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Symmetry determination and nonlinearization of a nonlinear time-fractional partial differential equation

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We first show that the infinitesimal generator of Lie symmetry of a time-fractional partial differential equation (PDE) takes a unified and simple form, and then separate the Lie symmetry condition into two distinct parts, where one is a linear time-fractional PDE and the other is an integer-order PDE that dominates the leading position, even completely determining the symmetry for a particular type of time-fractional PDE. Moreover, we show that a linear time-fractional PDE always admits an infinite-dimensional Lie algebra of an infinitesimal generator, just as the case for a linear PDE and a nonlinear time-fractional PDE admits, at most, finite-dimensional Lie algebra. Thus, there exists no invertible mapping that converts a nonlinear time-fractional PDE to a linear one. We illustrate the results by considering two examples.

1. Introduction

Fractional calculus, popularly regarded as stemming from a question discussed by two great mathematicians, Hopital and Leibniz in 1695, has attracted considerable attention due to its deep applications in the fields of science and engineering [1,2]. A fractional partial differential equation is the main tool used to describe natural phenomena and more adequate than the previously used integer order partial differential equation (PDE) in some ways, such as anomalous diffusion, quasi-chaotic dynamical systems, the dynamics of complex material or porous media and random walks with memory (see [3–5] and references therein). For example, diffusion of tracer particles in complex liquids is anomalous where subdiffusion and superdiffusion

emerge and are more suitably described by a fractional PDE [6]. Thermal stresses in a solid with a heat source varying harmonically in time are investigated in the framework of fractional thermoelasticity [7]. Hence many scholars and research groups are dedicated to studying fractional PDEs from the points of both theories and applications [1,2,8–10].

It is well known that Lie symmetry group theory provides powerful techniques for the analysis of PDEs and some significant symmetry properties of PDEs have been found, such as obtainment of similarity solutions and conservation laws, linearization of nonlinear PDEs, construction of a numerical difference scheme and so on [11–13]. Thus, it is a natural way to consider whether or not Lie symmetry group theory and methods are effective for fractional PDEs and, just as has been done for PDEs, such ideas are performed by many scholars and a number of significant results are obtained, we refer to [14–20] and references therein.

As an application, symmetry method is an effective way to determine whether the given nonlinear PDEs can be transformed into some linear PDEs [21,22]. In [21], based upon the group analysis of PDEs, Bluman and Kumei proposed a symmetry-based algorithm to discriminate the existence of invertible mapping connecting a given system of nonlinear PDEs with a linear one. The key point for determining the existence of such invertible mappings is that a system of linear PDEs always admits an infinite-dimensional Lie algebra of an infinitesimal generator. However, whether the results are suitable for the linearization of nonlinear fractional PDE (1.2) is still unknown.

Furthermore, there exist several problems in the application of Lie symmetry group theory to a fractional PDE. The first one is the definition of fractional derivatives, where there exist several definitions which are not necessarily equivalent to each other [2,23,24]. For example, the Riemann–Liouville derivative of a constant is not zero while the Caputo derivative of a constant is zero. Thus, one must first choose a suitable definition of a fractional derivative and then study a fractional PDE under the selected circumstance, which leads to the case that there exist different types of Lie symmetries for the same time-fractional PDE [25]. The second one is the obtainment of a symmetry determining system. The procedure is mechanical, just like for a PDE, but contains fractional derivative operations which do not facilitate being performed on a computer. In the majority of cases, the ultimate simplified symmetry determining system mainly contains integer order derivative operations and can be handled by the programmes for a PDE, while the fractional derivative operations are usually involved in the last step and used to check the final results. In [26], the authors presented a Maple package to automatically determine Lie point symmetry of fractional differential equations under the assumption μ given in (1.9) vanishes, which has no rigorous proof. The third one is the application of the Lie symmetry method to fractional PDE. The reduced equation of a fractional PDE via Lie symmetry is generally expressed by the Erdelyi–Kober fractional differential operator, which cannot be further solved via analytical methods. Although the power series method is adopted to solve the reduced equations, there exist some misunderstandings, for example, the definition domain of Gamma function is neglected. Therefore, the studies associated with symmetry group theory and methods for a fractional PDE are definitely worth exploring further.

In this paper, we aim at the second problem and consider a time-fractional PDE involving Riemann–Liouville fractional partial derivative

$$\partial_t^\alpha u = E(x, t, u, u_1, u_2, u_3, \dots, u_l), \quad 0 < \alpha < 1, \quad (1.1)$$

where $u = u(x, t)$ is the dependent variable of two independent variables x, t , integer order derivatives $u_i = \partial^i u / \partial x^i$, $i = 0, 1, \dots, l$ and $u_0 = u$, $\partial_t^\alpha u$ is the Riemann–Liouville fractional partial derivative of $u = u(x, t)$ and defined as

$$\partial_t^\alpha u = \frac{\partial^\alpha u}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t (t-s)^{n-\alpha-1} u(s, x) ds, & 0 \leq n-1 < \alpha < n, \\ \frac{\partial^n u}{\partial t^n}, & \alpha = n, \end{cases}$$

where the gamma function $\Gamma(z) = \int_0^\infty e^{-zt} t^{z-1} dt$. Note that nonlinear equation (1.1) means its right side function E is nonlinear in u and its x -derivatives.

In order to facilitate Lie symmetry analysis of equation (1.1), according to whether the terms contain u and its derivatives, we rewrite equation (1.1) as the following form:

$$\partial_t^\alpha u = F(x, t, u, u_1, u_2, u_3, \dots, u_l) + G(x, t), \quad 0 < \alpha < 1, \quad (1.2)$$

where each term in $F = F(x, t, u, u_1, u_2, u_3, \dots, u_l)$ contains u or its x -derivatives or both of them, and the remainder in E are collected in $G = G(x, t)$, which means that G is only a function of x and t .

Assume that equation (1.2) is admitted by a one-parameter Lie group of invertible transformation

$$x^* = x^*(x, t, u; \epsilon), \quad t^* = t^*(x, t, u; \epsilon), \quad u^* = u^*(x, t, u; \epsilon), \quad (1.3)$$

with the group parameter ϵ . According to the Lie theory, it is necessary to expand (1.3) in Taylor's series around ϵ

$$x^* = x + \epsilon \xi(x, t, u) + o(\epsilon), \quad t^* = t + \epsilon \tau(x, t, u) + o(\epsilon), \quad u^* = u + \epsilon \eta(x, t, u) + o(\epsilon),$$

where

$$\xi(x, t, u) = \left. \frac{dx^*}{d\epsilon} \right|_{\epsilon=0}, \quad \tau(x, t, u) = \left. \frac{dt^*}{d\epsilon} \right|_{\epsilon=0}, \quad \eta(x, t, u) = \left. \frac{du^*}{d\epsilon} \right|_{\epsilon=0},$$

and finally to write explicitly the transformation with the expression of all the infinitesimals as the infinitesimal generator

$$X = \xi(x, t, u) \partial_x + \tau(x, t, u) \partial_t + \eta(x, t, u) \partial_u, \quad (1.4)$$

which satisfies two conditions, $\tau|_{t=0} = 0$ and

$$\text{Pr}^{(\alpha, l)} X (\partial_t^\alpha u - F - G)|_{\{\partial_t^\alpha u - F - G = 0\}} = 0. \quad (1.5)$$

