Influence of non-Newtonian power law rheology on inertial migration of particles in channel flow

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ABSTRACT

In this paper, the inertial migration of particles in the channel flow of power-law fluid is numerically investigated. The effects of the power-law index (n), Reynolds number (Re), blockage ratio (k), and channel aspect ratio (AR) on the inertial migration of particles and equilibrium position are explored. The results show that there exist two stages of particle migration and four stable equilibrium positions for particles in the cross section of a square channel. The particle equilibrium positions in a rectangular channel are much different from those in a square channel. In shear-thinning fluids, the long channel face equilibrium position and two kinds of particle trajectories are found at low Re . With increasing Re , the short channel face equilibrium position turns to be stable, multiequilibrium positions, and three kinds of particle trajectories along the long wall start to form. Only two stable equilibrium positions exist in shear-thickening fluids. The equilibrium positions are getting closer to the channel centerline with increasing n and k and with decreasing Re . The inertial focusing length L_2 in the second stage of particle migration is much longer than inertial focusing length L_1 in the first stage. In the square channel, L_2 is decreased with increasing Re and k and with decreasing n . In the rectangular channel, L_2 is the shortest in the shear-thinning fluid.

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I. INTRODUCTION

The fast developing nonlinear technique in biomicrofluidics is promising to precisely control the motion of particles and cells for focusing,¹ concentration,² and separation.³ Recently, the inertial effects and non-Newtonian fluids^{4,5} are getting much attention for their capabilities to achieve an extreme throughput continuously without the need for external forces.⁶ Therefore, some fundamental issues such as the number of stable equilibrium positions and the effect of rheology on the inertial migration in the channel will be crucial for the development of biomicrofluidic chips and medical particles/cell diagnostics.

Although the property of inertial migration of particles toward specific lateral equilibrium positions in the channel has been extensively studied experimentally,^{7,8} numerically,^{4,5} and theoretically,⁹ the process of particle migration and the number of stable equilibrium positions in fluids remain unclear, especially in non-Newtonian fluids.

For neutrally buoyant particles migrated in a square channel of Newtonian fluids, many experimental and numerical results have shown that two stages of the inertial migration process were formed, and there were four or eight equilibrium positions at

various blockage ratios and Reynolds numbers.¹⁰ Matas *et al.*¹¹ pointed out that particles would migrate to the Segrè-Silberberg ring, which was shifted toward the wall under higher Re , while another equilibrium ring was found closer to the centerline with further increasing Re . Chun and Ladd¹⁰ presented that particles focused at eight equilibrium positions in the cross section when $Re = 100$; the unstable equilibrium positions were accumulated at four corners of the cross section at $Re = 500$, while the equilibrium positions at the centerline of walls were formed at $Re = 1000$. Abbas *et al.*¹² found that particles were located at four stable equilibrium positions on the centerline of walls, and two stages of inertial migration were formed. Miura *et al.*¹³ explored the inertial migration of spherical particles with the range of $Re = 100$ –1200. They observed four equilibrium positions focused at the centerline of walls under low Re , while eight equilibrium positions were located at four corners and the centerline of walls at high Re . Ahmed *et al.*¹⁴ reported measurements of hydrodynamic mobility of confined polymeric particles, cancer cells, and vesicles. They found that the mobility of vesicles was higher in a square channel than in a circular tube, and the mobility of cancer cells was higher than

rigid particles but lower than vesicles. They also explained further the differences in the mobility of the three systems by considering their shape deformation and surface flow on the interface. For a rectangular channel ($AR = W/H \neq 1$, where W is the channel width and H is the channel height) of Newtonian fluid, more controversies and different phenomena have been reported. For example, researchers have claimed that there were only two equilibrium positions located at the center of two long walls.⁶ Gossett *et al.*¹⁵ found two stable equilibrium positions centered at the long walls and two unstable ones at the center of short walls. Bhagat *et al.*¹⁶ reported the presentation of six or even eight equilibrium positions in a rectangular channel. Ciftlik *et al.*¹⁷ discovered the complex variations in equilibrium positions situated along the long and short walls when $Re = 75$ –2000. Liu *et al.*¹⁸ observed two equilibrium positions at the center of long walls under low Re , and another two stable equilibrium positions were presented at the center of short walls when Re was larger than the critical value.

For non-Newtonian fluid, the motion of neutrally buoyant particles in the viscoelastic fluid has been increasingly studied since the work of Leshansky *et al.*¹⁹ For a variety of viscoelastic polymer solutions with the properties of elasticity, the coupling of fluid elasticity and shear-thinning played a significant role in the particle migration. The gradients of normal stress differences in viscoelastic fluid yielded an elastic lift, which pushed particles toward the low shear rate regions when the fluid inertia was negligible. Kim and Kim²⁰ found that the viscoelastic force pushed the particle toward the corners and channel centerline, while the fluid inertia produced an inertial lift force (or wall repulsive force) to drive the particle out of the corners. Yu *et al.*⁵ discovered that the equilibrium positions of particles were formed at the corner, wall centerline, diagonal line, and channel centerline by the effects of inertial and fluid elasticity, the fluid elasticity drove the particles toward the channel centerline, and the blockage ratio would influence particle migration significantly. D'Avino *et al.*²¹ found that three aligned particles migrated to the center of a circular channel in strong shear-thinning elastic fluids and the distance between the neighboring particles would increase up to a stable value, and the shear-thinning effect was considered to be the main reason for driving particles toward the walls. Del Giudice *et al.*⁸ showed that particles would migrate to the centerline of a square microchannel of strongly shear-thinning elastic fluids when the inertial effect was negligible. Raffiee *et al.*²² studied the effects of deformability, elasticity, inertia, and size on the cell motion. The results showed that the equilibrium position of the cell was on the channel diagonal line, in contrast to that of rigid particles, which was on the center of the channel faces for the same range of Reynolds number. The addition of polymers in microfluidic devices could be used to enhance the throughput in the cell focusing and separation devices at a low cost. Raoufi *et al.*¹ explored the effects of elasticity accompanying channel cross-sectional geometry and sample flow rates on the focusing phenomenon in elastoinertial systems. The results revealed that increasing the aspect ratio weakens the elastic force more than inertial force, causing a transition from one focusing position to two. Increasing the angle of a channel corner caused the elastic force to push the particles more efficiently toward the channel center.

