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On ballistic deposition process on a strip

Toufik Mansour^{*}, Reza Rastegar[†], Alexander Roitershtein[‡]

^{*}Department of Mathematics, University of Haifa, 199 Abba Khoushy Ave, 3498838 Haifa, Israel;

[†]Occidental Petroleum Corporation, Houston, TX 77046 and Departments of Mathematics and Petroleum Engineering, University of Tulsa, OK 74104, USA - Adjunct Professor;

[‡]Department of Statistics, Texas A&M University, College Station, TX 77843, USA;

Abstract

We revisit the model of the ballistic deposition studied in [5] and prove several combinatorial properties of the random tree structure formed by the underlying stochastic process. Our results include limit theorems for the number of roots and the empirical average of the distance between two successive roots of the underlying tree-like structure as well as certain intricate moments calculations.

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1 Introduction

Packing models arise in a variety of applied fields, including microscopic processes in physics, chemistry, and biology, and macroscopic ecological and sociological systems. One of the first proposed classes of packing processes are random sequential adsorption (RSA) models describing a process of deposition of thin disks (segments) placed at random one after another on a surface. When an attempt to deposit a new segment would result in an overlap with previously deposited one, this attempt is rejected. In statistical mechanics and biology models of this type are fundamental to the description of the irreversible deposition of macro-molecules, colloidal particles, viruses, polymer particles, and bacteria onto a surface. The model goes back to [39, 46], see, for instance, [10, 18, 44, 52, 56] for a review of classical results and [8, 9, 11, 12, 13, 30] and references therein for some very recent progress.

The RSA packing model is generalized to the ballistic deposition (BD) processes where, in contrast to the RSA model, the segments are “thick” and they do not get rejected but stick to the first point of contact, which might be either the surface or other segments [6, 20, 44, 48, 49]. Thus the shape formed by the deposited particles not only expands on the surface but also grows vertically as a complex multilayered conglomerate. Similarly to the RSA, there is a vast literature concerned with various versions of the basic deposition processes on continuum and lattice substrates, most of it is a numerical simulation study. The BD models date back to [57] and [54], where a variation of the model was proposed to describe sedimentation and aggregation in colloids. Models of this type have been applied to study formation, morphology, and surface roughness of sedimentary rocks [22] and thin films [21, 34].

We remark that a random deposition model, motivated by a cooperative sequential adsorption (CSA) [18, 45] rather than RSA, has been recently considered in [16, 35, 51], see also a related ballistic deposition model proposed in [3]. Arguably, one of the most fascinating features of the BD models is that they are believed to belong to the Kardar-Parisi-Zhang (KPZ) universality class [1, 23, 29, 32, 36, 53], see also [28, 31, 33, 48, 38] and references therein for some recent work in this direction for various types of BP models. For a general class of BD models [42, 44, 43, 50] established the existence of an asymptotic growth speed, thermodynamic limits, and asymptotically Gaussian fluctuations for the height and surface width of the random interface formed by the deposited particles. The main difficulty in the analysis of BD models is that local interactions of a deposited particle within a neighborhood of its projected location on the surface propagate into long-range spatial correlations and non-Markovian evolution of the model [15]. The $(1 + 1)$ -dimensional deposition process we consider in this paper was studied in [5] as an analytically tractable variation of the diffusion limited aggregation model (DLA). For a compact review of DLA models in mathematical literature, we refer the reader to the recent article [4].

We next describe the model that we are concerned with in this paper. For $K \in \mathbb{Z}$, let $[K]$ denote $\{1, \dots, K\}$ if $K \in \mathbb{N}$, and an empty set otherwise. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Informally speaking, we consider the x -axis in an $\mathbb{R}^2$ plane, at each instance of time $n \in \mathbb{N}_0$ we choose one site k on the lattice substrate $[K]$, independently of the history and uniformly over $[K]$, and drop a solid rectangular particle of length 1 and height 1 vertically from above, with its center aiming at k . The particle will instantly fall down and stops upon touching the axis or a particle previously deposited within the neighbour set $I(k)$ which is defined as follows:

$$I(k) = \begin{cases} \{K, 1, 2\} & \text{if } k = 1 \\ \{k - 1, k, k + 1\} & \text{if } 2 \leq k \leq K - 1, \\ \{K - 1, K, 1\} & \text{if } k = K. \end{cases} \quad (1)$$

See Figure 1 for a graphical example.

More precisely, let $(X_n)_{n \in \mathbb{N}_0}$ be a sequence of i. i. d. random variables sampled uniformly from $[K]$. Let $(H_n)_{n \in \mathbb{N}_0}$ be a sequence of random functions $H_n: [K] \rightarrow \mathbb{N}_0$ representing the height of the deposited structure at each location at time n . Formally, set

$$H_0(k) = 0 \quad \forall k \in [K], \quad (2)$$

and consider a Markov chain $H_n = (H_n(1), \dots, H_n(K))$ of vectors in the state space $\mathbb{N}_0^K$, defined recursively as follows:

$$H_n(k) = \begin{cases} H_{n-1}(k) & \text{if } X_n \neq k \\ \max_{j \in I(k)} H_{n-1}(j) + 1 & \text{otherwise,} \end{cases} \quad (3)$$

where $k \in [K]$ and the sets $I(k)$ are introduced in (1). We refer to the Markov chain $(X_n, H_n)_{n \in \mathbb{N}_0}$ as a *ballistic deposition on a strip*. Note that the cyclic rule (1) effectively turns $[K]$ into a K -dimensional discrete torus in which K is identified with zero.

Figure 2 shows the outcome of 40,000 iterations of this process simulated numerically for a strip with $K = 500$. A random number of tree-like structures (connected components) grow and merge through the process. We refer to these structures as *trees* even though they are not trees in a classical sense. Through several coupling arguments, a lower and an upper bounds for $\max_{k \in [K]} H_n(k)$ are calculated in [5]. Our simulations warrant

Conjecture. *With probability one, for all $j \in \mathbb{N}$,*

$$\lim_{n \rightarrow \infty} \frac{H_n(j)}{n} = \lim_{n \rightarrow \infty} \left(\max_{k \in [K]} \frac{H_n(k)}{n} \right) \sim \frac{4}{K},$$

where the notation $a_K \sim b_K$ stands for $\lim_{K \rightarrow \infty} \frac{a_K}{b_K} = 1$.

The goal of this paper is to study the configuration of particles deposited directly on the surface, i. e. *roots* of the trees formed by the deposited particles. More precisely, we focus on the probability distributions of the number of the particles eventually located on the surface (Section 2) and distances between them (Section 3). This information can serve as a basis for future investigation of the process as an evolving in time conglomerate of trees. Though problems of this type were intensively investigated for RSA models, to the best of our knowledge there is no previous work addressing the issue in the context of ballistic depositions. In terms of the principle object of study (but not the methods), the closest to our line of inquiry work that we are aware of is [55], where the formation of the first layer is studied for a significantly different “ballistic deposition with restructuring” model. A monolayer ballistic deposition model on a 2-dimensional continuum is considered in [41].

The main results of this paper are stated in Theorem 2.1 (exact moments, weak law of large numbers, and a CLT for the number of roots), Theorem 3.1 (limit theorem for the empirical

average of a gap between two successive roots), and Theorems 3.3 and 3.5 (exact moments for the distribution of the number of gaps of a given length between two successive roots) together with weak laws of large numbers implied by the latter (Corollaries 3.4 and 3.6). See also Remark 2.2 concerning large deviation estimates, Berry-Essen Bounds, and a local CLT accompanying the CLT obtained in Theorem 2.1 as well as a conjecture regarding a CLT for the number of gaps of a given length and their joint distribution stated at Section 3.5.

Our proofs rely on the analysis of recursive equations for underlying generating functions. Most of our moment calculations are exact rather than asymptotic. Some of the calculations are computationally intensive, we believe that the method developed in Section 3 in order to handle the computational complexity may be of independent interest.

2 Number of roots

In this section we study the distribution of the number of particles located directly on the surface. We refer to particles located on the surface as roots. The set of locations of the roots at time n is defined as

$$\mathcal{R}_n^{[K]} = \{k \in [K] : H_i(k) = 1 \text{ for some } i \leq n\}.$$

We denote by $\mathcal{R}^{[K]}$ the set of all roots eventually formed by the deposition process. That is,

$$\mathcal{R}^{[K]} = \lim_{n \rightarrow \infty} \mathcal{R}_n^{[K]} = \{k \in [K] : H_i(k) = 1 \text{ at some time } i \in \mathbb{N}\}. \quad (4)$$

The convergence of the sequence $\mathcal{R}_n^{[K]}$ to $\mathcal{R}^{[K]}$ is granted because the sequence is formed by non-decreasing subsets a finite set $[K]$.

