Approximate analytical time-domain Green's functions for the Caputo fractional wave equation

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The Caputo fractional wave equation [Geophys. J. R. Astron. Soc. **13**, 529–539 (1967)] models power-law attenuation and dispersion for both viscoelastic and ultrasound wave propagation. The Caputo model can be derived from an underlying fractional constitutive equation and is causal. In this study, an approximate analytical time-domain Green's function is derived for the Caputo equation in three dimensions (3D) for power law exponents greater than one. The Green's function consists of a shifted and scaled maximally skewed stable distribution multiplied by a spherical spreading factor $1/(4\pi R)$. The approximate one dimensional (1D) and two dimensional (2D) Green's functions are also computed in terms of stable distributions. Finally, this Green's function is decomposed into a loss component and a diffraction component, revealing that the Caputo wave equation may be approximated by a coupled lossless wave equation and a fractional diffusion equation. © 2016 Acoustical Society of America. [<http://dx.doi.org/10.1121/1.4960549>]

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I. INTRODUCTION

Extensive measurement of the attenuation coefficient in human and mammalian tissue in the ultrasonic range has revealed a power law dependence on frequency^{1–3} of the form $\alpha(\omega) = \alpha_0|\omega|^y$, where $\omega = 2\pi f$ is the angular frequency, f is the frequency, and α_0 is the attenuation constant. This relationship is valid over frequencies of diagnostic importance.¹ For most tissue, the power law exponent y typically ranges between 1 and 1.5 within this frequency range, although exponents slightly less than one have also been reported.^{2,4,5} This power-law behavior cannot be modeled by standard acoustic wave equations, such as the linear Stokes wave equation⁶ or the nonlinear Khokhlov-Zabolotskaya-Kuznetsov wave equation;⁷ both of these models assume a thermoviscous dissipation mechanism with a power law exponent of $y = 2$ in the low-frequency limit. To address this deficiency, researchers have proposed numerous phenomenological wave equations that utilize temporal and/or spatial convolution operators. Many of these operators can be expressed using fractional calculus operators,⁸ such as the Riemann-Liouville derivative, Caputo derivative, and/or the fractional Laplacian. We refer to the models that involve either time and/or space-fractional derivatives as fractional partial differential equations (FPDEs).

Within the biomedical acoustics community, the Szabo wave equation⁹ and the Chen-Holm model¹⁰ were some of the earlier FPDEs proposed to model power-law attenuation and the resulting frequency-dependent phase-speed, or dispersion. For $y \neq 1$, the Szabo wave equation adds a single time-fractional Riemann-Liouville derivative to the lossless

wave equation, whereas the Chen-Holm model adds a space-fractional Laplacian term to the lossless wave equation. More recent models, such as the power-law wave equation (PLWE),¹¹ the Treeby-Cox model,¹² and the “all-frequency” model proposed by Holm and Näsholm¹³ utilize two time-fractional or space-fractional derivatives to model power-law attenuation over a finite frequency band, where some of these are reviewed in Ref. 14. Recently, the fractional time-derivative operator in the Szabo and PLWE has been interpreted in terms of an underlying stochastic process for the time variable.¹⁵

Fractional calculus models entered diagnostic ultrasound models starting in the 1990s, yet fractional derivatives were used to model dispersive wave propagation¹⁶ in viscoelastic materials, such as the crust of the Earth, as early as 1967. The Loshkin-Rok model,¹⁷ which was later applied to propagation in porous media,¹⁸ is another early FPDE that models wave propagation in dispersive media. During the 1970s, fractional calculus was used to model anomalous damping in solid mechanics problems,¹⁹ primarily using fractional differential equations (FDEs) with respect to time. This FDE approach was popularized within the viscoelastic community by Ref. 20.

The Caputo fractional wave equation, given by Eq. (9) in Ref. 16, was derived from a time-fractional Kelvin-Voigt constitutive model that incorporates a time-dependent history in the stress-strain relationship using a singular integro-differential equation. This wave equation was also evaluated in Ref. 21 as the basis for a finite-element model, independent of Ref. 16. Unlike both the Szabo⁹ and PLWE¹¹ models, the loss term in the Caputo model evaluates a mixture of both time and space derivatives. The Caputo model is related to fractal ladder models for tissue²² and a time-convolutional

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constitutive equation.²³ A fractional constitutive equation can also be derived from a parallel arrangement of springs and dashpots, which is termed a Maxwell-Wiechert description²⁴ or a generalized Maxwell body.²⁵ Other papers have extended the original Caputo model with additional fractional derivative terms¹³ and connected the fractional operator to multiple-relaxation models.²⁶ Reference 27 has also proposed a time-fractional FPDE with multiple terms to capture power-law attenuation over a wide frequency band.

Recent analysis of the Caputo wave equation within the biomedical community has been limited to the frequency-domain;²³ moreover, no analytical time-domain solution to this model has been published. Analytical solutions are useful for understanding the underlying physics and also for verifying numerical models based on finite elements,²¹ k -space methods,¹² or pseudospectral methods.²⁸ An analytical solution for power-law wave propagation was derived in Ref. 11, where a Green's function solution to the power law fractional partial differential equation (FPDE) was derived; this Green's function consists of a scaled maximally skewed stable distribution²⁹ convolved in time with an outgoing spherical wave. Using properties of stable distributions, the Green's function was shown to be causal for power-law exponents $0 \leq y < 1$; however, for exponents $1 \leq y \leq 2$, the Green's function was demonstrated to be noncausal. Recent numerical evidence suggests that the noncausal behavior is only apparent in the extreme nearfield³⁰ for realistic values of the attenuation constant α_0 .