The previous studies mainly focused on the inertial migration of particles in the Newtonian fluid and viscoelastic fluid. In

practice, especially in bioengineering, shear-thinning has been demonstrated to be the dominant behavior in blood. A comprehensive exploration of inertial migration of particles in inelastic power-law fluid is still lacking. Nie and Lin²³ concluded that the shear-thinning and large inertial effect contributed to the motion of circular particles passing a fixed cylinder in a shear flow. Zhu *et al.*²⁴ simulated the thermosolutal convection of power-law fluid in a three-dimensional porous media and found that the shear-thinning fluid improved heat and mass transfer more efficiently than shear-thickening fluids. Ouyang *et al.*²⁵ studied the hydrodynamic interaction between two circular swimmers in shear-thinning fluid and found that the swimmers were more likely to rotate during the period of a colliding process than that in shear-thickening fluid. Recently, Li and Xuan²⁶ experimentally studied both the individual and combined effects of fluid inertia and the shear-thinning property on the inertial migration of particles in a rectangular channel under a relatively low inertial effect. They observed that the number of particle equilibrium positions was a strong function of channel dimension, k and n .

Based on the literature mentioned above, we can see that the inertial migration of neutrally buoyant particles in a rectangular channel flow of power-law fluids is still lacking. Therefore, the lattice Boltzmann method is used to study the inertial migration of particles in a channel of power-law fluids in the present study. Moreover, the effects of Reynolds number, power-law index, channel aspect ratio, and blockage ratio on the inertial migration of particles are systematically explored. The number of stable equilibrium positions and the inertial focusing length are determined. The conclusions would be helpful to understand the inertial migration of particles in power-law fluids and pave the way for designing the efficient biomicrofluidic devices.

II. NUMERICAL METHOD AND THEORETICAL MODEL

For the incompressible flow of power-law fluids, the continuity and momentum equation can be written as²³

$$\nabla \cdot \mathbf{u} = 0, \quad (1)$$

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{f} + \nabla \cdot \boldsymbol{\tau}, \quad (2)$$

where ρ , $\mathbf{u}$, p , and $\mathbf{f}$ are the fluid density, velocity vector, pressure, and body force, respectively, and $\boldsymbol{\tau}$ is the extra stress tensor, which can be expressed by $\boldsymbol{\tau} = \mu \dot{\boldsymbol{\gamma}}$, where μ is the dynamic viscosity and $\dot{\boldsymbol{\gamma}}$ is the rate-of-strain tensor.

A. Lattice Boltzmann method

For the past two decades, the lattice Boltzmann method (LBM) has been proved to be an efficient numerical method to study the multiphase flow, which can recover Eqs. (1) and (2) via the Chapman-Enskog expansion.²⁷ Unlike the traditional computational schemes based on the continuum fluid assumption, the LBM is a statistical approach in which the fluid is replaced by fractious particles, and the macroscopic properties of fluids can be evaluated by the particle distribution function. In the present study,

a three-dimensional (D3Q19) single-relaxation-time LBGK model with external forces is used for its high efficiency and precision,²⁸

$$f_i(\mathbf{x} + \Delta t \mathbf{e}_i, t + \Delta t) = f_i(\mathbf{x}, t) + \frac{1}{\tau_f} [f_i^{eq}(\mathbf{x}, t) - f_i(\mathbf{x}, t)] + F_p \Delta t, \quad (3)$$

$$f_i^{eq}(\mathbf{x}, t) = \rho w_i \left[1 + \frac{3}{c^2} \mathbf{e}_i \cdot \mathbf{u} + \frac{9}{2c^4} (\mathbf{e}_i \cdot \mathbf{u})^2 - \frac{3\mathbf{u}^2}{2c^2} \right], \quad (4)$$

$$F_p = \left(1 - \frac{1}{2\tau} \right) \frac{(\mathbf{e}_i - \mathbf{u}) \cdot \mathbf{F}_b}{c_s^2} f_i^{eq}(\mathbf{r}, t), \quad (5)$$

where Δt is the unit lattice time and $\Delta t = 1$; τ_f is the relaxation time; $f_i(\mathbf{x}, t)$ is the distribution function with velocity $\mathbf{e}_i$ at position $\mathbf{x}$ and time t ; $f_i^{eq}(\mathbf{r}, t)$ is the equilibrium distribution function; c_s is the speed of sound ($c_s^2 = c^2/3$ with c being the lattice speed); w_i is the weighting coefficient with $w_0 = 1/3$, $w_{1,\dots,6} = 1/18$, and $w_{7,\dots,18} = 1/36$; F_p is the driving force; and $\mathbf{F}_b$ is the body force.

The discrete velocities in 19 directions are

$$[\mathbf{e}_i, \quad i = 0, 1, \dots, 18] = c \begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \end{bmatrix}. \quad (6)$$

The fluid density and velocity are given by

$$\rho = \sum f_i, \quad \mathbf{u} = \frac{1}{\rho} \sum f_i \mathbf{e}_i + \frac{\Delta t}{2\rho} \mathbf{F}_b. \quad (7)$$

Boundary treatment at the solid-fluid interface is quite important to calculate the motion of a neutrally buoyant particle. Following the popular bounce-back rule²⁹ for a moving wall by the momentum-exchange method, the boundary node is placed on the links connecting the interior and exterior nodes,

$$f_i(\mathbf{x}, t + \Delta t) = f_i(\mathbf{x}, t_+) - 2B_i(\mathbf{e}_i \cdot \mathbf{u}_b), \quad (8)$$

where t_+ is the postcollision time; i' and i are the reflected and incident directions, respectively; $B_i = 3\rho w_i/c^2$; and $\mathbf{u}_b = \boldsymbol{\Omega} \times \mathbf{x}_b + \mathbf{u}_0$, where $\boldsymbol{\Omega}$ is the angular velocity and $\mathbf{x}_b = \mathbf{x} + \mathbf{e}_i/2 - \mathbf{x}_0$, with $\mathbf{x}_0$ and $\mathbf{u}_0$ being the position and the translational velocity of the particle, respectively.

The hydrodynamic force and torque on the solid particles exerted by fluids at $\mathbf{x}_b$ are calculated by

$$\mathbf{F}_h \left(\mathbf{x} + \frac{1}{2} \mathbf{e}_i, t \right) = 2\mathbf{e}_i [f_i(\mathbf{x}, t_+) - B_i(\mathbf{e}_i \cdot \mathbf{u}_b)], \quad (9)$$

$$\mathbf{T}_h \left(\mathbf{x} + \frac{1}{2} \mathbf{e}_i, t \right) = \mathbf{x}_b \times \mathbf{F}_h. \quad (10)$$

The added force and torque exerted on the particle due to the covered and uncovered fluid node can be calculated by³⁰

$$\mathbf{F}_c(\mathbf{x}, t) = \rho(\mathbf{x}, t) \mathbf{u}(\mathbf{x}, t), \quad (11)$$

$$\mathbf{F}_u(\mathbf{x}, t) = -\rho(\mathbf{x}, t) \mathbf{u}(\mathbf{x}, t). \quad (12)$$

