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A Multitaper Frequency-Domain Bootstrap Method

Seong-Eun Kim [Member, IEEE],

Department of Electronics and Control Engineering, Hanbat National University, Daejeon 34158, South Korea

Demba Ba,

School of Engineering and Applied Sciences, Harvard University, Cambridge, MA 02138 USA

Emery N. Brown [Fellow, IEEE]

Picower Institute for Learning and Memory, Institute for Medical Engineering and Science, Institute for Data Systems and Society, Massachusetts Institute of Technology, Cambridge, MA 02139 USA, and also with the Department of Anesthesia, Critical Care, and Pain Medicine, Massachusetts General Hospital, Harvard Medical School, Boston, MA 02114 USA

Abstract

Spectral properties of the electroencephalogram (EEG) are commonly analyzed to characterize the brain's oscillatory properties in basic science and clinical neuroscience studies. The spectrum is a function that describes power as a function of frequency. To date inference procedures for spectra have focused on constructing confidence intervals at single frequencies using large sample-based analytic procedures or jackknife techniques. These procedures perform well when the frequencies of interest are chosen before the analysis. When these frequencies are chosen after some of the data have been analyzed, the validity of these conditional inferences is not addressed. If power at more than one frequency is investigated, corrections for multiple comparisons must also be incorporated. To develop a statistical inference approach that considers the spectrum as a function defined across frequencies, we combine multitaper spectral methods with a frequency-domain bootstrap (FDB) procedure. The multitaper method is optimal for minimizing the bias-variance tradeoff in spectral estimation. The FDB makes it possible to conduct Monte Carlo based inferences for any part of the spectrum by drawing random samples that respect the dependence structure in the EEG time series. We show that our multitaper FDB procedure performs well in simulation studies and in analyses comparing EEG recordings of children from two different age groups receiving general anesthesia.

Keywords

Confidence interval; frequency-domain bootstrap (FDB); multitaper methods; spectra resampling

I. INTRODUCTION

THE analysis of oscillatory patterns in the electroencephalogram (EEG) plays an important role in basic science and clinical neuroscience problems, ranging from defining stages of sleep, and characterizing performance of subjects on cognitive tasks to assessing the functional status of patients in coma and characterizing the arousal states of patients receiving general anesthesia [1], [2]. By convention, EEG signals are commonly described in terms of frequency bands, namely slow (<0.2 Hz), delta (0.2–4 Hz), theta (4–8 Hz), alpha (8–12 Hz), beta (12–30 Hz), and gamma (> 30 Hz). Therefore, a widely used paradigm is to characterize the oscillatory properties of a given brain state in terms of the power of the frequency bands from an estimated spectrum. Fourier-transform-based nonparametric approaches are the standard spectrum estimation techniques [3]. Hence, statistical inference drawn from these analyses depends critically on the properties of the spectrum estimates.

To make inferences in elementary statistical problems a model is fit to the data and inferences are made by characterizing the uncertainty in the estimated model parameters [4], [5]. Uncertainty in the model parameters is characterized by using analytic formulas derived from the particular theoretical approach that is applied. Bootstrap methods are also used to characterize model parameter uncertainty either parametrically by drawing random samples from the estimate of the assumed statistical model or nonparametrically by resampling the data directly [6] – [9]. Although these approaches are the bedrock of standard statistical inference, there are challenges to apply in nonparametric spectral analyses of time series.

First, the spectrum is not a simple parameter in a probability model. Rather, it is a function that describes how power in a signal changes in relation to frequency. Second, standard spectral estimation methods apply the Fourier transform to the time series that has been preprocessed with one or more tapers [3]. A common practice is to construct confidence intervals using asymptotically derived analytic formula or jackknife methods not for the entire spectrum but rather for selected frequencies in the spectrum [3], [10], [11]. This approach is fine if the frequencies of interest have been identified beforehand. However, it does not take account of the joint distribution among the estimated spectral ordinates and the issues of multiple comparisons and conditional inferences when the frequencies of interest are identified after analyzing the data. Finally, classic bootstrap methods, which require drawing samples that are independent and identically distributed, cannot be applied to time-series data like EEG recordings where there is always strong serial dependence [12], [13]. Block resampling methods that have been devised for this problem require setting arbitrary data partitions.

To develop a statistical inference approach that takes account of the fact that the spectrum is a function across frequencies we combine two ideas. First, we apply multitaper spectral methods to estimate the spectrum from EEG time series. This spectrum estimation approach is known to provide the optimal bias-variance tradeoff for a given frequency resolution [3], [14]–[16]. Second, the Fourier transform of a stationary time series is approximately distributed as a multivariate complex Gaussian random variable with zero mean and diagonal covariance matrix [17]. The true spectral ordinates at the given frequencies define the diagonal elements. Hence, we replace the true spectrum with its multitaper estimate to

construct a frequency-domain bootstrap (FDB) procedure that can be used to make inferences for any function of the spectrum [12], [18]–[20].