In this paper, we extend the methodology developed in Ref. 11 to the Caputo wave equation. In Sec. II, we review the Caputo wave equation and its attenuation and dispersion relationships, and then we nondimensionalize the Caputo wave equation in terms of a single small parameter ε . Background on the class of stable distributions is also provided for our later calculations. In Sec. III, we calculate an approximate 3D Green's function for the nondimensionalized wave equation. We then utilize a complex-plane analysis of the poles of the transfer function, which was first developed for viscoelastic problems.^{19,31} The approximate solution found using both of these methods consists of a shifted and scaled maximally skewed stable distribution,²⁹ where the scaling parameter depends on the time. The 1D and 2D Green's functions are also calculated in terms of maximally skewed stable distributions. The Green's function is numerically evaluated in Sec. IV. These results indicate that for typical tissue parameters, the approximate Caputo solution is nearly identical to the previously published power-law solution. Section V uses the approximate non-causal Green's function to relate the Caputo wave equation to previous models, such as the space-fractional diffusion equation,³² the PLWE,¹¹ and the Szabo wave equation,⁹ while Sec. VI provides conclusions.

II. CAPUTO WAVE EQUATION AND STABLE DISTRIBUTIONS

The Caputo wave equation¹⁶ is written as

$$\nabla^2 \psi - \frac{1}{c_0^2} \frac{\partial^2 \psi}{\partial t^2} + \tau^\beta \frac{\partial^\beta \psi}{\partial t^\beta} = 0, \quad (1)$$

where $\psi(x, t)$ is an acoustic variable (e.g., pressure, velocity potential), c_0 is a reference speed of sound, τ is a fractional relaxation time, and $0 < \beta \leq 1$ is the order of the Caputo derivative defined by

$$\frac{d^\beta \psi}{dt^\beta} = \frac{1}{\Gamma(1-\beta)} \int_0^\infty \frac{d\psi(t-t')}{dt'} t'^{-\beta} dt', \quad (2)$$

where $\Gamma(z)$ is the gamma function.³³ The Caputo derivative given by Eq. (2) was introduced by Caputo in Eq. (5) of Ref. 16. If all initial conditions are zero, then the Caputo derivative is equivalent to the Riemann-Liouville fractional derivative⁸

$$\frac{d^\beta \psi}{dt^\beta} = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^\infty \psi(t-t') t'^{-\beta} dt'. \quad (3)$$

For the remainder of the paper, we shall assume zero initial conditions, allowing us to use Eq. (3) in the third term of Eq. (1). Within the biomedical modeling community, the Riemann-Liouville derivative is generally assumed in treatment of Eq. (1) (see Refs. 21 and 23). The Riemann-Liouville fractional derivative has the Fourier transform property

$$\mathcal{F}\left(\frac{\partial^\beta \psi(t)}{\partial t^\beta}\right) = (-i\omega)^\beta \mathcal{F}[\psi(t)]. \quad (4)$$

For $\beta = 1$, Eq. (1) reduces to the well-known Stokes wave equation [Eq. (8.2.4) in Ref. 6]

$$\nabla^2 \psi - \frac{1}{c_0^2} \frac{\partial^2 \psi}{\partial t^2} + \tau \frac{\partial \psi}{\partial t} = 0, \quad (5)$$

which governs linear wave propagation in a viscous and/or thermally conducting medium; the relaxation time τ is proportional to the dynamic viscosity and/or thermal conductivity.

A. Dispersion relationship

The power-law behavior of Eq. (1) is briefly summarized; more details are available in Refs. 23 and 22. Applying a Fourier-Fourier transform to Eq. (1) yields

$$-k^2 + \frac{\omega^2}{c_0^2} - k^2 \tau^\beta (-i\omega)^\beta = 0. \quad (6)$$

Solving for the wavenumber $k(\omega)$ yields

$$k(\omega) = \frac{\omega}{c_0 \sqrt{1 + (-i\tau\omega)^\beta}}. \quad (7)$$

In the low frequency limit, the binomial approximation is applied, yielding the attenuation coefficient $\alpha(\omega)$

$$\alpha(\omega) \rightarrow \text{Im } k(\omega) = \frac{\tau^{y-1} \cos(\pi y/2)}{2c_0} \omega^y, \quad (8)$$

where $y = \beta + 1$. Defining

$$\alpha_0 = \tau^{y-1} |\cos(\pi y/2)| / (2c_0), \quad (9)$$

yields the attenuation constant.^{21,22} Likewise, the phase velocity $c(\omega)$ is computed from the real part of Eq. (7), yielding

$$\frac{1}{c(\omega)} = \frac{\text{Re } k(\omega)}{\omega} = \frac{1}{c_0} + \alpha_0 \tan\left(\frac{\pi y}{2}\right) |\omega|^{y-1}, \quad (10)$$

which is identical to the phase speed predicted by the PLWE given by Eq. (8) in Ref. 11. Also note that Eq. (10) agrees with the phase velocities computed via the Kramers–Krönig relations^{34,35} and is in agreement with experimental data provided in Refs. 34 and 36–38.