Total force and torque on the particle are given by

$$\mathbf{F} = \sum \mathbf{F}_h \left(\mathbf{x} + \frac{1}{2} \mathbf{e}_i, t \right) + \sum \mathbf{F}_c(\mathbf{x}, t) + \sum \mathbf{F}_u(\mathbf{x}, t), \quad (13)$$

$$\mathbf{T} = \sum \mathbf{T}_h \left(\mathbf{x} + \frac{1}{2} \mathbf{e}_i, t \right) + \sum \mathbf{T}_c(\mathbf{x}, t) + \sum \mathbf{T}_u(\mathbf{x}, t). \quad (14)$$

B. Power-law fluid

In the Oswald-De Waele power-law model, the viscosity can be expressed as²⁴

$$\mu = m |\dot{\gamma}|^{n-1}, \quad (15)$$

$$|\dot{\gamma}| = \sqrt{2 \left(\frac{\partial u}{\partial x} \right)^2 + 2 \left(\frac{\partial v}{\partial y} \right)^2 + 2 \left(\frac{\partial w}{\partial z} \right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)^2}, \quad (16)$$

where m is the power-law consistency; n is the power-law index, and $n = 1$, $n < 1$, and $n > 1$ correspond to the case of Newtonian, shear-thinning, and shear-thickening fluids, respectively.

The local velocity derivatives in the mainstream region are calculated by the fourth-order finite-difference scheme,

$$\frac{\partial u}{\partial x} = \frac{2}{3\Delta x}(u_{i+1,j,k} - u_{i-1,j,k}) + \frac{1}{12\Delta x}(u_{i+2,j,k} - u_{i-2,j,k}) + O(\Delta x^4), \quad (17)$$

$$\frac{\partial v}{\partial y} = \frac{2}{3\Delta x}(u_{i,j+1,k} - u_{i,j-1,k}) + \frac{1}{12\Delta x}(u_{i,j+2,k} - u_{i,j-2,k}) + O(\Delta x^4), \quad (18)$$

$$\frac{\partial w}{\partial z} = \frac{2}{3\Delta x}(u_{i,j,k+1} - u_{i,j,k-1}) + \frac{1}{12\Delta x}(u_{i,j,k+2} - u_{i,j,k-2}) + O(\Delta x^4). \quad (19)$$

The second-order finite-difference scheme is adopted to calculate the local velocity derivatives in the boundary region,

$$\frac{\partial u}{\partial x} = \frac{-3u_{i,j,k} + 4u_{i+1,j,k} - u_{i+2,j,k}}{2\Delta x} + O(\Delta x^2), \quad (20)$$

$$\frac{\partial v}{\partial y} = \frac{-3u_{i,j,k} + 4u_{i,j+1,k} - u_{i,j+2,k}}{2\Delta x} + O(\Delta x^2), \quad (21)$$

$$\frac{\partial w}{\partial z} = \frac{-3u_{i,j,k} + 4u_{i,j,k+1} - u_{i,j,k+2}}{2\Delta x} + O(\Delta x^2). \quad (22)$$

For the power-law fluids, the instantaneous and local relaxation times in the LBM for all the nodes are calculated as $\tau_f = 3\mu/(\rho c^2 \Delta t) + 0.5$.

C. Repulsive force model

A short-range repulsive force model proposed by Glowinski *et al.*³¹ is adopted to avoid unphysical overlaps when the particle is close to the walls,

$$f_r = \begin{cases} \frac{C_m}{\varepsilon} \left(\frac{d - d_{\min} - \Delta r}{\Delta r} \right)^2 \mathbf{e}_r, & d \leq d_{\min} + \Delta r, \\ (0, 0), & d > d_{\min} + \Delta r, \end{cases} \quad (23)$$

where $C_m = MU^2/a$ is the characteristic force, M is the particle mass, U is the velocity, a is the radius, $\varepsilon = 10^{-4}$ is a positive coefficient, d is the distance between the particle center and the walls, $d_{\min} = 2a$, $\Delta r = 2\Delta x$ represents two lattices when the repulsive force exists in the simulation, and $\mathbf{e}_r$ is the direction vector from particle's center to the walls.

D. Problem definition

Figure 1 shows a neutrally buoyant particle migrating in a rectangular channel. In the computation, the periodic boundary condition with a constant pressure gradient is introduced in the

FIG. 1. Schematic diagram of a neutrally buoyant particle migrating in a rectangular channel.

x -direction, and no-slip boundary condition is applied on the channel walls. The trajectory of the particle is calculated with the range of $Re = 65$ – 250 and $n = 0.6$, 1.0 , and 1.2 (corresponding to the most common practical scenario). The channel height, width, and length are denoted by H , W , and L , respectively. The channel aspect ratio is $AR = W/H$, and the channel hydraulic diameter is $D_h = 2WH/(W + H)$. The particle diameter is denoted by D , and the blockage ratio is $k = D/H$. The Reynolds number is defined as $Re = \rho U_0^{2-n} D_h^n / \mu$, where ρ and U_0 are the fluid density and the mean velocity at the inlet, respectively.

III. VALIDATION

A. Velocity profile for the flow

Figure 2 shows the comparison of the simulated velocity profile with the theoretical solutions^{28,32} for the power-law fluid with $n = 0.6$, 1.0 , and 1.2 in the rectangular channel ($AR = 1.0, 2.0$). The excellent agreements are obtained. Figure 2(a) shows that the velocity profile for shear-thinning fluid is flatter than that for Newtonian fluid and shear-thickening fluid near the center of the square channel ($AR = 1.0$). This phenomenon is more obvious in the rectangular channel ($AR = 2.0$) as shown in Fig. 2(b) where the velocity profile in the long side is flatter than that in the short side, and the deviation of the velocity profile between long and short sides is larger at higher power-law index.

B. Equilibrium position

Figure 3 shows the comparison of the equilibrium positions of particles in a square channel between the present results ($n = 1.0$, $k = 0.16$) and the experimental data [$(k = 0.11)$,¹² $(k = 0.16)$,³² $(k = 0.125, 0.175)$ ³³]. We can see that the present results are in fairly good agreement with the experimental data. The equilibrium positions of the particle are getting closer to the wall with increasing Re , which is consistent with the experimental observations in the square channel³³ and circular tube.⁷

FIG. 2. Comparison of the simulated velocity profile with the theoretical solution^{28,32} in square and rectangular channels for different power-law indices. (a) AR = 1.0. (b) AR = 2.0.

FIG. 3. Comparison of equilibrium positions between the present results and the experimental data.

