The Equivalence of Linear Gaussian Connectivity Techniques

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Abstract: The theoretical basis of linear Gaussian connectivity methods for the analysis of fMRI data is examined in this article, resulting in a clarification of methodological dependencies between techniques. In particular, Granger causality connectivity procedures, which describe instantaneous and directed influence between sets of voxel timeseries, are shown to be remappings of correlation-based metrics. Furthermore, the statistical inference tests applied to pairwise Granger causality measures are theoretically shown to be equivalent to inference tests applied to correlation-based metrics. These results are demonstrated empirically using receiver operating characteristic curves derived from vector autoregressive models of various lags, sample size, and noise covariance values. The equivalence of linear Granger causality and correlation-based methods, in both metric and test statistic, renders linear Granger causality a restatement of traditional data-driven methodologies in the context of brain connectivity studies. Furthermore, the equivalence highlights the centrality of partial correlation and partial variance in linear connectivity analyses and bridges the gap between functional and effective connectivity techniques. Consequently, rather than a distinction rooted in methodological difference, the dichotomy between functional and effective connectivity methods is ultimately a function of model configuration realized in choices such as the selection of nodes, the choice to model instantaneous and/or directed influence, and the choice to employ many bivariate models or a single multivariate model. While these theoretical results may be unsurprising to the reader with advanced statistical knowledge, they highlight the importance of a clear understanding of the theoretical basis of connectivity analysis methods for human brain mapping researchers. *Hum Brain Mapp* 34:1999–2014, 2013. © 2012 Wiley Periodicals, Inc.

Key words: Granger causality; partial correlation; partial variance; correlation; fMRI connectivity; inference test

INTRODUCTION

Functional MRI connectivity analysis techniques can be categorized into data-driven methods and model-based methods, commonly termed functional and effective connectivity methods [Friston et al, 1993], respectively. Data-driven connectivity techniques most commonly extract information from whole-brain correlation or covariance maps [Friston, 1994; Rogers et al., 2007] and therefore

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do not incorporate a notion of directional influence, purportedly since correlation is a scalar measure of total linear connectivity between two voxels [Friston et al., 1993; Rogers et al., 2007; Sato et al., 2006b]. Correlative connectivity measures are described as “ambiguous with respect to underlying directed interactions” [Roebroeck et al., 2005, p. 230]. In contrast, model-based methods, such as structural equation modeling (SEM) [Gonzalez-Lima and McIntosh, 1994; McIntosh and Gonzalez-Lima, 1994] and dynamic causal modeling (DCM) [Friston et al., 2003], employ some form of a priori knowledge to examine specific networks in the brain, based on a parametric model of the relationships between network nodes [Friston, 1994].

Marrelec et al. [2005] examined the relationship between data-driven and model-based connectivity techniques and demonstrated that, while unmediated correlation fails to capture the complexity of interactions modeled in SEM, conditional correlation that mediates relationships between network nodes is sufficient to express the relationships that arise in a linear Gaussian SEM analysis. Their work provided the underpinnings of a unified view of connectivity analysis techniques that transcends the functional/effective dichotomy. However, it did not consider measures describing lagged influence, such as Granger causality.

A concurrent stream of connectivity analysis techniques arose in the medical imaging literature based on the concepts of Granger causality. Granger [1969] defined the axiom of causality, that cause must precede effect, in the context of timeseries by asserting that a timeseries y causally influences x if one is better able to predict x knowing the past of y . Granger causality, also termed Granger-Wiener causality with appropriate deference to the seminal work of Wiener [1956], refers to the quantification of causality in stochastic timeseries through analysis of the variance of the residual processes. Geweke [1982] extended Granger’s application of causality to autoregressive models by the introduction of both a directed and an instantaneous Granger causality measure, which together form a measure of total linear dependence between two timeseries.

The application of Geweke’s Granger causality (GGC) measures to fMRI data seemingly enlarged the pool of fMRI connectivity analysis techniques, “promising further insights” [Zhou et al., 2009, p. 2198] into neural connectivity by providing methods that were simultaneously functional, taking a whole-brain approach, and effective, by their dependence on a parametric model [Abler et al., 2005; Goebel et al., 2003; Roebroeck et al., 2005]. Although linear Granger causality measures are known to be limited in their ability to infer causality from fMRI timeseries [Smith et al., 2011], their uptake in fMRI analyses has been rapid. The breadth of GGC techniques to-date spans conditional GGC measures [Chen et al., 2006; Zhou et al., 2009], partial GGC measures [Guo et al., 2008], frequency domain implementations [Chen et al., 2006; Demirci et al., 2009; Deshpande et al., 2009; Sato et al., 2009; Zhou et al., 2009], and Bayesian [Valdes-Sosa, 2004], time-varying [Sato et al., 2006a] and nonlinear [Li et al., 2010] methods.

Although similarities between GGC and correlation-based connectivity methods have been acknowledged [e.g., Rogers et al. [2010, p. 1062] note that “partial Granger causality is based on the concept of partial covariance...”) the precise nature of the relationship between the two has remained oblique. The primary objective of this article is to ascertain whether connectivity identified by GGC measures can be equally assessed using lagged or partial correlation measures. We investigate the relationship between GGC and correlation-based connectivity methods through an examination of how GGC measures can be exactly expressed as remappings of correlation-based quantities. Furthermore, we show that the statistical inference tests applied to GGC and correlation-based connectivity are equivalent. Our aim is to clearly present the theoretical basis of linear connectivity analysis methods for human brain mapping researchers and, in particular, to establish the place of Granger causality as a surrogate for correlation-based techniques.

The article is organized as follows. Necessary theory is introduced in the section entitled Theoretical Background, including definitions of autoregressive models, GGC measures and their statistical inference tests. The Theoretical Results section contains the main contributions of the article, that GGC measures can be expressed as nonlinear functions of correlation-based quantities, and that the statistical inference tests of GGC measures are equivalent to those applied to their correlation-based counterparts. The theoretical results are then verified empirically, as provided in the subsequent Methods and Results sections. The article then concludes with a discussion regarding implications of the equivalence mappings between GGC measures and correlation-based measures for fMRI connectivity.

Theoretical Background

Notation

The notation x_t is used to denote the value of a real-valued timeseries x at sampled time instant t . The past p samples of x_t are denoted by $x_{t-1}^{(p)} = [x_{t-1}, \dots, x_{t-p}]'$.