In equation (1.5), $\text{Pr}^{(\alpha, l)} X$ denotes the prolongation of the generator X in (1.4) and is given by

$$\text{Pr}^{(\alpha, l)} X = X + \eta^\alpha \frac{\partial}{\partial(\partial_t^\alpha u)} + \sum_{i=1}^l \eta^{(i)} \frac{\partial}{\partial u_i}, \quad (1.6)$$

where $\eta^{(i)}$ is the usual i th-order prolongation of η with respect to x and formulated by [12,13]

$$\eta^{(i)} = D_x^i (\eta - \xi u_1 - \tau u_t) + \xi u_{i+1} + \tau u_{it}, \quad (1.7)$$

where $u_{it} = \partial^{i+1} u / \partial x^i \partial t$, $i = 0, 1, \dots, l$ and $\eta^{(0)} = \eta$, and the symbols D_x and D_t indicate the total derivatives with respect to x and t , respectively,

$$D_x = \partial_x + u_1 \partial_u + u_2 \partial_{u_1} + u_{1t} \partial_{u_t} + \dots$$

and

$$D_t = \partial_t + u_t \partial_u + u_{1t} \partial_{u_1} + u_{tt} \partial_{u_t} + \dots,$$

and $D_t^0(u) = u$, $D_t^{n+1} = D_t(D_t^n)$ and similar for D_x . The symbol η^α is associated with the Riemann-Liouville fractional partial derivative and is determined by [17,20,25]

$$\begin{aligned} \eta^\alpha &= \partial_t^\alpha \eta + [\eta_u - \alpha D_t(\tau)] \partial_t^\alpha u - u \partial_t^\alpha (\eta_u) + \mu - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^n(\xi) \partial_t^{\alpha-n}(u_1) \\ &+ \sum_{n=1}^{\infty} \left[\binom{\alpha}{n} \partial_t^n(\eta_u) - \binom{\alpha}{n+1} D_t^{n+1}(\tau) \right] \partial_t^{\alpha-n}(u), \end{aligned} \quad (1.8)$$

where

$$\left. \begin{aligned} \binom{\alpha}{n} &= \frac{(-1)^{n-1} \alpha \Gamma(n-\alpha)}{\Gamma(1-\alpha) \Gamma(n+1)} \\ \mu &= \sum_{n=2}^{\infty} \sum_{m=2}^n \sum_{k=2}^m \sum_{r=0}^{k-1} \binom{\alpha}{n} \binom{n}{m} \binom{k}{r} \frac{1}{k!} \frac{t^{m-\alpha} (-u)^r}{\Gamma(n+1-\alpha)} \partial_t^m(u^{k-r}) \partial_t^{n-m}(\partial_u^k \eta). \end{aligned} \right\} \quad (1.9)$$

and

After inserting expressions (1.7) and (1.8) into condition (1.5), we split (1.5) with respect to fractional as well as integer-order partial derivatives of u and annihilate the coefficients to zero, then obtain an over-determined linear system for the unknown functions ξ , τ and η . The linear system together with the condition $\tau|_{t=0} = 0$ is called the *symmetry determining system* of equation (1.2). However, solving the symmetry determining system is a cumbersome task and faces two obvious obstacles. One is that the unknown functions ξ , τ and η are functions of three variables, which makes the symmetry determining system very complicated, the other is that the symmetry determining system involves mixed operations of integer-order and fractional partial derivatives. Therefore, it is reasonable to start with these two points to simplify the symmetry determining system.

In this paper, we first show that the unknown functions ξ , τ and η take a unified and simple form, and then present two distinct conditions to determine the Lie symmetry of equation (1.2), where one is a linear time-fractional PDE and the other is an integer-order PDE. The results definitely simplify the symmetry determining system and make the system easier to be solved. In particular, we show that the assumption in previous literature, such as references [15,18,19,26], i.e. μ given in (1.9) vanishes, is correct for the type of equation (1.2).

The remainder of the article is arranged as follows: in §2, after reviewing several required properties, we first give an explicit form of the infinitesimal operators of Lie symmetry admitted by equation (1.2), and then present two simple conditions to determine Lie symmetries of equation (1.2). We also show that there exists no invertible mapping which converts the nonlinear time-fractional equation (1.2) into the linear one. In §3, we use the results to study two time-fractional PDEs including the time-fractional KdV-type equation and time-fractional diffusion-convection equation where in particular the second one contains two arbitrary functions, thus a complete Lie symmetry classification is shown rigorously. The last section concludes the results.

2. Main results

We first review some related properties of the Riemann–Liouville fractional partial derivative [23]. By the definition of the Riemann–Liouville fractional partial derivative, for a power function t^γ ,

$$\partial_t^\alpha t^\gamma = \begin{cases} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha}, & \gamma > \alpha - 1, \\ 0, & \gamma = \alpha - 1. \end{cases}$$

Then the Riemann–Liouville fractional partial derivative of a constant c is not zero and equals $c t^{-\alpha} / \Gamma(1-\alpha)$, thus for a constant c , $\partial_t^\alpha c = 0$ if and only if $c = 0$, which is different with the usual integer-order derivatives. The Riemann–Liouville fractional partial derivative of the sum and product of two functions $u = u(x, t)$ and $v = v(x, t)$ is formulated as follows:

$$\left. \begin{aligned} \partial_t^\alpha (a u + b v) &= a \partial_t^\alpha (u) + b \partial_t^\alpha (v) \\ \partial_t^\alpha (u v) &= \sum_{k=0}^{\infty} \binom{\alpha}{k} \partial_t^k u \partial_t^{\alpha-k} v, \end{aligned} \right\} \quad (2.1)$$

and

where a, b are two constants, u, v and all their derivatives are continuous in $[0, t]$.

Thus alternatively, let $v = 1$, then by (2.1) the Riemann–Liouville fractional partial derivative can be expressed as

$$\partial_t^\alpha u = \begin{cases} \sum_{k=0}^{\infty} \binom{\alpha}{k} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} \frac{\partial^k}{\partial t^k} u(x, t), & n-1 < \alpha < n, \\ \frac{\partial^n u}{\partial t^n}, & \alpha = n. \end{cases} \quad (2.2)$$

(a) The general form of Lie symmetry

Theorem 2.1. *If the infinitesimal generator X in (1.4) leaves equation (1.2) invariant, then X must take the form*

$$X = \xi(x)\partial_x + \tau(t)\partial_t + \eta(x, t, u)\partial_u, \quad (2.3)$$

where $\tau(t)|_{t=0} = 0$ and $\eta_{uu} = 0$.

Proof. We show the theorem by analysing the structure of condition (1.5) on the space with the base set $\{\partial_t u, \partial_t^2 u, \dots; \partial_t u_1, \partial_t^2 u_1, \dots\}$. On the solution manifold of equation (1.2), expanding condition (1.5) yields

$$\eta^\alpha - \tau(t, x, u)(F_t + G_t) - \xi(t, x, u)(F_x + G_x) - \sum_{i=0}^l F_{u_i} \eta^{(i)} = 0, \quad (2.4)$$

where $\eta^{(i)}$ is given by (1.7) and η^α by (1.8), $F_t = \partial F / \partial t$ and similar for F_x and F_{u_i} .

Then substituting (1.7) and (1.8) into equation (2.4) and vanishing the coefficients of time-fractional partial derivatives of u_1 , one obtains

$$\partial_t^{\alpha-n} u_1 : \binom{\alpha}{n} D_t^n(\xi) = 0,$$

which holds for $n = 1, 2, \dots$. Then for $n = 1$ we have $D_t(\xi) = \xi_t + u_t \xi_u = 0$ which implies $\xi_t = \xi_u = 0$, i.e. $\xi = \xi(x)$. Next, consider the coefficient of $u_{(l-1)t} = \partial^l u / \partial x^{l-1} \partial t$ in equation (2.4), which uniquely appears in $\eta^{(l)} = D_x^l(\eta - \xi u_1 - \tau u_t) + \xi u_{l+1} + \tau u_{lt}$, where

$$D_x^l(\tau u_t) = \sum_{j=0}^l \binom{l}{j} D_x^j \tau D_x^{l-j} u_t,$$

then the coefficient of $u_{(l-1)t}$ in equation (2.4) is $F_{u_l} D_x \tau = 0$ which means $D_x \tau = \tau_x + u_x \tau_u = 0$ since $F_{u_l} \neq 0$. Thus, we obtain $\tau_x = \tau_u = 0$, i.e. $\tau = \tau(t)$. Note that since $\xi = \xi(x)$, $\tau = \tau(t)$, the prolongation formula $\eta^{(i)}$ can be simplified to $\eta^{(i)} = D_x^i(\eta - \xi u_1) + \xi u_{i+1}$. Then equation (2.4) becomes

$$\begin{aligned} & \partial_t^\alpha \eta - u \partial_t^\alpha(\eta_u) + \sum_{n=1}^{\infty} \left[\binom{\alpha}{n} \partial_t^n(\eta_u) - \binom{\alpha}{n+1} D_t^{n+1}(\tau) \right] \partial_t^{\alpha-n}(u) + \mu \\ & + [\eta_u - \alpha D_t(\tau)] \Delta - \tau(t) \Delta_t - \xi(x) \Delta_x - \sum_{i=0}^l F_{u_i} \eta^{(i)} = 0, \end{aligned} \quad (2.5)$$

where $\Delta = F + G$.