B. Nondimensionalized equation

Since the Green’s function calculation in Sec. III hinges on a perturbation analysis, we express Eq. (1) in nondimensional units, allowing a small parameter ϵ to be identified. Assume the source function has the bulk of its acoustical energy concentrated about a characteristic frequency f_0 . Then the timescale of interest is $T = 1/f_0$ and the length scale is $L = c_0 T$. Define the nondimensionalized time as $t' = t/T$ and length as $\mathbf{r}' = \mathbf{r}/L$; applying these definitions to Eq. (1) yields

$$\nabla'^2 \psi - \frac{\partial^2 \psi}{\partial t'^2} + \left(\frac{\tau}{T}\right)^\beta \frac{\partial \psi}{\partial t'^\beta} \nabla'^2 \psi = 0, \quad (11)$$

where ∇'^2 denotes differentiation with respect to $\mathbf{r}'$. Note that we have assumed that the length scales are the same in all three spatial dimensions, which is consistent with the isotropic Caputo wave equation.

Using Eq. (9), a small parameter ϵ is identified as

$$\epsilon = \left(\frac{\tau}{T}\right)^\beta = \frac{2\alpha_0 c_0 f_0^\beta}{\sin(\pi\beta/2)}. \quad (12)$$

Typical values for human tissue (e.g., breast fat) are $\beta = 0.5$ (corresponding to the power-law exponent $y=1.5$), $c_0 = 1500$ m/s, $\alpha_0 = 0.086$ Np/cm/MHz^{1.5}, and $f_0 = 1.0$ MHz, yielding $\epsilon = 0.0365$, implying that the third term in Eq. (11) is small relative to the first two terms, so from a physical point of view, the terms which model propagation are much larger than the third term in Eq. (11) which model attenuation and dispersion (loss). Hence, there is a separation of timescales between wave propagation and attenuation, where propagation occurs on the timescale $T = 1/f_0$, while attenuation occurs on a much slower timescale τ , and the ratio of these timescales to the β power is given by Eq. (12). Note that Eq. (12) has the same scaling with respect to frequency as the smallness parameter given by Eq. (20) in Ref. 9; however, the constant factor in Eq. (12) differs from Szabo’s result.

C. Maximally skewed stable distributions

In this section, maximally skewed stable distributions are reviewed, which are needed to analytically evaluate the

inverse Fourier transforms in Sec. III. Stable distributions may be defined in terms of a Fourier integral³⁹ using the parameters $\theta = (y, q, \sigma, \mu)$ in

$$p_\theta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \exp(i\mu k + \sigma^y \Omega_{y,q}(k)) dk, \quad (13)$$

where

$$\Omega_{y,q}(k) = -|k|^y \left[1 - iq \text{sgn}(k) \tan\left(\frac{\pi y}{2}\right) \right] \quad (14)$$

and where $0 \leq y \leq 2$ is the stability parameter, $-1 \leq q \leq 1$ is the skewness parameter, $\sigma > 0$ is the scaling parameter, and μ is the location parameter. (In the statistics literature, e.g., Ref. 39, the stability parameter is commonly denoted by α and the skewness parameter by β .) Equation (14) is not valid for $y=1$; in this special case, a separate expression exists.³⁹ In general, stable distributions do not have probability density functions expressible in terms of elementary functions, although closed-form expressions exist for special cases of y and q . For example, for $y=2$ and any β , Eq. (13) is a normal density. Reference 11 tabulates special cases for $y = 1/3, 1/2, 2/3$, and $3/2$ with $q = 1$.

Stable distributions enjoy a *scaling property* where

$$p_{(y,q,\sigma,\mu)}(x) = \frac{1}{\sigma^{1/y}} p_{(y,q,1,\mu)}\left(\frac{x}{\sigma^{1/y}}\right). \quad (15)$$

We limit our attention to *maximally skewed stable distributions* where $q = 1$ with a location parameter $\mu = 0$. Since the calculations below require evaluating integrals along the half-axis $[0, \infty)$, we exploit trigonometric identities, yielding

$$p_{(y,1,\sigma,0)}(x) = \frac{1}{\pi} \int_0^\infty \exp(-k^y \sigma^y) \times \cos\left[kx - \tan\left(\frac{\pi y}{2}\right) k^y \sigma^y\right] dk. \quad (16)$$

Finally, for $1 < y \leq 2$ and a specific scaling parameter $\sigma^y = |\cos(\pi y/2)|$, we have a one parameter family of functions denoted by

$$f_y(x) = p_{(y,1,\sigma,0)}(x) = \frac{1}{\pi} \int_0^\infty \exp[k^y \cos(\pi y/2)] \cos\left[kx + \sin\left(\frac{\pi y}{2}\right) k^y\right] dk. \quad (17)$$