The sample covariance between two timeseries, x and y , is denoted by $\text{cov}(x_t, y_t)$, where covariance has the standard definition,

$$\text{cov}(x_t, y_t) = \frac{1}{T-1} \sum_{t=1}^T (x_t - \bar{x})(y_t - \bar{y}). \quad (1)$$

The sample variance of x_t is $\text{var}(x_t) = \text{cov}(x_t, x_t)$. Sample estimates of model parameters are identified by the $\hat{\cdot}$ symbol, asymptotic equivalence in distribution is denoted $\cong$, and $\sim$ denotes the prediction of a stochastic variate. Correlation refers to instantaneous (lag-0) correlation, unless a specific lag is prepended to the term.

Autoregressive models

Autoregressive (AR) models in their simplest form are a sub-class of the linear Gaussian model (LGM) class, for

which the predictor contains lagged components of itself in a self (auto) regression. Vector autoregressive (VAR) models are a multivariate generalization of AR models. The VAR model order, p , is determined by the greatest lag present in the predictor. Let the VAR(p) of x and y be defined as

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} \mathbf{a} & \mathbf{c} \\ \mathbf{b} & \mathbf{d} \end{bmatrix}' \begin{bmatrix} \mathbf{x}_{t-1}^{(p)} \\ \mathbf{y}_{t-1}^{(p)} \end{bmatrix} + \begin{bmatrix} e_t \\ u_t \end{bmatrix}, \quad \begin{bmatrix} e_t \\ u_t \end{bmatrix} \sim N(0, \Sigma), \quad (2)$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ are the $p \times 1$ vectors of autoregressive coefficients, and e_t and u_t are the white noise processes that drive the states x_t and y_t , respectively. The covariance matrix, Σ , is defined as

$$\Sigma = \begin{bmatrix} \varepsilon & \delta \\ \delta & \eta \end{bmatrix}. \quad (3)$$

The coefficient matrix of the VAR(p) model is typically estimated using maximum likelihood estimation (MLE) that, for Gaussian distributed noise, is equivalent to least squares estimation [Lupton, 1993],

$$\begin{bmatrix} \hat{\mathbf{a}} & \hat{\mathbf{c}} \\ \hat{\mathbf{b}} & \hat{\mathbf{d}} \end{bmatrix} = \text{cov} \left(\begin{bmatrix} \mathbf{x}_{t-1}^{(p)} \\ \mathbf{y}_{t-1}^{(p)} \end{bmatrix} \right)^{-1} \text{cov} \left(\begin{bmatrix} \mathbf{x}_{t-1}^{(p)} \\ \mathbf{y}_{t-1}^{(p)} \end{bmatrix}, \begin{bmatrix} x_t \\ y_t \end{bmatrix} \right). \quad (4)$$

Sample estimates of the state noise processes are given by

$$\begin{bmatrix} \hat{e}_t \\ \hat{u}_t \end{bmatrix} = \begin{bmatrix} x_t \\ y_t \end{bmatrix} - \begin{bmatrix} \hat{\mathbf{a}} & \hat{\mathbf{c}} \\ \hat{\mathbf{b}} & \hat{\mathbf{d}} \end{bmatrix}' \begin{bmatrix} \mathbf{x}_{t-1}^{(p)} \\ \mathbf{y}_{t-1}^{(p)} \end{bmatrix}, \quad (5)$$

leading to estimation of the covariance matrix as,

$$\hat{\Sigma} = \text{cov} \left(\begin{bmatrix} \hat{e}_t \\ \hat{u}_t \end{bmatrix} \right). \quad (6)$$

AR(p) models are the trivial VAR(p) case for which $\delta = 0$ and $\mathbf{b} = \mathbf{c} = \mathbf{0}$, which decouples the timeseries into two AR(p) models. In the fMRI context, this decoupling implies spatial independence of voxel timeseries x and y . Alternatively, if a VAR(p) model is employed, dependence between voxels can be modeled using both instantaneous and lagged influence [Goebel et al., 2003], from which directional influence can be inferred.

In the remainder of the manuscript, components of AR(p) models are identified with subscript A , while components of a VAR(p) model are identified with subscript V .

Granger causality measures of connectivity

GGC measures [Geweke, 1982] applied to the VAR(p) model in Eqs. (2) and (3) assess linear dependence between timeseries x and y . Directed Granger causality measures (DGC), $F_{y \rightarrow x}$ and $F_{x \rightarrow y}$, quantify directed influence from y to x and from x to y , respectively, while the instantaneous Granger causality measure (IGC), $F_{x,y}$, quan-

tifies contemporaneous influence between the two timeseries. Together, the GGC measures provide an assessment of the total linear dependence between the two timeseries,

$$F_{x,y} = F_{x,y} + F_{x \rightarrow y} + F_{y \rightarrow x}, \quad (7)$$

where

$$\text{IGC} : F_{x,y} = \ln \frac{\hat{\eta}_V \hat{\varepsilon}_V}{|\hat{\Sigma}_V|} = \ln \frac{\text{var}(e_{t,V}) \text{var}(u_{t,V})}{\left| \text{cov} \left(\begin{bmatrix} e_{t,V} \\ u_{t,V} \end{bmatrix} \right) \right|^2}, \quad (8)$$

and

$$\text{DGC} : \begin{cases} F_{y \rightarrow x} = \ln \frac{\hat{\varepsilon}_A}{\hat{\varepsilon}_V} = \ln \frac{\text{var}(e_{t,A})}{\text{var}(e_{t,V})}, \\ F_{x \rightarrow y} = \ln \frac{\hat{\eta}_A}{\hat{\eta}_V} = \ln \frac{\text{var}(u_{t,A})}{\text{var}(u_{t,V})}. \end{cases} \quad (9)$$

Conditional instantaneous Granger causality (CIGC) and conditional directed Granger causality (CDGC) measures [Geweke, 1984] provide robustness to the effects of mediated influence through assessment of linear dependence between x and y in the presence of a third timeseries, z . Let V_{xz} be the bivariate VAR(p) model containing x and z , and V_{xyz} be the multivariate VAR(p) model of x , y , and z . Then

$$\text{CIGC} : F_{x,y|z} = \ln \frac{\text{var}(e_{t,V_{xyz}}) \text{var}(u_{t,V_{xyz}})}{|\text{cov} \left(\begin{bmatrix} e_{t,V_{xyz}} \\ u_{t,V_{xyz}} \end{bmatrix} \right)|^2}, \quad (10)$$

and

$$\text{CDGC} : F_{y \rightarrow x|z} = \ln \frac{\text{var}(e_{t,V_{xz}})}{\text{var}(e_{t,V_{xyz}})}. \quad (11)$$