Now we turn to $\eta = \eta(x, t, u)$. Observe that t -derivatives of u in equation (2.5) uniquely occur in $\partial_t^{\alpha-n}(u)$ and μ , where all the terms in the former are linear and nonlinear terms emerge in the latter. In particular, the term u_t^2 uniquely occurs in μ with $m = 2$. Then we separate the terms with $m = 2$ from μ and rewrite μ as the following form:

$$\begin{aligned} \mu &= \sum_{n=2}^{\infty} \sum_{r=0}^1 \binom{\alpha}{n} \binom{n}{2} \binom{2}{r} \frac{1}{2!} \frac{t^{n-\alpha}(-u)^r}{\Gamma(n+1-\alpha)} \partial_t^2(u^{2-r}) \partial_t^{n-2}(\eta_{uu}) \\ &+ \sum_{n=3}^{\infty} \sum_{m=3}^n \sum_{k=2}^m \sum_{r=0}^{k-1} \binom{\alpha}{n} \binom{n}{m} \binom{k}{r} \frac{1}{k!} \frac{t^{n-\alpha}(-u)^r}{\Gamma(n+1-\alpha)} \partial_t^m(u^{k-r}) \partial_t^{n-m}(\partial_u^k \eta) \\ &= \sum_{n=2}^{\infty} \binom{\alpha}{n} \binom{n}{2} \frac{t^{n-\alpha}(u_t)^2}{\Gamma(n+1-\alpha)} \partial_t^{n-2}(\eta_{uu}) \\ &+ \sum_{n=3}^{\infty} \sum_{m=3}^n \sum_{k=2}^m \sum_{r=0}^{k-1} \binom{\alpha}{n} \binom{n}{m} \binom{k}{r} \frac{1}{k!} \frac{t^{n-\alpha}(-u)^r}{\Gamma(n+1-\alpha)} \partial_t^m(u^{k-r}) \partial_t^{n-m}(\partial_u^k \eta). \end{aligned} \quad (2.6)$$

For the coefficient of u_t^α in (2.6), let $p = n - 2$, then it becomes

$$\begin{aligned} & \sum_{p=0}^{\infty} \binom{\alpha}{p+2} \binom{p+2}{2} \frac{t^{p+2-\alpha}}{\Gamma(p+3-\alpha)} \partial_t^p (\eta_{uu}) \\ &= \frac{1}{2} \alpha (\alpha - 1) \sum_{p=0}^{\infty} \binom{\alpha-2}{p} \frac{t^{p+2-\alpha}}{\Gamma(p+3-\alpha)} \partial_t^p (\eta_{uu}) \\ &= \frac{1}{2} \alpha (\alpha - 1) \partial_t^{\alpha-2} (\eta_{uu}), \end{aligned} \quad (2.7)$$

where property (2.2) is used. Annihilating the coefficient of u_t^α to zero gives $\partial_t^{\alpha-2} (\eta_{uu}) = 0$, which implies $\eta_{uu} = C(x, u) t^{\alpha-3}$ with an undetermined function $C(x, u)$. Next, we show $C(x, u) = 0$. Observe that

$$\sum_{r=0}^{k-1} \binom{k}{r} (-u)^r \partial_t^m (u^{k-r}) = \sum_{r=0}^{k-1} (-1)^r \frac{1}{r!} \sum \frac{m! k!}{a_1! \dots a_m! a!} u^{k-a} \prod_{j=1}^m \left(\frac{\partial_t^j u}{j!} \right)^{a_j}, \quad (2.8)$$

where a_j are non-negative integers and the second sum works on $a_1 + \dots + a_m = a \leq k$, $a_1 + 2a_2 + \dots + ma_m = m$. Then, in expression (2.8), the total degree of each term is k . In particular for $k = 2$, we separate the case of $k = 2$ from the last summation of μ in (2.6) and rearrange it as the form

$$\begin{aligned} \mu &= \sum_{n=3}^{\infty} \sum_{m=3}^n \sum_{r=0}^1 \binom{\alpha}{n} \binom{n}{m} \binom{2}{r} \frac{1}{2} \frac{t^{n-\alpha} (-u)^r}{\Gamma(n+1-\alpha)} \partial_t^m (u^{2-r}) \partial_t^{n-m} (\eta_{uu}) \\ &+ \sum_{n=3}^{\infty} \sum_{m=3}^n \sum_{k=3}^m \sum_{r=0}^{k-1} \binom{\alpha}{n} \binom{n}{m} \binom{k}{r} \frac{1}{k!} \frac{t^{n-\alpha} (-u)^r}{\Gamma(n+1-\alpha)} \partial_t^m (u^{k-r}) \partial_t^{n-m} (\partial_u^k \eta) \\ &= C(x, u) \sum_{n=3}^{\infty} \sum_{m=3}^n \binom{\alpha}{n} \binom{n}{m} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} \partial_t^{n-m} (t^{\alpha-3}) \left[\sum_{i=1}^{m-1} \binom{m-1}{i} \partial_t^i u \partial_t^{m-i} u \right] \\ &+ \sum_{n=3}^{\infty} \sum_{m=3}^n \sum_{k=3}^m \sum_{r=0}^{k-1} \binom{\alpha}{n} \binom{n}{m} \binom{k}{r} \frac{1}{k!} \frac{t^{n-\alpha} (-u)^r}{\Gamma(n+1-\alpha)} \partial_t^m (u^{k-r}) \partial_t^{n-m} (\partial_u^k \eta). \end{aligned}$$

By the uniqueness of nonlinear terms $\partial_t^i u \partial_t^{m-i} u$ in equation (2.5), we obtain their coefficients $C(x, u) = 0$ and then $\eta_{uu} = 0$. It completes the proof. ■

By theorem 2.1, since $\eta_{uu} = 0$ and each term in (1.9) contains $\partial^k \eta / \partial u^k$ with $k \geq 2$, then $\mu = 0$, which confirms the assumption $\mu = 0$ in previous literature. Further analysis of equation (2.5) will give explicit expressions of $\tau(t)$ and $\eta(x, t, u)$ in (2.3).

Corollary 2.2. In operator (2.3), the infinitesimal $\tau = \tau(t)$ is expressed explicitly as

$$\tau = \chi_2 t^2 + \chi_1 t, \quad (2.9)$$

and $\eta = \eta(x, t, u)$ is given by

$$\eta = \begin{cases} \omega(x)u + \rho(x, t), & \chi_2 = 0, \\ \frac{1}{2}(\alpha - 1)(2\chi_2 t + \chi_1)u + \omega(x)u + \rho(x, t), & \chi_2 \neq 0, \end{cases} \quad (2.10)$$

where χ_1 and χ_2 are integral constants, $\omega(x)$ and $\rho(x, t)$ are undetermined functions of their arguments, respectively.