The remainder of this letter is organized as follows. Section II reviews multitaper spectral estimation, and Section III presents the proposed multitaper frequency-domain bootstrap (MFDB) method. The simulation results are presented and analyzed in Section IV, and conclusions are provided in Section V.

II. MULTITAPER SPECTRAL ESTIMATION

Suppose a discrete-time signal $x_k, k = 0, 1, \dots, N-1$, consists of N observations from a mean zero second-order stationary time series. Let us employ M orthonormal tapers that are known as discrete prolate spheroidal sequences or Slepian sequences [3]: $h_k^{(1)}, h_k^{(2)}, \dots, h_k^{(M)}$ such that

$$\sum_{k=0}^{N-1} h_k^{(i)} h_k^{(j)} = \delta_{ij}. \quad (1)$$

The m th tapered spectral estimate is then

$$\hat{S}^{(m)}(\omega_j) = \|X^{(m)}(\omega_j)\|^2 \quad (2)$$

where $X^{(m)}(\omega_j)$ is the discrete Fourier transform for m th tapered signals

$$X^{(m)}(\omega_j) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} h_k^{(m)} x_k \exp(-i\omega_j k) \quad (3)$$

and $\omega_j = 2\pi j/N$ for $j = 0, 1, \dots, N-1$. This yields the multitaper spectrum estimate

$$\hat{S}(\omega_j) = \frac{1}{M} \sum_{m=1}^M \hat{S}^{(m)}(\omega_j). \quad (4)$$

The multitaper method can be regarded as the decomposition of the spectral representation of the time signal over a set of orthogonal basis functions. In this regard, $X^{(m)}(\omega_j)$ and $\hat{S}^{(m)}(\omega_j)$ for $m = 1, 2, \dots, M$ are called the eigencoefficients and eigenspectra, respectively [21], [22]. The multitaper estimate is approximately unbiased if only the number of tapers is significantly less than $[2\alpha] - 1$, where α is the time-bandwidth product [16]. Additionally, if the true PSD is uniformly continuous, the variance can be upper bounded and is reduced by a factor of M compared to the periodogram estimate [3]. In addition, it has been proved that the multitaper method always performs better than Welch's method [23].

For confidence bounds of the multitaper spectrum estimate, the local frequency ensemble can be used to estimate jackknife confidence intervals [24], [25]. The idea of the jackknife is to create different estimates by, in turn, leaving out a tapered spectrum estimate. This creates a set of spectral estimates that forms an empirical distribution. As the number of tapers decreases, more deviations can be expected to be observed between the empirical estimate and the asymptotic estimate, with the empirical confidence intervals being larger than the asymptotic ones [3], [24].

III. MULTITAPER FREQUENCY-DOMAIN BOOTSTRAP

A. Statistical Properties

We describe the MFDB method. The Fourier coefficients of the observed time series x_k are as follows:

$$X(\omega_j) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k \exp(-i\omega_j k). \quad (5)$$

The Fourier coefficients can be represented as

$$X(\omega_j) = X_R(\omega_j) + iX_I(\omega_j) \quad (6)$$

where X_R and X_I are a real part and imaginary part, respectively. The real and imaginary parts of the Fourier coefficients are asymptotically independent and normally distributed [26]

$$X_R(\omega_j) \text{ and } X_I(\omega_j) \sim \mathcal{N}(0, S(\omega_j)/2) \quad (7)$$

for $j = 0, 1, \dots, N-1$. We assume that $S(\omega_j)$ is the true spectral density. The necessary and sufficient conditions for the asymptotic independence and normality of Fourier coefficients have been proved for all finite subsets for a large class of linear as well as nonlinear processes [26], [27].