Partial directed Granger causality (PDGC) [Guo et al., 2008] is an extension of CDGC that removes any additional covariance between the source and target variables, addressing concerns regarding potentially unmodeled exogenous inputs. Let w_t denote the residual of z_t after regression onto the VAR(p) predictor. Then

$$\text{PDGC} : F_{y \rightarrow x|z} = \ln \frac{\text{var}(e_{t,V_{xy}} | w_{t,V_{xz}})}{\text{var}(e_{t,V_{xyz}} | w_{t,V_{xyz}})}. \quad (12)$$

While the PDGC notation of “ $|$ ” is somewhat cumbersome, it is also intuitive in the sense that “ $|z$ ” denotes regressing onto z 's past, while “ $.z$ ” refers to the subsequent removal of any remaining instantaneous covariance between the residuals of x_t and z_t .

Statistical inference tests of Granger causality measures

The degrees of freedom, N , associated with statistical inference testing of GGC measures for timeseries length T

and model order p , is given by $N = T - 2p - 1$ [Bressier and Seth, 2011]. Test variates are constructed by scaling the GGC measures by the degrees of freedom, $S_{x,y} = NF_{x,y}$, and $S_{y \rightarrow x} = NF_{y \rightarrow x}$. The distribution of each test statistic is described under a null hypothesis of zero influence as follows.

The null hypothesis of zero instantaneous influence, $H_0 : F_{x,y} = 0$, is assessed using an IGC test variate with distribution [Geweke, 1982]

$$S_{x,y} \sim \chi^2(1). \quad (13)$$

The null hypothesis of zero directed influence, $H_0 : F_{y \rightarrow x} = 0$, is assessed using a DGC test variate with asymptotic $\chi^2(p)$ distribution [Geweke, 1982]

$$S_{y \rightarrow x} \sim \chi^2(p). \quad (14)$$

The test statistics of the conditional measures follow the same distributions with appropriate adjustment of degrees of freedom.

THEORETICAL RESULTS

Having defined the set of GGC measures and their statistical inference tests, we present the primary contribution of this article: that all such measures are remappings of traditional correlation-based connectivity measures. We subsequently show that the GGC inference tests provide the same information as inference tests of their counterpart correlation-based measures. Mathematical derivations of all functional relationships that follow are detailed in Appendices A and B.

GGC Measures as Functions of Correlation

Measuring instantaneous influence

As outlined above, IGC [Eq. (8)] quantifies instantaneous influence in the presence of lagged samples. Partial correlation, in the context of a VAR model, measures the correlation between x and y having removed the influence of lagged samples,

$$\text{corr}(x_t, y_t | \mathbf{x}_{t-1}^{(p)}, \mathbf{y}_{t-1}^{(p)}) = \frac{\text{cov}(e_{t,V}, u_{t,V})}{\text{var}(e_{t,V})\text{var}(u_{t,V})}. \quad (15)$$

A direct comparison of Eqs. (8) and (15) leads to the expression of IGC as a function of partial correlation,

$$F_{x,y} = -\ln\left(1 - \text{corr}(x_t, y_t | \mathbf{x}_{t-1}^{(p)}, \mathbf{y}_{t-1}^{(p)})^2\right). \quad (16)$$

Measuring directed influence

A similar result follows for DGC [Eq. (9)] assessing directed influence between x and y ; partial variance quantifies the variance remaining in x after removal of the

influence of lagged samples. In the bivariate VAR(p) model context, partial variance is given by [Greene, 2008]

$$\text{var}(x_t | \mathbf{x}_{t-1}^{(p)}, \mathbf{y}_{t-1}^{(p)}) = \text{var}(e_{t,V}), \quad (17)$$

while for the univariate AR(p) model [Greene, 2008],

$$\text{var}(x_t | \mathbf{x}_{t-1}^{(p)}) = \text{var}(e_{t,A}). \quad (18)$$

A direct comparison of Eqs. (9) and (17) leads to the expression of DGC as a remapping of partial variance,

$$F_{y \rightarrow x} = \ln \frac{\text{var}(x_t | \mathbf{x}_{t-1}^{(p)})}{\text{var}(x_t | \mathbf{x}_{t-1}^{(p)}, \mathbf{y}_{t-1}^{(p)})}. \quad (19)$$

The relationship between DGC and partial variance can also be expressed as a relationship between DGC and multiple correlation. Multiple correlation, for a VAR(p) model, quantifies the proportion of signal variance in x accounted for by a vector of lagged timeseries measurements [Hamilton, 1994],

$$\text{mcorr}\left(x_t, \begin{bmatrix} \mathbf{x}_{t-1}^{(p)} \\ \mathbf{y}_{t-1}^{(p)} \end{bmatrix}\right) = \sqrt{\frac{\text{cov}(x_t, \tilde{x}_{t,V})}{\text{var}(x_t)}}, \quad (20)$$

where $\tilde{x}_{t,V}$ denotes the prediction of x_t using the VAR(p) model. A similar expression holds for univariate AR(p) models,

$$\text{mcorr}(x_t, \mathbf{x}_{t-1}^{(p)}) = \sqrt{\frac{\text{cov}(x_t, \tilde{x}_{t,A})}{\text{var}(x_t)}}. \quad (21)$$

Straightforward algebraic manipulation leads to the expression of DGC as a function of multiple correlation,

$$F_{y \rightarrow x} = \ln \frac{1 - \text{mcorr}(x_t, \mathbf{x}_{t-1}^{(p)})^2}{1 - \text{mcorr}\left(x_t, \begin{bmatrix} \mathbf{x}_{t-1}^{(p)} \\ \mathbf{y}_{t-1}^{(p)} \end{bmatrix}\right)^2}. \quad (22)$$