Proof. Substituting $\xi = \xi(x)$, $\tau = \tau(t)$ and $\eta_{uu} = 0$ into equation (2.5) and vanishing the coefficients of different order time-fractional partial derivatives of u , one obtains

$$\partial_t^{\alpha-n} u : \binom{\alpha}{n} \partial_t^n (\eta_u) - \binom{\alpha}{n+1} D_t^{n+1}(\tau) = 0, \quad n = 1, 2, \dots \quad (2.11a)$$

and

$$\text{Else: } \partial_t^\alpha \eta - u \partial_t^\alpha (\eta_u) + [\eta_u - \alpha D_t(\tau)] \Delta - \tau \Delta_t - \xi \Delta_x - \sum_{i=0}^l F_{u_i} \eta^{(i)} = 0. \quad (2.11b)$$

Since $\tau = \tau(t)$, we rewrite equation (2.11a) as the following form:

$$\text{and} \quad \left. \begin{aligned} \frac{\partial}{\partial t} (\eta_u) - \frac{1}{2} (\alpha - 1) \tau'' &= 0, \quad n = 1 \\ \frac{\partial^n}{\partial t^n} (\eta_u) - \frac{\alpha - n}{n+1} \frac{\partial^{n+1}}{\partial t^{n+1}} (\tau) &= \frac{\partial^{n-1}}{\partial t^{n-1}} \left[\frac{\partial}{\partial t} (\eta_u) - \frac{\alpha - n}{n+1} \tau'' \right] = 0, \quad n \geq 2. \end{aligned} \right\} \quad (2.12)$$

In (2.12), we use the partial derivative on τ in order to write the second equation uniformly. Inserting $\partial \eta_u / \partial t = (\alpha - 1) \tau'' / 2$ into the second equations gives

$$\left(\frac{\alpha - 1}{2} - \frac{\alpha - n}{n+1} \right) \frac{\partial^{n-1}}{\partial t^{n-1}} (\tau'') = 0, \quad n = 2, 3, \dots$$

In particular, for $n = 2$, the above equation gives $\tau^{(3)} = d^3 \tau(t) / dt^3 = 0$. Together with the condition $\tau(t)|_{t=0} = 0$ we obtain $\tau = \chi_2 t^2 + \chi_1 t$ with two integral constants, χ_1 and χ_2 . Then solving the first equation in system (2.12) yields two different cases.

- (i) One is $\tau'' = 0$, i.e. $\chi_2 = 0$, then $\partial_t(\eta_u) = 0$, and by considering the condition $\eta_{uu} = 0$ in theorem 2.1, we obtain

$$\tau = \chi_1 t, \quad \eta = \omega(x)u + \rho(x, t),$$

where, hereinafter, $\omega(x)$ and $\rho(x, t)$ are two undetermined functions.

- (ii) The other is $\tau'' \neq 0$, i.e. $\chi_2 \neq 0$, then integrating the first equation in system (2.12) with respect to t once gives $\eta_u - \tau'(\alpha - 1)/2 = \omega(x)$ since $\eta_{uu} = 0$. Then we obtain

$$\eta = \left[\frac{1}{2} (\alpha - 1) (2\chi_2 t + \chi_1) + \omega(x) \right] u + \rho(x, t),$$

which makes the second equation in (2.12) hold identically. Then the undetermined functions $\omega(x)$ and $\rho(x, t)$ are determined by equation (2.11b). The proof ends. ■

In general, we assume $\eta = \gamma \tau' u + \omega(x)u + \rho(x, t)$, where $\gamma = (\alpha - 1)/2$ if $\tau'' \neq 0$, otherwise $\gamma = 0$. Such an assumption will facilitate the real computations and program writing. By means of the above results, we give two simple conditions to determine Lie symmetries of equation (1.2).

Theorem 2.3. *Following the above notations, Lie symmetries of equation (1.2) are determined by*

$$\text{and} \quad \left. \begin{aligned} \partial_t^\alpha \rho + (\eta_u - \alpha \tau') G - \xi G_x - \tau G_t - \sum_J F_{u_i} \partial_x^i \rho &= 0 \\ (\eta_u - \alpha \tau') F - \xi F_x - \tau F_t - \sum_J F_{u_i} (\eta^{(i)} - \partial_x^i \rho) - \sum_{I \setminus J} F_{u_i} \eta^{(i)} &= 0, \end{aligned} \right\} \quad (2.13)$$

where the sets $I = \{\text{all terms effective in } F\}$, $J = \{\text{the terms in } I \text{ which are linear in } u_i\}$ and $I \setminus J = \{\text{the terms contained in } I \text{ but not in } J\}$.

Proof. By the proof of theorem 2.1, since η is linear in u , then $\mu = 0$ and on the space (x, t, u) , one has

$$\partial_t^\alpha \eta - u \partial_t^\alpha (\eta_u) = \partial_t^\alpha (\eta - u \eta_u) = \partial_t^\alpha \rho,$$

thus equation (2.11b) becomes

$$\partial_t^\alpha \rho + (\eta_u - \alpha \tau')(F + G) - \xi(F_x + G_x) - \tau(F_t + G_t) - \sum_{i=0}^l F_{u_i} \eta^{(i)} = 0, \quad (2.14)$$

where $\xi = \xi(x)$, τ and η are given by (2.9) and (2.10), respectively. Moreover, equation (2.11a) holds identically with the ξ , τ and η in corollary 2.2. Thus, the Lie symmetry admitted by equation (1.2) is uniquely determined by equation (2.14).

Let $I = \{\text{all terms effective in } F\}$. Since $\xi = \xi(x)$, $\tau = \tau(t)$ and $\eta_{uu} = 0$, we can further separate equation (2.14) according to the terms involving u and its x -derivatives or not. Since each term in F contains at least one u_i ($i = 0, 1, \dots, l$) and

$$\eta^{(i)} = D_x^i (\eta - \xi u_1) + \xi u_{i+1} = D_x^i (\eta) - D_x^i (\xi u_1) + \xi u_{i+1},$$

then, except for $(\eta_u - \alpha \tau')G$, ξG_x , τG_t and $\partial_t^\alpha \rho$, the terms without containing u and its x -derivatives only possibly emerge in $\sum_{i=0}^l F_{u_i} \partial_x^i \rho$, thus we divide the terms in F into two parts, where one is the set $J = \{\text{the terms in } I \text{ which are linear in } u_i\}$ and the other is the set $I \setminus J = \{\text{the terms contained in } I \text{ but not in } J\}$. Thus by considering whether the terms contain u and its x -derivatives, we divide equation (2.14) into two equations in (2.13). The proof ends. ■

It is worth saying that the separation of equation (2.14) is about u and its x -derivatives since the unknown functions $\xi(x)$, $\tau(t)$, $\omega(x)$ and $\rho(x, t)$ are independent of u and its x -derivatives. Such a technique for separation, which is different to the algorithm for constructing symmetry determining equations of PDEs where the separation is performed with respect to x -derivatives of u , will generate a more simple symmetry for determining equations which are relatively more easily solved.

Further simplifications can be done for some particular type of equations. In general, most of the time-fractional PDEs take the form of equation (1.2) with $G(x, t) = 0$, then in conditions (2.13), the first condition becomes

$$\partial_t^\alpha \rho - \sum_J F_{u_i} \partial_x^i \rho = 0, \quad (2.15)$$

while the second one is the usual PDE which determines the Lie symmetry admitted by equation (1.2), thus the role of equation (2.15) is to check the solutions. Therefore, the Lie symmetry of equation (1.2) with $G(x, t) = 0$ is mainly determined by the second equation in system (2.13), and several Lie symmetry packages for PDEs can be directly used to determine the Lie symmetry of equation (1.2).