C. Particle trajectory

The trajectory of a particle migrating in a square channel of Newtonian fluids is calculated. The length along the x -direction for each period is changed from $140\Delta x$ to $240\Delta x$ ($\Delta x = 1$) for ensuring the calculated results do not change with the length along the x -direction. As shown in Fig. 4(a), the present results are in good agreement with those obtained by Lashgari *et al.*,³⁴ and the mesh convergence tests are conducted; i.e., the deviation of results between $200\Delta x$ and $240\Delta x$ is negligible. Therefore, we take the length along the x -direction as $200\Delta x$ in the square channel and $240\Delta x$ in the rectangular channel, respectively, in the computation. In Fig. 4(b), the mesh convergence tests are also conducted along the y - and z -direction. The deviations of results between $80\Delta y$ and $100\Delta y$ and $80\Delta z$ and $100\Delta z$ are negligible. Therefore, we take the length along the z -direction as $80\Delta z$ and the length along the y -direction as $80\Delta y$ in the square channel and $160\Delta y$ in the rectangular channel, respectively, in the computation.

IV. RESULTS AND DISCUSSION

The particle trajectories are depicted in the cross section of a square channel or a rectangular channel as shown in Figs. 5, 6, and 9–11 where the dotted gray lines represent the channel

FIG. 4. Comparison of present results of the particle trajectory with those obtained by Lashgari *et al.*³⁴ and mesh convergence tests. (a) Comparison of the particle trajectory and mesh convergence test along the x -direction. (b) Mesh convergence test along the y - and z -direction.

FIG. 5. Effect of the power-law index on the particle trajectories in a square channel with various initial positions ($Re = 150$). (a) In (y, z) plane ($n = 1.2$). (b) In (x, y) and (x, z) plane ($n = 1.2$). (c) In (y, z) plane ($n = 1.0$). (d) In (x, y) and (x, z) plane ($n = 1.0$). (e) In (y, z) plane ($n = 0.6$). (f) In (x, y) and (x, z) plane ($n = 0.6$).

diagonal lines and the centerlines of walls, and the light blue bold dotted lines represent that particle surface will touch the walls ($y/W = 0$ or $z/H = 0$) or centerlines ($y/W = 0.5$ or $z/H = 0.5$). In order to intuitively depict how fast the particle migrates in the channel, the trajectories are marked by circles with every 30 000 calculation steps.

A. Particle migration in a square channel

1. Effect of the power-law index n on the particle trajectory

The particle trajectories in the cross section of a square channel for various initial positions are shown in Fig. 5 where the

particle trajectories are depicted only in the lower-left quadrant of the section because of the geometric symmetry of the channel. On the left side of Figs. 5(a), 5(c), and 5(e), the solid line is z/H axis, and the particle will reach the wall surface when its center is at the dashed line.

Figures 5(a), 5(c), and 5(e) show that the particle migrates first toward the same ring that is nearly parallel to the wall at around 0.6 times from the center axis, and the position in the z -direction is changeless, which is consistent with the well-known Segré-Silberberg effects.⁷ Choi *et al.*³² and Shichi *et al.*³³ termed this ring as the pseudo-Segré-Silberberg ring (pSS ring) in a square channel and attributed it as the tubular pinch effect. Afterward, the particle moves along the pSS ring toward the centerline of the

FIG. 6. Effect of Re on the particle trajectory with various initial positions in the (y, z) plane ($k=0.2$). (a) $n=1.2$, $Re=65$. (b) $n=1.2$, $Re=250$. (c) $n=1.0$, $Re=65$. (d) $n=1.0$, $Re=250$. (e) $n=0.6$, $Re=65$. (f) $n=0.6$, $Re=250$.

channel face equilibrium (CFE) position. Due to the symmetry of the square channel, the particle migrates along a straight line toward the channel corner equilibrium (CCE) position when the particle is released directly on the diagonal line. Both the CFE and CCE are presented, the former is stable regardless of Re and n , and the equilibrium position is closer to the channel centerline with increasing n .

The particle trajectories in the (x, y) plane (dotted line) and the (x, z) plane (solid line), as shown in Figs. 5(b), 5(d), and 5(f), correspond to Figs. 5(a), 5(c), and 5(e) with the same color. It shows that there are two stages of particle migration along the x -direction for various initial positions. At the beginning, the positions in the y -direction (dotted line) and z -direction (solid line) are

changed rapidly, and the position in the z -direction will be stable at the end of the first stage [the filled blue circle and additional vertical dotted line of the initial position $(y, z) = (0.432, 0.36)$ are shown as an example]. Then, the position in the y -direction is getting close to the centerline of the wall finally [the open blue circle and additional vertical dotted line of the initial position $(y, z) = (0.432, 0.36)$ are also shown as an example]. The x -axial values of additional vertical dotted lines, as shown in Figs. 5(b), 5(d), and 5(f), show that the inertial focusing lengths in the second stage are about ten times ($n \geq 1$) and several times ($n < 1$) larger than that in the first stage. The inertial focusing length is decreased with decreasing n .

FIG. 7. Relationship between the equilibrium positions in the z -direction and Re in a square channel for different Re , k , and n . (a) Stable CFE. (b) Unstable CCE.

2. Effect of the Reynolds number on the particle trajectory

In order to study the influence of Re on the process of particle migration, the particle trajectories for various Reynolds number ($Re = 65, 100, 150, 200$, and 250) are calculated, and the results for $Re = 65$ and 250 are shown in Fig. 6. We can see that the migration velocity of the particle in the (y, z) plane is increased with increasing Re , and with decreasing n , the pSS ring and two stages of particle migration exist even at high Re . There are four stable CFE and four unstable CCE in the cross section of a square channel when $Re = 65$ – 250 . Recently, Shichi *et al.*³³ experimentally found that there exists intermediate equilibrium (IME) position along the pSS ring in a square channel of Newtonian fluid ($Re \leq 240$, $k = 0.125$), and the critical Reynolds number for the transition of the equilibrium position was 260–400 from the CFE to the IME and 450–600 from the IME to the CCE. The critical Reynolds number is higher when the particle is larger. Therefore, the IME is not found within the present range of Re and k .

3. Equilibrium position of the particle in a square channel

The relationship between the equilibrium positions in the z -direction (z_{eq}/H) and Re for CFE and CCE are plotted in Fig. 7. In Fig. 7(a), the values of z_{eq}/H in Newtonian fluid ($n = 1$) are consistent well with the results obtained by Yuan *et al.*,³⁵ Lashgari *et al.*,³⁴ and Su *et al.*³⁶ The values of z_{eq}/H exhibit a monotonic decrease with

increasing Re in power-law fluid and have a rapid reduction in shear-thickening fluid for both stable CFE and unstable CCE. The monotonically decreasing rate of z_{eq}/H for the CFE will eventually be stable in shear-thinning fluid when the Reynolds number further increases. The values of z_{eq}/H in shear-thickening fluid are closer to the centerline than that in Newtonian fluid and shear-thinning fluid, and the shear-thinning effect will drive the particle farther away from the channel centerline. The equilibrium positions in the z -direction are closer to the centerline with increasing k .