Measuring conditional and partial influences

Conditional GGC measures extend IGC and DGC to account for influence mediated by a third timeseries, z . Given the above demonstration of IGC and DGC as functions of partial and multiple correlation, it is no surprise that similar relationships exist for CIGC, $F_{x,y|z}$, and CDGC, $F_{y \rightarrow x|z}$, as follows:

$$F_{x,y|z} = \ln\left(1 - \text{corr}(x_t, y_t | \mathbf{x}_{t-1}^{(p)}, \mathbf{y}_{t-1}^{(p)}, \mathbf{z}_{t-1}^{(p)})^2\right), \quad (23)$$

$$F_{y \rightarrow x|z} = \ln \frac{\text{var}(x_t | \mathbf{x}_{t-1}^{(p)}, \mathbf{z}_{t-1}^{(p)})}{\text{var}(x_t | \mathbf{x}_{t-1}^{(p)}, \mathbf{y}_{t-1}^{(p)}, \mathbf{z}_{t-1}^{(p)})}, \quad (24)$$

and

$$F_{y \rightarrow x|z} = \ln \frac{1 - \text{mcorr} \left(x_t, \begin{bmatrix} \mathbf{x}_{t-1}^{(p)} \\ \mathbf{z}_{t-1}^{(p)} \end{bmatrix} \right)^2}{1 - \text{mcorr} \left(x_t, \begin{bmatrix} \mathbf{x}_{t-1}^{(p)} \\ \mathbf{y}_{t-1}^{(p)} \\ \mathbf{z}_{t-1}^{(p)} \end{bmatrix} \right)^2}. \quad (25)$$

To complete the set of GGC measures, we demonstrate that partial directed Granger causality can be expressed in a succinct correlation-based form. Recall that PDGC, $F_{y \rightarrow x|z}$, removes the influence of a third timeseries, z , as in CDGC, and additionally removes any remaining covariance between the noise processes. Algebraic manipulation of the relevant terms results in the relationship,

$$F_{y \rightarrow x|z} = F_{y \rightarrow x|z} + F_{x \rightarrow z|y} - F_{x \rightarrow z}. \quad (26)$$

The equivalence between GGC and correlation-based statistical inference tests

Having established that GGC measures are nonlinear functions of partial and multiple correlation, it is reasonable to question whether statistical inference tests of GGC measures provide additional sensitivity and/or specificity in comparison with traditional tests of correlation-based measures. It is our conjecture that this is not the case; the respective inference tests provide the same information, thereby rendering GGC connectivity methodology equivalent to correlation-based approaches. The derivations of all of the following relationships are in Appendix B.

Testing instantaneous influence

Consider the null hypothesis of zero instantaneous influence between x and y after removing influence from past samples,

$$H_0 : \rho_{x_t, y_t | \mathbf{x}_{t-1}^{(p)}, \mathbf{y}_{t-1}^{(p)}} = 0,$$

where ρ denotes true correlation. Under this hypothesis, the IGC test statistic, $S_{x,y}$, is distributed according to a $\chi^2(1)$ distribution [Eq. (13)]. Inference testing of partial correlation, on the other hand, typically employs Fisher's z-transform [Fisher, 1924],

$$r(\text{corr}(x_t, y_t | \mathbf{x}_{t-1}^{(p)}, \mathbf{y}_{t-1}^{(p)})) = \frac{1}{2} \ln \frac{1 + \text{corr}(x_t, y_t | \mathbf{x}_{t-1}^{(p)}, \mathbf{y}_{t-1}^{(p)})}{1 - \text{corr}(x_t, y_t | \mathbf{x}_{t-1}^{(p)}, \mathbf{y}_{t-1}^{(p)})}, \quad (27)$$

with r asymptotically Gaussian,

$$r(\text{corr}(x_t, y_t | \mathbf{x}_{t-1}^{(p)}, \mathbf{y}_{t-1}^{(p)})) \sim N \left(r(\rho_{x_t, y_t | \mathbf{x}_{t-1}^{(p)}, \mathbf{y}_{t-1}^{(p)}}), \frac{1}{N} \right). \quad (28)$$

Here $r(\rho_{x_t, y_t | \mathbf{x}_{t-1}^{(p)}, \mathbf{y}_{t-1}^{(p)}})$ denotes Fisher's z-transformation of the true partial correlation, $\rho_{x_t, y_t | \mathbf{x}_{t-1}^{(p)}, \mathbf{y}_{t-1}^{(p)}}$. The primary purpose

of Fisher's z-transform in this context is to render the variance of partial correlation independent of the true partial correlation. For notational simplicity, we will henceforth refer to $r(\text{corr}(x_t, y_t | \mathbf{x}_{t-1}^{(p)}, \mathbf{y}_{t-1}^{(p)}))$ as r .

A comparison of Eqs. (13) and (28), using the change of variables theorem [Papoulis and Pillai, 2002], demonstrates that the statistical inference tests for IGC and partial correlation are approximately equivalent,

$$p_{\chi^2}(S_{x,y}) \cong p_N(r) \left| \frac{dr}{dS_{x,y}} \right|. \quad (29)$$

Testing directed influence

The null hypothesis of zero directed influence from $\mathbf{y}_{t-1}^{(p)}$ to x_t is

$$H_0 : \mathbf{b} = \mathbf{0},$$

which follows immediately from the VAR(p) model definition [Eqs.(2) and (3)]. Under this hypothesis DGC is tested for significance via the test statistic $S_{y \rightarrow x}$, asymptotically distributed as a $\chi^2(p)$ distribution [Eq. (14)]. Having derived DGC as both a function of partial variance [Eq. (19)] and a function of multiple correlation [Eq. (22)], we require test statistics for partial variance and multiple correlation, which will be denoted q and h , respectively, and their distributions.

As an AR(p) model is a reduced parameter model nested within the VAR(p) class, it is known that [Greene, 2008]

$$q = \frac{\frac{\text{var}(e_{t,A}) - \text{var}(e_{t,V})}{p}}{\frac{\text{var}(e_{t,V})}{N}} \sim F(p, N), \quad (30)$$

and

$$h = \frac{\text{mcorr} \left(x_t, \begin{bmatrix} \mathbf{x}_{t-1}^{(p)} \\ \mathbf{y}_{t-1}^{(p)} \end{bmatrix} \right)^2 - \text{mcorr} \left(x_t, \mathbf{x}_{t-1}^{(p)} \right)^2}{1 - \text{mcorr} \left(x_t, \begin{bmatrix} \mathbf{x}_{t-1}^{(p)} \\ \mathbf{y}_{t-1}^{(p)} \end{bmatrix} \right)^2} \frac{N}{p} \sim F(p, N). \quad (31)$$