Equation (17) is consistent with the maximally skewed stable distribution used in Sec. 3b of Ref. 11. The skewed, stable *cumulative distribution function* (CDF) $F_y(t)$ is defined via

$$F_y(x) = \int_{-\infty}^x f_y(x') dx'. \quad (18)$$

III. GREEN’S FUNCTION CALCULATION

In this section, we calculate Green’s functions for the nondimensionalized Eq. (11) defined by

$$\nabla^2 g - \frac{\partial^2 g}{\partial t^2} + \epsilon \frac{\partial^\beta g}{\partial t^\beta} \nabla^2 g = -\delta(t)\delta(\mathbf{r}'), \quad (19)$$

where the primes have been dropped and all initial conditions are zero. We utilize a complex plane analysis first developed in the Soviet Union to analyze oscillations in a viscoelastic material;³¹ this method was later rediscovered in Ref. 40. Let $G(\mathbf{k}, t)$ denote the spatial Fourier transform of $g(\mathbf{r}, t)$, and let $\hat{G}(\mathbf{k}, s)$ denote the Laplace transform of $G(\mathbf{k}, t)$ with respect to time. The Laplace transform is applied to the time variable to facilitate analysis of the poles of the Fourier–Laplace transform.

First, apply a three dimensional (3D) spatial Fourier transform to Eq. (19), yielding the FDE

$$\frac{\partial^2 G}{\partial t^2} + \epsilon k^2 \frac{\partial^\beta G}{\partial t^\beta} + k^2 G = \delta(t). \quad (20)$$

Equations of this form are used to model oscillations in viscoelastic materials; in particular, Eq. (20) is sometimes termed the fractional Kelvin–Voigt model.¹⁹ An exact series solution may be derived [Eq. (4.25) in Ref. 41]; however, this series involves derivatives of the Mittag–Leffler function $E_{a,b}(z)$. Although inverse Fourier transform relations for $E_{a,b}(z)$ exist, they involve the Fox- H function [see, for example, Eq. (14.7) in Ref. 42]. Since we are not aware of any existing libraries for the numerical evaluation of the Fox- H function, this suggests that the resulting series is not useful from a computational point of view; moreover, the resulting series does not yield any physical insight. Hence, we seek an approximate solution using the complex plane analysis originally developed in Ref. 31 along with a perturbation analysis to locate the poles in the complex plane.

A. Complex plane analysis

Applying a Laplace transform to Eq. (20) yields

$$\hat{G}(k, s) = \frac{1}{w(s; k)}, \quad (21)$$

where the characteristic polynomial $w(s; k)$ is given by

$$w(s; k) = s^2 + \epsilon k^2 s^\beta + k^2. \quad (22)$$

In Ref. 31 and later in Theorem 13 of Ref. 40 it was shown that Eq. (22) has exactly two zeros in the complex s -plane; moreover, these zeros are simple, conjugate, and located in the open left half-plane for $0 < \beta \leq 1$. The Laplace transform is then evaluated by deforming the Bromwich contour around the two zeros, resulting in a residue contribution and an improper integral along the real axis. For details, see Ref. 31. Letting s_+ be the zero in the upper half-plane, Eq. (21) is inverted into the (k, t) domain via the residue theorem in terms of two components $G_1(k, t)$ and $G_2(k, t)$

$$G(k, t) = G_1(k, t) + G_2(k, t), \quad (23)$$

where

$$G_1(k, t) = 2H(t)\text{Re}\left[\frac{e^{s_+t}}{w'(s_+; k)}\right], \quad (24a)$$

$$G_2(k, t) = \frac{1}{\pi} \int_0^\infty H_\beta(r; \epsilon, k) e^{-rt} dr, \quad (24b)$$

$$H_\beta(r; \epsilon, k) = \frac{\epsilon k^2 \sin(\pi\beta) r^\beta}{[r^2 + \epsilon k^2 r^\beta \cos(\pi\beta) + k^2]^2 + [\epsilon k^2 r^\beta \sin(\pi\beta)]^2}, \quad (24c)$$

where Eq. (24c) is the kernel function $H_\beta(r; \epsilon, k)$ [see Eq. (26) in Ref. 40 with $a = \epsilon k^2$, $b = k^2$, and $q = \beta$]. Physically, we see that $G_1(k, t)$ is a damped sinusoid, whereas $G_2(k, t)$ is nonoscillatory, positive, and decreasing as $t \rightarrow \infty$. Note that $H_1(r; \epsilon, k) = 0$, implying that $G_2(k, t)$ vanishes for frequency-squared attenuation.

Moreover, note that $H_\beta(0; \epsilon, k) = 0$; hence, for $t \gg 1$, $H_\beta(0; \epsilon, k) = \mathcal{O}(\epsilon)$ (see Ref. 43). Since we are only interested in the $\epsilon \ll 1$ regime, the $G_2(k, t)$ term is not considered in this analysis. The unit step function in Eq. (24a) arises from the following argument: for $t < 0$, we may close the contour in the upper-half plane. Since there are no poles, the integrand is continuous, and the integral is identically zero.

