Research

Cite this article: Argatov I, Papangelo A, Ciavarella M. 2020 Elliptical adhesive contact under biaxial stretching. *Proc. R. Soc. A* **475**: 20190507.
<http://dx.doi.org/10.1098/rspa.2019.0507>

Received: 7 August 2019

Accepted: 19 December 2019

Subject Areas:

mechanical engineering

Keywords:

adhesion, elliptical contact, incompressible substrate, fracture mode mixity

Author for correspondence:

M. Ciavarella

e-mail: mciava@poliba.it

Elliptical adhesive contact under biaxial stretching

I. Argatov¹, A. Papangelo^{2,3} and M. Ciavarella^{2,3}

¹Faculty of Technology and Society, Malmö University, 205 06 Malmö, Sweden

²Department of Mechanics, Mathematics and Management, Politecnico di Bari, Via Orabona 4, Bari 70125, Italy

³Hamburg University of Technology, Department of Mechanical Engineering, Am Schwarzenberg-Campus 1, Hamburg 21073, Germany

AP, 0000-0002-0214-904X; MC, 0000-0001-6271-0081

Adhesive contact of the Hertzian indenter with an incompressible elastic substrate bi-directionally stretched along the indenter principal planes of curvature is considered in the Johnson–Kendall–Roberts theoretical framework. An approximate model is constructed by examining energy release rate conditions only on the edges of the minor and major axes of the contact ellipse. The effect of weak coupling between fracture modes I and II is introduced using a phenomenological mode-mixity function. This study was motivated by the need to model a passive–adhesive mechanism in cell mechanics on stretchable substrates.

1. Introduction

Contact mechanics with adhesion of elastic bodies has been applied to cell adhesion, originally in the form of the simple Johnson–Kendall–Roberts (JKR) theory [1], where it was shown that, unless it is disrupted, the cytoskeleton provides cells with a three-dimensional structure that is appropriate for JKR theory to be valid [2]; this includes that the material is relatively soft for the contact size to be regulated by what is essentially a Griffith energy criterion (for a recent review of adhesion, see [3]).

The mechanical properties of cells are related to their actin cytoskeletal structure, and both elastic and viscoelastic properties evolve, for example, with cancer malignancy (stiffness increasing by possibly an order of magnitude [4] and similar trends are followed by viscosities), leading to potential methods for cancer

diagnosis. Including viscoelastic material behaviour in adhesive contact models is not trivial [5,6], and an elastic model gives a limiting solution for a viscoelastic body in response to cyclical load at either a sufficiently low or a sufficiently high frequency. Other authors have studied the effect of increased stiffness induced in elastomeric materials by prestretch applied before contact is established [7–9] due to finite deformations of the elastomers. However, it has been found [10] that cell stretch to about twice the initial cell size still gave a force response that was strongly linear, reversible and repeatable.

Hence, even though linear elastic contact mechanics is only a simple attempt to model the cell mechanics of adhesion to substrates, its simplicity has attracted the interest of the scientific community because of significant interest from many fields of biology in the response of living cells to mechanical forces [11–15].

In particular, various experiments [16–21] have cultured cells on a periodically stretched substrate, finding that they tend to reorient themselves almost perpendicularly to the stretch direction. There is a certain threshold of strain, which was described first by Dartsch & Hämmerle [17], below which there is no sensitivity (about 2%), so that cells ultimately randomly orient, except in directions where the axial strains are above their thresholds [22]. At levels of strains above 5–6%, it was found that almost all cells had reoriented themselves [23]. Chen & Gao [24] first introduced a simple two-dimensional contact mechanics model to capture some features of this behaviour of cells on stretched substrates. Being only a two-dimensional model, they could not model the orientation of cells, but they did find that the response to strains in terms of the size of the contact area showed a distinctly sigmoidal dependence on the applied surface strain, so that two thresholds could indeed be qualitatively identified.

More recently, Waters *et al.* [25] developed a model for a perfectly axisymmetric configuration, i.e. with equibiaxial strains applied to a substrate, and also realized some experiments on model convex glass lenses in contact with polydimethylsiloxane sheet substrates, subjected to equibiaxial stretch. Results similar to those of Chen and Gao [26] were found, but a significant additional ingredient was introducing the mode-mixity dependence of the work of adhesion at the contact edge: without this ‘empirical’ corrective factor, the effect of substrate stretch would be too large in terms of effective change of contact areas, as found already in a related problem of contact in the presence of tangential forces [27–33]. Chen *et al.* [13] considered the more complex scenario of cell reorientation under biaxial stretch by replacing the cell with a linear spring that adheres to the substrate at its extremities. A more sophisticated model was later introduced by Xu *et al.* [14], who described the cell by means of a ‘tensegrity bar model’ and derived the cell orientation by energy minimization, similar to what was proposed by Livne *et al.* [15].

Here, we keep to the assumptions of a linear elastic, isotropic material, but attempt to give some insights about the orientation of the contact upon application of a bi-directional strain within the framework of JKR theory, which has been not addressed so far. In particular, we shall consider what is known classically as the Hertzian indenter (parabolic surface with two principal curvatures $1/R_1$ and $1/R_2$), in adhesive JKR contact with a substrate (figure 1), which, after contact is established, is stretched along two axes with strains ε_x , ε_y —the only assumption is therefore that there is no applied shear stretch and that the principal directions of in-plane loading must coincide with the principal planes of curvature of the indenter. Then, we shall obtain an approximate solution imposing energy release rate conditions on the edges of the major and minor axes of the contact ellipse.

The rest of the paper is organized as follows. In §2, we outline the JKR model-based theoretical framework for adhesive elliptical contact of an ellipsoidal rigid indenter with an incompressible elastic half-space under biaxial stretch. Special attention is paid to the case of the force-controlled indentation. Provided the contact force and the substrate strains are known, the resulting system is reduced to two nonlinear equations with respect to the major semi-axis and eccentricity of the contact area. In §3, we apply the developed theory to solving a special problem of elliptical adhesive contact between a spherical indenter and a uniaxially stretched substrate. In §4, we discuss the results and formulate our conclusions. In appendix D, we consider two-dimensional adhesive contact of a rigid cylinder with a stretched incompressible elastic half-plane. It is

Figure 1. Schematic of the adhesive contact between a Hertzian probe and an elastic half-space under bi-directional stretch. (Online version in colour.)

shown that the two-dimensional case can be treated as a limit situation of the developed three-dimensional model.

2. Theory

(a) Stress intensity factors at the vertices of the elliptical contact area

Let the surface of an elastic half-space under a punch's base undergo a vertical displacement

$$w(x, y) = \delta - \frac{x^2}{2R_1} - \frac{y^2}{2R_2}, \quad (2.1)$$

over an elliptical area of contact inside the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad (2.2)$$

Then, according to Galin's theorem [34], the pressures needed for this can be expressed as

$$p(x, y) = (p_0 - p_{20}x^2 - p_{02}y^2) \left(1 - \frac{x^2}{a^2} - \frac{y^2}{a^2(1-e^2)} \right)^{-1/2}, \quad (2.3)$$

where $e = \sqrt{1 - b^2/a^2}$ is the ellipse's eccentricity (it is assumed that $b \leq a$). The coefficients p_0 , p_{20} and p_{02} can be expressed in terms of δ , R_1 , R_2 , a and e (see equations (A 1)–(A 3) in appendix A).

