Note: Diffusion-limited annihilation in cavities

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Many chemical reactions in various natural and technological processes occur in cavities containing a small number of molecules. Examples include reactions in enzymatic fuel cells, nanoreactors, and nanopores,^{1,2} drug delivery devices,³ cells and cellular subdomains, and^{4,5} micelles and bubbles^{6,7} to mention just a few. One of the distinctive features of reactions in cavities is the important role of fluctuations.^{8,9} Therefore, the description of the reaction kinetics in terms of the rate equations for the concentrations should be used with care. The present paper reports on the results of a Brownian dynamics simulation study of the simplest reaction in a cavity: diffusion-limited annihilation of two spherical particles of radius b occurring at their first contact. This reaction is studied in spherical and cubic cavities of the same volume over a wide range of the cavity size. The analysis is performed for three cases: (1) both particles diffuse with the same diffusivity, denoted by D , (2) one particle is fixed at the center of the cavity, and the second diffuses with doubled diffusivity, $2D$, that guarantees the same relative diffusivity of the reaction partners, (3) the same as case (2) with the immobile particle located on the wall of the spherical cavity and at the center of one of the faces of the cubic cavity.

The focus is on the mean particle lifetime and the validity of the single exponential approximation of the particle survival probability.

When the immobile particle is fixed at the center of a spherical (*sph*) cavity of radius R [case (2)], one can find an exact analytical solution for the mean lifetime $\tau(r|R)$, as a function of the initial distance r of the diffusing particle from the cavity center,¹⁰

$$\tau(r|R) = \frac{1}{6D} \left[2(R-b)^3 \left(\frac{1}{b} - \frac{1}{r} \right) - r^2 + b^2 \right]. \quad (1)$$

Averaging this mean lifetime over the uniform distribution of the particle starting point inside the cavity, one obtains the mean lifetime, denoted by $\tau_{sph}(R)$, which can be written as

$$\tau_{sph}(R) = \frac{3}{[(R-b)^3 - b^3]} \int_b^{R-b} \tau(r|R) r^2 dr = \frac{R^3}{12bD} F_2(R/b), \quad (2)$$

where $F_2(x)$ is a dimensionless function of the ratio $x = R/b$ given by

$$F_2(x) = \frac{5(x-1)^5 - 8(x-1)^4 - 16(x-1)^3 + 8(x-1)^2 + 16(x-1) + 32}{5x^3[(x-1)^2 + x]}, \quad x \geq 3, \quad (3)$$

and the subscript 2 indicates the case number. This function monotonically increases from zero at $x = 3$ (minimum cavity radius, $R = 3b$, corresponds to two contacting particles) to unity at $x \rightarrow \infty$. According to Eqs. (2) and (3), the asymptotic behavior of the mean particle lifetime in a large spherical cavity ($R \rightarrow \infty$) is $R^3/(12bD)$. Thus, Eq. (2) represents the mean lifetime as the product of its asymptotic behavior, $R^3/(12bD)$, and function $F_2(R/b)$ which describes how the mean lifetime approaches its asymptotic form.

One can find the above-mentioned asymptotic behavior of the mean lifetime assuming that the particle survival probability, $S(t)$, decays as a single exponential, $S(t) = \exp(-kt)$,

with the unimolecular rate constant k given by the ratio of the Smoluchowski rate constant, k_{Sm} , and the cavity volume, $V_{cav} = 4\pi R^3/3$ (the ratio $1/V_{cav}$ plays the role of the concentration of diffusing particles in the Smoluchowski theory),

$$S(t) = \exp(-k_{Sm}t/V_{cav}), \quad k_{Sm} = 16\pi bD. \quad (4)$$

Here, in the expression for k_{Sm} , we have taken into account the fact that the relative diffusivity of the two particles is $2D$ and the contact radius (the distance between the particle centers when they are in contact) is $2b$. As follows from Eq. (4),

$$\tau_{sph}(R)\Big|_{R \rightarrow \infty} = \frac{V_{cav}}{k_{Sm}} = \frac{R^3}{12bD}. \quad (5)$$

Similar arguments suggest that the asymptotic estimates in Eqs. (4) and (5) work not only when one particle is fixed at the center of a large spherical cavity ($R \rightarrow \infty$) but also in case (1), where both particles diffuse with diffusivity D . The situation changes when one of the particles is fixed on the wall of a large cavity [case (3)]. The reason is that here only one half of the surface of the immobile particle is available for the contact. As a consequence, the mean lifetime in this case is twice longer than its counterpart in Eq. (5) for the immobile particle placed at the center of the cavity. Thus, as $R \rightarrow \infty$, the mean lifetimes in cases (1) and (2) are $R^3/(12bD)$, while in case (3), the mean lifetime is $R^3/(6bD)$. Similar reasoning leads to the same asymptotic estimates for the mean lifetimes in large cubic cavities of the same volume as their spherical counterparts.