In this paper we are concerned with $R^{[K]} = \text{Card}(\mathcal{R}^{[K]})$. The evolution of the sequence $R_n^{[K]} = \text{Card}(\mathcal{R}_n^{[K]})$ will be studied by the authors in more detail elsewhere. Figure 3 shows the empirical distribution of $R^{[K]}$ obtained in simulations for $K = 100, 300, 500$, and 1500 .

The simulations suggest that the random variable $R^{[K]}$ is asymptotically normal as K approaches infinity. The corresponding formal statement is the content of the following theorem. Heuristically, one can expect a CLT to hold because $R^{[K]} = \sum_{i=1}^K \eta_i^{(K)}$, where $\eta_i^{(K)} = \mathbf{1}_{\{i \text{ is a root}\}}$ is the indicator of the event $\{i \in \mathcal{R}^{[K]}\}$, and the random variables $(\eta_i^{(K)})_{K \in \mathbb{N}, i \in [K]}$ form a uniformly mixing triangular array, the middle bulk of which is stationary asymptotically. Here and henceforth, $\mathbf{1}_A$ stands for the indicator function of the event A . For generic examples of limit theorems for triangular arrays of this type see [14, 37, 40]. Though our proof doesn't use any of limit theorems for mixing arrays, and our asymptotic analysis of the characteristic function of $R^{[K]}$ relies on recursions obtained through the use of underlying combinatorial structure rather than on a direct exploiting of mixing properties, the weak dependence and approximate stationarity of the indicators

seems to be a good intuitive way to understand the asymptotic normality of a properly normalized sequence $R^{[K]}$ (cf. [7]).

For a random variable X , we denote its mean and variance by, respectively, $E(X)$ and $\sigma^2(X)$. In addition, we denote by $\mathcal{L}(X)$ its probability distribution. We use the notation $\lim_{K \rightarrow \infty} \mathcal{L}(X_K) = Y$ to indicate the convergence in distribution of a sequence of random variables X_K , $K \in \mathbb{N}$, to a random variable Y , as K tends to infinity.

Theorem 2.1. *The following holds true for $R^{[K]}$:*

- i. $E(R^{[K]}) = \frac{K}{3}$ for all $K \geq 3$.
- ii. $\sigma^2(R^{[3]}) = 0$, $\sigma^2(R^{[4]}) = \frac{1}{9}$, $\sigma^2(R^{[5]}) = \frac{2}{9}$, $\sigma^2(R^{[6]}) = \frac{4}{15}$, and

$$\sigma^2(R^{[K]}) = \frac{2K}{45} \quad \forall K \geq 7.$$

- iii. Let $\tilde{R}^{[K]} = \frac{R^{[K]} - E(R^{[K]})}{\sigma(R^{[K]})}$. Then

$$\lim_{K \rightarrow \infty} \mathcal{L}(\tilde{R}^{[K]}) = \mathcal{N}(0, 1),$$

where $\mathcal{N}(0, 1)$ is a standard normal random variable.

Remark 2.2. *In order to prove the limit theorem for $R^{[K]}$ we employ a version of Hwang's general CLT (quasi-power theorem) [25]. We refer an interested reader to Section IX in [19] for a comprehensive account of the quasi-power theorem and its history. In fact, general results available in [25, 26] can be used to obtain more detailed information about the limiting behavior of $R^{[K]}$ than it is given in part (ii) of Theorem 2.1. More specifically, it is not hard to verify that our key estimate given in (14) implies that $R^{[K]}$ satisfies the conditions of both Theorem 1 in [25] and Theorem 1 in [26]. An application of these results yields large deviation estimates, local central limit theorem, and Berry-Essen type estimates for the distribution of $R^{[K]}$. In particular, it turns out that the rate of convergence to the normal distribution in part (iii) of Theorem 2.1 is of order $n^{-1/2}$. We omit the details, and instead refer the reader to the statement of the results in [25, 26]. Hwang's theory produces the asymptotic form of $E(R^{[K]})$ and $\sigma^2(R^{[K]})$ as a byproduct. Therefore, the proof of the limit theorem in part (iii) is in fact independent from the computation in parts (i) and (ii). The latter are included because they give the exact values of the expectation and variance, and hence may be of independent interest.*

Proof of Theorem 2.1.

(i) Consider a slight modification of the underlying process H_n which is formally obtained by replacing the definition of $I(k)$ in (1) with

$$I(k) = \begin{cases} \{0, 1, 2\} & \text{if } k = 1 \\ \{k-1, k, k+1\} & \text{if } 2 \leq k \leq K-1, \\ \{K-1, K, K+1\} & \text{if } k = K. \end{cases} \quad (5)$$

and the initial condition in (2) by the following one:

$$H_0(0) = H_0(K+1) = 1 \quad \text{and} \quad H_0(k) = 0 \quad \text{for } k \in [K]. \quad (6)$$

Thus the ballistic deposition in the auxiliary process occurs on the same lattice substrate $[K]$ and according to the same rule (3) as in the original one, with the only two exceptions being that (i) two particles are placed before the process starts at the external boundary $\{0, K+1\}$, and (ii) by virtue of (5), the surface represented by the interval $[K]$ is not anymore cyclic, cf. (1). Note that according to our definition, similarly to the original process, particles in the auxiliary one are never deposited outside of the interval $[K]$ after time zero. Informally, the auxiliary process on the substrate $[K]$ (ignoring the initial particles at 0 and $K+1$) coincides with the original cyclic one, observed on the substrate $[K+1]$ *after* the arrival of the first particle (ignoring the first particle).

Let R_K be the limiting number of roots, i. e. the analogue of $R^{[K]}$ in (4), in the auxiliary process. Observe that (now counting two initial particles in the auxiliary process and the first particle in the original one)

$$\mathcal{L}(R^{[K]}) = \mathcal{L}(R_{K-1} + 1),$$

and hence it suffices to analyze R_K . The first-step decomposition of R_K translates into the following distributional recursion:

$$\begin{aligned} \mathcal{L}(R_K) = & \mathcal{L}\left(\mathbf{1}_{\{Y_K=1\}} R_{K-1}^{(1)} + \sum_{j=2}^{K-1} \mathbf{1}_{\{Y_K=j\}} (R_{j-1}^{(1,j)} + R_{K-j}^{(2,j)} + 1) \right. \\ & \left. + \mathbf{1}_{\{Y_K=K\}} R_{K-1}^{(K)}\right), \end{aligned} \quad (7)$$

where

- $R_0 = R_1 = R_2 = 0$,
- Y_K is the location on the surface of the first particle,
- $\mathcal{L}(R_k) = \mathcal{L}(R_k^{(1)}) = \mathcal{L}(R_k^{(1,j)}) = \mathcal{L}(R_k^{(2,j)}) = \mathcal{L}(R_k^{(K)})$ for all $k \in [K]$,
- $R_K, R_k^{(1)}, R_k^{(1,j)}, R_m^{(2,j)}, R_k^{(K)}$ and Y_K are independent of each other for all values of the arguments k, j, m , and K .

Let $L_K(z) = E\left(z^{R_K}\right)$, $z \in \mathbb{C}$, be the generating function of R_K with the domain in the complex plane. Note that $L_0(z) = L_1(z) = L_2(z) = 1$ and $L_3(z) = \frac{1}{3}(2+z)$. Since $R_K \leq K$, the generating function is well defined and analytic in $\mathbb{C}$. In particular, due to the initial condition (6), $L_K(0) = P(R_K = 0) = \frac{2^{K-3}}{K!}$. It follows from (7) that for $K \geq 3$,

$$\begin{aligned} L_K(z) &= E\left(z^{R_K}\right) = \frac{1}{K} \left(E\left(z^{R_{K-1}}\right) + E\left(z^{R_{K-1}}\right) + z \sum_{j=2}^{K-1} E\left(z^{R_{j-1}}\right) E\left(z^{R_{K-j}}\right) \right) \\ &= \frac{1}{K} \left(2L_{K-1}(z) + z \sum_{j=2}^{K-1} L_{j-1}(z) L_{K-j}(z) \right). \end{aligned} \quad (8)$$

In order to calculate the first moment of R_K , we take the derivative at $z = 1$ on both sides of the identity in (8), and obtain

$$E(R_K) = \frac{1}{K} \left(2E(R_{K-1}) + (K-2) + \sum_{j=2}^{K-1} [E(R_{j-1}) + E(R_{K-j})] \right).$$

Therefore, since $E(R_0) = E(R_1) = E(R_2) = 0$, for $k \geq 3$ we get

$$KE(R_K) = 2 \sum_{j=3}^{K-1} E(R_j) + (K-2).$$

Subtracting from this identity

$$(K-1)E(R_{K-1}) = 2 \sum_{j=3}^{K-2} E(R_j) + (K-3)$$

and solving the resulting first-order linear recursion

$$KE(R_K) = (K+1)E(R_{K-1}) + 1$$

with the boundary condition $E(R_2) = 0$, we obtain that for $K \geq 3$,

$$E(R_K) = \frac{K-2}{3} \quad \text{and} \quad E(R^{[K]}) = E(R_{K-1}) + 1 = \frac{K}{3}. \quad (9)$$

(ii) Similarly, to calculate the second moment of R_K and $R^{[K]}$, we take the second derivative at $z = 1$ in both sides of (8), and obtain

$$K\{E(R_K^2) - E(R_K)\} = 2\{E(R_{K-1}^2) - E(R_{K-1})\} + 2 \sum_{j=1}^{K-2} \{E(R_j^2) - E(R_j)\} + 4 \sum_{j=1}^{K-2} E(R_j) \\ + 2 \sum_{j=1}^{K-2} E(R_j)E(R_{K-j-1}).$$

Therefore, for $K \geq 3$,

$$KE(R_K^2) = 2 \sum_{j=1}^{K-1} E(R_j^2) + 2 \sum_{j=1}^{K-2} E(R_j) \{E(R_{K-j-1}) + 1\} + \frac{K^2 - 4K + 6}{3}. \quad (10)$$

It is easy to check directly that $E(R_3^2) = \frac{1}{3}$ and $E(R_4^2) = \frac{2}{3}$. Then, using (10) we get $E(R_5^2) = \frac{19}{15}$.