If $\beta = 1$, the zeros may be found analytically using the quadratic formula. However, for $0 < \beta < 1$, the zeros of Eq. (22) cannot be readily found in closed form. Rather, we seek a perturbation solution⁴⁴ by noting that for $\epsilon = 0$, we have the solution $s_+ = ik$. Assuming that $\epsilon \ll 1$, we seek zeros of the form $s_+ = ik + \epsilon s_1 + \epsilon^2 s_2 \dots$. Inserting this ansatz into the characteristic polynomial yields

$$2iks_1\epsilon + (2iks_2 + s_1^2)\epsilon^2 + k^2\epsilon(ik)^\beta \left(1 + \frac{\epsilon s_1}{ik} + \frac{\epsilon^2 s_2}{ik}\right)^\beta = 0. \quad (25)$$

To proceed, we approximate the final factor using a binomial series expansion $(1+z)^\beta = 1 + \beta z + \dots$, yielding a hierarchy of equations:

$$2iks_1 + k^2(ik)^\beta = 0, \quad (26a)$$

$$2iks_2 = -s_1^2 + \beta(ik)^{\beta+1}s_1. \quad (26b)$$

Solving Eq. (26a) yields the $\mathcal{O}(\epsilon)$ term $s_1 = 1/2(ik)^{\beta+1}$ and the approximation

$$s_+ \approx ik + \frac{\epsilon}{2}(ik)^{\beta+1}. \quad (27)$$

Since Eq. (27) are *approximate* zeros of the characteristic polynomial, an error of order $\mathcal{O}(\epsilon^2)$ is introduced by using this approximation. To evaluate this error, we numerically calculated the exact zeros of Eq. (22) using a wavenumber $k = 2\pi$ and varying ϵ from 0 to 0.1. These “exact” zeros are compared with Eq. (27) in Fig. 1. Additional terms in the perturbation series may be calculated if necessary. For $\beta = 0.5$, the approximate zeros are virtually identical to the exact zeros. For $\beta = 0.14$, there is a small deviation as ϵ approaches 0.1. Nevertheless, the approximate zeros are very close to the “exact” zeros for all parameters examined.

B. Inverse 3D Fourier transform

Continuing with the derivation, Eq. (27) is inserted into Eq. (24a), and approximating $w'(s_+) \approx 2ik$ yields

$$G(k, t) = H(t) \operatorname{Re} \left[\frac{\exp(ikt + \epsilon t(ik)^{\beta+1}/2)}{ik} \right]. \quad (28)$$

Applying Euler's formula to $(ik)^{\beta+1} = -\sin(\beta\pi/2)k^{\beta+1} + i\cos(\beta\pi/2)k^{\beta+1}$ and calculating the real part of the exponential yields

$$G(k, t) = \frac{H(t)}{k} \exp \left[-\frac{\epsilon}{2} t k^{\beta+1} \sin(\pi\beta/2) \right] \times \sin \left[kt + \frac{\epsilon t}{2} k^{\beta+1} \cos(\pi\beta/2) \right]. \quad (29)$$

We are now in a position to invert the Fourier transform using the machinery developed in the Appendix of Ref. 45. The 3D inverse Fourier transform is expressed in spherical coordinates (k, θ_k, ϕ_k) as

$$g(R, t) = \frac{1}{8\pi^3} \int_0^\infty \int_0^\pi \int_0^{2\pi} G(\mathbf{k}, t) \exp(ikR \cos \theta_k) \times \sin \theta_k k^2 d\phi_k d\theta_k dk, \quad (30)$$

where $\theta_k = \cos^{-1}(k_z/k)$ and $\phi_k = \tan^{-1}(k_y/k_x)$. Since $G(\mathbf{k}, t)$ does not have any angular dependence, the integrals with respect to ϕ_k and θ_k are readily evaluated, producing

$$g(R, t) = \frac{1}{2\pi^2 R} \int_0^\infty k \sin(kR) G(k, t) dk. \quad (31)$$

Inserting Eq. (29) into Eq. (31) yields

$$g(r, t) = \frac{H(t)}{2\pi^2 R} \int_0^\infty \exp \left[-\frac{\epsilon t}{2} k^{\beta+1} \sin(\pi\beta/2) \right] \times \sin \left[kt + \frac{\epsilon t}{2} k^{\beta+1} \cos(\pi\beta/2) \right] \sin(kR) dk. \quad (32)$$

Identifying $y = \beta + 1$ yields

$$g(r, t) = \frac{H(t)}{2\pi^2 R} \int_0^\infty \exp \left[\frac{\epsilon t}{2} k^y \cos(\pi y/2) \right] \times \sin \left[kt + \frac{\epsilon t}{2} k^y \sin(\pi y/2) \right] \sin(kR) dk. \quad (33)$$

Applying the sum difference formula for the cosine, along with the scaling property given by Eq. (15), and identifying the maximally skewed distribution given by Eq. (17) yields

$$g(R, t) = \frac{H(t)}{4\pi R} \frac{1}{(\epsilon/2)^{1/y}} \left[f_y \left(\frac{t-R}{(\epsilon/2)^{1/y}} \right) - f_y \left(\frac{t+R}{(\epsilon/2)^{1/y}} \right) \right]. \quad (34)$$

In the quadratic case of $y=2$, Eq. (34) reduces to two Gaussian terms; compare, for instance, to the result in Sec.

FIG. 1. Exact zeros of Eq. (22) using a wavenumber $k = 2\pi$ compared with the approximate formula given by Eq. (27). The small parameter ϵ is varied from 0 to 0.1.