A direct comparison of the distributions for DGC, partial variance, and multiple correlation via change of variables demonstrates asymptotic equivalence, denoted by $\cong$, as follows,

$$p_{\chi^2}(S_{y \rightarrow x}) \cong p_F(q) \left| \frac{dq}{dS_{y \rightarrow x}} \right|, \quad (32)$$

$$p_{\chi^2}(S_{y \rightarrow x}) \cong p_F(h) \left| \frac{dh}{dS_{y \rightarrow x}} \right|. \quad (33)$$

Methods

Probability distribution generation

The equivalence relationship between the IGC and partial correlation test variates [Eq. (29)] was validated

numerically using MATLAB. Probability distributions were generated according to the known tests for IGC [Eq. (13)] and Fisher's z -transformation of partial correlation [Eq. (28)]. To incorporate the effect of degrees of freedom on the $\chi^2(1)$ distribution of IGC, the range of $F_{x,y}$ was made constant and scaled by various degrees of freedom values, $N \in \{50, 200, 350, 500\}$. The equivalence relationship [Eq. (29)] was then derived. The differences between the known and derived distributions for IGC, as a function of timeseries length, were calculated.

The equivalence between DGC and partial variance [Eq. (32)] was similarly validated numerically using MATLAB. Probability distributions were generated for the known $\chi^2(p)$ distribution of the DGC test statistic, for $p \in \{1, 2, 3, 4\}$. The effect of degrees of freedom on the $\chi^2(p)$ distribution of $S_{y \rightarrow x}$ was modeled by fixing the range of $F_{y \rightarrow x}$ and scaling by degrees of freedom values, $N \in \{50, 200, 350, 500\}$. The distribution of the DGC test was then derived from partial variance, as described in Eq. (32). The differences between the known and derived distributions for DGC were calculated.

Synthetic data generation and analysis

MATLAB was used to generate simulation data for bivariate VAR models. Totally, 5,000 datasets were generated for each unique model configuration based on model order, p , noise covariance, δ , timeseries length, T , and directed influence regression coefficient vector, $\mathbf{b}$, such that $p \in \{1, 2, 3, 4\}$, $T \in \{50, 100, 200\}$, $\delta \in \{0, 0.2, 0.4, 0.6\}$, and $\mathbf{b} \in \{0, \text{random}\}$. Non-zero regression coefficients were randomly generated and stability of the resulting model validated before use.

Receiver operating characteristic (ROC) curves were generated for each statistical inference test to compare the sensitivity and specificity of influence measures across the bivariate VAR model configurations. For instantaneous influence, a null hypothesis of zero instantaneous influence was defined by $H_0 : \delta = 0$, while an alternative hypothesis was one of $H_1 : \delta = 0.2$, $H_1 : \delta = 0.4$, or $H_1 : \delta = 0.6$. ROC curves were generated for IGC, tested for significance using a $\chi^2(1)$ test [Eq. (13)], and for correlation and partial correlation, both tested for significance using Fisher's z -transformation [Eq. (28)]. For directed influence measures, a null hypothesis of zero directed influence was defined by $H_0 : \mathbf{b} = \mathbf{0}$, while the alternative hypothesis was defined by $H_1 : \mathbf{b} \neq \mathbf{0}$. ROC curves were generated for DGC, tested as a $\chi^2(p)$ variate [Eq. (14)], and for partial variance and multiple correlation, both tested using F distributed statistics [Eqs. (30) and (31)].

For each model configuration, the associated datasets were used to generate empirical null distributions for each measure of dependence using three different resampling procedures: randomization [e.g., Cordes et al., 2000; Deshpande et al., 2008], permutation [e.g., Kayser et al., 2009], and a bootstrapping surrogate analysis [Theiler et al., 1990] [e.g., Guo et al., 2008; Roebroek et al., 2005], which

we will term bootstrapping. For each resampling method, the influence contained in a dataset was eliminated in a process applicable to that method: for randomization, the phase of each timeseries was randomized to retain the power spectrum while destroying temporal information; for permutation, the time point labels were exchanged by transposing the first and second halves of one of the timeseries; for bootstrapping, the maximum likelihood parameters were determined and a new dataset generated using the MLE model parameters, but with the parameters controlling influence modified to satisfy the null hypothesis. The influence measures were calculated for the modified datasets satisfying the null hypothesis and the results from 5,000 datasets were amalgamated to create an empirical null distribution for each influence measure. The significance of an influence measure calculated for a dataset was determined by the percentage of values in the empirical null distribution that were further from the null hypothesis than the given value.

For each influence measure, both instantaneous and directed, three ROC curves were generated using the empirical null distributions resulting from the three different resampling techniques: randomization, permutation, and bootstrapping.

RESULTS

The relationships detailed in the Theoretical Results section demonstrate that GGC measures of linear dependence evaluate the same aspects of a LGM as partial correlation and partial variance, for instantaneous and directed influence, respectively. We now present numerical findings that verify the equivalence between statistical distributions, together with the results from applying GGC and correlation-based measures to synthetic data.

Validation of Statistical Inference Test Equivalence

The equivalence between probability density functions of $S_{x,y}$ (Fig. 1A) and Fisher's z -transform of partial correlation (Fig. 1B) is demonstrated across a range of timeseries lengths using a transformation of variable according to Eq. (29) (Fig. 1C,D). The error (difference) between actual and derived probability density functions is calculated (Fig. 1E,F). Errors between the actual and derived probability density functions for $S_{x,y}$ and Fisher's z -transform of partial correlation have magnitudes on the order of 10^{-3} and 10^{-4} , respectively, diminishing with increasing degrees of freedom; clearly the statistical inference tests for partial correlation and IGC are asymptotically equivalent with timeseries length.

Asymptotic equivalence between probability density functions of $S_{y \rightarrow x}$ and the inference test for partial variance, q [Eq. (30)], was verified as follows. The probability distribution of $S_{y \rightarrow x}$, as a function of model order, p , and

Figure 1.

Equivalence of instantaneous influence test distributions. **A:** Distribution of the IGC test statistic, $p_{\chi^2}(S_{x,y})$ (Eq. (13)). **B:** Distribution of Fisher's z-transform of partial correlation, $p_N(r)$ (Eq. (28)). **C:** $p_N(r) \left| \frac{dr}{dS_{x,y}} \right|$. **D:** $p_{\chi^2}(S_{x,y}) \left| \frac{dS_{x,y}}{dr} \right|$. **E:** Difference between A and C. **F:** Difference between B and D.

degrees of freedom, N (Fig. 2A), was derived using a change of variables according to Eq. (32) (Fig. 2B). The equivalence relationship was verified by the error between the actual and derived probability density functions of $S_{y \rightarrow x}$, with magnitudes on the order of 10^{-3} and diminishing with degrees of freedom (Fig. 2C). Furthermore, for order $p = 2$ the distributions are identical for all values of N , while further increases in p increase the magnitude of error. This finding was predicted by the theory, as detailed fully in the derivation of Eq. (32), found in Appendix B.