It follows from (10) that for $K \geq 6$,

$$KE(R_K^2) = 2 \sum_{j=1}^{K-1} E(R_j^2) + 2 \sum_{j=3}^{K-3} \left(\frac{j-2}{3} \cdot \frac{K-j}{3} \right) + \frac{K^2 - 2K - 2}{3}. \quad (11)$$

In particular, $E(R_6^2) = \frac{188}{90}$. For $K \geq 7$, subtracting from (11) the identity

$$(K-1)E(R_{K-1}^2) = 2 \sum_{j=1}^{K-2} E(R_j^2) + 2 \sum_{j=3}^{K-4} \left(\frac{j-2}{3} \cdot \frac{K-j-1}{3} \right) + \frac{K^2 - 4K + 1}{3}$$

yields the first-order linear recursion

$$KE(R_K^2) = (K+1)E(R_{K-1}^2) + \frac{K^2 + K - 9}{9}, \quad K \geq 7,$$

with the boundary condition $E(R_6^2) = \frac{188}{90}$. Let $E(R_K^2) = \alpha_K(K+1)$. Then $\alpha_6 = \frac{188}{630}$, and for $K \geq 7$ we get

$$\alpha_K = \alpha_{K-1} + \frac{K^2 + K - 9}{9K(K+1)} = \alpha_{K-1} + \frac{1}{9} - \left(\frac{1}{K} - \frac{1}{K+1} \right).$$

Iterating and taking in account that $\alpha_6 = \frac{188}{630}$, we obtain

$$\alpha_K = \frac{188}{630} + \frac{K-6}{9} - \left(\frac{1}{7} - \frac{1}{K+1} \right),$$

and hence

$$\sigma^2(R^{[K+1]}) = \sigma^2(R_K) = \alpha_{K+1}^2(K+1)^2 - \frac{(K-2)^2}{9} = \frac{2}{45}(K+1),$$

as desired.

(iii) To show that the CLT holds for $R^{[K]}$ we will verify that the conditions of Hwang's quasi-power theorem hold for $L_K(z)$ (the version of this general combinatorial CLT given in Theorem IX.8 of [19] will be sufficient for the purpose). Toward this end, consider the generating function

$$L(x, z) = \sum_{K=1}^{\infty} L_K(z)x^K$$

with the domain in $\mathbb{C}^2$. Since $R_K \leq K$, the function is well defined at least for all $(x, z) \in \mathbb{C}^2$ such that $|x| < \min\{1, |z|^{-1}\}$. We will be interested in the behavior of $L(x, z)$ in an open $\mathbb{C}^2$ -neighborhood of $(x, z) = (0, 1)$ where $L(x, z)$ is well defined and analytic as a function of x for each fixed z .

Substituting (8) into the definition of $L(x, z)$ gives

$$L(x, z) = x + x^2 + \sum_{K=3}^{\infty} \frac{1}{K} \left(2L_{K-1}(z) + z \sum_{j=2}^{K-1} L_{j-1}(z)L_{K-j}(z) \right) x^K.$$

Taking the partial derivative with respect to x on both sides, we obtain the following inhomogeneous Riccati equation:

$$\begin{aligned} \frac{\partial L(x, z)}{\partial x} &= 1 + 2x + \sum_{K=3}^{\infty} \left(2L_{K-1}(z) + z \sum_{j=2}^{K-1} L_{j-1}(z)L_{K-j}(z) \right) x^{K-1} = 1 + 2 \sum_{K=1}^{\infty} L_K(z)x^K \\ &+ z \left(\sum_{j=1}^{\infty} L_j(z)x^j \right) \left(\sum_{K=j+2}^{\infty} L_{K-j-1}(z)x^{K-j-1} \right) = 1 + 2L(x, z) + zL(x, z)^2 \end{aligned}$$

with the initial condition $L(0, z) = 0$. The solution $L(x, z)$ in a neighborhood of $(x, z) = (0, 1)$ is given by [24]

$$L(x, z) = \begin{cases} \frac{\tanh(x\sqrt{1-z})}{\sqrt{1-z} - \tanh(x\sqrt{1-z})} & \text{if } z \neq 1, \\ \frac{x}{1-x} & \text{if } z = 1. \end{cases}$$

Since the singularity at $z = 1$ is removable, we will simply write

$$L(x, z) = \frac{\tanh(x\sqrt{1-z})}{\sqrt{1-z} - \tanh(x\sqrt{1-z})} = \frac{\tanh(x\sqrt{z-1})}{\sqrt{z-1} - \tanh(x\sqrt{z-1})} =: \frac{Q(x, z)}{P(x, z)}. \quad (12)$$

The poles of $L(x, z)$ for $z \neq 1$ are in the form $\rho_m(z) = \frac{\tan^{-1}(\sqrt{z-1}) + \pi m}{\sqrt{z-1}}$, $m \in \mathbb{Z}$. Since

$$\lim_{z \rightarrow 1} \frac{\tan^{-1}(\sqrt{z-1})}{\sqrt{z-1}} = 1,$$

there exists a complex punctured neighborhood U of 1 and a real number $r > 1$ such that $|\rho_0(z)| < r$ and $|\rho_m(z)| > r$ for all $m \in \mathbb{N}$ and $z \in U$.

By the residue theorem, for $z \in U$ we have

$$\frac{1}{2\pi i} \int_{|x|=r} L(x, z) \frac{dx}{x^{K+1}} = \text{Res}_0(L(x, z)x^{-K-1}) + \text{Res}_{\rho_0(z)}(L(x, z)x^{-K-1}).$$

Since $\text{Res}_0(L(x, z)x^{-K-1}) = L_K(z)$, we get

$$L_K(z) = -\text{Res}_{\rho_0(z)}(L(x, z)x^{-K-1}) + \frac{1}{2\pi i} \int_{|x|=r} L(x, z) \frac{dx}{x^{K+1}}. \quad (13)$$

By virtue of (12),

$$\text{Res}_{\rho_0(z)}(L(x, z)x^{-K-1}) = \lim_{x \rightarrow \rho_0(z)} (x - \rho_0(z))L(x, z)x^{-K-1} = \lim_{x \rightarrow \rho_0(z)} \frac{Q(x, z)x^{-K-1}}{\frac{\partial P(x, z)}{\partial x}} = -\frac{\rho_0(z)^{-K-1}}{z},$$

where in order to compute the partial derivative in the denominator we used the fact that $\tan(\rho_0(z)\sqrt{z-1}) = \sqrt{z-1}$. Since $L(x, z)$ is continuous and therefore bounded on the closure of $\{(x, z) \in \mathbb{C}^2 : |x| = r, z \in U\}$, we obtain from (13) that there exists a function g on U such that

$$L_K(z) = \frac{\rho_0(z)^{-K-1}}{z} + g(z)r^{-K}, \quad \text{where } \sup_{z \in U} |g(z)| < \infty. \quad (14)$$

It is now a simple routine to verify that the conditions of the quasi-power theorem (see Theorem IX.8 in [19]) are satisfied for $R^{[K]}$. The quasi-power theorem implies the CLT, and hence the proof of part (iii) of the theorem is complete. $\square$

3 Distance between two adjacent roots

In this section we investigate the random vector compound of distances between adjacent roots in the set $\mathcal{R}^{[K]}$. Our main results here are stated in Theorems 3.1, 3.3 and 3.3 below, see also a simulation-supported conjecture which is formulated in Section 3.5.