Corollary 2.4. *If $G(x, t) = 0$ and J is empty, then the Lie symmetry admitted by equation (1.2) is uniquely determined by the second equation in (2.13).*

Proof. If $G(x, t) = 0$ and J is empty, then the first equation in (2.13) becomes $\partial_t^\alpha \rho = 0$, which implies $\rho = C(x)t^{\alpha-1}$ with an undetermined function $C(x)$ by the property of the Riemann–Liouville fractional partial derivative [23]. The proof ends. ■

(b) Non-existence of invertible mapping for linearization

Quite similar to the results for a linear PDE, a linear time-fractional PDE always admits an infinite-dimensional Lie algebra of an infinitesimal generator, which plays the key role in determining the condition of linearization.

Lemma 2.5. *If equation (1.2) is linear, then it always admits a Lie group whose associated infinitesimal generator is $X = \rho(t, x)\partial_u$, where $\rho(t, x)$ satisfies linear equation (1.2) with $G(t, x) = 0$.*

Proof. It suffices to prove $X = \rho(t, x)\partial_u$ satisfying

$$\text{Pr}^{(\alpha, l)} X (\partial_t^\alpha u - F - G) |_{\{\partial_t^\alpha u - F - G = 0\}} = 0, \quad (2.16)$$

under the condition that $\rho(t, x)$ satisfies linear equation (1.2) with $G(t, x) = 0$.

Observe that

$$\text{Pr}^{(\alpha, l)} X = X + \partial_t^\alpha [\rho(t, x)] \frac{\partial}{\partial(\partial_t^\alpha u)} - \sum_{j=0}^l D_x^j [\rho(t, x)] \partial_{u_j},$$

then inserting it into equation (2.16), we obtain

$$\partial_t^\alpha [\rho(t, x)] - \sum_{j=0}^l D_x^j [\rho(t, x)] \partial_{u_j} (F) = 0. \quad (2.17)$$

Since equation (1.2) is linear, then equation (2.17) can be obtained by replacing $u(t, x)$ with $\rho(t, x)$ in equation (1.2) and setting $G(t, x) = 0$. Thus, if $\rho(t, x)$ satisfies linear equation (1.2) with $G(t, x) = 0$, then (2.16) holds. The proof ends. ■

Alternatively, one can prove lemma 2.5 by checking that $\xi = \tau = 0$, $\eta = \rho(t, x)$ satisfy the two symmetry determining conditions in theorem 2.3.

Lemma 2.6. *A scalar nonlinear time-fractional PDE does not admit infinite-dimensional Lie algebra of infinitesimal generators.*

Proof. The proof is equivalent to showing that system (2.13) can explicitly determine the infinitesimals τ , ξ and η of each Lie subalgebra admitted by equation (1.2).

By corollary 2.2, there exist three possible infinite-dimensional Lie subalgebras $X^1 = \xi(x)\partial_x$, $X^2 = \omega(x)u\partial_u$ and $X^3 = \rho(t, x)\partial_u$. Next, we discuss the three Lie subalgebras, respectively.

For X^1 , substituting $\tau = \eta = 0$ and $\xi = \xi(x)$ into system (2.13) yields

$$\xi G_x = 0, \quad \xi F_x + \sum_i F_{u_i} [-D_x^i (\xi u_1) + \xi u_{i+1}] = 0. \quad (2.18)$$

If $G_x \neq 0$, then $\xi(x) = 0$ from the first equation in (2.18). If $G_x = 0$, then by separating the second equation with respect to u and its x -derivatives, one obtains over-determined determining equations for $\xi(x)$, which will determine the specific form of $\xi(x)$.

For X^2 , taking $\tau = \xi = 0$ and $\eta = \omega(x)u$ into system (2.13) gives

$$\omega G = 0, \quad \omega F - \sum_i F_{u_i} [D_x^i (\omega u)] = 0, \quad (2.19)$$

If $G \neq 0$, then $\omega(x) = 0$ by the first equation in (2.19). If $G = 0$, then similar to X^1 , we get the explicit expression of $\omega(x)$ by solving the corresponding over-determined determining equations.

For X^3 , by inserting $\tau = \xi = 0$ and $\eta = \rho(t, x)$ into system (2.13) we get

$$\partial_t^\alpha \rho - \sum_j F_{u_j} \partial_x^j \rho = 0 \quad (2.20a)$$

and

$$\sum_{l \in J} F_{u_i} \partial_x^l \rho = 0. \quad (2.20b)$$

First consider equation (2.20b). We rearrange F as the form $F = \sum_{s=0}^\lambda f_s(t, x) F_s(t, x, u_0, u_1, \dots, u_l)$ where λ is the number of all terms effective in F and for each term, $f_s(t, x)$ only contains t and x , $F_s(t, x, u_0, u_1, \dots, u_l)$ collects the remainder and may contain the inseparable term of u_i with t, x , then equation (2.20b) can be explicitly expressed as

$$\sum_{s=0}^\lambda f_s(t, x) \left(\frac{\partial F_s}{\partial u_0} \rho + \frac{\partial F_s}{\partial u_1} \partial_x \rho + \dots + \frac{\partial F_s}{\partial u_l} \partial_x^l \rho \right) = 0. \quad (2.21)$$

Let T_i^s denote the set of all terms in $\partial F_s / \partial u_i$, where $i = 0, 1, \dots, l$, $s = 0, 1, \dots, \lambda$. Observe that some terms in different T_i^s may be identical except for the coefficients, thus we put the same terms

as an equivalent class and assume the number of such inequivalent classes is r . Meanwhile, since $\rho(t, x)$ is independent of u_i , by considering the inequivalent classes we separate equation (2.21) into r cases and denote the number of each case by d_j with $j = 1, \dots, r$, then obtain

$$\left. \begin{aligned} f_{j_0^1} \partial_x^{j_0^1} \rho + f_{j_1^1} \partial_x^{j_1^1} \rho + \dots + f_{j_{d_1}^1} \partial_x^{j_{d_1}^1} \rho &= 0, \\ f_{j_0^2} \partial_x^{j_0^2} \rho + f_{j_1^2} \partial_x^{j_1^2} \rho + \dots + f_{j_{d_2}^2} \partial_x^{j_{d_2}^2} \rho &= 0 \\ &\dots \\ f_{j_0^r} \partial_x^{j_0^r} \rho + f_{j_1^r} \partial_x^{j_1^r} \rho + \dots + f_{j_{d_r}^r} \partial_x^{j_{d_r}^r} \rho &= 0, \end{aligned} \right\} \quad (2.22)$$

and

where non-negative integers m, k, j_m^k satisfy $0 \leq m \leq \max\{d_1, \dots, d_r\}$, $1 \leq k \leq r$ and $0 \leq j_m^k \leq l$, $f_{j_m^k}^k = f_{j_m^k}^k(t, x)$ corresponds to some $f_s(t, x)$. Note that there exist two extreme cases, where one is that all terms in $T = \bigcup_{s=0}^{\lambda} \bigcup_{i=0}^l T_i^s$ are identical, and the other is that any two terms in T are different.

System (2.22) consists of r linear ordinary differential equations of order at most l with variable coefficient $f_{j_m^k}^k$ where t is regarded as a parameter variable. Following the solution structure of higher-order linear ordinary differential equations, the number of function linear independent solutions is at most l [27], and such solutions contain at most l undetermined functions of t . Then, inserting such solutions into equation (2.20a) yields a system which will completely determine the l functions of t . The above procedure gives the specific expression of $\rho(t, x)$ and thus nonlinear equation (1.2) does not admit infinite-dimensional Lie algebra. The proof ends. ■

Theorem 2.7. *There exists no invertible mapping which transforms nonlinear time-fractional equation (1.1) into a linear one.*

Proof. By lemma 2.5, a linear time-fractional PDE always admits an infinite-dimensional Lie algebra while by lemma 2.6 the Lie algebra admitted by a nonlinear time-fractional PDE are at most finite-dimensional. Observe that any point transformation maps a point transformation into another point transformation and any contact transformation maps a contact transformation into another contact transformation [21]. Then it is impossible to find an invertible mapping connecting an infinite-dimensional Lie algebra with a finite-dimensional one, thus there exists no invertible mapping converting a nonlinear time-fractional PDE to a linear one. The proof ends. ■

3. Two examples

(a) Time-fractional KdV-type equation

Consider the time-fractional KdV-type equation

$$\partial_t^\alpha u + uu_x + \frac{1}{2}u_{xxx} + \frac{\nu}{2t}u = 0, \quad 0 < \alpha < 1, \quad (3.1)$$

where equation (3.1) with $\nu = 0, 1, 2$ corresponds to time-fractional KdV, cylindrical KdV and spherical KdV equations, respectively. The classical KdV equation is originated from the Dutch mathematicians Korteweg and de Vries (KdV) studying shallow water waves, while equation (3.1) may be useful to construct the compressive and rarefactive electrostatic potential pulses associated with the broadband electrostatic noise-type emissions [28].