By integrating the contact pressure (2.3) over the contact area bounded by the ellipse (2.2), we readily obtain the contact force

$$P = 2\pi a^2 \sqrt{1-e^2} \left\{ p_0 - \frac{a^3}{3} (p_{20} + p_{02}(1-e^2)) \right\}. \quad (2.4)$$

Let the equation for an ellipse (2.2) be parameterized by $x = a \cos \theta$, $y = a \sqrt{1-e^2} \sin \theta$. Then, the stress intensity factor (SIF) produced by the contact pressure distribution (2.3) is as follows [35]:

$$K_I(\theta) = -\frac{\sqrt{\pi}(1-e^2)^{1/4}a^{1/2}}{(1-e^2 \cos^2 \theta)^{1/4}} \left\{ p_0 - a^2 [p_{20} \cos^2 \theta + p_{02}(1-e^2) \sin^2 \theta] \right\}. \quad (2.5)$$

By Mossakovskii's reciprocity theorem [36], the contact force can be evaluated as

$$\delta = \frac{\mathbf{K}(e)}{\pi E^* a} P + \frac{a^2}{6} \left(\frac{1}{R_1} + \frac{1-e^2}{R_2} \right). \quad (2.6)$$

We note that if the value of δ is given *a priori*, then equation (2.6) yields the value of P .

Figure 2. Schematic of the adhesive contact between a spherical probe and an elastic half-space under unidirectional stretch.

Furthermore, the tangential stresses inside the elliptical contact under biaxial stretch with strains ε_x and ε_y are given as follows [37]:

$$\begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{Bmatrix} d_{11}x \\ d_{22}y \end{Bmatrix} \left(1 - \frac{x^2}{a^2} - \frac{y^2}{a^2(1-e^2)} \right)^{-1/2}, \quad (2.7)$$

where the coefficients d_{11} and d_{22} can be expressed in terms of ε_x , ε_y , a and e (see equation (B 4) in appendix B).

The SIFs of the stress field (2.7) near the vertices of the elliptical contact area are

$$K_{II}|_{x=\pm a} = \pm \frac{\sqrt{\pi} E^*}{2} \frac{a^{1/2}}{\sqrt{1-e^2}} \tilde{d}_{11} \quad \text{and} \quad K_{II}|_{y=\pm b} = \pm \frac{\sqrt{\pi} E^*}{2} a^{1/2} (1-e^2)^{1/4} \tilde{d}_{22}. \quad (2.8)$$

The dimensionless factors $\tilde{d}_{11}$ and $\tilde{d}_{22}$ depend solely on ε_x , ε_y and e as

$$\tilde{d}_{11} = \varepsilon_x \delta_{12}(e) + \varepsilon_y \delta_{12}(e) \quad \text{and} \quad \tilde{d}_{22} = \varepsilon_x \delta_{21}(e) + \varepsilon_y \delta_{22}(e), \quad (2.9)$$

where the coefficients $\delta_{ij}(e)$ are given by equations (B 7) and (B 8) (see appendix B).

(b) Adhesive elliptical contact under stretching

In our analysis, we assume that, first, a normal load is applied to the Hertzian indenter to establish some contact area with an elastic substrate, and, second, a biaxial stretch is superimposed by respective loading of the substrate (figure 2). For simplicity, we consider the situation of an absolutely rigid indenter in contact with an incompressible substrate, so that the first-stage loading does not induce any tangential stresses beneath the indenter. This approach can be generalized for uncoupled contact systems, for which Dundurs' second dissimilarity elastic constant does not vanish.

Following [32], we apply the approximate JKR theory for elliptical contacts [35] and assume that the values of the equivalent SIF,

$$K_{eq} = \sqrt{K_I^2 + K_{II}^2}, \quad (2.10)$$

at the ends of the major and minor axes should be the same and equal to $K_{Ic} \sqrt{f_{II}(\psi_2)}$, that is,

$$K_{eq} = K_{Ic} \sqrt{f_{II}(\psi_2)}, \quad (2.11)$$

where K_{Ic} is the critical mode I SIF, $f_{II}(\psi_2)$ is the mode-mixity function [38], and ψ_2 is the phase angle defined as $\psi_2 = \arctan(K_{II}/K_I)$. We also recall that, by standard linear elastic fracture mechanics arguments, $G_{Ic} = K_{Ic}^2/(2E^*)$ is the mode I surface energy density.

In our analysis, we confine ourselves to the following one-parametric forms of the mode mixity [27,38]:

$$f_b(\psi) = 1 + \tan^2[(1 - \lambda)\psi]. \quad (2.12)$$

Here, $\lambda \in [0, 1]$ is a mode-mixity parameter. It is readily seen that, if $\lambda = 1$, we have $f_b(\psi) \equiv 1$, so that equation (2.11) reduces to the Griffith surface energy criterion of ideal brittle fracture. On the other hand, if $\lambda = 0$, we have $f_b(\psi) = (K_I^2 + K_{II}^2)/K_I^2$, so that equation (2.11) simplifies as $K_I = K_{Ic}$ and the mode II SIF K_{II} does not affect the adhesive mechanism.

In what follows, we make use of the following dimensionless notation [32]:

$$\tilde{a} = \frac{\xi a}{R_e}, \quad \tilde{b} = \frac{\xi b}{R_e}, \quad \tilde{\delta} = \frac{\xi^2 \delta}{R_e}, \quad \tilde{P} = \frac{P}{R_e G_{Ic}} \quad \text{and} \quad \gamma = \sqrt{\frac{R_2}{R_1}}, \quad (2.13)$$

where

$$R_e = \sqrt{R_1 R_2} \quad \text{and} \quad \xi = \left(\frac{E^* R_e}{G_{Ic}} \right)^{1/3}. \quad (2.14)$$

Correspondingly, the dimensionless SIFs can be defined as

$$\tilde{K}_I = \frac{\xi^{3/2} K_I}{\sqrt{\pi} R_e^{1/2} E^*} \quad \text{and} \quad \tilde{K}_{II} = \frac{\xi^{3/2} K_{II}}{\sqrt{\pi} R_e^{1/2} E^*},$$

whereas, according to equation (2.6), the dimensionless indenter displacement will be

$$\tilde{\delta} = \frac{\mathbf{K}(e)}{\pi \tilde{a}} \tilde{P} + \frac{\tilde{a}^2}{6} \left(\gamma + \frac{(1 - e^2)}{\gamma} \right). \quad (2.15)$$

Recall that, in contact problems, either the contact force or the indenter displacement should be specified in advance. In what follows, we assume that the value of the dimensionless contact force $\tilde{P}$ is specified.

From (2.8) and (2.9), it follows that

$$\tilde{K}_{II}|_{\tilde{x}=\pm\tilde{a}} = \pm \frac{\tilde{a}^{1/2} \xi}{2(1 - e^2)^{1/2}} [\varepsilon_x \delta_{12}(e) + \varepsilon_y \delta_{12}(e)] \quad (2.16)$$

and

$$\tilde{K}_I|_{\tilde{y}=\pm\tilde{b}} = \pm \frac{1}{2} (1 - e^2)^{1/4} \tilde{a}^{1/2} \xi [\varepsilon_x \delta_{21}(e) + \varepsilon_y \delta_{22}(e)], \quad (2.17)$$

where $\tilde{x} = x/a$, $\tilde{y} = y/a$ are dimensionless coordinates.

On the other hand, the dimensionless mode I SIFs are given by (see equation (C 6))

$$\tilde{K}_I|_{\tilde{x}=\pm\tilde{a}} = -\tilde{a}^{1/2} \left(\frac{\tilde{P}}{2\pi \tilde{a}^2 \sqrt{1 - e^2}} + \tilde{a} \mathcal{F}_{20}(e, \gamma) \right) \quad (2.18)$$

and

$$\tilde{K}_{II}|_{\tilde{y}=\pm\tilde{b}} = -(1 - e^2)^{1/4} \tilde{a}^{1/2} \left(\frac{\tilde{P}}{2\pi \tilde{a}^2 \sqrt{1 - e^2}} + \tilde{a} \mathcal{F}_{02}(e, \gamma) \right), \quad (2.19)$$

where the functions $\mathcal{F}_{20}(e, \gamma)$ and $\mathcal{F}_{02}(e, \gamma)$ are defined by equations (C14) and (C15) in appendix C.