Let $r_1 < \dots < r_{R^{[K]}}$ be the ordered locations of the roots. Let

$$D_K = (D_{1,K}, D_{2,K}, \dots, D_{K-1,K})$$

be a $(K-1)$ -vector whose i -th component $D_{i,K}$ counts the number of pairs of consecutive roots with the distance between them equal to i . That is,

$$D_{i,K} = \sum_{j=1}^{R^{[K]}-1} \mathbf{1}_{\{r_{j+1} - r_j = i+1\}} + \mathbf{1}_{\{r_{R^{[K]}} - r_1 = K-i-1\}}.$$

In what follows we focus on the study of D_K . The section is divided into subsections as follows. Section 3.1 is devoted to a discussion of the asymptotic behavior of certain “mean-field” and empiric averages of $D_{i,K}$. A central limit theorem for the empiric average is derived as a corollary to Theorem 2.1, the result is stated in Theorem 3.1. In Section 3.2 we are concerned with first moments of the random variables $D_{i,K}$. In order to compute the moments we implement an approach similar to the one we used in the previous section in order to analyze $R^{[K]}$. The recursive equations that we obtain in this section are considerably more complex, and we believe that our method of solving them is of independent interest.

The general method that we develop in Section 3.2 is applied in Section 3.3 to obtain an exact formula for the first and second moments of $D_{1,K}$ (Theorem 3.3) and in Section 3.4 to obtain similar explicit results for moments of $D_{i,K}$ with arbitrary K and $i \in [2, 7]$ (Theorem 3.5). In principle, the method allows to obtain similar results recursively for an arbitrary value of the parameter i . Since both $E(D_{i,K})$ and $\sigma^2(D_{i,K})$ turns out to be linear in K , a byproduct of the above theorems are weak laws of large numbers stated as Corollary 3.4 (the case $i=1$) and Corollary 3.6 (the case $i \in [2, 7]$). In Section 3.5 we state a conjecture regarding the asymptotic normality of the vector D_K . The result is supported by our simulations, but unfortunately we were unable to prove it analytically.

We remark that the results in Section 3.4 are not as complete as the results for the number of roots in Section 2. However, the exact computation of moments for probabilistic combinatorial structures is a rather common line of inquiry in combinatorics, in particular with the goal of proving limit theorems for dependent variables in mind, cf. [27, 47]. We therefore consider our Theorem 3.5 as a first step in the study of a challenging subject and hope that our proof method not only is of interest on its own in general, but also can be further developed to prove the results conjectured later in Section 3.5.

3.1 Average gap between two successive roots

We begin with a simple observation regarding certain averages of the distance between two consecutive roots. For $i \in [K-1]$ let

$$q_i = \left(\sum_{i=1}^{K-1} E(D_{i,K}) \right)^{-1} E(D_{i,K}) \quad \text{and} \quad p_i = \left(\sum_{i=1}^{K-1} D_{i,K} \right)^{-1} D_{i,K}.$$

Intuitively, p_i and q_i represent, respectively, the empirical and a “mean-field” frequency of pairs of roots with distance i between them. The fact that $\sum_{i=1}^{K-1} D_{i,K} = R^{[K]}$ along with (9) imply that

$$q_i = \frac{3E(D_{i,K})}{K} \quad \text{and} \quad p_i = \frac{D_{i,K}}{R^{[K]}}.$$

Observe that

$$K = R^{[K]} + \sum_{i=1}^{K-1} i D_{i,K}. \quad (15)$$

Therefore, by virtue of (9), for any $K \geq 3$ we have

$$\sum_{i=1}^{K-1} i E(D_{i,K}) = \frac{2K}{3}, \quad \text{or, equivalently,} \quad \langle D_{\cdot,K} \rangle_q := \sum_{i=1}^{K-1} i q_i = 2, \quad (16)$$

where $\langle D_{\cdot,K} \rangle_q$ is the average distance between consecutive roots in the above “mean-field” model. Similarly, in view of (15),

$$\langle D_{\cdot,K} \rangle_p := \sum_{i=1}^{K-1} i p_i = \frac{K}{R^{[K]}} - 1 \quad (17)$$

As a corollary to Theorem 2.1 we derive the following result for the empirical average.

Theorem 3.1. *For $K \in \mathbb{N}$ let $T^{[K]} = \sqrt{K}(\langle D_{\cdot,K} \rangle_p - 2)$. Then*

$$\lim_{K \rightarrow \infty} \mathcal{L}(T^{[K]}) = \mathcal{N}\left(0, \frac{5}{18}\right).$$

where $\mathcal{N}\left(0, \frac{5}{18}\right)$ is a normal random variable with mean zero and variance $\frac{5}{18}$.

Figure 4 below show results of numerical simulations for $\langle D_{\cdot,K} \rangle_p$.

The proof of the theorem is a standard routine, we will only outline the argument for the sake of completeness. Taking in account (17), write for an arbitrary $x \in \mathbb{R}$ and any $K \in \mathbb{N}$ large enough (specifically, we need $3\sqrt{K} + x > 0$),

$$\begin{aligned} P\left(\sqrt{K}(\langle D_{\cdot,K} \rangle_p - 2) \leq x\right) &= P\left(\frac{K}{R^{[K]}} - 3 \leq \frac{x}{\sqrt{K}}\right) = P\left(\frac{R^{[K]}}{K} - \frac{1}{3} \geq \frac{\sqrt{K}}{3\sqrt{K} + x} - \frac{1}{3}\right) \\ &= P\left(\frac{R^{[K]} - K/3}{\sqrt{K}} \geq -\frac{x\sqrt{K}}{3(3\sqrt{K} + x)}\right). \end{aligned}$$

Taking the limit as K tends to infinity and inserting $\sigma(R^{[K]}) = \frac{1}{3}\sqrt{\frac{2K}{5}}$, yields

$$\lim_{K \rightarrow \infty} P\left(\sqrt{K}(\langle D_{\cdot, K} \rangle_p - 2) \leq x\right) = \lim_{K \rightarrow \infty} P\left(\frac{K/3 - R^{[K]}}{\sigma(R^{[K]})} \leq x\sqrt{\frac{5}{18}}\right),$$

as required.

In particular, Theorem 3.1 implies the weak law of large numbers for $\langle D_{\cdot, K} \rangle_p$:

$$\langle D_{\cdot, K} \rangle_p \xrightarrow{P} 2, \quad \text{as } K \rightarrow \infty,$$

where $\xrightarrow{P}$ indicates the convergence in probability. Interestingly enough, this law of large numbers is consistent with the “mean-field” (16). Heuristically, this may be explained by the CLT for $D_{i,K}$ stated as a conjecture in Section 3.5, which implies that for large values of K , with high probability the value of $D_{i,K}$ is close to its expectation. This is also consistent with the heuristic Ginzburg criterion [2] asserting that a mean-field approximation may work suitably in the situation when the variance of the underlying parameter is of a smaller order than its average square. In our case, in view of (16),

$$E\left[\left(\sum_{i=1}^{K-1} iD_{i,K}\right)^2\right] \geq \left[E\left(\sum_{i=1}^{K-1} iD_{i,K}\right)\right]^2 = \frac{4K^2}{9},$$

while by Theorem 2.1, $\sigma^2\left(\sum_{i=1}^{K-1} iD_{i,K}\right) = \sigma^2(R^{[K]}) = \frac{2K}{45}$.

3.2 Moments of $D_{i,K}$: general recursions

Our numerical simulations strongly suggest that for all $i \in \mathbb{N}$, a properly scaled $D_{i,K}$ converges to a normal law as K tends to infinity. A formal conjecture in this regard is stated in Section 3.5 below. See Figure 5 for a histogram of the empirical distribution of $D_{i,K}$ for several values of i and K obtained in a simulation of 200,000 simulations of the model. In this section we devise a method for estimation of the first two moments of $D_{i,K}$. We believe that both the expectation and the variance of this random variable grow linearly with K (see Section 3.5), in what follows we will verify this conjecture analytically for all $i \geq 7$.

Fix $i \in \mathbb{N}$ and assume that $K > i$. Recall the heights $H_n(k)$ from (2) and (3). Let $s, l, r, k \in \mathbb{N}$ be such that

$$2 \leq s \leq s+l < s+k-r < k+s < K.$$

Some of the above inequalities are trivially hold for natural numbers, they are illustrated altogether in Fig. 6. Assume that at some time $T \in \mathbb{N}$ we have (see Fig. 6)

$$\begin{aligned}
H_T(j) &= 1 \text{ if } j \in \{s-1, s+k\}, \\
H_T(j) &> 1 \text{ if } j \in \{s, s+1, \dots, s+l-1\} \cap \{s+k, s+1, \dots, s+k-1\}, \\
H_T(j) &= 0 \text{ if } j \in \{s+l, \dots, s+k-r-1\}.
\end{aligned}$$

Note that we do not specify the values of $H_T(j)$ for $j < s-1$ and $j > s+k$. In words, at time T we have a root at site $s-1$ followed by a block of particles not touching the ground of length l , followed by an interval empty of particles, which is followed by a block of particles not touching the ground of length r , and ends with a root at point $s+k$.