4. Effects of Re , n , and k on the inertial focusing length in a square channel

The inertial focusing length is one of the most useful parameters for designing the microchannels to meet the needs of separation and sorting of particles. Therefore, the effects of Re , n , and k on the inertial focusing length are analyzed. The lift force induced by the inertia or fluid rheology was found to be responsible for the particle migration toward specific lateral equilibrium positions.³⁷

The trajectories of particle migrating from $(y/W, z/H) = (0.432, 0.36)$ in power-law fluid for different Re in the (x, y) plane (dotted line) and the (x, z) plane (solid line) are shown in Fig. 8(a) where the area between the green vertical dotted line (39 – 50) is the inertial focusing length of the first stage (L_1/H). Figures 5, 6, and 8(a) show that the inertial focusing length along the pSS ring toward the stable CFE in the second stage (L_2/H) is longer than that in the first stage (L_1/H). The inertial focusing length is obviously different in shear-thinning fluid, Newtonian fluid, and shear-thickening fluid. For

FIG. 8. Effects of Re , n , and k on the inertial focusing length. (a) Trajectories in the (x, y) plane and the (x, z) plane ($k = 0.2$). (b) Trajectories in the (x, y) plane.

clarity, the trajectories of the particle in the (x, y) plane are depicted in Fig. 8(b) where the stable second stages are represented with the open symbols and additional vertical dotted lines. In Newtonian fluid, it can be estimated that the values of L_2/H are equal to 100, 160, 235, and 320 with $k=0.2$ when $Re=250, 150, 100$, and 65, respectively. If the blockage ratio is further increased to $k=0.25$ when $Re=150$, the value of L_2/H is decreased from 160 to 120 in Newtonian fluid, from 260 to 180 in shear-thickening fluid, and from 135 to 75 in shear-thinning fluid. The reason is that the inertial lift force is proportional to the fourth power or third power of the particle diameter, while only the first power for the Stokes drag force ($F_s = 3\pi\mu DU$). As a result, the competition of two forces results in a larger migrating velocity for a larger particle and accelerates the cross-lateral migration of the particle to the CFE. It needs longer inertial focusing length to reach the stable second stage with increasing n and with decreasing Re .

Based on the above discussion, we can conclude that the value of L_1/H is not changed rapidly with increasing n , Re , and k . The inertial focusing length in the second stage (L_2/H) is about ten times ($n \geq 1$) and several times ($n < 1$) larger than L_1/H . The value of L_2/H is decreased with increasing Re and k and with decreasing n . Therefore, the shear-thinning fluid is beneficial for quickly focusing particles to the equilibrium positions in a square channel and can improve the efficiency for particle separation, screening, and counting in applications.

B. Particle migration in a rectangular channel

1. Effect of the power-law index n on the particle trajectory

The different phenomena of the inertial migration of particles in a rectangular channel of shear-thinning fluid are still not studied thoroughly. Next, we study the inertial migration of particles in a rectangular channel ($AR = 2.0$) of power-law fluid.

Figure 9 shows the particle trajectories for different initial positions in the lower-left quadrant of the cross section with $k=0.25$, $Re=150$ and $n=0.6, 1.0$, and 1.2, respectively. In Newtonian fluid and shear-thickening fluid, particles migrate first toward a pSS ring and then move along the pSS ring toward the long channel face equilibrium position (LCFE) regardless of their initial positions. The difference in the particle trajectory between a square and rectangular channel is that the particle migration in the (y, z) plane in a rectangular channel is much slower than that in a square channel at the same Re . Particles will not migrate along a straight line to the CCE if it is released directly on the diagonal line, which means that the unstable CCE has vanished. Similar phenomena in Newtonian fluid are also observed in experiments as shown in the blue dotted box of Fig. 9(d).¹⁷

For shear-thinning fluid, the pSS ring is also formed as shown in Fig. 9(c). For particle trajectories, the number of equilibrium positions and the location of the pSS ring are much different from that of Newtonian fluid and shear-thickening fluid in a rectangular channel and that of power-law fluid in a square channel. Three gray bold dotted dashed lines in Fig. 9(c) divide the cross section into four parts, and the particle will migrate to the corresponding equilibrium positions in each part. If the particle is released near the short channel face, it will move to the centerline of the short channel face equilibrium position (SCFE) as shown with the green color. There are three different kinds of trajectories if the particle is released at the right side of the SCFE: (1) as denoted with the orange color in MLCFE, the particle on the right (located in the centerline) moves vertically along the long wall centerline, and equilibrium position is in the centerline because of the symmetry of the channel. The particles in the middle of MLCFE move to the same equilibrium position, while the particle on the left (near the LLCFE) moves to another equilibrium position; (2) as denoted with the blue color, the particle will move right along the pSS ring to a fixed equilibrium position away from the centerline of the long wall when it is released near the

FIG. 9. Effect of the power-law index on particle trajectories in a rectangular channel ($AR=2.0$) for different initial positions in the (y, z) plane. (a) $n=1.2$, $k=0.25$, $Re=150$. (b) $n=1.0$, $k=0.25$, $Re=150$. (c) $n=0.6$, $k=0.25$, $Re=150$. (d) Experimental results in Newtonian fluid¹⁷ (above) [Reproduced with permission from Ciftlik *et al.*, *Small* 9(16), 2764–2773 (2013). Copyright 2013 Wiley-VCH Verlag GmbH & Co. KGaA, Weinheim] and in xanthan solution with a strong shear-thinning effect²⁶ (below) [Reproduced with permission from Li and Xuan, *Microfluid. Nanofluid.* 23(4), 54 (2019). Copyright 2019 Springer-Verlag GmbH Germany].

SCFE, which is named as the right long channel face equilibrium position (RLCFE); (3) as denoted with the red color, the particle will move left along the pSS ring to other fixed equilibrium position away from the centerline of the long wall when it is released between the RLCFE and the MLCFE, which is named as the left long channel face equilibrium position (LLCFE). Therefore, there exist multiequilibrium positions in a rectangular channel of shear-thinning fluid. Recently, Li and Xuan²⁶ found experimentally that particles focused at multiequilibrium positions along the long walls in xanthan solution with a strong shear-thinning effect, and the green fluorescent bands in the green solid box of Fig. 9(d) became wider, which supports our present results.