Furthermore, the tapered Fourier transforms constitute a statistical ensemble for the Fourier transform of the time signal at each frequency. They are uncorrelated random variables having the same variance when tapers are orthonormal [17]. By (7) and Brillinger's theorem [17], for orthonormal tapers, the Fourier coefficients of the m th tapered time series have the following distribution:

$$X_R^{(m)}(\omega_j) \text{ and } X_I^{(m)}(\omega_j) \sim \mathcal{N}(0, \hat{S}(\omega_j)/2) \quad (8)$$

where $\hat{S}(\omega_j)$ is the multitaper spectral estimate and

$$X^{(m)}(\omega_j) = X_R^{(m)}(\omega_j) + iX_I^{(m)}(\omega_j) \text{ for } j = 0, 1, \dots, N-1 \text{ and } m = 1, 2, \dots, M.$$

B. Resampling Spectrum

From the properties of the Fourier coefficients, the periodogram ordinates, $P(\omega_j) = X_R(\omega_j)^2 + X_I(\omega_j)^2$, are asymptotically independent and distributed as chi-squared with two degrees of freedom times the true spectral density, $S(\omega_j)$, i.e., $P(\omega_j) \sim S(\omega_j)\chi^2_2/2$ [26]. Most FDB methods have been developed based on resampling the periodogram ordinates using these statistical properties. However, we resample the Fourier coefficients rather than the periodogram ordinates. Note that the Fourier coefficients have the following symmetry property:

$$\begin{aligned} X_R^{(m)}(\omega_{N-j}) &= X_R^{(m)}(\omega_j) \\ X_I^{(m)}(\omega_{N-j}) &= -X_I^{(m)}(\omega_j) \end{aligned} \quad (9)$$

which enables use of only $L = \lfloor N/2 \rfloor$ Fourier coefficients for necessary information.

The proposed MFDB procedure is as follows.

Step 1: Estimate the spectral power $\hat{S}(\cdot)$ using the multitaper spectral estimation of (4).

Step 2: Estimate the residuals of Fourier coefficients and put them together into a vector $\{\hat{s}_l^{(m)} : 0 \leq l \leq 2L-1\}$

$$\hat{s}_j^{(m)} = \frac{X_R^{(m)}(\omega_j)}{\sqrt{\hat{S}(\omega_j)/2}}, \quad \hat{s}_{L+j}^{(m)} = \frac{X_I^{(m)}(\omega_j)}{\sqrt{\hat{S}(\omega_j)/2}}$$

for $j = 0, 1, \dots, L-1$ and $m = 1, 2, \dots, M$.

Step 3: Standardize the residuals

$$\hat{s}_l^{(m)} = \frac{\hat{s}_l^{(m)} - \frac{1}{2L} \sum_{n=0}^{2L-1} \hat{s}_n^{(m)}}{\sqrt{\frac{1}{2L} \sum_{t=0}^{2L-1} \left(\hat{s}_t^{(m)} - \frac{1}{2L} \sum_{i=0}^{2L-1} \hat{s}_i^{(m)} \right)^2}}$$

for $l = 0, 1, \dots, 2L-1$.

Step 4: Draw independent bootstrap replicates $s_l^{(m)*}$, $l = 0, 1, \dots, 2L-1$, from the empirical distribution of $\hat{s}_l^{(m)}$.

Step 5: Obtain the bootstrapped Fourier coefficients by

$$X^{(m)*}(\omega_j) = s_j^{(m)*} \sqrt{\hat{S}(\omega_j)/2} + i s_{L+j}^{(m)*} \sqrt{\hat{S}(\omega_j)/2} \quad (10)$$

where $j = 0, 1, \dots, L-1$.

Step 6: Compute the bootstrapped spectral density by

$$\hat{S}^*(\omega_j) = \frac{1}{M} \sum_{m=1}^M \|X^{(m)*}(\omega_j)\|^2 \quad (11)$$

where $j = 0, 1, \dots, L-1$.

Step 7: Repeat steps 1–6 B times, where B is the number of bootstrap samples.

The rescaling in step 3 avoids an unnecessary bias at the resampling stage. By using the properties of the asymptotic distribution of Fourier coefficients, it is possible to modify the procedure by replacing the empirical distribution of $s_j^{(m)}$ with independent and standard Gaussian variables in step 4.

The practical approach is to directly compute the $1 - \alpha\%$ confidence interval from the $\alpha/2$ th and $1 - \alpha/2$ th percentiles of the distribution. This approach can be used regardless of the distribution of the bootstrap estimates.

C. MFDB Applied to Group Time-Series Analysis

Assume that we have a sample of independent time series generated from the same population. We can compute confidence intervals for the group spectrum by pooling spectrum estimates and weighted by their associated MFDB measures of uncertainty across the individuals in the sample. By analogy with Section III-B, we can also use this approach to compute confidence intervals and to make inferences about any function of the spectrum such as the difference in the spectrum between two groups. We illustrate this approach in Section IV-B.