We define $\mathcal{D}_{l,s,j,i}$ as the number of pairs of consecutive roots in the interval $[s-1, s+k]$ with the distance i between them. It is not hard to verify that the distribution of $\mathcal{D}_{l,s,j,i}$ is independent of T and the configuration of particles at time T . In particular,

$$\mathcal{L}(D_{iK}) = \mathcal{L}(\mathcal{D}_{0,0K-1,i}). \quad (18)$$

In what follows we derive and study a system of equations for $\mathcal{D}_{l,s,j,i}$ and then extract an appropriate information for D_{iK} from these equations. The above construction is considerably more involved comparing to the auxiliary process exploited in Section 2. The reason why we are using this construction is that the distribution of $\mathcal{D}_{l,r,K,i}$ does vary with l and r because of the effect of the corner, and hence l and r should be taken into consideration in some way. By the corner effect we mean that the distribution of the distance between the corner root $s-1$ and next to it root within the interval depends on the values of the parameter l . Similarly, the distribution of the distance between the corner root $s+k$ and next to it root within the interval depends on r .

For all $k \geq l+r+3$,

$$\begin{aligned}
\mathcal{L}(\mathcal{D}_{l,r,k,i}) &= \mathcal{L}\left(\mathbf{1}_{\{Y=1\}} \mathcal{D}_{l+1,r,k,i}^{(1,1)} + \sum_{j=2}^{k-r-l-1} \mathbf{1}_{\{Y=j\}} \right. \\
&\quad \left. (\mathcal{D}_{l,0,j+l-1,i}^{(1,j)} + \mathcal{D}_{0,r,k-j-l,i}^{(2,j)}) + \mathbf{1}_{\{Y=k-r-l\}} \mathcal{D}_{l,r+1,k,i}^{(1,k)} \right)
\end{aligned} \quad (19)$$

where

- Y is distributed uniformly over the interval of integers $[1, k-r-l]$,
- $\mathcal{L}(\mathcal{D}_{l,r,k,i}) = \mathcal{L}(\mathcal{D}_{l,r,k,i}^{(1,j)}) = \mathcal{L}(\mathcal{D}_{l,r,k,i}^{(2,j)})$ for all $k \in [K]$, and $l, r \in \mathbb{N}$,
- $\mathcal{D}_{l,r,k,i}, \mathcal{D}_{l,r,k,i}^{(1,j)}, \mathcal{D}_{l,r,k,i}^{(2,j)}$ and Y_K are independent of each other for all values of the arguments l, r, k, i and j .

In addition, we have

$$\mathcal{L}(\mathcal{D}_{l,r,k,i}) = \mathbf{1}_{\{k=i\}}, \quad \text{if } l+r \in \{k, k-1, k-2\}, \quad (20)$$

for the initial condition of the system. See Figure 7 for a visual explanation of (19).

One can rewrite (19) as

$$E(u^{\mathcal{D}_{l,r,k,i}}) = \frac{1}{k-l-r} \left(E(u^{\mathcal{D}_{l+1,r,k,i}}) + \sum_{j=2}^{k-r-l-1} E(u^{\mathcal{D}_{l,0,j+l-1,i}}) E(u^{\mathcal{D}_{0,r,k-j-l,i}}) + E(u^{\mathcal{D}_{l,r+1,k,i}}) \right) \quad (21)$$

for all $l+r \leq k-3$. Similarly, (20) can be written as

$$E(u^{\mathcal{D}_{l,r,k,i}}) = u^{\mathbf{1}_{\{k=i\}}}, \quad l+r \in \{k, k-1, k-2\}. \quad (22)$$

To solve the system of equations (21) and (22) for all possible k, l, r , we define the following generating function with the domain in $\mathbb{C}^4$:

$$D_i(x, y, z, u) = \sum_{K=3}^{\infty} \sum_{l=0}^{K-3} \sum_{r=0}^{K-3-l} E(u^{\mathcal{D}_{l,r,K,i}}) y^l z^r x^{K-l-r}. \quad (23)$$

Notice that

$$\begin{aligned} D_i(0, y, z, u) &= 0, \\ D_i(x, y, 0, u) &= \sum_{K=3}^{\infty} \sum_{l=0}^{K-3} E(u^{\mathcal{D}_{l,0,K,i}}) x^{K-l} y^l, \\ D_i(x, 0, z, u) &= \sum_{K=3}^{\infty} \sum_{r=0}^{K-3} E(u^{\mathcal{D}_{0,r,K,i}}) x^{K-r} z^r, \\ D_i(x, 0, 0, u) &= \sum_{K=3}^{\infty} E(u^{\mathcal{D}_{0,0,K,i}}) x^K = \sum_{K=3}^{\infty} E(u^{\mathcal{D}_{i,K+1}}) x^K. \end{aligned} \quad (24)$$

Here we used the usual convention $0^0 = 1$. Inserting (21) and (22) into (23) we obtain

$$\begin{aligned} D_i(x, y, z, u) &= \sum_{K=3}^{\infty} \sum_{l+r \leq K-3} \frac{1}{K-l-r} \times \left(E(u^{\mathcal{D}_{l+1,r,K,i}}) \right. \\ &\quad \left. + \sum_{j=2}^{K-r-l-1} E(u^{\mathcal{D}_{l,0,j+l-1,i}}) E(u^{\mathcal{D}_{0,r,K-j-l,i}}) + E(u^{\mathcal{D}_{l,r+1,K,i}}) \right) x^{K-l-r} y^l z^r. \end{aligned} \quad (25)$$

Let $A_i(x, y, z, u) = \sum_{K=3}^{\infty} \sum_{l+r \leq K-3} \frac{1}{K-l-r} E\left(u^{\mathcal{D}_{l+1, r, K, i}}\right) x^{K-l-r} y^l z^r$ to be the first summation term (25). Then

$$\begin{aligned}
 y \frac{\partial A_i(x, y, z, u)}{\partial x} &= \sum_{K=3}^{\infty} \sum_{l=0}^{K-3} \sum_{r=0}^{K-3-l} E\left(u^{\mathcal{D}_{l+1, r, K, i}}\right) x^{K-(l+1)-r} y^{l+1} z^r \\
 &= \sum_{K=3}^{\infty} \sum_{l=1}^{K-2} \sum_{r=0}^{K-2-l} E\left(u^{\mathcal{D}_{l, r, K, i}}\right) x^{K-l-r} y^l z^r \\
 &= \sum_{K=3}^{\infty} \sum_{l=0}^{K-3} \sum_{r=0}^{K-3-l} E\left(u^{\mathcal{D}_{l, r, K, i}}\right) x^{K-l-r} y^l z^r + x^2 \sum_{K=3}^{\infty} \sum_{l=0}^{K-3} u^{1\{K=i\}} y^l z^{K-2-l} \\
 &\quad - \sum_{K=3}^{\infty} \sum_{r=0}^{K-3} E\left(u^{\mathcal{D}_{0, r, K, i}}\right) x^{K-r} z^r - \sum_{K=3}^{\infty} u^{1\{K=i\}} x^2 z^{K-2} + \sum_{K=3}^{\infty} u^{1\{K=i\}} x^2 y^{K-2} \\
 &= D_i(x, y, z, u) - D_i(x, 0, z, u) + x^2 \sum_{K=3}^{\infty} u^{1\{K=i\}} \left(\sum_{l=0}^{K-3} y^l z^{K-2-l} - z^{K-2} + y^{K-2} \right)
 \end{aligned} \tag{26}$$

Let $B_i(x, y, z, u) = \sum_{K=3}^{\infty} \sum_{l+r \leq K-3} \frac{1}{K-l-r} E\left(u^{\mathcal{D}_{l, r+1, K, i}}\right) x^{K-l-r} y^l z^r$ be the third summation term in (25). Then

$$\begin{aligned}
 z \frac{\partial B_i(x, y, z, u)}{\partial x} &= D_i(x, y, z, u) - D_i(x, y, 0, u) \\
 &\quad + x^2 \sum_{K=3}^{\infty} u^{1\{K=i\}} \left(\sum_{r=0}^{K-3} y^{K-2-r} z^r - y^{K-2} + z^{K-2} \right).
 \end{aligned} \tag{27}$$

Finally, we investigate the second summand in (25). Let

$$C_i(x, y, z, u) = \sum_{K=3}^{\infty} \sum_{l+r \leq K-3} \frac{1}{K-l-r} \sum_{j=2}^{K-r-l-1} E\left(u^{\mathcal{D}_{l, 0, j+l-1, i}}\right) E\left(u^{\mathcal{D}_{0, r, K-j-l, i}}\right) x^{K-l-r} y^l z^r.$$