2.10 of Ref. 46 (which contains a typo) or Eq. (A10) in Ref. 45. The first term in Eq. (34) corresponds to an outgoing spherical wave, while the second term corresponds to an incoming spherical wave. The second term, which is not physically relevant, results from approximations made in neglecting the term $G_2(k, t)$ in Eq. (23) and utilizing the approximate zeros in Eq. (27). In addition, the stable density $f_y(z)$ decays as z^{-y-1} as $z \rightarrow \infty$, and $t+R$ is large compared with $t-R$; for these reasons, we discard the incoming term, yielding the approximation

$$g(R, t) = \frac{H(t)}{4\pi R} \frac{1}{(\epsilon/2)^{1/y}} f_y \left[\frac{t-R}{(\epsilon/2)^{1/y}} \right]. \quad (35)$$

Finally, we restore the physical units, utilizing Eq. (9) and identifying $S_y = |\cos(\pi y/2)|$

$$g(R, t) = \frac{H(t)}{4\pi R} \frac{1}{(S_y \alpha_0 c_0 t)^{1/y}} f_y \left(\frac{t-R/c_0}{(S_y \alpha_0 c_0 t)^{1/y}} \right). \quad (36)$$

Equation (36), which is an approximate time-domain Green's function for the Caputo wave equation, is a shifted and scaled maximally skewed stable PDF with stable index y . Equation (36) assumes $\epsilon \ll 1$, implying that this approximate solution is valid for small attenuation constants α_0 and

low frequencies ω . This solution models longitudinal wave propagation in media with a power-law exponent $1 < y \leq 2$. This approximate solution is causal due to the Heaviside unit step function, which arises during the inversion of the Laplace inversion of $\hat{G}(k, s)$.

C. 1D and 2D Green's functions

Although most biomedical applications are concerned with wave propagation and absorption in three dimensions, simplified 1D and 2D models may be useful, for instance, in the verification of a numerical code. Hence, we provide approximate 1D and 2D Green's functions for the Caputo wave equation based on Eq. (36) in this section.

The 3D and 1D Green's functions for isotropic operators (such as the Laplacian) are related via⁴⁷

$$g_{3D}(R, t) = -\frac{1}{2\pi R} \frac{\partial g_{1D}}{\partial x} \Big|_{|x|=R}. \quad (37)$$

Integrating Eq. (37) yields

$$g_{1D}(x, t) = -\int_0^{|x|} 2\pi R g_{3D}(R, t) dR. \quad (38)$$

Identifying the maximally skewed stable CDF $F_y(t)$ defined via Eq. (18) allows Eq. (38) to be evaluated as

$$g_{1D}(x, t) = \frac{c_0 H(t)}{2} F_y \left(\frac{t - |x|/c_0}{(S_y \alpha_0 c_0 t)^{1/y}} \right). \quad (39)$$

As $\alpha_0 \rightarrow 0$, $F_y(z) \rightarrow H(z)$, yielding the 1D Green's function for the lossless wave equation.⁴⁶ Properties of the maximally skewed stable CDF are available in Chap. 2 of Ref. 29. Like the stable PDF, $F_y(z)$ may be numerically evaluated via the STABLE toolbox.⁴⁸

The 2D Green's function is calculated by integrating the 3D result along the z -axis, sometimes known as the *method of descent*.⁴⁹ Letting $R = \sqrt{r^2 + z^2}$ in Eq. (36), where $r = \sqrt{x^2 + y^2}$ is the radial distance to the origin, yields

$$g_{2D}(r, t) = 2 \int_0^\infty g_{3D}(r, z, t) dz. \quad (40)$$

Inserting Eq. (36) into Eq. (40) and making a change of variable yields

$$g_{2D}(r, t) = \frac{H(t)}{2\pi(S_y \alpha_0 c_0 t)^{1/y}} \int_{-\infty}^{t-r/c_0} f_y \left[\frac{v}{(S_y \alpha_0 c_0 t)^{1/y}} \right] q(v) dv, \quad (41)$$

where the "wake function" $q(v)$ is defined via

$$q(v) = 1/\sqrt{(t-v)^2 - (r/c_0)^2}. \quad (42)$$

Unlike the lossless Green's function given by

$$g_{2D}(r, t) = \frac{H(t - r/c_0)}{2\pi\sqrt{t^2 - (r/c_0)^2}}. \quad (43)$$

Eq. (41) is finite at the arrival time $t = r/c_0$ due to the smoothing effect of the stable PDF $f_y(t)$. In addition, Eq. (41) is nonzero for $t < r/c_0$ since $f_y(t)$ has support on $(-\infty, \infty)$ for $y > 1$. Since $f_y(t)$ is a C^∞ , or smooth, function,²⁹ it follows that the 2D Green's function given by Eq. (41) also C^∞ by the smoothing property of convolutions.⁵⁰

IV. NUMERICAL RESULTS

In the following numerical simulation, the maximally skewed stable distribution $f_y(t)$ is evaluated using the STABLE toolbox for MATLAB.⁴⁸ For special values of the power law exponent $y = 3/2$ and 2, $f_y(t)$ exists in closed form; these special cases are tabulated in Ref. 11 and textbooks devoted to stable distributions.²⁹ Alternatively, the acoustic field (e.g., velocity potential) may be evaluated numerically via an inverse Fourier transform.^{30,51}

To determine the validity of the approximate Green's function given by Eq. (36), we plot the smallness parameter ϵ given Eq. (12) as a function of frequency in Fig. 2. The smallness parameter is evaluated for (1) a fat-like medium with $y = 1.5$, $c_0 = 0.1432$ cm/ μ s, and $\alpha_0 = 0.086$ Np/cm/MHz^{1.5} and (2) a liver-like medium with $y = 1.14$, $c_0 = 0.1569$ cm/ μ s, and $\alpha_0 = 0.0459$ Np/cm/MHz^{1.14}. For both sets of parameters, $\epsilon < 0.15$ for frequencies less than 10 MHz; for frequencies beyond 10 MHz, ϵ grows as $\sqrt{f_0}$ for the fat-like medium, whereas the parameter ϵ grows more slowly for the liver-like medium. Note that ϵ becomes unbounded as $y \rightarrow 1$ (or $\beta \rightarrow 0$), indicating that the assumptions behind these approximate Green's functions are violated in the limiting case of linear with frequency attenuation.

