0.300	−0.353	−0.353	−2.334	−2.334
numerical results by using SCBEM ($a/L = 0.05$)						
0°	−0.277	−0.277	−0.342	−0.342	−2.246	−2.246
30°	0.274	0.274	−0.374	−0.374	−0.700	−0.700
60°	0.472	0.472	0.018	0.018	1.860	1.860
90°	0.315	0.315	0.343	0.343	2.991	2.991
120°	−0.028	−0.028	0.450	0.450	2.467	2.467
150°	−0.387	−0.387	0.198	0.198	0.104	0.104
180°	−0.277	−0.277	−0.342	−0.342	−2.246	−2.246
numerical results by using FEM ($a/L = 0.05$)						
0°	−0.286	−0.273	−0.307	−0.326	−2.172	−2.232
30°	0.263	0.260	−0.354	−0.368	−0.714	−0.718
60°	0.466	0.459	0.021	0.009	1.782	1.786
90°	0.320	0.314	0.306	0.327	2.865	2.891
120°	−0.024	−0.017	0.423	0.438	2.355	2.379
150°	−0.373	−0.370	0.196	0.200	0.073	0.067
180°	−0.286	−0.273	−0.308	−0.326	−2.173	−2.233

contour plot we now make $w = w_{,1} = w_{,2} = 0$ at $\hat{x}$ by adding a proper rigid body translation and rotation to the entire laminate and the results are shown in figure 8.

Similar to the previous discussions, we now consider a concentrated moment whose $\hat{m}_1 = \hat{m}_2 = \hat{m}$ applied at $\hat{x} = (0.5a, 0.25a)$ inside the inclusion. Figures 9 and 10 show the contour plots of β_1 and w for [45/0/45/−45] laminate, whereas table 2 shows the results of stress resultants, bending moments and deflection along the interface. Again, the continuity of the negative slopes and deflection, and the constant difference of the deflection between two approaches verify the derivation in §§4 and 5.

(b) Example 2: an unsymmetric laminated plate subjected to a uniform bending moment

Although the continuity has been checked in Example 1, the correctness of the newly derived Green’s functions is hardly to be verified due to the lack of comparison data. Here, we employ this function into SCBEM and implement it to a case whose analytical solution has been obtained. For this purpose, we now consider an unsymmetric laminate [45/0/45/−45] with steel inclusion subjected to a uniform bending moment M^∞ at two opposite sides as shown in figure 11.

The analytical solution for this problem with an infinite laminate has been obtained in [34]. In addition, the finite-element method (FEM) through the commercial software ANSYS is also implemented here to compare with SCBEM. To simulate the infinite laminate in SCBEM and FEM, we let $a/L = 0.05$. In FEM, 87 028 linearly shell elements (SHELL181) and 87 145 nodes are constructed. While in SCBEM, the mesh consists of only 40 linear elements and 48 nodes along the outer square. Figure 12 and table 3 show the results of stress resultants and bending moments along the interface. From figure 12, we see that the results among the analytical solution, SCBEM and FEM are almost the same. Therefore, the corrected Green's functions in §4, the newly derived Green's function of deflections in §5, the evaluation method suggested in §6, and the newly developed SCBEM in §7 are all verified through this simple example. Furthermore, from table 3 we observe that the traction continuity condition is precisely satisfied both in analytical solutions and in SCBEM, while FEM cannot satisfy such continuity. Moreover, the result of SCBEM is closer to the analytical solutions than FEM, and hence, SCBEM is much more accurate than FEM for the inclusion problems.

9. Conclusion

With the correction made in §4 and the evaluation of complex logarithmic functions introduced in §6, the unexpected discontinuity of the existing solutions obtained in [53] can now be eliminated. Moreover, Green's functions of deflection, which requires the integrals of function vectors, can be obtained explicitly as presented in §5, or numerically by the standard Gaussian quadrature rule with the integral path shown in figure 5. Due to the dominance in accuracy and efficiency, we strongly recommend employing the explicit closed-form solutions obtained in (5.2)–(5.4) to calculate the deflection instead of the numerical evaluation mentioned in (2.8). With all the explicit closed solutions of Green's functions presented in this paper, the special fundamental solutions for the coupled stretching–bending analysis of the unsymmetric laminates with elastic inclusions can be readily obtained to develop the SCBEM. Since the continuity condition across the interface has been satisfied exactly by the special fundamental solutions, no meshes are required along the inclusion interface for SCBEM. The numerical results also show that SCBEM is more accurate than FEM.