Then

$$\begin{aligned}
 \frac{\partial C_i(x, y, z, u)}{\partial x} &= \sum_{K=3}^{\infty} \sum_{l=0}^{K-3} \sum_{r=0}^{K-3-l} \sum_{j=2}^{K-r-l-1} E\left(u^{\mathcal{D}_{l, 0, j+l-1, i}}\right) E\left(u^{\mathcal{D}_{0, r, K-j-l, i}}\right) x^{K-r-l-1} y^l z^r \\
 &= \sum_{K=3}^{\infty} \sum_{l=0}^{K-3} \sum_{r=0}^{K-3-l} \sum_{j=2+l}^{K-r-l-1} E\left(u^{\mathcal{D}_{l, 0, j-1, i}}\right) x^{j-1-l} y^l \left(E\left(u^{\mathcal{D}_{0, r, K-j, i}}\right) x^{K-j-r} z^r \right).
 \end{aligned}$$

Changing the order of the summations, we obtain

$$\begin{aligned}
 \frac{\partial C_i(x, y, z, u)}{\partial x} &= \left(\sum_{j=2}^{\infty} \sum_{l=0}^{j-2} E(u^{\mathcal{D}_{l,0,j-1,i}}) x^{j-1-l} y^l \right) \\
 &= \left(\sum_{K=j+1}^{\infty} \sum_{r=0}^{K-j-1} E(u^{\mathcal{D}_{0,r,K-j,i}}) x^{K-j-r} z^r \right) \left(\sum_{j=1}^{\infty} \sum_{l=0}^{j-1} E(u^{\mathcal{D}_{l,0,j,i}}) x^{j-l} y^l \right) \\
 &\quad \left(\sum_{j=1}^{\infty} \sum_{l=0}^{j-1} E(u^{\mathcal{D}_{l,0,j,i}}) x^{j-l} y^l \right) \left(\sum_{K=1}^{\infty} \sum_{r=0}^{K-1} E(u^{\mathcal{D}_{0,r,K,i}}) x^{K-r} z^r \right) \\
 &= \left(D_i(x, y, 0, u) + u^{1\{i=1\}} x + \sum_{j \geq 2} u^{1\{i=j\}} x y^{j-2} (y+x) \right) \\
 &\quad \times \left(D_i(x, 0, z, u) + u^{1\{i=1\}} x + \sum_{j \geq 2} u^{1\{i=j\}} x z^{j-2} (z+x) \right).
 \end{aligned} \tag{28}$$

Taking the derivative with respect to x on both sides of (25) and combining the result with (26), (27), and (28), yields the following system of equations:

$$\begin{aligned}
 yz \frac{\partial D_i(x, y, z, u)}{\partial x} &= \\
 & z \left(D_i(x, y, z, u) - D_i(x, 0, z, u) + x^2 \sum_{K=3}^{\infty} u^{1\{K=i\}} \left(\sum_{l=0}^{K-3} y^l z^{K-2-l} - z^{K-2} + y^{K-2} \right) \right) \\
 & + y \left(D_i(x, y, z, u) - D_i(x, y, 0, u) + x^2 \sum_{K=3}^{\infty} u^{1\{K=i\}} \right. \\
 & \quad \left. \left(\sum_{r=0}^{K-3} y^{K-2-r} z^r - y^{K-2} + z^{K-2} \right) \right) \\
 & + yz \left(D_i(x, y, 0, u) + u^{1\{i=1\}} x + \sum_{j \geq 2} u^{1\{i=j\}} x y^{j-2} (y+x) \right) \\
 & \times \left(D_i(x, 0, z, u) + u^{1\{i=1\}} x + \sum_{j \geq 2} u^{1\{i=j\}} x z^{j-2} (z+x) \right)
 \end{aligned} \tag{29}$$

3.3 Moments of $D_{i,K}$: case $i = 1$

When $i = 1$, (29) reduces to

$$\begin{aligned}
 D_1(x, y, z, u) &= \frac{(u-1)^2 yz - u(u-1)(y+z) + u^2 + 2}{3(1-y)(1-z)} x^3 + \frac{(1-u)(y+z) + 2u + 1}{3(1-y)(1-z)} x^4 \\
 & + \frac{2u(u-1)^2 yz - (u-1)(2u^2 + 3)(y+z) + 7 + 6u + 2u^3}{15(1-y)(1-z)} x^3 + \dots
 \end{aligned}$$

An inspection of the terms in right-hand side reveals that $D_1(x, y, z, u)$ may have the following form:

$$D_1(x, y, z, u) = \sum_{K \geq 3} \frac{a_K(u)yz + b_K(u)(y+z) + c_K(u)}{(1-y)(1-z)} x^K, \quad (30)$$

where $a_K(u)$, $b_K(u)$, $c_K(u)$ are polynomials in u . In what follows we confirm this guess. First observe that

$$D_1(x, y, z, 1) = \frac{x^3}{(1-x)(1-y)(1-z)} = \sum_{K \geq 3} \frac{x^K}{(1-y)(1-z)}, \quad (31)$$

which implies that $a_K(1) = 0$, $b_K(1) = 0$ and $c_K(1) = 1$, for all $K \geq 3$. Similarly, letting $y = z = 0$, we obtain from the identity

$$c(x, u) = D_1(x, 0, 0, u) = \sum_{K \geq 3} c_K(u) x^K, \quad (32)$$

that

$$c_K(u) = E\left(u^{\mathcal{D}_{0,0,K,1}}\right) = E\left(u^{\mathcal{D}_{1,K+1}}\right). \quad (33)$$

Next, we substitute the functional form (30) of $D_1(x, y, z, u)$ into (29). After a few simple algebraic manipulations, grouping, and comparing coefficients on both sides (29), we arrive to the following system of recurrence equations:

$$(K+1)a_{K+1}(u) = (1-u)^2 \mathbf{1}_{\{K=2\}} + \sum_{j=3}^{K-3} b_j(u)b_{K-j}(u) + 2(1-u)b_{K-1}(u), \quad (34)$$

$$\begin{aligned} (K+1)b_{K+1}(u) &= a_K(u) + b_K(u) + \sum_{j=3}^{K-3} b_j(u)c_{K-j}(u) + ub_{K-1}(u) + b_{K-2}(u) \\ &+ (1-u)c_{K-1}(u) + u(1-u)\mathbf{1}_{K=2} + (1-u)\mathbf{1}_{\{K=3\}}, \end{aligned} \quad (35)$$

$$\begin{aligned} (K+1)c_{K+1}(u) &= 2b_K(u) + 2c_K(u) + \sum_{j=3}^{K-3} c_j(u)c_{K-j}(u) + 2uc_{K-1}(u) + 2c_{K-2} \\ &(u) + (2+u^2)\mathbf{1}_{\{K=2\}} + 2u\mathbf{1}_{\{K=3\}} + \mathbf{1}_{\{K=4\}}, \end{aligned} \quad (36)$$

for all $K \geq 2$.

Using induction, one can now verify that $a_K(u)$, $b_K(u)$, $c_K(u)$ are all polynomials of degree $\frac{2K-1-3(-1)^K}{4}$ for $K \geq 3$. Several first values of these polynomials are given in Table 1.

Remark 3.2. The above recurrence equations can be equivalently written as the following system of differential equations. Define

$$A(x, u) = \sum_{K \geq 3} a_K(u)x^K \quad \text{and} \quad B(x, u) = \sum_{K \geq 3} b_K(u)x^K.$$

Then by multiplying each equation in (34)–(36) by x^K and summing over $K \geq 2$, we get

$$\frac{\partial}{\partial x} A(x, u) = (1-u)^2 x^2 + B(x, u)(B(x, u) + 2(1-u)x),$$

$$\frac{\partial}{\partial x} A(x, u) = (1-u)^2 x^2 + B(x, u)(B(x, u) + 2(1-u)x),$$

$$\frac{\partial}{\partial x} C(x, u) = 2B(x, u) + 2C(x, u) + C^2(x, u) + 2x(u+x)C(x, u) + (2+u^2)x^2 + 2ux^3 + x^4.$$

Even though we were unable to solve (34), (35), and (36) directly, we can leverage them to compute the first two moments of $D_{1,K}$ for an arbitrary $K \geq 3$.

Theorem 3.3. The following holds true for $D_{1,K}$:

$$\text{i.} \quad E(D_{1,4}) = \frac{2}{3} \text{ and}$$

$$E(D_{1,K}) = \frac{2K}{15} \quad \forall K \geq 5. \quad (37)$$

$$\text{ii.} \quad \sigma^2(D_{1,4}) = \frac{8}{9}, \sigma^2(D_{1,5}) = \frac{2}{9}, \sigma^2(D_{1,6}) = \frac{24}{25}, \sigma^2(D_{1,7}) = \frac{184}{225}, \sigma^2(D_{1,8}) = \frac{1588}{1575}, \text{ and}$$

$$\sigma^2(D_{1,K}) = \frac{1772K}{14175} \quad \forall K \geq 9. \quad (38)$$

Since $\sigma^2(D_{1,K}) \sim E(D_{1,K})$ as $K \rightarrow \infty$, Chebyshev's inequality yields

Corollary 3.4. $\frac{15D_{1,K}}{2K} \xrightarrow{P} 1$ as K tends to infinity.