Application of GGC and Correlation-Based Measures

To gain insight into how the GGC and correlation-based connectivity measures perform, we examined the statistical inference tests available for each measure using synthetic data.

The three resampling techniques, randomization, permutation, and bootstrapping, showed negligible difference across all inference measures and VAR models, to the extent that it was not possible to differentiate between the individual empirical ROC curves. We have, therefore, only included the results for a single resampling method, randomization, to improve figure clarity.

ROC curves illustrating the performance of instantaneous influence measures are shown in Figure 3. The effects of both noise covariance, δ (Fig. 3A), and timeseries length,

T (Fig. 3B), on the performance of each influence measure was examined using VAR(0) datasets. Performance was found to be strongly and positively correlated with both noise covariance and sample size. Importantly, a single ROC curve is visible for each distinct model examined, clearly demonstrating the negligible difference between the instantaneous tests and the similar dependence on noise covariance and timeseries length. Furthermore, the close agreement between analytic and empirical tests supports the veracity of the analytic distributions, while the agreement between the empirical results of testing each measure reinforces the asymptotic equivalence between statistical inference tests for IGC and partial correlation [Eq. (29)].

A slight performance difference between empirical and analytic tests was observed for large noise covariance values, diminishing with sample size, suggesting an asymptotic equivalence between empirical and analytic tests. This effect is illustrated in Figure 3C,D, which both use an enlarged false positive axis for ROC curves generated from VAR(4) datasets with 50 and 200 samples, respectively. The strong performance of partial correlation and IGC in the presence of lagged influence demonstrates how effectively directed influence was regressed out when model order was assumed known. Conversely, (lag-0) correlation does not model directed influence so its performance was, predictably, severely impacted for non-zero model orders (Fig. 3C,D).

Figure 2.

Equivalence of directed influence test distributions. Probability density functions derived from model orders, $p \in \{1, 2, 3, 4\}$, indicated by row. **A.** Distribution of the DGC test statistic, $p_{\chi^2}(S_{y \rightarrow x})$ (Eq. (14)). **B.** $p_F(q) \left| \frac{dq}{dS_{y \rightarrow x}} \right|$. **C.** Difference between A and B.

The ROC curves generated from directed influence tests are shown in Figure 4. For each model, zero directed influence was tested using the χ^2 DGC test, the F distributed multiple correlation test, and the F distributed test for partial variance. For each distinct model, a single ROC curve is visible for all inference tests, both analytic and empirical (Fig. 4A,B), and all measures showed identical dependence on sample size. The agreement between the analytic measures supports equivalence of the directed influence tests [Eqs. (32) and (33)], while that of the empirical null distributions substantiates the mapping between directed influence measures [Eqs. (19) and (22)]. Furthermore, the negligible difference between theoretical and empirical tests verifies the analytic expressions.

Zoomed-in axes are shown for ROC curves generated from VAR(4) datasets to highlight the marginal difference between the probability density functions of $S_{y \rightarrow x}$ and the test variates for both partial variance and multiple correlation (Fig. 4C,D). The increased error between the probability density functions generated with $N = 50$ (Fig. 4C) and $N = 200$ (Fig. 4D) supports the theoretical finding of asymptotic equivalence with time series length [Eq. (32)]. The ROC curves of directed influence tests did not change with noise covariance, which is expected because the measures are functions of lagged influence, expressed in the regressors of a LGM.

The ROC curves for both instantaneous and directed influence measures confirm the theoretical findings. The ROC curves for conditional measures exhibit the same behavior as those for the nonconditional measures as the only change is the addition of regression term(s) for the third variate (data not shown).

DISCUSSION

Linear connectivity analyses based on the concepts of Granger causality take on characteristics of both data-driven functional connectivity methods and model-based effective connectivity methods [Goebel et al., 2003]. The data-driven method class is comprised of correlation-based techniques including partial correlation and partial variance, which measure the degree of linear association between variates [Marrelec et al., 2006; Tana et al., 2008; Wang and Xia, 2007]. Instances of the model-based method class are attributed with the ability to reveal directional dependencies in the data [Abler et al., 2005; Eichler, 2005; Goebel et al., 2003]. In this article we have demonstrated that Geweke's measures of Granger causality [Geweke, 1982], along with conditional and partial Granger causality measures, can all be expressed as functions of correlation-based quantities. Furthermore, we have shown that the relevant correlation-based and

Figure 3.

Sensitivity and specificity characteristics of instantaneous influence for bivariate VAR models, comparing IGC (green), partial correlation (PC, red) and correlation (corr, blue). Legend is applicable to all figures. Where a single ROC plot is visible for a model, all inference tests performed with negligible difference. Note the enlargement of axis in C and D to enable distinction between analytic and empirical inference tests. **A:** VAR(0) datasets with $T = 50$, and three difference noise covariance values, such that: $H_0 : \delta = 0$ inner curve, $H_1 : \delta = 0.2$; middle curve,

$H_1 : \delta = 0.4$; outer curve, $H_1 : \delta = 0.6$. **B:** VAR(0) datasets with $H_0 : \delta = 0$, $H_1 : \delta = 0.2$ and timeseries length: inner curve, $T = 50$; middle curve, $T = 100$; outer curve, $T = 200$. **C:** VAR(4) data, with $T = 50$, and $H_0 : \delta = 0$, $H_1 : \delta = 0.6$. Note that the curves for IGC and partial correlation are indistinguishable. **D:** VAR(4) data, with $T = 200$, and $H_0 : \delta = 0$, $H_1 : \delta = 0.6$. IGC and partial correlation are indistinguishable. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

GGC statistical inference tests are, in both theory and practice, equivalent. These results call into question the utility of Granger causality in connectivity analyses, beyond that elucidated using traditional correlation-based metrics.

It should be noted that while the use of correlation-based measures imply an instance of the LGM class, application of GGC measures are not restricted by this implicit model assumption. For a non-Gaussian noise process comprised of independent and identically distributed samples,

Figure 4.