Data accessibility. The code and numerical results can be found in the electronic supplementary material.

Authors' contributions. Most of the works were done by C.-W.H. under the guidance of C.H. As to the manuscript, §§2 and 3 are contributed by C.H., while all the other sections were written initially by C.-W.H. and revised later by C.H.

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Appendix A. Details of the constant vectors $\mathbf{v}_1$, $\mathbf{v}_{-1}$, $\mathbf{v}_0$, $\mathbf{k}_{-1}$, $\mathbf{k}_1$, $\mathbf{c}_k$ and $\mathbf{d}_k$.

The constant vectors appeared in (3.5a) and (3.6b) can be obtained from the solutions provided in [53] and are presented as follows.

(a) Forces/moments outside the inclusion

The vectors $\mathbf{c}_k$ and $\mathbf{d}_k$ of (3.5a) are defined as

$$\left. \begin{aligned} \mathbf{d}_k &= \mathbf{A}_1^{-1} \left\{ \bar{\mathbf{A}}_2 \bar{\mathbf{c}}_k + \mathbf{A}_2 \langle (\gamma_\alpha^*)^k \rangle \mathbf{c}_k - \bar{\mathbf{A}}_1 \bar{\mathbf{e}}_k^- \right\} \\ &= \mathbf{B}_1^{-1} \left\{ \bar{\mathbf{B}}_2 \bar{\mathbf{c}}_k + \mathbf{B}_2 \langle (\gamma_\alpha^*)^k \rangle \mathbf{c}_k - \bar{\mathbf{B}}_1 \bar{\mathbf{e}}_k^- \right\}, \\ \mathbf{c}_k &= \{ \mathbf{G}_0 - \bar{\mathbf{G}}_k \bar{\mathbf{G}}_0^{-1} \mathbf{G}_k \}^{-1} \{ \mathbf{t}_k - \bar{\mathbf{G}}_k \bar{\mathbf{G}}_0^{-1} \bar{\mathbf{t}}_k \}, \\ \mathbf{t}_k &= -i \mathbf{A}_1^{-T} \mathbf{e}_k^-, \quad k = 1, 2, \dots, \infty. \end{aligned} \right\} \quad (\text{A } 1)$$

In the above,

$$\mathbf{G}_0 = \{\bar{\mathbf{M}}_1 + \mathbf{M}_2\}\mathbf{A}_2, \quad \mathbf{G}_k = \{\mathbf{M}_1 - \mathbf{M}_2\}\mathbf{A}_2\langle(\gamma_\alpha^*)^k\rangle, \quad (\text{A } 2a)$$

and

$$\mathbf{e}_k^- = \frac{1}{2\pi i} \left\{ \left\langle -\frac{\hat{\zeta}_\alpha^{-k}}{k} \right\rangle \mathbf{A}_1^T \hat{\mathbf{p}} + \hat{f}_3 \langle q_k^- (\hat{\zeta}_\alpha) \rangle \mathbf{A}_1^T \mathbf{i}_3 + \hat{m}_3 \langle c_{4\alpha} \hat{\zeta}_\alpha^{-k} \rangle \mathbf{A}_1^T \mathbf{i}_2 \right\}, \quad (\text{A } 2b)$$

where

$$\mathbf{M}_1 = -i\mathbf{B}_1\mathbf{A}_1^{-1}, \quad \mathbf{M}_2 = -i\mathbf{B}_2\mathbf{A}_2^{-1} \quad (\text{A } 3a)$$

and

$$q_1^- (\hat{\zeta}_\alpha) = c_\alpha \left\{ \frac{\gamma_\alpha}{2\hat{\zeta}_\alpha^2} + \ln(-c_\alpha \hat{\zeta}_\alpha) \right\} \quad (\text{A } 3b)$$

and

$$q_k^- (\hat{\zeta}_\alpha) = \frac{c_\alpha}{k\hat{\zeta}_\alpha^k} \left\{ \frac{\gamma_\alpha}{(k+1)\hat{\zeta}_\alpha} - \frac{\hat{\zeta}_\alpha}{k-1} \right\}, \quad k \neq 1.$$