(i) The original method

Assume that the infinitesimal generator X given in (1.4) leaves equation (3.1) invariant, then the Lie infinitesimal invariant criterion for equation (3.1) is

$$\text{Pr}^{(\alpha, 3)} X \left(\partial_t^\alpha u + uu_x + \frac{1}{2}u_{xxx} + \frac{\nu}{2t}u \right)_{|(3.1)} = 0, \quad (3.2)$$

where $\Pr^{(\alpha,3)}$ is given by (1.6). The usual method is first to assume $\mu = 0$, then equation (3.2) becomes

$$\eta^\alpha + \eta u_x + u \eta^{(1)} - \frac{\nu}{2t^2} \tau u + \frac{\nu}{2t} \eta + \frac{1}{2} \eta^{(3)} = 0, \quad (3.3)$$

where η^α is given by (1.8) with $\mu = 0$, $\eta^{(1)}$ and $\eta^{(3)}$ by (1.7). Then, separating equation (3.3) with respect to time-fractional derivatives of u , yields

$$\left. \begin{aligned} \binom{\alpha}{n} D_t^n(\xi) &= 0, \quad n = 1, 2, \dots, \\ \binom{\alpha}{n} \partial_t^n(\eta u) - \binom{\alpha}{n+1} D_t^{n+1}(\tau) &= 0, \quad n = 1, 2, \dots \\ \partial_t^\alpha \eta - u \partial_t^\alpha(\eta u) - [\eta u - \alpha D_t(\tau)] \left(u u_x + \frac{1}{2} u_{xxx} + \frac{\nu}{2t} u \right) \\ &\quad + \eta u_x + u \eta^{(1)} - \frac{\nu}{2t^2} \tau u + \frac{\nu}{2t} \eta + \frac{1}{2} \eta^{(3)} = 0. \end{aligned} \right\} \quad (3.4)$$

Then, separating the last equation in system (3.4) with respect to x - and t -derivatives of u generates a linear over-determined system, which is a very large and complicated system since ξ, τ and η depend on x, t and u .

(ii) Our method

By theorem 2.1 and corollary 2.2, we assume that the infinitesimal generator of Lie symmetry admitted by equation (3.1) takes the form $X = \xi \partial_x + \tau \partial_t + \eta \partial_u$, where $\xi = \xi(x)$, $\tau = \chi_2 t^2 + \chi_1 t$ and $\eta = \eta(x, t, u)$, is given by (2.10).

Then, by theorem 2.3, $I = \{uu_x, u_{xxx}, u/t\}$, $J = \{u_{xxx}, u/t\}$, $I \setminus J = \{uu_x\}$ and $G(x, t) = 0$, thus the infinitesimal generator of equation (3.1) is determined by

$$\left. \begin{aligned} \partial_t^\alpha \rho + \frac{1}{2} \rho_{xxx} + \frac{\nu}{2t} \rho &= 0 \\ [(\gamma - \alpha)\tau' + \omega]F + \eta u_x + u \eta^{(1)} - \frac{\nu}{2t^2} \tau u + \frac{\nu}{2t} (\eta - \rho) + \frac{1}{2} (\eta^{(3)} - \rho_{xxx}) &= 0, \end{aligned} \right\} \quad (3.5)$$

where $F = -uu_x - u_{xxx}/2 - \nu u/(2t)$,

$$\left. \begin{aligned} \eta &= (\gamma \tau' + \omega)u + \rho, \\ \eta^{(1)} &= (\gamma \tau' + \omega - \xi')u_1 + \omega' u + \rho_x \\ \eta^{(3)} &= (\gamma \tau' + \omega - 3\xi')u_3 + 3(\omega' - \xi'')u_2 + (3\omega'' - \xi^{(3)})u_1 + \omega^{(3)}u + \rho_{xxx}. \end{aligned} \right\} \quad (3.6)$$

Note that the prolongations (3.6) are more simple than the formulae (1.7) in the original method.

Inserting (3.6) into the second equation in system (3.5) and separating it with respect to u and its x -derivatives, we obtain Lie symmetry determining equations

$$\left. \begin{aligned} \partial_t^\alpha \rho + \frac{1}{2} \rho_{xxx} + \frac{\nu}{2t} \rho &= 0, \\ \alpha \tau' - 3\xi' &= (\alpha + \gamma)\tau' - \xi' + \omega = 0, \\ \omega' &= \xi'' = \rho - \frac{1}{2} \xi^{(3)} + \frac{3}{2} \omega'' = 0 \\ 2\rho_x + \frac{\nu}{t^2} (\alpha t \tau' - \tau) + \omega^{(3)} &= 0, \end{aligned} \right\} \quad (3.7)$$

and $\tau|_{t=0} = 0$.

Observe that system (3.7) consists of one linear time-fractional PDE and six PDEs, and the last six PDEs completely determine the Lie symmetries of equation (3.1) and the first time-fractional PDE is to check the solutions. System (3.7) is easier to solve, even by hand. The general solution

is divided into two cases: if $\nu \neq 0$, then equation (3.1) is admitted by $X = \partial_x$ while for $\nu = 0$, we obtain

$$\omega = -2\alpha c_1, \quad \tau = 3c_1 t, \quad \xi = \alpha c_1 x + c_2, \quad \gamma = \rho = 0,$$

where c_1 and c_2 are two arbitrary constants, and then equation (3.1) is admitted by the Lie symmetry $X = (\alpha c_1 x + c_2)\partial_x + 3c_1 t\partial_t - 2\alpha c_1 u\partial_u$.

By the infinitesimal generator $X_1 = \alpha x\partial_x + 3t\partial_t - 2\alpha u\partial_u$ which corresponds to X with $c_1 = 1$ and $c_2 = 0$, we reduce equation (3.1) with $\nu = 0$ into the form

$$\left(P_{3/\alpha}^{1-5/3\alpha, \alpha} U\right)(z) + UU' + \frac{1}{2}U''' = 0,$$

where the similarity variables $z = xt^{-\alpha/3}$, $U = U(z)$, and $P_{3/\alpha}^{1-5\alpha/3, \alpha}$ is the Erdélyi–Kober fractional differential operator [23].

(b) The time-fractional diffusion–convection equation

With the theoretical results in §2, we perform a complete Lie symmetry classification for the nonlinear time-fractional diffusion–convection equation

$$\partial_t^\alpha u = (P(u)u_x)_x + Q(u)u_x, \quad 0 < \alpha < 1, \quad (3.8)$$

where diffusion function $P(u)$ and advection function $Q(u)$ make equation (3.8) nonlinear. Equation (3.8) and its special cases are used to describe many particular phenomena, such as anomalous diffusion, random walkers in an expanding medium and transport dynamics in complex systems [3,4], and are studied with analytical and numerical methods (see [29,30] and references therein). For example, the linear fractional-type advection–diffusion equation is considered using the Lie symmetry method [29]. A discrete scheme is proposed to find numerical solutions of the linear time-fractional diffusion–convection equation in [30].