The above reasoning allows one to find the asymptotic behavior of the mean lifetime in large cavities but does not tell anything about the range of applicability of the obtained results. The available analytical solution for $\tau_{sph}(R)$ in case (2) shows that the mean lifetime approaches its asymptotic behavior very slowly. As follows from Eqs. (2) and (3),

$$\tau_{sph}(R) \approx \frac{R^3}{12bD} \left(1 - \frac{28}{5} \frac{b}{R} \right), \quad R \gg b. \quad (6)$$

Thus, the deviation of this mean lifetime from its asymptotic behavior becomes less than 3% when the cavity radius exceeds $187b$. To study how the mean lifetime approaches its asymptotic behavior, we run Brownian dynamics simulations in both spherical and cubic cavities. The radius of the spherical cavity was varied from $9b$ to $120b$, with the step $3b$, $R_n = 3nb$, $n = 3, 4, \dots, 40$. Cubic cavities had the same volumes as their spherical counterparts. The relation between the edge L_n of the cubic cavity and the radius R_n was $L_n = (4\pi/3)^{1/3} R_n$.

Comparison of the simulation results for cavities of equal volumes but different shapes shows that, as might be expected, the difference in the cavity shape results in differences in the mean lifetimes in small cavities. These differences decrease as the cavity size increases. The difference between the mean lifetimes in cavities of different shapes is less than 3% when $R_n \geq 15b$ in cases (1) and (3) and when $R_n \geq 36b$ in case (2). Thus, in case (2), function $F_2(x)$, given in Eq. (3), describes the slow approach of the mean lifetime to its asymptotic behavior both in spherical and cubic cavities.

Since in cases (1) and (3) the mean lifetimes in spherical and cubic cavities are close at $R_n \geq 15b$, we analyze how these mean lifetimes approach their asymptotic behavior only for spherical cavities. It is convenient to present the mean lifetimes obtained from the simulations as the products of the asymptotic dependencies, $R^3/(12bD)$ and $R^3/(6bD)$ in cases (1) and (3), respectively, and functions $F_{1,3}(x)$, $x = R/b$, which increase from zero to unity as x increases from $x = 2$ ($R = 2b$ for two contacting particles) to infinity. We found that functions $F_{1,3}(x)$ are well approximated by a simple formula

$$F_{1,3}(x) = 1 - \exp[-\alpha_{1,3}(x - 2)], \quad (7)$$

with $\alpha_1 = 0.08$ and $\alpha_3 = 0.14$. Function $F_1(x)$ approximates the simulation results with the relative error less than 2%. Function

$F_3(x)$ provides less accurate approximation: for $x \geq 12$, the maximum relative error is about 5% and for $x = 9$, the relative error is 8.5%. Comparison of functions $F_i(x)$, $i = 1, 2, 3$, shows that the mean lifetimes approach their asymptotic behavior in cases (1) and (3) significantly faster than in case (2): in cases (1) and (3), the difference between the mean lifetimes obtained from the simulations and given by the asymptotic estimates, $R^3/(12bD)$ and $R^3/(6bD)$, is less than 3% when $R > 45b$, while in case (2), this happens when $R > 187b$.

The simulation results are also used to analyze whether the particle annihilation kinetics is single-exponential or not. In studying this question, we take advantage of the fact that when the system survival probability decays as a single exponential, $S(t) = \exp(-kt)$; there is a simple relation between the moments of the system lifetime. In this case, the lifetime probability density is $\varphi(t) = -dS(t)/dt = k \exp(-kt)$, and the m -th moment of the lifetime is given by $\langle t^m \rangle = \int_0^\infty t^m \varphi(t) dt = m!/k^m$. This allows one to write the m -th moment in terms of the first moment $\langle t^m \rangle = m! \langle t \rangle^m$. Thus, when the kinetics is single-exponential, the ratio $\langle t^2 \rangle / \langle t \rangle^2$ is 2. We used the magnitude of this ratio as an indicator of whether the kinetics is single-exponential or not. The moment ratios obtained from the simulations show that the annihilation kinetics becomes single-exponential in relatively small cavities, $R > 15b$. Note that the mean particle lifetimes in such cavities are shorter than those given by the asymptotic estimate, and hence the unimolecular rate constant k is greater than the ratio k_{Sm}/V_{cav} .

In summary, one might expect that the survival probability of the annihilating particles decays as a single exponential, $S(t) = \exp(-kt)$, with the rate constant equal to the ratio of the Smoluchowski rate constant and the cavity volume, $k = k_{Sm}/V_{cav}$, in cases (1) and (2) and one half of the ratio in case (3). Our simulation results support this idea for sufficiently large cavities. Specifically, the radius R of the spherical cavity should exceed $45b$ in cases (1) and (3) and $187b$ in case (2), respectively, and the cubic cavity edge L should exceed $72b$ in cases (1) and (3) and $300b$ in case (2). In smaller cavities, the survival probabilities still decay as single exponentials but with the rate constants that are higher than the simple estimate k_{Sm}/V_{cav} in cases (1) and (2) and one half of this ratio in case (3). As the cavity size further decreases, the decay kinetics becomes multi-exponential.

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