(b) Forces/moments inside the inclusion

The vectors $\mathbf{v}_1, \mathbf{v}_{-1}, \mathbf{v}_0, \mathbf{k}_{-1}$ and $\mathbf{k}_1$ in (3.6b) are

$$\left. \begin{aligned} \mathbf{v}_0 &= \mathbf{A}_1^T \hat{\mathbf{p}} - 2\hat{f}_3 \{\mathbf{B}_1^T \mathbf{J}_{2R} + \mathbf{A}_1^T \mathbf{J}_{1R}\} \mathbf{i}_3, \\ \mathbf{v}_1 &= \mathbf{B}_1^T (\mathbf{g}_1 + i\mathbf{g}_2) + \mathbf{A}_1^T (\mathbf{h}_1 + i\mathbf{h}_2), \quad \mathbf{v}_{-1} = \mathbf{B}_1^T (\mathbf{g}_1 - i\mathbf{g}_2) + \mathbf{A}_1^T (\mathbf{h}_1 - i\mathbf{h}_2), \\ \mathbf{k}_1 &= \hat{f}_3 \{\mathbf{B}_1^T \mathbf{Q}_2 + \mathbf{A}_1^T \mathbf{Q}_1\} \mathbf{i}_3, \quad \mathbf{k}_{-1} = -\hat{f}_3 \{\mathbf{B}_1^T \bar{\mathbf{Q}}_2 + \mathbf{A}_1^T \bar{\mathbf{Q}}_1\} \mathbf{i}_3, \end{aligned} \right\} \quad (\text{A } 4)$$

where,

$$\left\{ \begin{matrix} \mathbf{g}_1 \\ \mathbf{h}_1 \end{matrix} \right\} = \hat{f}_3 \left\{ \begin{matrix} (\mathbf{E}_{2R} + \mathbf{F}_{2R}) \mathbf{i}_3 \\ (\mathbf{E}_{1R} + \mathbf{F}_{1R}) \mathbf{i}_3 \end{matrix} \right\} \quad \text{and} \quad \left\{ \begin{matrix} \mathbf{g}_2 \\ \mathbf{h}_2 \end{matrix} \right\} = \hat{f}_3 \left\{ \begin{matrix} (\mathbf{E}_{2I} - \mathbf{F}_{2I}) \mathbf{i}_3 \\ (\mathbf{E}_{1I} - \mathbf{F}_{1I}) \mathbf{i}_3 \end{matrix} \right\}. \quad (\text{A } 5)$$

In the above,

$$\left. \begin{aligned} \mathbf{E}_1 &= \mathbf{B}_2 \langle c_\alpha^* \rangle \mathbf{A}_2^T, \quad \mathbf{E}_2 = \mathbf{A}_2 \langle c_\alpha^* \rangle \mathbf{A}_2^T, \\ \mathbf{F}_1 &= \mathbf{B}_2 \langle c_\alpha^* \gamma_\alpha^* \rangle \mathbf{A}_2^T, \quad \mathbf{F}_2 = \mathbf{A}_2 \langle c_\alpha^* \gamma_\alpha^* \rangle \mathbf{A}_2^T, \\ \mathbf{J}_1 &= \mathbf{B}_2 \left\langle c_\alpha^* \left(\hat{\zeta}_\alpha^* + \frac{\gamma_\alpha^*}{\hat{\zeta}_\alpha^*} \right) \right\rangle \mathbf{A}_2^T, \quad \mathbf{J}_2 = \mathbf{A}_2 \left\langle c_\alpha^* \left(\hat{\zeta}_\alpha^* + \frac{\gamma_\alpha^*}{\hat{\zeta}_\alpha^*} \right) \right\rangle \mathbf{A}_2^T, \\ \mathbf{Q}_1 &= \mathbf{B}_2 \langle c_\alpha^* (\ln c_\alpha^* - 1) \rangle \mathbf{A}_2^T, \quad \mathbf{Q}_2 = \mathbf{A}_2 \langle c_\alpha^* (\ln c_\alpha^* - 1) \rangle \mathbf{A}_2^T, \end{aligned} \right\} \quad (\text{A } 6)$$