Proof of Theorem 3.3.

(i) Define $a'_K = \frac{d}{du} a_K(u) \big|_{u=1}$, $b'_K = \frac{d}{du} b_K(u) \big|_{u=1}$ and $c'_K = \frac{d}{du} c_K(u) \big|_{u=1}$. Note that $a_K(1) = b_K(1) = 0$ and $c_K(1) = 1$ for all $K \geq 3$. By differentiating (34) at $u = 1$, we obtain $a'_K = 0$.

Thus, in view of (35) and (36), we have for all $K \geq 4$,

$$(K+1)b'_{K+1} = -1 + \sum_{j=3}^K b'_j, \text{ and } (K+1)c'_{K+1} = 2b'_K + 2 + 2 \sum_{j=3}^K c'_j,$$

with $b'_3 = b'_4 = -1/3$ and $c'_3 = c'_4 = 2/3$. By induction, $b'_K = -1/3$ and $c'_K = \frac{2K+2}{15}$ for all K

4. Therefore, for $K \geq 4$,

$$\frac{d}{du}a_K(u)|_{u=1} = 0, \frac{d}{du}b_K(u)|_{u=1} = -\frac{1}{3}, \text{ and } \frac{d}{du}c_K(u)|_{u=1} = \frac{2(K+1)}{15}.$$

This along with (32) and (33) implies that

$$\sum_{K \geq 3} E(D_{1,K+1})x^K = \frac{\partial}{\partial u}D_1(x, 0, 0, u)|_{u=1} = \frac{2}{3}x^3 + \sum_{K \geq 4} \frac{(2K+2)}{15}x^K.$$

Then (18) gives the result for the expected values.

(ii) Similarly, in order to calculate the variance, we consider

$$a''_K = \frac{d^2}{du^2}a_K(u)|_{u=1}, \quad b''_K = \frac{d^2}{du^2}b_K(u)|_{u=1}, \quad \text{and} \quad c''_K = \frac{d^2}{du^2}c_K(u)|_{u=1}.$$

Note that $a_K(1) = b_K(1) = 0$ and $c_K(1) = 1$ for all $K \geq 3$. By differentiating the equations in (34)–(36) twice, letting $u = 1$, and using induction on K , we obtain that

$$a''_3 = \frac{2}{3}, a''_4 = 0, a''_5 = \frac{4}{15}, a''_k = \frac{2}{9} \text{ for } k \geq 6,$$

$$b''_3 = -\frac{2}{3}, b''_4 = 0, b''_5 = -\frac{8}{15}, b''_6 = -\frac{22}{45}, b''_K = \frac{8}{315} - \frac{4K}{45}, \text{ for } K \geq 7,$$

$$c''_3 = \frac{2}{3}, c''_4 = 0, c''_5 = \frac{4}{5}, c''_6 = \frac{34}{45}, c''_7 = \frac{68}{63}, c''_K = \frac{2(K+1)(126K+67)}{14175}, \text{ for } K \geq 8.$$

Combining these equations with (32) and (33), we conclude that

$$\begin{aligned} \sum_{K \geq 3} E\left((D_{1,K+1})^2\right)x^K &= \frac{\partial^2}{\partial u^2}D_1(x, 0, 0, u)|_{u=1} = \frac{2}{3}x^3 + \frac{4}{5}x^5 + \frac{34}{45}x^6 + \frac{68}{63}x^7 \\ &+ \sum_{K \geq 8} \frac{2(K+1)(126K+67)}{14175}x^K. \end{aligned}$$

Hence, the variance of $D_{1,K+1}$ for $K \geq 8$ is given by

$$\sigma^2(D_{1,K+1}) = \frac{2(K+1)(126K+67)}{14175} + \frac{2(K+1)}{15} - \frac{4(K+1)^2}{15^2} = \frac{1772(K+1)}{14175},$$

which completes the proof of part (ii) of the theorem. $\square$

3.4 Moments of $D_{i,K}$: case $i = 2$

Next, we focus on computing the first two moments of $D_{i,K}$ for $i = 2$. Our method as presented in the previous subsection in finding moments of $D_{1,K}$ allows in principle to compute the moments recursively for any $i \in \mathbb{N}$. To illustrate the method, we will provide a detailed calculation for another case, namely $i = 2$ and state the results for $3 \leq i \leq 7$. For the sake of simplicity, we assume $K \geq 31$. Starting from $K = 31$ the computation of moment is generic for all values, the computation for lower values of K can be carried out in a similar way, but would be complicated by the necessity to consider multiple special cases.

Theorem 3.5. Suppose that $K \geq 31$. Then the following holds true:

$$\begin{aligned} E(D_{2,K}) &= \frac{K}{9}, & E(D_{3,K}) &= \frac{2K}{35}, & E(D_{4,K}) &= \frac{K}{45}, \\ E(D_{5,K}) &= \frac{4K}{567}, & E(D_{6,K}) &= \frac{K}{525}, & E(D_{7,K}) &= \frac{2K}{4455}, \\ \sigma^2(D_{2,K}) &= \frac{32K}{405}, & \sigma^2(D_{3,K}) &= \frac{119732K}{2837835}, & \sigma^2(D_{4,K}) &= \frac{12154K}{637875}, \\ \sigma^2(D_{5,K}) &= \frac{649555688K}{97692469875}, & \sigma^2(D_{6,K}) &= \frac{5967328K}{3192564375}, & \sigma^2(D_{7,K}) &= \frac{191501338988K}{428772250281375}. \end{aligned}$$

Similarly to Corollary 3.4 we have

Corollary 3.6. For $i \in [2, 7]$, $\frac{D_{i,K}}{E(D_{i,K})} \xrightarrow{P} 1$ as K tends to infinity.

Proof of Theorem 3.5. For integer $i \geq 2$, define

$$E_i(x, y, z) = \frac{\partial}{\partial u} D_i(x, y, z, u)|_{u=1} \quad \text{and} \quad F_i(x, y, z) = \frac{\partial^2}{\partial u^2} D_i(x, y, z, u)|_{u=1}.$$

Let now $i = 2$. Differentiating (29) with respect to u and letting $u = 1$, we obtain

$$\begin{aligned} yz \frac{\partial}{\partial x} E_2(x, y, z) &= z(E_2(x, y, z, u) - E_2(x, 0, z, u)) + y(E_2(x, y, z, u) - E_2(x, y, 0, u)) \\ &+ yz(E_2(x, y, 0) + x(x+y)) \left(\frac{x^3}{(1-x)(1-z)} + x + x(x+z) + \frac{xz(z+x)}{1-z} \right) \\ &+ yz \left(\frac{x^3}{(1-x)(1-y)} + x + x(x+y) + \frac{xy(y+x)}{1-y} \right) (E_2(x, 0, z) + x(x+z)) \end{aligned}$$

and

$$\begin{aligned}
yz \frac{\partial F_2(x, y, z)}{\partial x} &= z(F_2(x, y, z) - F_2(x, 0, z)) + y(F_2(x, y, z) - F_2(x, y, 0)) + yzF_2(x, y, 0) \\
&\left(\frac{x^3}{(1-x)(1-z)} + x + x(x+z) + \frac{xz(z+x)}{1-z} \right) + 2yz(E_2(x, y, 0) + x(x+y))(E_2(x, 0, z) + x(x+z)) \\
&+ yz \left(\frac{x^3}{(1-x)(1-y)} + x + x(x+y) + \frac{xy(y+x)}{1-y} \right) F_2(x, 0, z),
\end{aligned}$$

where we used the fact that $D_2(x, y, z, 1) = \frac{x^3}{(1-x)(1-y)(1-z)}$. Leveraging any computational mathematics software such as Maple, one can verify that the solution of these equations are given by, respectively,

$$3(1-y)(1-z)(1-x)^2 E_2(x, y, z) = -\left(y^2 + z^2 - y - z\right)x^3 + \left(y^2 + z^2 - 2y - 2z + 2\right)x^4 + (y+z-2)x^5 + \frac{1}{3}x^6.$$

and

$$\begin{aligned}
F_2(x, y, z) &= 405(1-x)^3 F_2(x, y, z) = 27yzx^3 + 270(-3yz + y + z)x^4 + 54(17yz - 15y - 15z + 5)x^5 \\
&- \frac{18(28y^2z^2 - 79y^2z - 79yz^2 + 56y^2 + 175yz + 56z^2 - 101y - 101z + 45)}{(1-y)(1-z)}x^6 \\
&+ \frac{18(8y^2z^2 - 36y^2z - 36yz^2 + 38y^2 + 115yz + 38z^2 - 94y - 94z + 61)}{(1-y)(1-z)}x^7 \\
&- \frac{18(y^2z^2 - 9y^2z - 9yz^2 + 15y^2 + 45yz + 15z^2 - 53y - 53z + 48)}{(1-y)(1-z)}x^8 \\
&- \frac{6(3y^2z + 3yz^2 - 10y^2 - 30yz - 10z^2 + 55y + 55z - 71)}{(1-y)(1-z)}x^9 - \frac{6(y^2 + 3yz + z^2 - 11y - 11z + 22)}{(1-y)(1-z)}x^{10} \\
&- \frac{6(y+z-4)}{(1-y)(1-z)}x^{11} - \frac{2}{(1-y)(1-z)}x^{12}.
\end{aligned}$$