IV. RESULTS

A. Numerical Experiment

A simulated experiment was conducted to illustrate the applicability of the proposed techniques. For example, we generated a time series of length 1024 through the sixth-order autoregressive (AR) model, which was previously used for a comparison of the spectral estimation [16]. The AR model is of the form

$$x_t = 3.9515x_{t-1} - 7.8885x_{t-2} + 9.7340x_{t-3} - 7.7435x_{t-4} + 3.8078x_{t-5} - 0.9472x_{t-6} + v_t \quad (12)$$

where v_t is independent zero-mean Gaussian noise with a unit variance.

In Fig. 1, the jackknife confidence intervals are compared [24] with MFDB confidence intervals for the multitaper spectral estimation of the time series obtained by the AR model [see (12)]. The number of tapers is chosen as four, which corresponds to a spectral resolution of $F_s/(128)$, where F_s is a sampling rate in hertz. Fig. 1(a) shows the true spectrum (green curve), the multitaper estimate (black curve), 95% jackknife confidence intervals (magenta and cyan dashed curves), and 95% MFDB confidence intervals (red and blue

dashed curves). Fig. 1(b) and (c) shows magnified views of the two frequencies between 0.14–0.18 and 0.44–0.5. As shown in Fig. 1, the MFDB confidence intervals are narrower compared with the jackknife multitaper confidence intervals.

To obtain an asymptotic confidence interval, we conducted a Monte Carlo (MC) simulation, where the number of MC replications was taken to be 1,000. We randomly chose five samples from 1,000 replications and computed the confidence intervals of the multiple spectral estimates, as described in Section III-C. In Fig. 2, the MFDB confidence intervals of five samples clearly agree with the MC confidence intervals. Using only a small number of independent samples, it is possible to accurately estimate the asymptotic confidence intervals.

B. Application of MFDB to Analysis of the EEG of Anesthetized Children

We tested the MFDB method using EEG data recorded from 30 infants, 0–6 months of age receiving sevoflurane general anesthesia at Boston Children’s Hospital, Boston, MA, USA [28]. The infants were divided into two groups: 0–3 months, 11 infants and 4–6 months, 19 infants. The objective of the analysis was to compare the differences in the spectral properties between the two groups. We selected 5 min of artifact-free EEG data for each child and computed the spectrum and the MFDB estimate of uncertainty. We computed a weighted-average (median) spectrum for each group and the associated 95% confidence interval for each group [see Fig. 3(a)] as well as the 95% confidence interval for the median difference between the two groups [see Fig. 3(b)] using 2000 bootstrap samples. Because the lower bounds of the 95% confidence interval is greater than zero at each frequency, we infer that the spectra of the two groups are quite different [see Fig. 3(b)].

For the infants in the 4–6 months age group, we used the MFDB procedure to assess whether there is a frontal predominance of alpha power (7–16 Hz) by evaluating the power differences between a frontal EEG lead (FPz) and an occipital EEG lead (Oz). The two spectra overlap [see Fig. 4(a)]. However, there is an appreciable increase in frontal alpha power compared with the occipital alpha as seen in the 95% confidence intervals for the difference spectrum [see Fig. 4(b)].

V. CONCLUSION

Spectral analysis is widely used to characterize the oscillatory dynamics of neural signals in the frequency domain. We presented a MFDB procedure for statistical inference by simulating the approximate distribution of any function of the spectrum. Our bootstrap procedure uses multitaper spectral methods to compute the spectrum estimate. In simulated experiments, the MFDB confidence intervals were more accurate than the jackknife multitaper confidence intervals. Moreover, the MFDB accurately estimated the asymptotic confidence intervals using a relatively small number of bootstrap samples. These findings coupled with our successful use of this method to make inferences regarding the difference in EEG power of anesthetized children of different age groups suggest that the MFDB will likely be a useful tool for time-series inference in neuroscience and in other fields.

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Fig. 1.

(a) True spectrum (green curve) and its multitaper spectral estimates (black curve). Comparisons between 95% confidence intervals of the MFDB and jackknife method. (b), (c) Magnified views of the two frequency ranges between 0.14–0.18 and 0.44–0.5.

Fig. 2. True spectrum (black curve) and comparisons between 95% confidence bounds of MC replications and the MFDB method.

Fig. 3. Comparisons of the median spectral densities of infants 0–3 months and 4–6 months of age. (a) MFDB median spectrum for the two groups along with 95% confidence intervals (shaded regions). (b) Upper (red) and lower (blue) confidence bounds for the difference in median spectra between the two groups.

Fig. 4.

Comparisons of the median spectral densities for a frontal (red) and an occipital (blue) electrode for the 4–6 months age group. (a) MFDB median spectrum for the two electrode sites along with 95% confidence intervals (shaded regions). (b) Upper (red) and lower (blue) confidence bounds for the difference in median spectra between the two electrode sites.