Sensitivity and specificity characteristics of directed influence for bivariate VAR models, under the hypothesis $H_0 : \mathbf{b} = 0$ $H_0 : \mathbf{b} = 0$, $H_1 : \mathbf{b} \neq 0$, comparing multiple correlation (MC, red), partial variance (PV, blue), and DGC (green). Where a single curve is visible all inference tests performed with negligible difference. Legend is applicable to all panels. **A:** Bivariate VAR(2) datasets with $T = 100$, and three different model orders: inner curve, $p = 2$; middle curve, $p = 1$; outer curve, $p = 4$. **B:** VAR(2) datasets for three

different timeseries length: inner curve, $T = 50$; middle curve, $T = 100$; outer curve, $T = 200$. **C:** VAR(4) data, with $T = 50$. The curves for multiple correlation and partial variance are indistinguishable. **D:** VAR(4) data, with $T = 200$, with results for multiple correlation and partial variance indistinguishable. Note zoomed axes in C and D highlight differences between methods. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

the GGC test variates can still be described by an asymptotic chi-square distribution [Geweke, 1982]. Thus for highly non-Gaussian noise that may be present in experimental BOLD data, it is anticipated that the GGC inference tests would remain more robust than the correlation-based

equivalents. Theoretical analysis of robustness to model/test mismatch, and analysis of the necessarily ill-posed multivariate models that extend from the bivariate models considered herein, are beyond the scope of this article and the subject of ongoing research.

A corollary of the equivalence results demonstrated in this article is the ubiquity of partial correlation and partial variance in linear connectivity methods. Partial correlation and partial variance can describe both directed and instantaneous influence, optionally in the presence of exogenous inputs for a LGM of arbitrary dimensionality and order, limited only by the fMRI timeseries length. In the time domain, correlation is a trivial case of partial correlation, and the equivalence between multiple correlation [Wang and Xia, 2007] and partial variance is well known [Greene, 2008; Whittaker, 2009]. Frequency domain linear connectivity methods fall under the same umbrella, as Marrelec et al. [2005] have shown that correlation, coherence [Sun et al., 2004], and mutual information [Salvador et al., 2005] extract the same information from a LGM. Marrelec et al. [2006] introduced spatial partial correlation as a fMRI connectivity method, applying it to models incorporating only instantaneous influence, which were thus unable to discern directionality. They observed that spatial partial correlation has both data-driven attributes, as it is correlation-based, and model-based attributes, such as the requirement to perform node selection. Marrelec et al. [2007, p. 1188] later concluded that spatial partial correlation “provides a pertinent first step to bridge the gap that has endured between functional and effective connectivity”. This article augments this statement, by establishing equivalence between GGC analysis procedures and their spatio-temporal correlation-based counterparts.

SEM is a model-driven method that employs a priori knowledge to determine the structure of a connectivity network, and then applies a linear regression model to estimate the strength of connections via regression coefficients [McIntosh and Gonzalez-Lima, 1994]. SEM employs a goodness-of-fit test to guide node selection that can alternatively be approached using a likelihood ratio test [Kline, 2005; Schumacker and Lomax, 2004]. Under the assumption of Gaussian noise, the likelihood ratio test is equivalent to the DGC test [Geweke, 1982] and is, therefore, asymptotically equivalent to the partial variance test by Eq. (32), further bridging the gap between classes of connectivity methods. However, SEM differs from the linear connectivity measures discussed herein in its use of a scalar regression coefficient to denote influence from a source to a target region. To maintain a scalar description of influence from the source to the target the inclusion of lagged influence is precluded, as it requires multiple regression terms for the source. SEM is, therefore, not applicable to all instances of the LGM class. Furthermore, regression coefficients should not be used to designate the strength of linear association because they are unit dependent [Eichler, 2005]. Harrison et al. [2003] used regression coefficients to designate connection strength in a VAR(p) model incorporating multiple influence regressors. Consequently, there were multiple connection coefficients for each source. The graphical tabulation of the multiple estimated coefficients is both cumbersome to summarize for large models and, we believe, potentially misleading;

the coefficients of a VAR(p) model form characteristic polynomials, the roots (poles) of which determine system behavior, not the absolute weights of individual coefficients.

Conditional and/or partial connectivity methods, whether presented as Granger causality or correlation-based, and similarly SEM, apply to a single multivariate model that necessitates a node selection process due to the small timeseries length relative to the number of voxels. In contrast, seed correlation and pairwise Granger causality methods apply many bivariate models in the search for connectivity networks, thereby eliminating the need for node selection but also rescinding the capacity to remove mediated influence. It is this a priori classification of network model type, in conjunction with the choice to model instantaneous and/or directed influence via selection of model order, that differentiates linear connectivity methods, rather than the distinguishing feature being a methodological difference in influence measure or statistical inference test.

CONCLUSION

The equivalence of Granger causality connectivity methods and correlation-based connectivity methods has been delineated in this article. Geweke's measures of Granger causality provide a framework in which to describe linear connectivity, but offer no further insight beyond that offered by correlation-based techniques. We have clarified distinctions, both perceived and real, between linear connectivity methods in their application to instantiations of the linear Gaussian model class, and we have highlighted the centrality of correlation-based measures in all linear connectivity methods. While these theoretical results may be unsurprising to the reader with advanced statistical knowledge, they highlight the importance of a clear understanding of the theoretical basis of connectivity analysis methods for human brain mapping researchers. Our conclusion is that assignment of a linear Gaussian connectivity analysis method into one or more of the data-driven, model-based, and Granger causality classes is ultimately a function of node selection and model configuration, rather than a distinction rooted in methodological difference.