Finally, the fracture criterion (2.10), (2.11) takes the following non-dimensionalized form:

$$\tilde{K}_I^2 + \tilde{K}_{II}^2 = \frac{2}{\pi} f_{II}(\psi_2), \quad (2.20)$$

where

$$\psi_2 = \arctan \frac{\tilde{K}_{II}}{\tilde{K}_I}. \quad (2.21)$$

Thus, given the values of loading factors $\tilde{P}$, ε_x and ε_y , the contact area parameters $\tilde{a}$ and e can be evaluated from the system of two equations

$$\begin{cases} \tilde{K}_I^2 + \tilde{K}_{II}^2 = \frac{2}{\pi} f_{II} \left(\arctan \frac{\tilde{K}_{II}}{\tilde{K}_I} \right), & \tilde{x} = \tilde{a}, \\ \tilde{K}_I^2 + \tilde{K}_{II}^2 = \frac{2}{\pi} f_{II} \left(\arctan \frac{\tilde{K}_{II}}{\tilde{K}_I} \right), & \tilde{y} = \tilde{a} \sqrt{1 - e^2}, \end{cases} \quad (2.22)$$

where the dimensionless SIFs $\tilde{K}_I$ and $\tilde{K}_{II}$ at the ellipse's vertices are given by equations (2.18), (2.19) and (2.16), (2.17), respectively.

Observe that, for an elliptical indenter case under both the normal load acted on the indenter and the bi-directional strains loaded on the incompressible substrate, assumption (2.22) would lead to a constraint relationship among the normal load, the loading strain in the x -direction and that in the y -direction, if the contact area were fixed. However, the region of established contact is not known *a priori*, and system (2.22) serves to determine a unique elliptical approximation for the true contact area.

3. Results

(a) Elliptical adhesive contact of a rigid sphere with a stretched substrate

(i) Zero normal loading

We put $R_1 = R_2 = R$, so that $\gamma = 1$. Moreover, we consider a practically important special case of zero normal loading, when $\tilde{P} = 0$. Then, the resulting system (C 4), (C 5), (C 9) and (C 10) reduces to the system

$$\begin{cases} \tilde{a}^3 F_{20}(e)^2 + \frac{\xi^2 \tilde{a} \tilde{d}_{11}^2}{4(1 - e^2)} = \frac{2}{\pi} f_{II}(\psi_2|_{\tilde{x}=\tilde{a}}) \\ (1 - e^2)^{1/2} \tilde{a} \left\{ \tilde{a}^2 F_{02}(e)^2 + \frac{\xi^2}{4} \tilde{d}_{22}^2 \right\} = \frac{2}{\pi} f_{II}(\psi_2|_{\tilde{x}=\tilde{b}}), \end{cases} \quad (3.1)$$

and

where $F_{20}(e) = \mathcal{F}_{20}(e, 1)$ and $F_{02}(e) = \mathcal{F}_{02}(e, 1)$ (see equations (C 16) and (C 17)), $\tilde{d}_{11}$ and $\tilde{d}_{22}$ are given by (2.9), and

$$\psi_2|_{\tilde{x}=\tilde{a}} = \arctan \frac{\xi \tilde{d}_{11}}{2(1 - e^2)^{1/2} \tilde{a} F_{20}(e)} \quad \text{and} \quad \psi_2|_{\tilde{x}=\tilde{b}} = \arctan \frac{\xi \tilde{d}_{22}}{2 \tilde{a} F_{02}(e)}. \quad (3.2)$$

To fix our ideas, we set $\varepsilon_x = 0$, so that

$$\tilde{d}_{11} = \varepsilon_y \delta_{12}(e) \quad \text{and} \quad \tilde{d}_{22} = \varepsilon_y \delta_{22}(e), \quad (3.3)$$

where $\delta_{12}(e)$ and $\delta_{22}(e)$ are given by (B 7) and (B 8).

Thus, for a given value of the strain ε_y , the system of equations (3.1) can be resolved for $\tilde{a}$ and e .

According to equations (2.16)–(2.19), the dimensionless SIFs can be evaluated as follows:

$$\tilde{K}_I|_{\tilde{x}=\pm\tilde{a}} = -\tilde{a}^{3/2} F_{20}(e) \quad \text{and} \quad \tilde{K}_I|_{\tilde{y}=\pm\tilde{b}} = -(1 - e^2)^{1/4} \tilde{a}^{3/2} F_{02}(e) \quad (3.4)$$

and

$$\tilde{K}_{II}|_{\tilde{x}=\pm\tilde{a}} = \pm \frac{\xi \varepsilon_y \tilde{a}^{1/2}}{2(1 - e^2)^{1/2}} \delta_{12}(e) \quad \text{and} \quad \tilde{K}_{II}|_{\tilde{y}=\pm\tilde{b}} = \pm \frac{1}{2} (1 - e^2)^{1/4} \xi \varepsilon_y \tilde{a}^{1/2} \delta_{22}(e). \quad (3.5)$$

It can be shown that $F_{20}(e)$ is strictly negative for $e \in [0, 1)$, whereas $F_{02}(e)$ changes sign from negative to positive at $e_c \approx 0.861$ (note that this value does not depend on the mode-mixity function). This means that there is a critical value of ε_y beyond which higher stretch of the elastic incompressible substrate cannot be sustained by the contact interface. Indeed, for $e < e_c$ the SIF $\tilde{K}_I$ at the semi-axes b would be negative, i.e. there would be compenetration between the half-space and the indenter. In turn, this critical value ε_{yc} determines the critical value of the major

Figure 3. Variation of the contact area eccentricity (a) and the contact area major axis (b) as functions of the substrate uniaxial stretch in the case of zero normal loading. (Online version in colour.)

contact semi-axis $\tilde{a}_c$, both of which will depend on the mode-mixity parameter λ (figure 3). And it is of interest to observe that $\tilde{a}_c$ is inversely proportional to $(\xi \varepsilon_y)^2$, as it directly follows from equation (3.1)₂.

(ii) Compressive normal loading

The critical state of the adhesive contact under a fixed compressive normal load and variable uniaxial stretching is determined by the condition

$$\tilde{K}_I|_{\tilde{y}=\tilde{b}} = 0, \quad (3.6)$$

from where, in the light of (2.21), it immediately follows that

$$\psi_2|_{\tilde{y}=\tilde{b}} = \frac{\pi}{2}. \quad (3.7)$$

For a spherical indenter ($\gamma = 1$), equations (3.6) and (2.19) imply the critical major semi-axis as

$$\tilde{a}_C = \sqrt[3]{-\frac{\tilde{P}}{2\pi\sqrt{1-e^2}F_{02}(e)}}. \quad (3.8)$$

Hence, in view of (3.6) and (3.7), equation (2.22)₂ yields the critical stretch

$$(\xi \varepsilon_y)_C = \frac{2}{\delta_{22}(e)} \left(\frac{2f_{II}(\pi/2)}{\pi \tilde{a}_C \sqrt{1-e^2}} \right)^{1/2}, \quad (3.9)$$

where $\tilde{a}_C$ is given by (3.8).

Finally, the substitution of (3.8) and (3.9) into equation (2.22)₁ leads to a nonlinear equation for evaluating the critical contact area eccentricity e_C , as shown in figure 4a. After that, being given the value of e_C as a function of $\tilde{P}$, equations (3.8) and (3.9) allow the values of $\tilde{a}_C$ and $(\xi \varepsilon_y)_C$ as functions of $\tilde{P}$ to be determined (see figure 4b and figure 5, respectively).

In contrast to the axisymmetric case, the shape of the critical contact area, when the contact interface can no longer sustain the substrate stretch, varies with the level of normal load, as can be seen from figure 4a. Otherwise, the trend of the critical substrate stretch against the contact force (figure 5) is quite similar. It should be emphasized that the results of the critical values given in figures 4 and 5 have been obtained using the mixed-mode function (2.12). Changing the mixed-mode function or the value of the parameter λ may lead to different results. For a given experimental set-up, one may use a single experiment to calibrate the value of λ or of the mixed-mode function itself, and then one could use the latter to make predictions.