To illustrate the similar behavior of the approximate Caputo solution given by Eq. (36) and the solution to the power law wave equation derived in Ref. 11, we evaluate the material impulse response function (MIRF) corresponding to both expressions. The MIRF is the 1D analog of Eq. (36), where the spherical spreading factor $1/(4\pi R)$ is removed. Figure 3 evaluates both expressions for power-law index $y = 1.5$, $c_0 = 1.5$ mm/ μ s, and $\alpha_0 = 0.01151$ Np/MHz/cm. The time-dependent solution is shown for three observation points: (a) 0.135 cm, (b) 0.675 cm, and (c) 1.35 cm. Both the (a) power-law and (b) Caputo solutions are skewed toward late times and have slowly decaying tails. There is a small deviation between the two solutions at $z = 0.135$ cm near the peak value; as the observation point z increases, the two solutions appear to converge. As the three panels illustrate, the two solutions are nearly identical, suggesting the near equivalence of the two models. Finally, both the power law and Caputo solutions may be compared to Fig. 1 in Ref. 51, where the material impulse response function is evaluated for the same parameters using an inverse fast Fourier transform.

V. DISCUSSION

A. Green's function decomposition and fractional diffusion

In the special case of frequency-squared attenuation ($y = 2$), Eq. (36) reduces to an approximate solution to the Stokes wave equation that was derived in Ref. 52. In Ref. 45,

FIG. 2. The smallness parameter ϵ given Eq. (12) plotted as a function of frequency for (1) a fat-like medium with $y = 1.5$, $c_0 = 0.1432$ cm/ μ s, and $\alpha_0 = 0.086$ Np/cm/MHz^{1.5} and (2) a liver-like medium with $y = 1.14$, $c_0 = 0.1569$ cm/ μ s, and $\alpha_0 = 0.0459$ Np/cm/MHz^{1.14}.

this Green's function for the Stokes wave equation was decomposed into a loss function $g_L(t, t')$ which accounts for the frequency-dependent loss and a diffraction function $g_D(R, t)$ which accounts for the propagation and spreading due to diffraction. The loss function $g_L(t, t')$ was then identified as a solution to the diffusion equation. In this section, we extend this analysis to the approximate Green's function for the Caputo wave equation given by Eq. (36).

Equation (36) admits the following decomposition:

$$g(R, t) = \int_{-\infty}^{\infty} \left[\frac{\delta(t - t' - R/c_0)}{4\pi R} \right] \times \left[\frac{H(t)}{(S_y \alpha_0 c_0 t)^{1/y} f_y \left(\frac{t'}{(S_y \alpha_0 c_0 t)^{1/y}} \right)} \right] dt'. \quad (44)$$

Thus, the Caputo Green's function is given by the nonstationary convolution

$$g(R, t) = g_L(t, t') \otimes g_D(R, t'), \quad (45)$$

where

$$g_L(t, t') = \frac{H(t)}{(S_y \alpha_0 c_0 t)^{1/y} f_y \left(\frac{t'}{(S_y \alpha_0 c_0 t)^{1/y}} \right)}, \quad (46a)$$

$$g_D(R, t') = \frac{\delta(t' - R/c_0)}{4\pi R}. \quad (46b)$$

Note that $g_D(R, t)$ is the 3D Green's function to the lossless wave equation given by Eq. (1) with $\tau = 0$, while Eq. (46a) is a solution to the space-fractional diffusion equation⁵³

$$\left(\frac{\partial}{\partial t} - S_y \alpha_0 c_0 \frac{\partial^y}{\partial t^y} \right) g_L(t, t') = 0, \quad (47)$$

where the operator $\partial^y / \partial t^y$ is a Riemann-Liouville fractional derivative defined by Eq. (3). In the viscous case

FIG. 3. The 1D material impulse response (MIRF) for the power-law wave equation and the Caputo wave equation. These solutions are evaluated for power-law index $y = 1.5$, $c_0 = 1.5$ mm/ μ s, and $\alpha_0 = 0.01151$ Np/MHz^{1.5}/cm at three observation points: (a) 0.135 cm, (b) 0.675 cm, and (c) 1.35 cm.

($y = 2$), Eq. (47) reduces to the diffusion equation. Analogous to the Stokes wave equation, Eq. (45) consists of a “fast” timescale t' over which wave propagation and diffraction occurs, and a “slow” timescale t over which dissipation and dispersion occurs. Thus, Eq. (45) is interpreted as a coupled wave/fractional diffusion system. Also, note that Eq. (46a) is causal on account of the Heaviside step function $H(t)$. In the limit of $y \rightarrow 2$, the classical diffusion equation is recovered.