By theorem 2.1 and corollary 2.2, assume the infinitesimal operator of Lie symmetry admitted by equation (3.8) takes the form (2.3), then separating the equation in theorem 2.3 with respect to different order x -derivatives of u , we obtain the symmetry determining equations

$$\left. \begin{aligned} (\omega''u + \rho_{xx})P + (\omega'u + \rho_x)Q - \partial_t^\alpha \rho &= 0, \\ (\gamma\tau'u + \omega u + \rho)P' + (\alpha\tau' - 2\xi')P &= 0, \\ (\gamma\tau'u + \omega u + \rho)P'' + [(\alpha + \gamma)\tau' - 2\xi' + \omega]P' &= 0 \\ \text{and} \quad 2(\omega'u + \rho_x)P' - (\xi'' - 2\omega')P + (\gamma\tau'u + \omega u + \rho)Q' - (\xi' - \alpha\tau')Q &= 0, \end{aligned} \right\} \quad (3.9)$$

where $\tau|_{t=0} = 0$, the functions $\xi = \xi(x)$, $\tau = \tau(t)$, $\omega = \omega(x)$ and $\rho = \rho(x, t)$ are to be determined after choosing $P = P(u)$ and $Q = Q(u)$. It is easy to verify that the second equation implies the third equation by taking a u -derivative of the second equation once, thus in fact system (3.9) consists of three equations by excluding the third equation.

For arbitrary functions P and Q , equation (3.8) is admitted by $X = \partial_x$. For some particular functions P and Q , the admitted Lie symmetries may be extended. We classify them as the following three classes. Note that $c_i (i = 1, \dots, 4)$ are integral constants unless otherwise noted.

(i) P is a constant

It corresponds to constant diffusion in equation (3.8). Without loss of generality, assume $P = 1$. Then system (3.9) becomes

$$\left. \begin{aligned} \partial_t^\alpha \rho - \rho_{xx} &= 0, \\ \alpha\tau' - 2\xi' &= 0, \\ (\omega'u + \rho_x)Q + \omega''u &= 0 \\ \text{and} \quad (\alpha\tau' - \xi')Q + (\gamma\tau'u + \omega u + \rho)Q' - \xi'' + 2\omega' &= 0. \end{aligned} \right\} \quad (3.10)$$

Since $\xi = \xi(x)$, $\tau = \tau(t)$, from the second equation we have $\xi = \alpha c_1 x + c_2$, $\tau = 2c_1 t$, and then $\gamma = 0$ by corollary 2.2. Since equation (3.8) is nonlinear, then Q is not a constant. Thus, the third equation implies $\omega' = \rho_x = 0$, which gives $\omega = c_3$, $\rho = \rho(t)$. Then $\rho = c_4 t^{\alpha-1}$ by the first equation. With these results, the last equation becomes $c_1 \alpha Q + (c_3 u + c_4 t^{\alpha-1}) Q' = 0$, thus $c_4 = 0$ and Q is a power function and let $Q = u^a (a \neq 0)$, then equation (3.8) with $P = 1$, $Q = u^a$ is admitted by $X = (c_1 \alpha x + c_2) \partial_x + 2c_1 t \partial_t - c_1 \alpha u / a \partial_u$.

(ii) Q is a constant

It means that equation (3.8) keeps constant advection. Similarly, we assume $Q = 1$. Then system (3.9) becomes

$$\left. \begin{aligned} \partial_t^\alpha \rho - \rho_x &= 0, \\ (\omega'' u + \rho_{xx}) P + \omega' u &= 0, \\ (\gamma \tau' u + \omega u + \rho) P' + (\alpha \tau' - 2\xi') P &= 0 \\ \text{and} \quad 2(\omega' u + \rho_x) P' - (\xi'' - 2\omega') P - \xi' + \alpha \tau' &= 0. \end{aligned} \right\} \quad (3.11)$$

Observe that P is a non-constant function since equation (3.8) is nonlinear. Thus the second equation in system (3.11) implies $\omega'' = 0$, $\rho_{xx} P + \omega' u = 0$, then we consider two cases.

- (I) If $P = Au$ with a constant A , then $A \rho_{xx} + \omega' = 0$ and the third equation becomes $(\gamma \tau' u + \omega u + \rho) + (\alpha \tau' - 2\xi') u = 0$, which means $\rho = 0$ and $\omega = 2\xi' - (\alpha + \gamma) \tau'$. The last equation implies $\xi'' - 4\omega' = \alpha \tau' - \xi' = 0$, which gives $\gamma = 0$, $\xi = c_2 \alpha x + c_3$, $\tau = c_2 t$ since $\xi = \xi(x)$, $\tau = \tau(t)$. Therefore, equation (3.8) with $P = u$, $Q = 1$ is admitted by $X = (c_2 \alpha x + c_3) \partial_x + c_2 t \partial_t + c_2 \alpha u \partial_u$.
- (II) If $P \neq Au$, then $\rho_{xx} P + \omega' u = 0$ means $\rho_{xx} = \omega' = 0$. Then the last equation gives $2\rho_x P' - \xi'' P = \alpha \tau' - \xi' = 0$, which means $\gamma = 0$, $\xi = c_2 \alpha x + c_3$, $\tau = c_2 t$ and $\rho_x = 0$. Inserting them into the third equation gives $(c_1 u + \rho) P' - c_2 \alpha P = 0$, which implies that $\rho = c_4$ and P takes the power function form and we set $P = u^b (b \neq 0)$. Moreover, by the first equation we obtain $\rho = 0$. Therefore, equation (3.8) with $P = u^b$, $Q = 1$ is admitted by $X = (c_2 \alpha x + c_3) \partial_x + c_2 t \partial_t + c_2 \alpha u / b \partial_u$ which includes the first case (I) as a special case.

(iii) Both P and Q are not constants

In this case, we will determine the function pair (P, Q) to make equation (3.8) admit more Lie symmetries. Since P and Q are not constants, then by corollary 2.4 we obtain $\rho = 0$. Then the symmetry determining equations are

$$\left. \begin{aligned} \omega'' P + \omega' Q &= 0, \\ u (\gamma \tau' + \omega) P' + (\alpha \tau' - 2\xi') P &= 0 \\ \text{and} \quad 2\omega' u P' - (\xi'' - 2\omega') P + u (\gamma \tau' + \omega) Q' - (\xi' - \alpha \tau') Q &= 0. \end{aligned} \right\} \quad (3.12)$$

We start with the second equation in system (3.12) and consider two cases. If $\gamma \tau' + \omega = 0$, then by considering $\tau = \tau(t)$ and $\omega = \omega(x)$ together with condition (2.10), we obtain $\omega = \gamma = 0$, $\tau = c_1 t$. Thus, system (3.12) becomes

$$\alpha c_1 - 2\xi' = \xi'' P + (\xi' - \alpha c_1) Q = 0,$$

which gives $\xi = c_2$, $c_1 = 0$. No further Lie symmetry is found, thus we consider $\gamma \tau' + \omega \neq 0$.

Since $(\gamma \tau' + \omega) P \neq 0$, the second equation in system (3.12) can be rewritten as

$$\frac{P'}{P} = \frac{2\xi' - \alpha \tau'}{(\gamma \tau' + \omega) u}. \quad (3.13)$$

Integrating equation (3.13) with respect to u once gives

$$\ln P = \frac{2\xi' - \alpha\tau'}{\gamma\tau' + \omega} \ln u + C,$$

which implies

$$\frac{2\xi' - \alpha\tau'}{\gamma\tau' + \omega} = \beta, \quad (3.14)$$

where C is the integral constant and β is a non-zero constant, thus P takes the power function form and let $P = u^\beta$. Without loss of generality, we assume $C = 0$, which does not affect the classification results. Taking such results into system (3.12) yields

$$\left. \begin{aligned} \xi''Q + \xi^{(3)}u^\beta &= 0 \\ u(\alpha\tau' - 2\xi')Q' + \beta(\xi' - \alpha\tau')Q - (3\beta + 4)\xi''u^\beta &= 0. \end{aligned} \right\} \quad (3.15)$$

and

Subcase I. $\xi'' = 0$.