Figure 4. Variation of the critical contact area eccentricity (a) and the critical contact area major axis (b) versus the compressive normal load. (Online version in colour.)

Figure 5. Variation of the critical substrate stretch versus the compressive normal load. (Online version in colour.)

Another important feature of the obtained solution (3.8), (3.9) is that it does not withstand the limit passage for $\lambda \rightarrow 0$, since $f_{II}(\pi/2) \rightarrow +\infty$ and the critical value $(\xi \varepsilon_y)_C$ goes to infinity. On the other hand, the limit case of mode decoupling $\lambda = 0$ has been introduced previously for $K_I \neq 0$ and, correspondingly, $\psi_2 \in [0, \pi/2)$. Therefore, the limit curves for $\lambda = 0$ in figure 4 will be meaningless.

(iii) Tensile normal loading

When the contact force becomes tensile, the punch detachment may occur under certain conditions. We define the pull-off force $\tilde{P}_c$ by the condition that $d\tilde{A}/d\varepsilon_y$ is infinite. It is clear that, for $\varepsilon_y = 0$, the critical force $\tilde{P}_c$ coincides with that predicted by the JKR theory.

In particular, when $\lambda = 1$, equations (C 4) and (C 5) reduce to the following system:

$$\left. \begin{aligned} \tilde{a} \left(\frac{\tilde{P}}{2\pi\tilde{a}^2\sqrt{1-e^2}} + \tilde{a}F_{20}(e) \right)^2 + \frac{\tilde{a}[\xi\varepsilon_y\delta_{12}(e)]^2}{4(1-e^2)} &= \frac{2}{\pi} \\ (1-e^2)^{1/2}\tilde{a} \left\{ \left(\frac{\tilde{P}}{2\pi\tilde{a}^2\sqrt{1-e^2}} + \tilde{a}F_{02}(e) \right)^2 + \frac{1}{4}[\xi\varepsilon_y\delta_{22}(e)]^2 \right\} &= \frac{2}{\pi} \end{aligned} \right\} \quad (3.10)$$

and

Figure 6. Variation of the pull-off force versus the substrate stretch. (Online version in colour.)

Here, $F_{20}(e)$, $F_{02}(e)$ and $\delta_{12}(e)$, $\delta_{22}(e)$ are given by (C 16), (C 17) and (B 7), (B 8), respectively (see appendix B).

When the quantity $\xi \varepsilon_y$ varies, while the non-dimensionalized contact force $\tilde{P}$ keeps a fixed value, both the non-dimensionalized major semi-axis $\tilde{a}$ and the eccentricity e change, respectively. Therefore, we will have

$$\frac{d\tilde{A}}{d\varepsilon_y} = 2\pi\tilde{a}\sqrt{1-e^2} \frac{\partial\tilde{a}}{\partial\varepsilon_y} - \frac{\pi\tilde{a}^2e}{\sqrt{1-e^2}} \frac{\partial e}{\partial\varepsilon_y}. \quad (3.11)$$

The pull-off force $\tilde{P}_c$ will be determined by the condition that $\partial\tilde{A}/\partial\varepsilon_y$ is infinite.

Let $\mathcal{L}_1(\tilde{P}, \tilde{a}, e, \varepsilon_y)$ and $\mathcal{L}_2(\tilde{P}, \tilde{a}, e, \varepsilon_y)$ be the left-hand sides of equations (3.10)₁ and (3.10)₂, respectively. Then, by differentiating these equations, we obtain

$$\frac{\partial\mathcal{L}_1}{\partial\varepsilon_y} + \frac{\partial\mathcal{L}_1}{\partial\tilde{a}} \frac{\partial\tilde{a}}{\partial\varepsilon_y} + \frac{\partial\mathcal{L}_1}{\partial e} \frac{\partial e}{\partial\varepsilon_y} = 0 \quad (3.12)$$

and

$$\frac{\partial\mathcal{L}_2}{\partial\varepsilon_y} + \frac{\partial\mathcal{L}_2}{\partial\tilde{a}} \frac{\partial\tilde{a}}{\partial\varepsilon_y} + \frac{\partial\mathcal{L}_2}{\partial e} \frac{\partial e}{\partial\varepsilon_y} = 0, \quad (3.13)$$

from where one can evaluate the partial derivatives $\partial\tilde{a}/\partial\varepsilon_y$ and $\partial e/\partial\varepsilon_y$.

Thus, to determine the critical value of $\xi \varepsilon_y$ and, respectively, the pull-off force $\tilde{P}_c$, the following equation can be added to equations (3.12) and (3.13):

$$\frac{\partial\mathcal{L}_1}{\partial e} \frac{\partial\mathcal{L}_2}{\partial\tilde{a}} - \frac{\partial\mathcal{L}_1}{\partial\tilde{a}} \frac{\partial\mathcal{L}_2}{\partial e} = 0. \quad (3.14)$$

Figure 6 shows an evident trend of reducing the absolute value of $\tilde{P}_c$ with increasing substrate stretch.

The developed mathematical model forecasts that the critical contact area eccentricity will increase with increasing substrate stretch (figure 7a). The variation of the half-diameter of the contact elliptical area versus the contact force is illustrated in figure 8a in the case of brittle fracture ($\lambda = 1$). The solution to the system of equations (3.12), (3.13) and (3.14) is presented in figure 8b by the double-dot-dashed line.

The effect of anisotropy of the substrate stretch reveals itself in the change of the major semi-axis with a gradual increase of $\varepsilon_y = 0$ (figure 7b). Contrary to the axisymmetric case, it is anticipated that a small tensile load $\tilde{P} < 0$ can strengthen the contact interface, so that a larger uniaxial substrate strain can be sustained. However, this finding should be verified experimentally.

Figure 7. Variation of the critical contact area eccentricity (a) and the critical contact area major axis (b) at the moment of detachment versus the substrate stretch. (Online version in colour.)

Figure 8. (a) Variation of the dimensionless major semi-axis of the contact area $\bar{a}$ as a function of the dimensionless contact force $\bar{P}$ for different levels of the substrate uniaxial stretch (the red dashed line corresponds to the critical state). (b) Variation of $\bar{a}$ as a function of the reduced substrate uniaxial stretch $\xi\epsilon_y$ (the dot-dashed line corresponds to the critical state, while the double-dot-dashed line corresponds to the unstable critical state for the pull-off force). (Online version in colour.)

(iv) Uniaxial stress-controlled stretch of the substrate

Now, let us assume that the deformation of an incompressible substrate is induced by a remotely applied stress, σ_y , acting in the y -direction. Then, we have

$$\epsilon_x = -\frac{\sigma_y}{2E} \quad \text{and} \quad \epsilon_y = \frac{\sigma_y}{E}. \quad (3.15)$$

The substitution of (3.15) into equation (2.9) yields

$$\tilde{d}_{11} = -\frac{\sigma_y}{E} \beta_{12}(e) \quad \text{and} \quad \tilde{d}_{22} = \frac{\sigma_y}{E} \beta_{22}(e), \quad (3.16)$$

where we have introduced the notation

$$\beta_{12}(e) = \frac{\mathbf{D}(e) - \mathbf{C}(e)}{2(\mathbf{E}(e)\mathbf{C}(e) + \mathbf{D}(e)\mathbf{B}(e))} \quad \text{and} \quad \beta_{22}(e) = \frac{\mathbf{C}(e) + 2\mathbf{D}(e)}{2(\mathbf{E}(e)\mathbf{C}(e) + \mathbf{D}(e)\mathbf{B}(e))}.$$

Note that it can be easily shown that $\beta_{12}(e) > 0$.