The generating functions $E_i(x, y, z)$ and $F_i(x, y, z)$ for all $i \geq 1$ can be in principle calculated in a similar fashion. We omit the details due to the length and complexity of expressions and only report the results for $E_i(x, 0, 0)$, and $F_i(x, 0, 0)$ (see Tables 2 and 3). With this information in hand we are now in a position to calculate the mean and variance of $D_{i,K}$. We accomplish this by virtue of (24), and the fact that $\sum_{K=3}^{\infty} E(D_{i,K})x^K = E_i(x, 0, 0)$ and

$$\sum_{K=3}^{\infty} E(D_{i,K}^2)x^K = F_i(x, 0, 0) + E_i(x, 0, 0). \quad \square$$

3.5 CLT for $D_{i,K}$

We conclude with a brief discussion of a central limit theorem for $D_{i,K}$. Using a similar argument as in case of $D_1(x, y, z, u)$, one can show that the generating function $D_i(x, y, z, u)$ has the form

$$D_i(x, y, z, u) = \sum_{K \geq 3} \frac{\sum_{0 \leq a \leq b \leq i^A K, i, a, b}(u) \left(y^a z^b + y^b z^a \right) + \sum_{a=0}^{i-1} B_{K, i, a}(u) (y^a + z^a) + C_{K, i}(u)}{(1-y)(1-z)} x^K,$$

and that the coefficients $A_{K,i,a,b}(u)$, $B_{K,i,a}(u)$ and $C_{K,i}(u)$ are polynomials.

We believe that this information can be leveraged to prove the following result.

Conjecture.

- i. For every $i \in \mathbb{N}$, there exist $K_i \in \mathbb{N}$ and positive $a_i, b_i \in \mathbb{Q}$ such that $E(D_{i,K}) = a_i K$ and $\sigma(D_{i,K}) = b_i K$ for all $K > K_i$.
- ii. For $i \geq 1$, let $\tilde{D}_{i,K} = \frac{D_{i,K} - E(D_{i,K})}{\sigma(D_{i,K})}$. Then $\lim_{K \rightarrow \infty} \mathcal{L}(\tilde{D}_{i,K}) = \mathcal{N}(0, 1)$, where $\mathcal{N}(0, 1)$ is a standard normal distribution.

The above stated result is supported by our intensive numerical simulations (cf. Figure 5).

In fact, we believe that a stronger conjecture is true. Set $\tilde{D}_{i,K} = 0$ for an integer $i > K$ and use the notation $\tilde{D}_K$ for the (infinite) vector $(\tilde{D}_{i,K})_{i \in \mathbb{N}}$. We have:

Conjecture. As K tends to infinity, $\tilde{D}_K$ converges weakly to a Gaussian process in the product space $\mathbb{R}^{\mathbb{N}}$.

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Figure 1:

An instance of deposition process at time $n = 30$. $H_{30}(4) = 6$.

(a) 1000 Iterations

(b) 10000 Iterations

(c) 20000 Iterations

(d) 40000 Iterations

Figure 2:

Simulation of the process with $K = 500$ for $n = 40,000$ iterations.

Figure 3:

Empirical distribution of the limit number of roots, $R^{[K]}$, for various values of K . Each histogram is based on a simulation of 200,000 runs.

Figure 4:
Empirical distribution of the empirical average $\langle D, K \rangle_p$ for various values of K . Each histogram is based on a simulation of 200,000 runs.

Figure 5:

Empirical distribution of $D_{i,K}$ for several values of i and $K=1500$, each obtained via a simulation of 200,000 runs.

Figure 6:

Example of configuration at time T in the auxiliary construction defining $\mathcal{D}_{l,r,k,i}$

Figure 7:

The picture on the top corresponds to $Y = 1$, in the middle corresponds to $Y \in \{2, \dots, k-l-r-1\}$, and in the bottom to $Y = k-r-l$.

Table 1:

The values of $a_K(u)$, $b_K(u)$, $c_K(u)$, for $K = 3, 4, 5, 6, 7$.

K	$a_K(u)$	$b_K(u)$	$c_K(u)$
3	$\frac{1}{3}(1 - 2u + u^2)$	$\frac{1}{3}(u - u^2)$	$\frac{1}{3}(2 + u^2)$
4	0	$\frac{1}{3}(1 - u)$	$\frac{1}{3}(1 + 2u)$
5	$\frac{1}{15}(2u - 4u^2 + 2u^3)$	$\frac{1}{15}(3 - 3u + 2u^2 - 2u^3)$	$\frac{1}{15}(7 + 6u + 2u^3)$
6	$\frac{1}{9}(1 - 2u + u^2)$	$\frac{1}{45}(4 + 7u - 11u^2)$	$\frac{1}{45}(20 + 8u + 17u^2)$
7	$\frac{1}{315}(18 - 36u + 35u^2 - 34u^3 + 17u^4)$	$\frac{1}{315}(45 - 2u - 43u^2 + 17u^3 - 17u^4)$	$\frac{1}{315}(98 + 132u + 68u^2 + 17u^4)$

Table 2:

$E_i(x, 0, 0)$, for $i = 1, 2, 3, 4, 5, 6, 7$

i	$E_i(\mathbf{x}, \mathbf{0}, \mathbf{0})$
1	$15(1-x)^2 E_1(x, 0, 0) = 2(x^2 - 5x + 5)$
2	$9(1-x)^2 E_2(x, 0, 0) = x^4(x^2 - 6x + 6)$
3	$105(1-x)^2 E_3(x, 0, 0) = 2x^3(3x^4 - 21x + 56x^2 - 70x + 35)$
4	$45(1-x)^2 E_4(x, 0, 0) = x^4(x^2 - 3x + 3)(x^2 - 5x + 5)$
5	$2835(1-x)^2 E_5(x, 0, 0) = 2x^5(10x^4 - 90x^3 + 279x^2 - 378x + 189)$
6	$1575(1-x)^2 E_6(x, 0, 0) = x^6(3x^4 - 30x^3 + 100x^2 - 140x + 70)$
7	$31185(1-x)^2 E_7(x, 0, 0) = 2x^7(7x^4 - 77x^3 + 275x^2 - 396x + 198)$

Table 3: $F_i(x, 0, 0)$, for $i = 1, 2, 3, 4, 5, 6, 7$

i	$F_i(x, 0, 0)$
1	$14175(1-x)^3 F_1(x, 0, 0) = 2x^3(4725 - 14175x + 19845x^2 - 16380x^3 + 8595x^4 - 2880x^5 + 580x^6 - 58x^7)$
2	$405(1-x)^3 F_2(x, 0, 0) = 2x^5(15 - 15x + 6x^2 - x^3)(3 - 3x + x^2)^2$
3	$14189175(1-x)^3 F_3(x, 0, 0) = 2x^7(1711710 - 5135130x + 6786780x^2 - 5118113x^3 + 2380287x^4 - 682864x^5 + 111636x^6 - 7974x^7)$
4	$637875(1-x)^3 F_4(x, 0, 0) = 2x^9(15525 - 46575x + 60255x^2 - 43695x^3 + 19200x^4 - 5100x^5 + 752x^6 - 47x^7)$
5	$97692469875(1-x)^4 F_5(x, 0, 0) = 4x^{11}(157260285 - 471780855x + 598855005x^2 - 418392270x^3 + 173906073x^4 - 42827760x^5 + 5728788x^6 - 318266x^7)$
6	$3192564375(1-x)^3 F_6(x, 0, 0) = 2x^{13}(969150 - 2907450x + 3627930x^2 - 2446595x^3 + 963525x^4 - 220570x^5 + 26940x^6 - 1347x^7)$
7	$428772250281375(1-x)^3 F_7(x, 0, 0) = 2x^{15}(9215899308 - 27647697924x + 33973625070x^2 - 22162777791x^3 + 8287091967x^4 - 1769271504x^5 + 198572308x^6 - 9026014x^7)$