B. Qualitative properties of the approximate Green's function

Using properties of maximally skewed stable PDFs, we can infer physically relevant properties from Eq. (36). First, $f_y(t)$ has support on the real line, implying that for a fixed R , $g(R, t) > 0$. Physically, there is infinite propagation speed of disturbances, similar to the Stokes wave equation⁵² and the diffusion equation;⁴⁹ this behavior agrees with Eq. (10), which predicts that the phase velocity $c(\omega) \rightarrow \infty$ as $\omega \rightarrow \infty$. Second, $f_y(t)$ decays faster than an exponential as $t \rightarrow \infty$, implying that any infinitely fast disturbance is vanishingly small. This behavior is also reflected by Eq. (8), which predicts that the high-frequency components traveling much faster than c_0 also experience very large attenuation. Third, $f_y(t) \rightarrow t^{-y-1}$ as $t \rightarrow \infty$. That is, the maximally skewed stable PDF has a *heavy-tail*, indicating that Eq. (36) has a power-law wake. All three of these properties are shared by the PLWE Green's function reported in Ref. 11. Unlike the PLWE Green's function which scales with respect to $R^{1/y}$, Eq. (36) scales with respect to $t^{1/y}$ and is causal due to the Heaviside function.

C. Comparison to the PLWE Green's function

The PLWE Green's function previously derived in Ref. 11 may be recovered as an approximation to the Caputo Green's function given by Eq. (36). Examining Eqs. (45) and (46a), observe that the wave attenuation scales with respect to the quantity $c_0 t$. Near the peak of the Green's function, $c_0 t \approx R$. Applying this approximation to the loss function yields $g_L(c_0 R, t')$; evaluating the integral in Eq. (46) yields

$$g(R, t) \approx \frac{1}{4\pi R (S_y \alpha_0 R)^{1/y}} f_y \left[\frac{t - R/c_0}{(S_y \alpha_0 R)^{1/y}} \right], \quad (48)$$

which corresponds to the previously derived power law Green's function derived in Ref. 11. Note that the Heaviside step function in Eq. (46a) evaluates to one since $c_0 R > 0$ for all observation points removed from the source. Since $f_y(t) > 0$ for all t for $y > 1$ (see Ref. 54), Eq. (48) is nonzero for $t \leq 0$. Hence, the time-domain Green's function for the power law wave equation is a noncausal approximation to the time-domain Green's function for the Caputo wave equation.

D. Numerical evaluations of approximate Green's functions

Numerical evaluations of time-domain Green's functions that describe power law attenuation are readily evaluated when expressed in terms of a stable distribution, which enables convenient calculations with the STABLE toolbox⁴⁸ or MATLAB's Statistics and Machine Learning Toolbox (R2016a).⁵⁵ We previously derived one such analytical Green's function for the power law wave equation,¹¹ and this expression is also an approximate analytical Green's function for the Szabo wave equation.³⁰ Thus, exact or approximate analytical Green's functions have been found

for the power law, Szabo, and Caputo wave equations, which all contain time-fractional derivatives. An analytical time-domain Green's function for the space-fractional wave equation that describes power law attenuation and dispersion¹² has not yet been found, so numerical evaluation of the time-domain Green's function, which requires additional numerical integration or other numerical calculations, is presently much more complicated for space-fractional wave equations.

E. Smallness parameter

The expression for the smallness parameter in Eq. (12) contains several terms that also appear in an expression given in Appendix B of Ref. 14, specifically the expression $2\alpha_0 c_0 / \cos(\pi y/2) \ll 1$, which is obtained from Eq. (5) and Eq. (B.3) from Ref. 14. Note that the units on the left hand side of the expression from Ref. 14 are $\text{Np} \cdot \text{s}^{y-1}$, but the expression on the left hand side should be unitless for the comparison with 1 on the right hand side. This problem is solved when the expression on the left hand side is multiplied by the frequency in Hz to the $y - 1$ power. For comparisons with unity, we prefer to multiply the relaxation time with the frequency in Hz, i.e., $f\tau \ll 1$, yielding $2\alpha_0 c_0 / \cos(\pi y/2) f^{y-1} \ll 1$, which solves the problem with the units and is also consistent with the smallness parameter defined in Eq. (12) of the present work.

VI. CONCLUSIONS

An approximate time-domain Green's function for the 3D Caputo wave equation is derived by reducing the Caputo wave equation to a fractional ordinary differential equation, and then applying a complex plane analysis developed in Refs. 31 and 40. This Green's function consists of a shifted and scaled maximally skewed stable distribution multiplied by a spherical spreading factor $1/(4\pi R)$. Unlike the previously derived power-law Green's function,¹¹ which is only causal for $0 \leq y < 1$, this approximate Green's function for the Caputo wave equation is causal for $y > 1$. Approximate 1D and 2D time-domain Green's functions for the Caputo wave equation are also constructed. Numerical evaluation of the causal Green's function reveals that it is virtually identical to the noncausal power-law Green's function derived in Ref. 11. The Green's function is decomposed into a loss component and a diffraction component, revealing that the Caputo wave equation may be approximated via a coupled lossless wave equation and a fractional diffusion equation. Finally, the noncausal power-law Green's function derived in Ref. 11 is recovered as an approximation to the causal Caputo Green's function.

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