Observe that in contrast to the equibiaxial (axisymmetric) case of zero normal loading, when the contact radius decreases to zero with increasing level of stretch, in the asymmetric case of uniaxial stretch, there is a critical size of the contact area beyond which no further decrease in

Figure 9. Variation of the contact area eccentricity (a) and the contact area major axis (b) as functions of the substrate uniaxial stress in the case of zero normal loading. (Online version in colour.)

Figure 10. Variation of the contact area in the strain-controlled (a) and stress-controlled (b) cases for zero normal loading. (Online version in colour.)

contact is possible (figure 9b). It is of interest to compare the cases of strain-controlled and stress-controlled stretches (figures 3 and 9). Owing to the significant Poisson's ratio effect, the critical diameter of the contact area in the latter case (figure 9b) is approximately half as much. This is readily seen from the horizontal asymptotes $\tilde{a} = \tilde{a}_c^\infty$ in figures 3b ($\tilde{a}_c^\infty = 1.554$) and 9b ($\tilde{a}_c^\infty = 0.778$), where

$$\tilde{a}_c^\infty = \lim_{\lambda \rightarrow 0} \tilde{a}_c(\lambda) = \left(\frac{\sqrt{2}[(1 - e_c^2)^{3/4} \beta_{22}(e_c) - \beta_{12}(e_c)]}{\sqrt{\pi}(1 - e_c^2)^{3/4} |F_{20}(e_c)| \beta_{22}(e_c)} \right)^{2/3}. \quad (3.17)$$

In the strain-controlled case, the value of $\tilde{a}_c^\infty$ is given by the same formula (3.17) with $\beta_{12}(e_c)$ and $\beta_{22}(e_c)$ being replaced by $\delta_{12}(e_c)$ and $\delta_{22}(e_c)$.

Figure 10a,b shows the plots of the relative contact area $\tilde{A} = \pi \sqrt{1 - e^2} \tilde{a}^2$ versus the dimensional measure of stretch in the strain- and stress-controlled cases, respectively.

(v) Axisymmetric loading

Recall [3] that the JKR-type models of adhesion are applicable for relatively large values of the Tabor parameter [39]

$$\mu_T = \left(\frac{RG_{Ic}^2}{E^* z_0^3} \right)^{1/3},$$

where z_0 is the range of attraction of adhesive forces.

In view of the notation (2.14), we can rewrite the above formula as

$$\mu_T = \frac{1}{\xi^2} \frac{R}{z_0}. \quad (3.18)$$

Therefore, if, for instance, it is required that $\mu_T \geq 5$, then equation (3.18) implies the upper bound

$$\xi \leq \sqrt{\frac{R}{5z_0}}.$$

This inequality determines the range of applicability of the developed mathematical model (e.g. figures 3, 6, 7 and 8b).

As a partial check of our model, we consider the case of axisymmetric loading under equibiaxial stretching. Let us assume that $R_1 = R_2 = R$ and $\varepsilon_x = \varepsilon_y = \varepsilon$, so that $e = 0$ and equations (2.9) and (B9) yield $\tilde{d}_{11} = \tilde{d}_{22} = (4/\pi)\varepsilon$. Further, since $\mathcal{E}_{20}(0, 1) = \mathcal{E}_{02}(0, 1) = 2/\pi$, as follows from equations (A 4), (A 5), (2.14)₃, and (C 7), (C 8), the system of two equations (C 9) reduces to $\tilde{p}_{20} = \tilde{p}_{02} = 2/(\pi\tilde{a})$ and its substitution into equation (C 10) yields $\tilde{p}_0 = \tilde{P}/(2\pi\tilde{a}^2) + 4\tilde{a}/(3\pi)$. Finally, equations (C 2) and (C 3) simplify as $\psi_2 = \arctan\{12\xi\varepsilon\tilde{a}^2/(3\tilde{P} - 4\tilde{a}^3)\}$, and the system of two equations (C 4) and (C 5) reduces to the following one:

$$\frac{2}{9\pi}\tilde{a}^3 + \frac{2\xi^2}{\pi}\varepsilon^2\tilde{a} + \frac{\tilde{P}^2}{8\pi\tilde{a}^3} - \frac{\tilde{P}}{3\pi} = f_{II}(\psi_2). \quad (3.19)$$

Equation (3.19) involves three variables, $\tilde{P}$, ε , and $\tilde{a}$, and for given parameters of loading $\tilde{P}$ and ε it allows us to determine the relative contact radius $\tilde{a}$. It can be verified that equation (3.19) corresponds to the previously obtained solution [26]. A detailed analysis of the axisymmetric model is done in [40].

It is of practical importance to have a simpler analytical tool to grasp the broader picture of adhesive spherical contact under biaxial stretching. With this in mind, taking into account the fact that the governing equations depend on the square of the substrate strain, we introduce the average strain as follows:

$$\begin{aligned} \bar{\varepsilon} &= \left(\frac{1}{\pi} \int_0^\pi \left(\frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\varphi \right)^2 d\varphi \right)^{1/2} \\ &= \sqrt{\frac{3}{8} \left(\varepsilon_x^2 + \frac{2}{3}\varepsilon_x\varepsilon_y + \varepsilon_y^2 \right)}. \end{aligned} \quad (3.20)$$

Here, ε_x and ε_y are the principal strains in the substrate.

It is suggested that the axisymmetric model (3.19), with ε being replaced by $\bar{\varepsilon}$ according to formula (3.20), can be used for initial parameter estimate guesses. To assess the efficiency of such predictions, we compare the variations in contact area shown in figure 10a,b with the axisymmetric case in the situation of zero normal loading (figure 11). Note that the ratio $\bar{\varepsilon}/\max\{\varepsilon_x, \varepsilon_y\}$ for the strain-controlled and stress-controlled cases is equal to 0.612 and 0.586, respectively. It should also be noted that, of course, the non-axisymmetric and axisymmetric models produce different *qualitative* predictions with increasing $\xi\bar{\varepsilon}$, since the latter model (see the dashed line in figure 11) forecasts a gradual decrease in the contact area with the substrate stretch, while the non-axisymmetric model implies the existence of a critical stretch.

It should be emphasized that the presented theory requires that the contact force has to be imposed first to establish a contact area, then the in-plane stretch is subsequently applied to weaken the interface. Therefore, any conclusion made may not hold if the above assumption is violated. For instance, it would be of interest to verify experimentally whether increasing a normal load, when an interface is already in equilibrium, will not make the system more vulnerable (see the conclusion at the end of §3a(iii)).

Figure 11. Variation in the contact area in the strain-controlled (red solid lines), stress-controlled (blue solid lines) and axisymmetric (dashed lines) cases for zero normal loading. (Online version in colour.)

4. Discussion

Observe that our analysis shows that the contact area between a spherical indenter and an elastic half-space (substrate) becomes elongated when the substrate is subjected to uniaxial tension and the major axis of the contact area is oriented perpendicularly to the tension direction. This mechanism can be regarded as passive, since the indenter shape does not change and the adhesion between the surfaces is not controlled during the substrate stretch.

In response to cyclic stretching, cultured cells certainly align away from the stretching direction, and the present model agrees with this qualitative finding. However, the precise cell alignment, the dependence on the stretching magnitude and rate, the role of the actin cytoskeleton and microtubules and many other aspects in the alignment response remain unclear. Some authors suggest that, upon application of uniaxial stretch, cells align at nearly 70° (oblique) from the load direction—and interpret this as the point at which the minimal deformation of a membrane occurs [41,42], or simply as the direction of no normal strain in the substrate [20]. Other authors report that uniaxial cyclic strain causes cells to reorient themselves perpendicularly to the direction of loading [21,43,44]. In some cases, authors report different directions for pure elongation (no Poisson's effect allowed) and pure stretching (negative strain in the direction perpendicular to loading) for the two types of stretching, but it must be borne in mind that the two results differed by only about 20° [45].