and the letters R and I denote, respectively, the real and imaginary parts of the matrix. And the vectors $\mathbf{c}_k$ and $\mathbf{d}_k$ of (3.6b) are defined as

$$\left. \begin{aligned} \mathbf{d}_k &= \mathbf{A}_1^{-1} \left\{ \bar{\mathbf{A}}_2 \bar{\mathbf{c}}_k + \mathbf{A}_2 \langle (\gamma_\alpha^*)^k \rangle \mathbf{c}_k + \mathbf{A}_2 \mathbf{e}_k^+ \right\} \\ &= \mathbf{B}_1^{-1} \left\{ \bar{\mathbf{B}}_2 \bar{\mathbf{c}}_k + \mathbf{B}_2 \langle (\gamma_\alpha^*)^k \rangle \mathbf{c}_k + \mathbf{B}_2 \mathbf{e}_k^+ \right\}, \\ \mathbf{c}_k &= \{\mathbf{G}_0 - \bar{\mathbf{G}}_k \bar{\mathbf{G}}_0^{-1} \mathbf{G}_k\}^{-1} \{\mathbf{t}_k - \bar{\mathbf{G}}_k \bar{\mathbf{G}}_0^{-1} \bar{\mathbf{t}}_k\}, \quad k = 1, 2, \dots, \infty \\ \mathbf{t}_k &= (\bar{\mathbf{M}}_2 - \bar{\mathbf{M}}_1) \bar{\mathbf{A}}_2 \bar{\mathbf{e}}_k^+, \end{aligned} \right\} \quad (\text{A } 7)$$

and

in which $\mathbf{G}_0, \mathbf{G}_k$ are defined in (A 2a), and

$$\begin{aligned} \mathbf{e}_k^+ &= \frac{1}{2\pi i} \left\{ -\frac{1}{k} \left\langle \langle \hat{\zeta}_\alpha^* \rangle^k + \left(\frac{\gamma_\alpha^*}{\hat{\zeta}_\alpha^*} \right)^k \right\rangle \mathbf{A}_2^T \hat{\mathbf{p}} + \hat{f}_3 \langle q_k^+ (\hat{\zeta}_\alpha^*) \rangle \mathbf{A}_2^T \mathbf{i}_3 \right. \\ &\quad \left. - \hat{m}_3 \left\langle c_{4\alpha}^* \left[\langle \hat{\zeta}_\alpha^* \rangle^k - \left(\frac{\gamma_\alpha^*}{\hat{\zeta}_\alpha^*} \right)^k \right] \right\rangle \mathbf{A}_2^T \mathbf{i}_2 \right\}, \end{aligned} \quad (\text{A } 8)$$

$$\left. \begin{aligned} q_1^+(\hat{\zeta}_\alpha^*) &= c_\alpha^* \left\{ \frac{1}{2} \left[\hat{\zeta}_\alpha^{*2} + \left(\frac{\gamma_\alpha^*}{\hat{\zeta}_\alpha^*} \right)^2 \right] + \gamma_\alpha^* (\ln c_\alpha^* + 1) \right\}, \\ q_k^+(\hat{\zeta}_\alpha^*) &= \frac{c_\alpha^*}{k} \left\{ \frac{1}{k+1} \left[(\hat{\zeta}_\alpha^*)^{k+1} + \left(\frac{\gamma_\alpha^*}{\hat{\zeta}_\alpha^*} \right)^{k+1} \right] - \frac{\gamma_\alpha^*}{k-1} \left[(\hat{\zeta}_\alpha^*)^{k-1} + \left(\frac{\gamma_\alpha^*}{\hat{\zeta}_\alpha^*} \right)^{k-1} \right] \right\}, \quad k \neq 1, \end{aligned} \right\} \quad (\text{A } 9a)$$

and

$$c_{4\alpha}^* = \frac{1}{c_\alpha^* (\hat{\zeta}_\alpha^* - \gamma_\alpha^* / \hat{\zeta}_\alpha^*)}. \quad (\text{A } 9b)$$

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