2. Effect of Re on the particle trajectory with $n = 0.6$

In order to understand the effect of Re on the multiequilibrium positions, Figure 10 shows the particle trajectories for different initial positions with $n = 0.6$, $k = 0.25$ and $Re = 65$, 100, 200, and 250. We can see that the two stages of particle migration are also formed, and the MLCFE still exists. While the equilibrium position is much different for different Re , the pSS rings are formed both along the y -axis and z -axis except the case of $Re = 65$. Figure 10(a) shows that only two kinds of trajectories (RLCFE and MLCFE) exist, the pSS ring along the long wall is very obvious at low Re , and the particle in the RLCFE moves right along the pSS ring to a fixed equilibrium position and is closer to the centerline of the long wall than that at high Re . What's more, the particle close to the short wall with the initial positions of $(y/W, z/H) = (0.18, 0.48)$, $(0.15, 0.40)$ can migrate to the LCFE. However, the SCFE begins to form with increasing Re to 100 as shown in Fig. 10(b), and the equilibrium positions of the particle with the same initial positions as denoted with the green color will be substable near the short wall on the unobvious pSS ring along the gray dotted line. In Figs. 9(c), 10(c), and 10(d), the particle

with the same initial positions near the short wall will migrate toward the short wall centerline; thus, the SCFE turns to be stable. The phenomenon is more obvious with increasing Re because the area occupied by the trajectories marked with the green color becomes large. Multiequilibrium positions and three kinds of trajectories along the long wall become obvious at higher Re , and the RLCFE and LLCFE are getting closer when Re is increased from 150 to 250. The equilibrium positions along the long wall are getting further away from the centerline of the long wall with increasing Re .

As shown in Fig. 2, the velocity profile of shear-thinning fluids is flatter than that of Newtonian fluid and shear-thickening fluid near the centerline of a rectangular channel. The formation of the SCFE and multiequilibrium positions along long walls can be explained as follows. The inertial effect is weak when Re is relatively small, and the inertial force pushes the particle toward the centerline of the long wall, but the reduced velocity curvature near the centerline can affect the shear gradient-induced inertial lift in magnitude or direction and increase the number of equilibrium positions along the long wall. Therefore, there are only the RLCFE and MLCFE without the CFE. It is more obvious that the particle shifts toward an equilibrium position away from the centerline of the long wall with increasing Re as shown in Figs. 9(c) and 10(b)–10(d).

3. Effect of Re on the particle trajectory with $n = 1.0$ and 1.2

In order to understand the influence of Re on the inertial migration of the particle in Newtonian fluid and shear-thickening fluid, the trajectories of the particle with different initial positions in a rectangular channel ($AR = 2.0$) with $k = 0.25$, $Re = 100$, 200, and 250 are analyzed. As shown in Fig. 11, the pSS ring is formed along the long walls, and two stages of inertial migration also exist. However, the equilibrium positions in long and short walls are

FIG. 10. Effect of Re on the particle trajectory in a rectangular channel ($AR = 2.0$) for different initial positions in the (y, z) plane ($n = 0.6$). (a) $Re = 65$. (b) $Re = 100$. (c) $Re = 200$. (d) $Re = 250$.

FIG. 11. Effect of Re on the particle trajectory in a rectangular channel ($AR=2.0$) for different initial positions in the (y, z) plane. (a) $n=1.0$, $Re=100$. (b) $n=1.2$, $Re=100$. (c) $n=1.0$, $Re=200$. (d) $n=1.2$, $Re=200$. (e) $n=1.0$, $Re=250$. (f) $n=1.2$, $Re=250$.

much different for different Re and n . When $Re=100$ and 150 , the particle with different initial positions in Newtonian fluid and shear-thickening fluid migrates along the pSS ring to the LCFE as marked with the blue color in Figs. 9(a), 9(b), 11(a), and 11(b). Therefore, the number of stable equilibrium positions is reduced to two, whereas particle trajectories toward the short walls and additional stable equilibrium positions (SCFE) marked with the green color emerge in Newtonian fluid when Re is increased to 200 and 250. The results of four stable equilibrium positions are consistent with the experimental results as shown in the blue solid box of Fig. 9(d). When Re is further increased to 250 in Newtonian fluid, the particle will migrate to the same position near the centerline of the long wall, and then the MLCFE occurs again as shown in Figs. 11(e) and 13(b). The particle in shear-thickening fluid with the same initial positions of $(y/W, z/H) = (0.15, 0.40)$, $(0.18, 0.48)$ will still migrate toward the LCFE at $Re=200$ as shown in Fig. 11(d). The substable equilibrium positions with the same initial positions occur again as marked with the green color in Figs. 11(d) and 13(a), and the particle positions are stable along the gray dotted line when Re is increased to 250.

The velocity profile along the long side in a rectangular channel is flatter than that along the short side, and the difference in the velocity profile between the two sides is larger at higher n . Therefore, the inertial force pushes the particle toward the LCFE when the

inertial effect is weak at low Re . Particle in shear-thickening fluid is more likely to shift toward the LCFE than in Newtonian fluid when the difference in the velocity profile between the two sides is larger. For the equilibrium position along the long wall at high Re , the lift force profile changes dramatically with increasing Re , and the inertial force pushes the particle toward the centerline of the long wall. Therefore, the reduced velocity curvature near the long wall centerline can also affect the lift force in magnitude or direction and push the particle to a position of force balancing near the centerline of the long wall as shown in Figs. 11(e) and 13(b). If the velocity profile is getting flatter with decreasing n , it will be harder for the particle to reach the LCFE, and the multiequilibrium positions of RLCFE and MCFE will appear in shear-thinning fluid as shown in Figs. 9(c), 9(d), and 10. The opposite is true; i.e., the particle is more likely to migrate to the LCFE for the thinner velocity profile in shear-thickening fluid as shown in Figs. 11(b), 11(d), and 11(f). Therefore, the particle migration behavior is in a complicated manner in a rectangular channel. In the cross section of a rectangular channel, there are two stable LCFE in the present study with Reynolds number changing from 65 to 250 in shear-thickening fluid, while in Newtonian fluid, there are two stable equilibrium positions (LCFE) when $Re \leq 150$, four stable equilibrium positions (two LCFE and two SCFE) when $Re=200$, and six stable equilibrium positions (four LCFE and two SCFE) when $Re=250$.

FIG. 12. Equilibrium position in a rectangular channel (AR = 2.0) for different Re and n . (a) LCFE. (b) SCFE.

4. Equilibrium position in a rectangular channel

The locations of the LCFE (z_{eq}/H) and SCFE (y_{eq}/W) are plotted against Re in Figs. 12(a) and 12(b). The values of z_{eq}/H and y_{eq}/W in equilibrium positions exhibit a monotonic decrease with increasing Re in Newtonian fluid and shear-thickening fluid. Figure 12(a) shows that the values of z_{eq}/H for shear-thinning fluid are the smallest and exhibit a monotonic decrease when $Re \leq 200$ and a slight increase with Re changing from 200 to 250, which is consistent with the results in a square channel. The LCFE is getting closer to the channel centerline with increasing n . Figure 12(b) shows that the stable SCFE depends on Re and n , and it is much easier to form the SCFE in shear-thinning fluid than that in

Newtonian fluid and shear-thickening fluid. The values of y_{eq}/W for the stable SCFE also decrease with increasing Re and with decreasing n . Therefore, the shear-thinning effect should be the primary reason for the equilibrium position shifting farther away from the channel centerline.