It is clear that our simple model, while showing some accurate treatment of the adhesion mechanics for elastic half-spaces, has strong simplifications and assumptions in other respects, and other models of cells' adhesion to a substrate may be more appropriate in predicting the cell orientation. Also, we are not modelling the case of an initially elongated contact (that is, before stretching), which would probably continuously change not only the size but also the orientation of the contact area. However, a random population of cells would have no initial preferential direction, and the present model therefore has no strong assumptions in these respects. The more subtle question about the oblique versus perpendicular preferential direction remains largely unanswered.

One of the model simplifications concerns approximate treatment of non-axisymmetric adhesive contact. In our analysis, we employed the approximate JKR-type model for elliptical adhesive contact proposed by Greenwood & Johnson [6], where it was assumed that (i) the SIF is the same at the elliptical contact border (while imposing the equivalence of the SIF only on the semi-axes) and (ii) the contact shape remains elliptical during the pull-off, which is actually not

exact. In the literature [46,47], there are numerical solutions to the adhesive contact problem for Hertzian-type contact geometry, which are based on the assumption that the surface interaction is governed by the Lennard–Jones potential. It is shown that the JKR model is valid for values of the Tabor parameter μ_T larger than 3. However, to the best of the authors' knowledge, the extension of the latter approach to the case of tangential loading has not been developed so far.

5. Conclusion

We have introduced a more complete adhesive model for the Hertzian problem under the JKR assumption, where stretching of a substrate is non-axisymmetric. Full results have been provided assuming a certain form of the mixed-mode interaction function, although the choice is a quite sensitive one, since the final instability of the contact depends qualitatively and quantitatively on it. The model suggests a possible effect of strengthening of the contact interface by imposing a small pre-pulling force acting on the indenter, so that the contact area can withstand a larger level of substrate stretching, while increasing the contact area eccentricity, before the moment of instability. It is shown that the JKR-type passive–adhesive mechanism results in orienting the elliptical contact area under a spherical probe in a direction perpendicular to the direction of substrate stretch. The developed extension to an elliptical Hertzian contact between a rigid indenter and an incompressible substrate that is subjected to in-plane tensions parallel with the free surface reflects the interest of the biomechanics community.

Data accessibility. This article has no additional data.

Authors' contributions. All authors conceived the study. I.A. carried out the theoretical analysis, created the figures and drafted the manuscript. I.A. and A.P. revised the manuscript. M.C. coordinated the study. All authors gave final approval for publication.

Competing interests. We declare we have no competing interest.

Funding. M.C. and A.P. acknowledge the support of the Italian Ministry of Education, University and Research under the programme 'Department of Excellence' Legge 232/2016 (grant no. CUP-D94I18000260001). A.P. is grateful to the DFG (German Research Foundation) for funding project PA 3303/1-1. A.P. acknowledges support from 'PON Ricerca e Innovazione 2014–2020-Azione I.2-D.D. n. 407, 27/02/2018, bando AIM (grant no. AIM1895471)'.

Acknowledgements. I.A. is grateful to the Teaching@Poliba programme for hospitality during his stay at the Polytechnic University of Bari, where this research was carried out. A.P. thanks Prof. M. Napolitano for his support with the 'Center of excellence in computational mechanics' funding during 2018.

Appendix A. Normal contact with an elliptical contact area

Based on the known solution [48], the following relations have been established [49] for the coefficients in the contact pressure distribution (2.3):

$$p_0 = \frac{E^*}{2a\sqrt{1-e^2}} \left\{ \frac{\delta}{\mathbf{K}(e)} - \frac{a^2}{2R_1} \left(\frac{1}{3\mathbf{K}(e)} - C_1(e)\sigma_1 - C_2(e)\sigma_2 \right) - \frac{a^2}{2R_2} \left(\frac{1-e^2}{3\mathbf{K}(e)} - C_1(e)(3\sigma_1-1)\sigma_1 + C_2(e)(1-3\sigma_2)\sigma_2 \right) \right\}, \quad (\text{A } 1)$$

$$p_{20} = \frac{E^*}{2a\sqrt{1-e^2}} \left\{ \frac{1}{2R_1} [C_1(e) + C_2(e)] + \frac{1}{2R_2} [C_1(e)(3\sigma_1-1)\sigma_1 - C_2(e)(1-3\sigma_2)] \right\} \quad (\text{A } 2)$$

$$\text{and } p_{02} = \frac{E^*}{2a\sqrt{1-e^2}} \left\{ \frac{1}{2R_1} \left(\frac{C_1(e)}{1-3\sigma_2} + \frac{C_2(e)}{3\sigma_1-1} \right) + \frac{1}{2R_2} \left(\frac{C_1(e)(3\sigma_1-1)}{1-3\sigma_2} + \frac{C_2(e)(1-3\sigma_2)}{3\sigma_1-1} \right) \right\}. \quad (\text{A } 3)$$

Figure 12. Schematic of the bonded contact between an elliptical inextensible patch and an elastic half-space under unidirectional tensile stress. (Online version in colour.)

Here, $\mathbf{K}(e)$ is the complete elliptic integral of the first kind, $3\sigma_{1,2} = 1 + e^2 \pm \sqrt{1 - e^2 + e^4}$,

$$C_1(e) = \frac{2(\sigma_1 - e^2)^2}{(3\sigma_1^2 - e^2)(1 - \sigma_1)[\mathbf{K}(e) - 3\sigma_2\mathbf{D}(e)]} \quad (\text{A } 4)$$

and

$$C_2(e) = \frac{2\sigma_1^2(3\sigma_1 - 1)^2}{(3\sigma_1^2 - e^2)(3\sigma_1^2 - e^2)[\sigma_1\mathbf{C}(e) + (2\sigma_1 - 1)\mathbf{D}(e)]}, \quad (\text{A } 5)$$

where $\mathbf{D}(e) = e^{-2}[\mathbf{K}(e) - \mathbf{E}(e)]$, $\mathbf{C}(e) = e^{-2}[2\mathbf{D}(e) - \mathbf{K}(e)]$ and $\mathbf{E}(e)$ is the complete elliptic integral of the second kind.

Note that equation (2.6) follows from Mossakovskii's reciprocity theorem [36], which states that

$$P = \frac{\pi E^* a}{\mathbf{K}(e)} \delta - \frac{E^*}{2b\mathbf{K}(e)} \iint_{\omega} \left(\frac{x^2}{2R_1} + \frac{y^2}{2R_2} \right) \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right)^{-1/2} dx dy,$$

where ω is the elliptical contact area defined by equation (2.2).

Appendix B. Tangential contact under biaxial stretch

According to the known solution [37], the tangential stresses beneath an elliptical inextensible membrane bonded to the surface of an isotropic elastic half-space with Poisson's ratio ν (figure 12), which is subjected to uniform stresses, p , remotely applied at angle φ to the major axis, can be evaluated as follows:

$$\tau_{xz}(x, y) = \left(\frac{a_1 Q_{01} - q_1 b_2 Q_{11}}{P_{10} Q_{01} - q_1^2 P_{11} Q_{11}} x + \frac{a_2 Q_{10} - q_1 b_1 P_{11}}{P_{01} Q_{10} - q_1^2 P_{11} Q_{11}} y \right) \frac{1}{l(x, y)} \quad (\text{B } 1)$$

and

$$\tau_{yz}(x, y) = \left(\frac{b_1 P_{01} - q_1 a_2 Q_{11}}{P_{01} Q_{10} - q_1^2 P_{11} Q_{11}} x + \frac{b_2 P_{10} - q_1 a_1 P_{11}}{P_{10} Q_{01} - q_1^2 P_{11} Q_{11}} y \right) \frac{1}{l(x, y)}. \quad (\text{B } 2)$$