It means $\xi = c_1x + c_2$, then the first equation in system (3.15) holds identically and the second equation becomes

$$u(\alpha\tau' - 2c_1)Q' + \beta Q(c_1 - \alpha\tau') = 0, \quad (3.16)$$

which implies Q must be a power function, thus we set $Q = u^\kappa$ with a constant κ . Then equation (3.16) becomes

$$c_1(\beta - 2\kappa) + \alpha\tau'(\kappa - \beta) = 0.$$

Then three cases arise.

- (1) If $\beta = \kappa$, then $c_1 = 0$. It means that τ' is arbitrary, thus there are two subcases to be considered.

One is $\tau'' = 0$, i.e. $\tau = c_3t$ by considering $\tau|_{t=0} = 0$. Then, by condition (2.10) and equation (3.14), we have $\gamma = 0$ and $\omega = -c_3\alpha/\beta$. Thus, equation (3.8) with $P = u^\beta$ and $Q = \delta u^\beta$ is admitted by $X = c_2\partial_x + c_3t\partial_t - c_3\alpha u/\beta\partial_u$.

The other is $\tau'' \neq 0$. By (2.10), we have $\gamma = (\alpha - 1)/2$. Then equation (3.14) becomes

$$\omega + \left[\frac{1}{2}(\alpha - 1) + \frac{\alpha}{\beta} \right] \tau' = 0.$$

If ω is a non-zero constant, then τ' is a constant and thus $\tau'' = 0$, it is contradicted. If $\omega = (\alpha - 1)/2 + \alpha/\beta = 0$, i.e., $\beta = 2\alpha/(1 - \alpha)$, then τ' is arbitrary. Thus by corollary 2.2, equation (3.8) with $P = u^\beta$ and $Q = \delta u^\beta$ is admitted by $X = c_2\partial_x + (c_3t^2 + c_4t)\partial_t + (\alpha - 1)(2c_3t + c_4)u/2\partial_u$.

- (2) If $\beta = 2\kappa$, then $\tau' = 0$, i.e. $\tau = 0$ since $\tau|_{t=0} = 0$. From equation (3.14) one has $\omega = c_1/\kappa$. Thus, equation (3.8) with $P = u^{2\kappa}$, $Q = u^\kappa$ is admitted by $X = (c_1x + c_2)\partial_x + c_1u/\kappa\partial_u$.
- (3) If $(\beta - 2\kappa)(\kappa - \beta) \neq 0$, then $\tau = c_1(\beta - 2\kappa)t/[\alpha(\beta - \kappa)]$ by $\tau|_{t=0} = 0$ and thus $\gamma = 0$ by (2.10). Substituting them into equation (3.14) gives $\omega = c_1/(\beta - \kappa)$. Hence equation (3.8) with $P = u^\beta$, $Q = u^\kappa$ is admitted by $X = (c_1x + c_2)\partial_x + c_1(\beta - 2\kappa)/[\alpha(\beta - \kappa)]t\partial_t + c_1/(\beta - \kappa)u\partial_u$, which includes (2) as a special case.

Subcase II. $\xi'' \neq 0$.

The first equation in system (3.15) can be rewritten as $Q/u^\beta = -\xi^{(3)}/\xi'' = \delta$, where δ is a non-zero constant. Thus $Q = \delta u^\beta$, $\xi = c_1e^{-\delta x} + c_2x + c_4$, which implies $c_1 \neq 0$ since $\xi'' \neq 0$. Substituting them into the second equation gives $2c_1(\beta + 2)\delta + \beta c_2e^{\delta x} = 0$, which means $c_2 = 0$, $\beta = -2$. Then from equation (3.14) we obtain $\omega = (2c_1\delta e^{-\delta x} + \alpha c_3)/2$. Thus equation (3.8) with $P = u^{-2}$, $Q = \delta u^{-2}$ is admitted by $X = (c_1e^{-\delta x} + c_4)\partial_x + c_3t\partial_t + (2c_1\delta e^{-\delta x} + \alpha c_3)u/2\partial_u$.

In summary, we start with the Lie symmetry conditions in theorem 2.3 for equation (3.8) and rigorously show the classification result is complete by analysing the structure of the simplified symmetry determining equations (3.9).

Table 1. A complete Lie symmetry classification of equation (3.8).

P	Q	infinitesimal operators
arbitrary	arbitrary	∂_x
1	u^κ	$\partial_x, \alpha x \partial_x + 2t \partial_t - \frac{\alpha}{\kappa} u \partial_u$
u^β	1	$\partial_x, \alpha x \partial_x + t \partial_t + \frac{\alpha}{\beta} u \partial_u$
$u^{2\alpha/(1-\alpha)}$	$u^{2\alpha/(1-\alpha)}$	$\partial_x, t \partial_t + \frac{\alpha-1}{2} u \partial_u, t^2 \partial_t + (\alpha-1) t u \partial_u$
u^β	δu^β	$\partial_x, t \partial_t - \frac{\alpha}{\beta} u \partial_u$
u^β	$u^\kappa (\beta \neq \kappa)$	$\partial_x, x \partial_x + \frac{\beta-2\kappa}{\alpha(\beta-\kappa)} t \partial_t + \frac{1}{\beta-\kappa} u \partial_u$
u^{-2}	δu^{-2}	$\partial_x, t \partial_t + \frac{\alpha}{2} u \partial_u, e^{-\delta x} (\partial_x + \delta u \partial_u)$

Theorem 3.1. A complete Lie symmetry classification of equation (3.8) is listed in table 1.

With the infinitesimal generators in table 1, one can convert equation (3.8) into a fractional or integer-order ordinary differential equation. For example, consider the case $P = Q = u^{2\alpha/(1-\alpha)}$, which by means of the generator $X = t^2 \partial_t + (\alpha - 1) t u \partial_u$, equation (3.8) is reduced to

$$U U'' + \frac{2\alpha}{1-\alpha} (U')^2 + U U' = 0, \quad (3.17)$$

where the similarity variables are $z = x$, $U(z) = t^{1-\alpha} u(x, t)$. The general solution of equation (3.17) is $U = c_2 [c_1(1 + \alpha) e^{-x} - c_1^\alpha]^{(1-\alpha)/(1+\alpha)}$, then we obtain a solution of equation (3.8) in the form

$$u(x, t) = c_2 t^{-2/3} [c_1(1 + \alpha) e^{-x} - c_1^\alpha]^{(1-\alpha)/(1+\alpha)}.$$

In particular, for $\alpha = 1/3$, we obtain $U = c_2 \sqrt{1 - 2c_1} e^{-x}$, then equation (3.8) has a solution $u = c_2 t^{\alpha-1} \sqrt{1 - 2c_1} e^{-x}$.

4. Conclusion

We perform a further Lie symmetry analysis of a time-fractional PDE with one dependent variable and two independent variables. The general structure of infinitesimal generator admitted by equation (1.2) is given. Then the symmetry determining system is further simplified and divided into two simple conditions which enable us to find the infinitesimal generators and to perform the whole procedure on a computer. Our results also provide a theoretical proof of the assumption in previous literature, where μ given in (1.9) must vanish. Furthermore, we show that a nonlinear time-fractional PDE cannot be linearized to a linear time-fractional PDE via invertible mappings since the dimensions of their admitted Lie algebras are different, where the former is at most finite-dimensional while the latter is always infinite-dimensional.

In addition, there exist several issues worthy of further exploration. For example, whether the results holds in the sense of the Caputo fractional derivative is still unknown, and how to extend the results to the system of multi-dimensional time-fractional PDEs. Such topics are in preparation and will be reported in our future work.

Data accessibility. This article has no additional data.

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