5. Effects of Re and n on the inertial focusing length in a rectangular channel

The inertial focusing length was seldom studied in a rectangular channel, even for the widely used channel in a microfluidic chip and cytometry for separation and counting. The present studies show that the inertial focusing length in a rectangular channel is

FIG. 13. Effects of n on the inertial focusing length ($Re = 250$). (a) Particle trajectory with different initial positions in the (x, y) and (x, z) plane ($n = 1.2$). (b) Particle trajectory with different initial positions in the (x, y) and (x, z) plane ($n = 1.0$). (c) Particle trajectory with different initial positions in the (x, y) and (x, z) plane ($n = 0.6$). (d) Comparison of the inertial focusing length in the (x, y) plane for different n and Re .

larger than that in a square channel, and the lower average shear rate for the flat velocity profile contributes to the longer inertial focusing length in a rectangular channel. While the effect of Re and n on the inertial focusing length is the issue, we need further study.

All kinds of equilibrium positions and particle trajectories as mentioned above coexist when $Re = 250$ in Newtonian fluid, shear-thinning fluid, and shear-thickening fluid. Therefore, the particle trajectories in the (x, y) plane (dotted line) and the (x, z) plane (solid line) with different initial positions are depicted in Figs. 13(a)–13(c) when $Re = 250$. We take the orange and blue lines in Fig. 13(b) as an example. When the dotted lines (y -direction) reach a steady value, the inertial focusing length is formed when the first stage is completed. Then, the solid lines (z -direction) reach the value of $z/H = 0.5$; i.e., the particle migrates toward the SCFE, and the inertial focusing length is formed when the second stage is completed. The effects of Re and n on the particle inertial focusing length are also compared in Fig. 13(d) when the particle ($k = 0.25$) is released from the initial position of $(y/W, z/H) = (0.35, 0.4)$. As shown in Fig. 13(d), the stable second stage of particle migration is marked with the open symbols and additional vertical dotted lines.

One of the key results of the present study is that the inertial focusing length in a rectangular channel of power-law fluid is much different from that in a square channel. The inertial focusing length to the LCFE (dotted line) is much larger than that to the SCFE (solid line) as shown in Figs. 13(b) and 13(c). The particle trajectories marked with the blue color in Fig. 13(d) show that the inertial focusing length in shear-thinning fluid is the smallest for the multiequilibrium positions along the long wall because the equilibrium position is getting further away the long wall centerline with increasing Re . As a result, the distance to the stable equilibrium position is reduced, and the inertial focusing length is also reduced subsequently. While in Newtonian fluid and shear-thickening fluid as shown in Fig. 13(d), the inertial focusing length to the LCFE is increased rapidly with increasing Re , which is contrary to that in a square channel. It is interesting to note that the inertial focusing length in Newtonian fluid is much larger than that in shear-thickening fluid. For the equilibrium position at the long wall for higher Re , the lift force profile is dramatically changed with increasing Re . As a result, the particle could not reach the long wall centerline, and the inertial focusing length in the flow direction is increased at high Re as shown in Figs. 13(b) and 11(e), while a larger difference in the velocity profile between the long and short side in shear-thickening fluid also accelerates the particle migration in the (y, z) plane to the LCFE. Therefore, a shorter inertial focusing length in the flow direction is needed in shear-thickening fluid than in Newtonian fluid. The opposite is true for the same velocity profile in a square channel, and the inertial focusing length is increased with increasing n as shown in Fig. 8.

V. SUMMARY

The inertial migration of particles in the channel of power-law fluid has been studied numerically using the three-dimensional lattice Boltzmann method. The present method is validated with both experiments and other simulations. The effects of the power-law index, Reynolds number, channel aspect ratio and blockage ratio are explored. The following conclusions are drawn.

For particle migration in a square channel with $AR = 1.0$.

- (1) The particle migrates first toward a pSS ring and then moves along the pSS ring toward the CFE except in the case that the particle is released directly on a diagonal line. Four stable CFE, four unstable CCE, and two stages of particle migration are presented.
- (2) Equilibrium positions of particles exhibit a monotonic decrease with increasing Re . Both stable CFE and unstable CCE in shear-thickening fluid show a rapid decrease, while the monotonic decrease rate of the CFE will be stable eventually in shear-thinning fluid with even further increasing Re . The equilibrium positions are closer to the centerline with increasing n and k .
- (3) The inertial focusing length L_1 in the first stage is not changed rapidly with increasing n , Re , and k , while the inertial focusing length L_2 in the second stage is much larger than L_1 , and L_2 is decreased with increasing Re and k and with decreasing n . Therefore, the shear-thinning fluid is beneficial for quickly focusing the particle to the equilibrium positions in a square channel.

For particle migration in a rectangular channel with $AR = 2.0$.

- (1) The CCE disappeared, and the equilibrium positions of particles are much different from that in a square channel. In shear-thinning fluid, the MLCFE is formed, and only two kinds of particle trajectories are found at low Re . While the SCFE turns to be stable, multiequilibrium positions and three kinds of particle trajectories along the long wall start to form with increasing Re . There exist two stable LCFE at low Re , while four and six stable equilibrium positions are formed in Newtonian fluid at $Re = 200$ and 250 , respectively. Only two stable LCFE are formed in shear-thickening fluid.
- (2) Equilibrium positions of particles are getting closer to the centerline with increasing n and shift toward the wall with increasing Re . The equilibrium position in shear-thinning fluid exhibits a monotonic decrease when $Re \leq 200$ and a slight increase with Re changing from 200 to 250. It is much easier to form the stable SCFE at low n and high Re .
- (3) The inertial focusing length in shear-thinning fluid is the shortest when multiequilibrium positions exist. While in Newtonian fluid and shear-thickening fluid, the inertial focusing length to the LCFE is increased rapidly with increasing Re , and the value of L_2 in Newtonian fluid is larger than that in shear-thickening fluid.

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REFERENCES

- ¹M. A. Raoofi *et al.*, "Experimental and numerical study of elasto-inertial focusing in straight channels," *Biomicrofluidics* **13**(3), 034103 (2019).
- ²Y. L. Chen and H. R. Jiang, "Particle concentrating and sorting under a rotating electric field by direct optical-liquid heating in a microfluidics chip," *Biomicrofluidics* **11**(3), 034102 (2017).