Here, we have introduced the notation

$$l(x, y) = \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{a^2(1 - e^2)}},$$

$$\left. \begin{aligned}
 P_{01} &= \frac{2b}{e^2}(q_1 + 1)[(e^2 - 1)\mathbf{K}(e) + \mathbf{E}(e)] + q_1 Q_{11}, \\
 P_{11} &= \frac{2b}{e^4}[2\mathbf{E}(e) - (2 - e^2)\mathbf{K}(e)], \\
 P_{10} &= \frac{2b}{e^2}[\mathbf{K}(e) - \mathbf{E}(e)] - q_1 P_{11}, \quad Q_{11} = (1 - e^2)P_{11}, \\
 Q_{01} &= \frac{2b}{e^2}[\mathbf{E}(e) - (1 - e^2)\mathbf{K}(e)] - q_1 Q_{11}, \\
 Q_{10} &= \frac{2b}{e^2}(q_1 + 1)[\mathbf{K}(e) - \mathbf{E}(e)] + q_1 P_{11}, \quad q_1 = \frac{\nu}{1 - \nu}, \\
 a_1 &= -\frac{p}{1 - \nu^2}(\cos^2 \varphi - \nu \sin^2 \varphi), \quad a_2 = -\frac{p}{2(1 - \nu)} \sin 2\varphi \\
 \text{and} \quad b_1 &= -\frac{p}{2(1 - \nu)} \sin 2\varphi, \quad b_2 = -\frac{p}{1 - \nu^2}(\sin^2 \varphi - \nu \cos^2 \varphi).
 \end{aligned} \right\} \quad (\text{B } 3)$$

Thus, the tangential stresses inside the elliptical contact under biaxial stretch with strains ε_x and ε_y are given by equations (2.7) with the coefficients

$$d_{11} = -\frac{E^*(\varepsilon_x Q_{01} - \varepsilon_y Q_{11})}{P_{10}Q_{01} - P_{11}Q_{11}} \quad \text{and} \quad d_{22} = -\frac{E^*(\varepsilon_y P_{10} - \varepsilon_x P_{11})}{P_{10}Q_{01} - P_{11}Q_{11}}. \quad (\text{B } 4)$$

Recall that for an incompressible substrate $E^* = (4/3)E$ and $q_1 = 1$.

The SIFs of the stress field (2.7) near the vertices of the elliptical contact area ω are

$$K_{\text{II}}|_{x=\pm a} = -\sqrt{2\pi} \lim_{|x| \rightarrow a} (a - |x|)^{1/2} \tau_{xz} = \mp \sqrt{\pi} d_{11} a^{3/2} \quad (\text{B } 5)$$

and

$$K_{\text{II}}|_{y=\pm b} = -\sqrt{2\pi} \lim_{|y| \rightarrow b} (b - |y|)^{1/2} \tau_{yz} = \mp \sqrt{\pi} d_{22} b^{3/2}. \quad (\text{B } 6)$$

From equations (B 4)–(B 6), we readily arrive at equation (2.8), where we have introduced the notation

$$\delta_{11}(e) = \frac{\mathbf{B}(e) + (1 - e^2)\mathbf{C}(e)}{\Delta(e)}, \quad \delta_{12}(e) = \frac{(1 - e^2)\mathbf{C}(e)}{\Delta(e)}, \quad (\text{B } 7)$$

$$\delta_{21}(e) = \frac{\mathbf{C}(e)}{\Delta(e)}, \quad \delta_{22}(e) = \frac{\mathbf{D}(e) + \mathbf{C}(e)}{\Delta(e)}, \quad (\text{B } 8)$$

and $\mathbf{B}(e) = \mathbf{K}(e) - \mathbf{D}(e)$,

$$\Delta(e) = \mathbf{E}(e)\mathbf{C}(e) + \mathbf{D}(e)\mathbf{B}(e), \quad (\text{B } 9)$$

so that the denominator of equation (2.9) is positive for any $e \in [0, 1]$.

Appendix C. Non-dimensionalization of the problem equations

In the light of (2.14) and (2.13), we put

$$\tilde{p}_0 = \frac{\xi p_0}{E^*}, \quad \tilde{p}_{20} = \frac{R_e^2 p_{20}}{\xi E^*} \quad \text{and} \quad \tilde{p}_{02} = \frac{R_e^2 p_{02}}{\xi E^*}. \quad (\text{C } 1)$$

Then, taking equations (2.5), (2.8) and (2.13) into account, (C 1), the phase angles at the ellipse's vertices, will be given by

$$\psi_2|_{\theta=0} = \arctan \frac{\xi \tilde{d}_{11}}{2(1 - e^2)^{1/2}(\tilde{p}_0 - \tilde{a}^2 \tilde{p}_{20})} \quad (\text{C } 2)$$

and

$$\psi_2|_{\theta=\pi/2} = \arctan \frac{\xi \tilde{d}_{22}}{2(\tilde{p}_0 - \tilde{a}^2 \tilde{p}_{02}(1 - e^2))}, \quad (\text{C } 3)$$

where $\tilde{d}_{11}$ and $\tilde{d}_{22}$ are given by equation (2.9).

Respectively, the equations

$$K_{\text{eq}}^2|_{\theta=0} = 2E^* G_{\text{Ic}} f_{\text{II}}(\psi_2(0)) \quad \text{and} \quad K_{\text{eq}}^2|_{\theta=\pi/2} = 2E^* G_{\text{Ic}} f_{\text{II}}\left(\psi_2\left(\frac{\pi}{2}\right)\right)$$

take the form

$$\tilde{a}(\tilde{p}_0 - \tilde{a}^2 \tilde{p}_{20})^2 + \frac{\xi^2 \tilde{a} \tilde{d}_{11}^2}{4(1 - e^2)} = \frac{2}{\pi} f_{\text{II}}(\psi_2(0)) \quad (\text{C } 4)$$

and

$$(1 - e^2)^{1/2} \tilde{a} \left\{ (\tilde{p}_0 - \tilde{a}^2(1 - e^2) \tilde{p}_{02})^2 + \frac{\xi^2}{4} \tilde{d}_{22}^2 \right\} = \frac{2}{\pi} f_{\text{II}}\left(\psi_2\left(\frac{\pi}{2}\right)\right), \quad (\text{C } 5)$$

where $\psi_2(0)$ and $\psi_2(\pi/2)$ are given by equations (C 2) and (C 3).

In particular, from equation (2.5), it follows that

$$\tilde{K}_{\text{I}}|_{\tilde{x}=\pm\tilde{a}} = -\tilde{a}^{1/2}(\tilde{p}_0 - \tilde{a}^2 \tilde{p}_{20})$$

and

$$\tilde{K}_{\text{I}}|_{\tilde{y}=\pm\tilde{b}} = -(1 - e^2)^{1/4} \tilde{a}^{1/2}(\tilde{p}_0 - \tilde{a}^2(1 - e^2) \tilde{p}_{02}), \quad (\text{C } 6)$$

where $\tilde{x} = x/a$ and $\tilde{y} = y/a$ are dimensionless coordinates.

Now, let us introduce the notation

$$\mathcal{E}_{20}(e, \gamma) = \frac{1}{2\sqrt{1 - e^2}} \left\{ \frac{\gamma}{2} [C_1(e) + C_2(e)] + \frac{1}{2\gamma} [C_1(e)(3\sigma_1 - 1) - C_2(e)(1 - 3\sigma_2)] \right\} \quad (\text{C } 7)$$

and

$$\mathcal{E}_{02}(e, \gamma) = \frac{1}{2\sqrt{1 - e^2}} \left\{ \frac{\gamma}{2} \left(\frac{C_1(e)}{1 - 3\sigma_2} + \frac{C_2(e)}{3\sigma_1 - 1} \right) + \frac{1}{2\gamma} \left(\frac{C_1(e)(3\sigma_1 - 1)}{1 - 3\sigma_2} + \frac{C_2(e)(1 - 3\sigma_2)}{3\sigma_1 - 1} \right) \right\}. \quad (\text{C } 8)$$

Then, in view of (2.14), (2.13), (C 1), (C 7) and (C 8), we can rewrite equations (A 2) and (A 3) as

$$\tilde{p}_{20} = \frac{1}{\tilde{a}} \mathcal{E}_{20}(e, \gamma) \quad \text{and} \quad \tilde{p}_{02} = \frac{1}{\tilde{a}} \mathcal{E}_{02}(e, \gamma). \quad (\text{C } 9)$$

In the same way, equation (2.4) can be represented in the form

$$\tilde{p}_0 = \frac{\tilde{P}}{2\pi \tilde{a}^2 \sqrt{1 - e^2}} + \frac{\tilde{a}^2}{3} [\tilde{p}_{20} + (1 - e^2) \tilde{p}_{02}]. \quad (\text{C } 10)$$

It is assumed that the value of $\tilde{P}$ is known. Then, for given values of ε_x and ε_y , the system of five equations (C 4), (C 5), (C 9) and (C 10) with respect to five unknowns $\tilde{a}$, e , $\tilde{p}_0$, $\tilde{p}_{20}$ and $\tilde{p}_{02}$ can be reduced to a system of two equations with respect to only two variables, $\tilde{a}$ and e , as follows.

First, by excluding $\tilde{p}_{20}$ and $\tilde{p}_{02}$ from equation (C 10) and using equation (C 9), we obtain

$$\tilde{p}_0 = \frac{\tilde{P}}{2\pi \tilde{a}^2 \sqrt{1 - e^2}} + \frac{\tilde{a}}{3} [\mathcal{E}_{20}(e, \gamma) + (1 - e^2) \mathcal{E}_{02}(e, \gamma)]. \quad (\text{C } 11)$$

In the same way, using equation (C 9), we obtain

$$\tilde{p}_0 - \tilde{a}^2 \tilde{p}_{20} = \frac{\tilde{P}}{2\pi \tilde{a}^2 \sqrt{1 - e^2}} + \tilde{a} \mathcal{F}_{20}(e, \gamma) \quad (\text{C } 12)$$

and

$$\tilde{p}_0 - \tilde{a}^2(1 - e^2) \tilde{p}_{02} = \frac{\tilde{P}}{2\pi \tilde{a}^2 \sqrt{1 - e^2}} + \tilde{a} \mathcal{F}_{02}(e, \gamma), \quad (\text{C } 13)$$

where

$$\mathcal{F}_{20}(e, \gamma) = \frac{1}{3} [(1 - e^2) \mathcal{E}_{02}(e, \gamma) - 2 \mathcal{E}_{20}(e, \gamma)] \quad (\text{C } 14)$$

and

$$\mathcal{F}_{02}(e, \gamma) = \frac{1}{3} [\mathcal{E}_{20}(e, \gamma) - 2(1 - e^2) \mathcal{E}_{02}(e, \gamma)], \quad (\text{C } 15)$$

and $\mathcal{E}_{20}(e, \gamma)$ and $\mathcal{E}_{02}(e, \gamma)$ are given by (C 7) and (C 8), respectively.

By setting $F_{20}(e) = \mathcal{F}_{20}(e, 1)$ and $F_{02}(e) = \mathcal{F}_{02}(e, 1)$, we obtain

$$F_{20}(e) = \frac{1}{4\sqrt{1-e^2}} \left(3\sigma_1(\sigma_1 - 1)C_1(e) - 2C_2(e)\sigma_2 - \sigma_2 + 3\sigma_2^2 \right) \quad (\text{C } 16)$$

and

$$F_{02}(e) = \frac{1}{4\sqrt{1-e^2}} \left(-3\sigma_1(2\sigma_1 - 1)C_1(e) + C_2(e)\sigma_2 + 2\sigma_2 - 6\sigma_2^2 \right). \quad (\text{C } 17)$$

Now, making use of equations (C 12) and (C 13), we can express the right-hand sides of equations (C 4), (C 5) in terms of $\tilde{P}$, $\tilde{a}$ and e .

Appendix D. Adhesive contact of a rigid cylinder with a stretched substrate

Let us now investigate the limit as $e \rightarrow 1$ and $\gamma \rightarrow 0$. By integrating the contact pressure (2.3) across an elongated elliptical area of contact, we evaluate the line load

$$\begin{aligned} L(x) &= b \int_{-\sqrt{1-\xi^2}}^{\sqrt{1-\xi^2}} \frac{p_0 - p_{20}a^2\xi^2 - p_{02}b^2\eta^2}{\sqrt{1-\xi^2-\eta^2}} d\eta \\ &= \pi b(p_0 - p_{20}x^2) - \frac{\pi}{2}p_{02}b^3 \left(1 - \frac{x^2}{a^2} \right), \end{aligned} \quad (\text{D } 1)$$

where $\xi = x/a$.

As R_2/R_1 tends to zero, equation (D 1) yields

$$L \simeq \pi b p_0 - \frac{\pi}{2} p_{02} b^3, \quad (\text{D } 2)$$

where

$$p_{02} = \frac{E^*}{2bR_2}. \quad (\text{D } 3)$$

At the same time, equations (2.5) and (B 6) imply that

$$K_I|_{y=\pm b} = -\sqrt{\pi}b^{1/2}(p_0 - p_{02}b^2) \quad (\text{D } 4)$$

and

$$K_{II}|_{y=\pm b} = \pm \frac{\sqrt{\pi}}{2} b^{1/2} E^* \varepsilon_y. \quad (\text{D } 5)$$

By excluding p_0 and p_{02} from equation (D 4) using the asymptotic relations (D 2) and (D 3), we get

$$K_I|_{y=\pm b} = \sqrt{\pi}b^{1/2} \left(\frac{E^*b}{4R_2} - \frac{L}{\pi b} \right). \quad (\text{D } 6)$$

Hence, substituting (D 5) and (D 6) into equation (2.11), we arrive at the equation

$$b^3 \left(1 - \frac{4R_2L}{\pi b^2 E^*} \right)^2 + 4R_2^2 b \varepsilon_y^2 - \frac{32G_{Ic}R_2^2}{\pi E^*} f_{II}(\psi_2) = 0, \quad (\text{D } 7)$$

where

$$\psi_2 = \mp \arctan \left\{ \frac{2R_2\varepsilon_y}{b} \left(1 - \frac{4R_2L}{\pi b^2 E^*} \right)^{-1} \right\}. \quad (\text{D } 8)$$

We note that, in the case $L = 0$ and $f_{II}(\psi) \equiv 1$, equation (D 7) (apart from notation) coincides with equation (4.8), which was derived by Chen & Gao [24].

In the light of (2.14) and (2.13), we introduce the notation

$$\xi_2 = \left(\frac{E^* R_2}{G_{1c}} \right)^{1/3}, \quad \tilde{b} = \frac{\xi_2 b}{R_2} \quad \text{and} \quad \tilde{L} = \frac{L}{\xi_2 G_{1c}}. \quad (\text{D } 9)$$

Then, the substitution of (D 9) into equation (D 7) yields

$$\tilde{b}^3 \left(1 - \frac{4\tilde{L}}{\pi \tilde{b}^2} \right)^2 + 4\tilde{b}(\xi_2 \varepsilon_y)^2 - \frac{32}{\pi} f_{II}(\psi_2) = 0, \quad (\text{D } 10)$$

while equation (D 8) implies

$$\psi_2 = \mp \arctan \left\{ 2\xi_2 \varepsilon_y \left(1 - \frac{4\tilde{L}}{\pi \tilde{b}^2} \right)^{-1} \right\}. \quad (\text{D } 11)$$

We note that the minus/plus sign in (D 11) can be dropped since the function $f_{II}(\psi)$ is even.

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