- ³B. Behdani *et al.*, "Shape-based separation of micro-/nanoparticles in liquid phases," *Biomicrofluidics* **12**(5), 051503 (2018).
- ⁴G. D'Avino *et al.*, "Particle migration due to viscoelasticity of the suspending liquid and its relevance in microfluidic devices," *Annu. Rev. Fluid Mech.* **49**(1), 341–360 (2017).
- ⁵Z. S. Yu *et al.*, "Equilibrium positions of the elasto-inertial particle migration in rectangular channel flow of Oldroyd-B viscoelastic fluids," *J. Fluid Mech.* **868**, 316–340 (2019).
- ⁶S. C. Hur *et al.*, "Sheathless inertial cell ordering for extreme throughput flow cytometry," *Lab Chip* **10**(3), 274–280 (2010).
- ⁷G. Segré and A. Silberberg, "Radial Poiseuille flow of suspensions," *Nature* **189**, 209–210 (1961).
- ⁸F. Del Giudice *et al.*, "Fluid viscoelasticity drives self-assembly of particle trains in a straight microfluidic channel," *Phys. Rev. Appl.* **10**(6), 064058 (2018).
- ⁹K. Hood *et al.*, "Inertial migration of a rigid sphere in three-dimensional Poiseuille flow," *J. Fluid Mech.* **765**, 452–479 (2015).
- ¹⁰B. Chun and A. J. C. Ladd, "Inertial migration of neutrally buoyant particles in a square duct: An investigation of multiple equilibrium positions," *Phys. Fluids* **18**(3), 031704 (2006).
- ¹¹J. P. Matas *et al.*, "Inertial migration of rigid spherical particles in Poiseuille flow," *J. Fluid Mech.* **515**, 171–195 (2004).
- ¹²M. Abbas *et al.*, "Migration of finite sized particles in a laminar square channel flow from low to high Reynolds numbers," *Phys. Fluids* **26**(12), 136–157 (2014).
- ¹³K. Miura *et al.*, "Inertial migration of neutrally buoyant spheres in a pressure-driven flow through square channels," *J. Fluid Mech.* **749**, 320–330 (2014).
- ¹⁴S. M. Ahmed *et al.*, "Hydrodynamic mobility of confined polymeric particles, vesicles, and cancer cells in a square microchannel," *Biomicrofluidics* **12**(1), 014114 (2018).
- ¹⁵D. R. Gossett *et al.*, "Inertial manipulation and transfer of microparticles across laminar fluid streams," *Small* **8**(17), 2757–2764 (2012).
- ¹⁶A. A. S. Bhagat *et al.*, "Inertial microfluidics for continuous particle filtration and extraction," *Microfluid. Nanofluid.* **7**(2), 217–226 (2009).
- ¹⁷A. T. Ciftlik *et al.*, "High throughput-per-footprint inertial focusing," *Small* **9**(16), 2764–2773 (2013).
- ¹⁸C. Liu *et al.*, "Inertial focusing of spherical particles in rectangular microchannels over a wide range of Reynolds numbers," *Lab Chip* **15**(4), 1168–1177 (2015).
- ¹⁹A. M. Leshansky *et al.*, "Tunable nonlinear viscoelastic "focusing" in a microfluidic device," *Phys. Rev. Lett.* **98**, 234501 (2007).
- ²⁰B. Kim and J. M. Kim, "Elasto-inertial particle focusing under the viscoelastic flow of DNA solution in a square channel," *Biomicrofluidics* **10**(2), 024111 (2016).
- ²¹G. D'Avino *et al.*, "Dynamics of pairs and triplets of particles in a viscoelastic fluid flowing in a cylindrical channel," *Comput. Fluids* **86**, 45–55 (2013).
- ²²A. H. Raffiee *et al.*, "Elasto-inertial migration of deformable capsules in a microchannel," *Biomicrofluidics* **11**(6), 064113 (2017).
- ²³D. M. Nie and J. Z. Lin, "Behavior of three circular particles in a confined power-law fluid under shear," *J. Non-Newtonian Fluid Mech.* **221**, 76–94 (2015).
- ²⁴Q. Y. Zhu *et al.*, "Three-dimensional numerical investigation on thermosolutal convection of power-law fluids in anisotropic porous media," *Int. J. Heat Mass Transfer* **104**, 897–917 (2017).
- ²⁵Z. Y. Ouyang *et al.*, "Hydrodynamic interaction between a pair of swimmers in power-law fluid," *Int. J. Non Linear Mech.* **108**, 72–82 (2019).
- ²⁶D. Li and X. Xuan, "The motion of rigid particles in the Poiseuille flow of pseudoplastic fluids through straight rectangular microchannels," *Microfluid. Nanofluid.* **23**(4), 54 (2019).
- ²⁷Y. H. Qian *et al.*, "Lattice BGK models for Navier-Stokes equation," *Europhys. Lett.* **17**, 479–484 (1992).
- ²⁸Z. L. Guo *et al.*, "Discrete lattice effects on the forcing term in the lattice Boltzmann method," *Phys. Rev. E* **65**(4), 046308 (2002).
- ²⁹A. J. C. Ladd, "Lattice-Boltzmann simulations of particle fluid suspensions," *J. Stat. Phys.* **104**(5–6), 1191–1251 (2001).
- ³⁰C. K. Aidun *et al.*, "Direct analysis of particulate suspensions with inertia using the discrete Boltzmann equation," *J. Fluid Mech.* **373**(373), 287–311 (1998).
- ³¹R. Glowinski *et al.*, "A fictitious domain approach to the direct numerical simulation of incompressible viscous flow past moving rigid bodies: Application to particulate flow," *J. Comput. Phys.* **169**(2), 363–426 (2001).
- ³²Y. S. Choi *et al.*, "Lateral and cross-lateral focusing of spherical particles in a square microchannel," *Lab Chip* **11**(3), 460–465 (2011).
- ³³H. Shichi *et al.*, "Inertial migration regimes of spherical particles suspended in square tube flows," *Phys. Rev. Fluids* **2**(4), 044201 (2017).
- ³⁴I. Lashgari *et al.*, "Inertial migration of spherical and oblate particles in straight ducts," *J. Fluid Mech.* **819**, 540–561 (2017).
- ³⁵C. Yuan *et al.*, "Inertial migration of single particle in a square microchannel over wide ranges of Re and particle sizes," *Microfluid. Nanofluid.* **22**(9), 102 (2018).
- ³⁶J. H. Su *et al.*, "Inertial migrations of cylindrical particles in rectangular microchannels: Variations of equilibrium positions and equivalent diameters," *Phys. Fluids* **30**, 032007 (2018).
- ³⁷E. S. Asmolov, "The inertial lift on a spherical particle in a plane Poiseuille flow at large channel Reynolds number," *J. Fluid Mech.* **381**, 63–87 (1999).