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APPENDIX A. DERIVATION OF GGC MEASURES AS CORRELATION-BASED FUNCTIONS

Derivation 1: Relationship Between IGC and Partial Correlation [Eq. (16)]

Rearranging Eq. (8) gives,

$$F_{x,y} = \ln \frac{\text{var}(e_{t,V})\text{var}(u_{t,V})}{\text{var}(e_{t,V})\text{var}(u_{t,V}) - \text{cov}(e_{t,V}, u_{t,V})^2} = -\ln \left(1 - \frac{\text{cov}(e_{t,V}, u_{t,V})^2}{\text{var}(e_{t,V})\text{var}(u_{t,V})} \right). \quad (\text{A1})$$

The result follows through employing the definition of partial correlation,

$$\text{corr}(x_t, y_t | \mathbf{x}_{t-1}^{(p)}, \mathbf{y}_{t-1}^{(p)}) = \frac{\text{cov}(e_{t,V}, u_{t,V})}{\text{var}(e_{t,V})\text{var}(u_{t,V})}. \quad (\text{A2})$$

It follows trivially from the equivalence mapping between IGC and partial correlation [Eq. (16)] that, for order-zero VAR models, instantaneous Granger causality is a remapping of correlation,

$$F_{x,y} = -\ln(1 - \text{corr}(x_t, y_t)^2).$$

Derivation 2: Relationship Between DGC and Partial Variance [Eq. (19)]

The equivalence of DGC to a log ratio of partial variances is an immediate consequence of the definition of partial variance in a LGM [Davidson and MacKinnon, 2004; Greene, 2008]. For completeness, using the definition of partial variance for the VAR model,

$$\text{var}(x_t | \mathbf{x}_{t-1}^{(p)}, \mathbf{y}_{t-1}^{(p)}) = \text{var}(e_{t,V}), \quad (\text{A3})$$

and for the AR model,

$$\text{var}(x_t | \mathbf{x}_{t-1}^{(p)}) = \text{var}(e_{t,A}), \quad (\text{A4})$$

it follows that

$$F_{y \rightarrow x} = \ln \frac{\text{var}(x_t | \mathbf{x}_{t-1}^{(p)})}{\text{var}(x_t | \mathbf{x}_{t-1}^{(p)}, \mathbf{y}_{t-1}^{(p)})} = \ln \frac{\text{var}(e_{t,A})}{\text{var}(e_{t,V})}. \quad (\text{A5})$$

Derivation 3: Relationship Between DGC and Multiple Correlation [Eq. (22)]

Recall that sample estimates are denoted by $\hat{\cdot}$, and the prediction of x_t using the VAR(p) (AR(p)) model is designated $\tilde{x}_{t,V}$ ($\tilde{x}_{t,A}$). Consider first an AR(p) model in which $\mathbf{x}_{t-1}^{(p)}$ is used to predict x_t , and coefficient vectors $\mathbf{b} = \mathbf{c} = \mathbf{0}$. From the properties of covariance [Whittaker, 2009],

$$\begin{aligned} \text{var}(\tilde{x}_{t,A}) &= \text{cov}(\tilde{x}_{t,A}, \tilde{x}_{t,A}) \\ &= \text{cov}(\tilde{x}_{t,A}, x_t - e_t) \\ &= \text{cov}(\tilde{x}_{t,A}, x_t) - \text{cov}(\tilde{x}_{t,A}, e_t) \\ &= \text{cov}(\tilde{x}_{t,A}, x_t), \end{aligned} \quad (\text{A6})$$

where the last line results from the orthogonality of the predictor and residual timeseries.

Applying this relationship to the definition of multiple correlation [Eq. (21)] gives,

$$\text{mcorr}(x_t, \mathbf{x}_{t-1}^{(p)}) = \sqrt{\frac{\text{var}(\tilde{x}_{t,A})}{\text{var}(x_t)}}. \quad (\text{A7})$$

The total signal variance can be decomposed into orthogonal components [Hamilton, 1994],

$$\text{var}(x_t) = \text{var}(\tilde{x}_{t,A}) + \text{var}(e_{t,A}), \quad (\text{A8})$$

so that multiple correlation can be expressed in terms of residual variance [Whittaker, 2009],

$$\text{mcorr}(x_t, \mathbf{x}_{t-1}^{(p)}) = \sqrt{1 - \frac{\text{var}(e_{t,A})}{\text{var}(x_t)}}. \quad (\text{A9})$$

Similarly, multiple correlation for the VAR model can be expressed as,

$$\text{mcorr}\left(x_t, \begin{bmatrix} \mathbf{x}_{t-1}^{(p)} \\ \mathbf{y}_{t-1}^{(p)} \end{bmatrix}\right) = \sqrt{1 - \frac{\text{var}(e_{t,V})}{\text{var}(x_t)}}. \quad (\text{A10})$$

Consequently, a ratio of squared multiple correlation estimates for the VAR(p) and AR(p) models can be written,

$$\frac{1 - \text{mcorr}\left(x_t, \mathbf{x}_{t-1}^{(P)}\right)^2}{1 - \text{mcorr}\left(x_t, \begin{bmatrix} \mathbf{x}_{t-1}^{(P)} \\ \mathbf{y}_{t-1}^{(P)} \end{bmatrix}\right)^2} = \frac{\text{var}(e_{t,A})}{\text{var}(e_{t,V})}. \quad (\text{A11})$$

The result follows from employing the equivalence relationship between DGC and partial variance [Eq. (19)] in conjunction with the definition of partial variance [Eqs. (A3) and (A4)].

In the following derivation of conditional Granger causality measures, the Frisch-Waugh-Lovell theorem is used, which states that regression can be applied sequentially, allowing the influence of a third signal to be removed prior to measuring dependence between two variates [Lovell, 2008].

Derivation 4: Relationship Between CIGC and Partial Correlation, [Eq. (23)]

The derivation follows directly from Eq. (16) and application of the Frisch-Waugh-Lovell theorem, by removing the influence of $\mathbf{z}_{t-1}^{(p)}$, and then subsequently applying Eq. (16).

Derivation 5: Relationship Between CDGC and Partial Variance [Eq. (24)]

The derivation follows directly from Eq. (19) and application of the Frisch-Waugh-Lovell theorem.

Derivation 6: Relationship Between CDGC and Multiple Correlation [Eq. (25)]

The derivation follows directly from Eq. (22) and application of the Frisch-Waugh-Lovell theorem.

Derivation 7: Relationship Between PDGC and Directed and Instantaneous Granger Causality Measures [Eq. (26)]

Recall that w_t denotes the residual of z_t . The result follows from direct substitution of the MLE estimation of the residual [Eqs. (4)–(6)] and the definition of partial correlation [Eq. (A2)] as follows,

$$\begin{aligned} F_{y \rightarrow x|z} &= \ln \frac{\text{var}(e_{t,V_{xz}}|w_t, V_{xz})}{\text{var}(e_{t,V_{xyz}}|w_t, V_{xyz})} \\ &= \ln \frac{\text{var}(e_{t,V_{xz}}) - \text{cov}(e_{t,V_{xz}}, w_t, V_{xz}) \text{var}(w_t, V_{xz})^{-1} \text{cov}(w_t, V_{xz}, e_{t,V_{xz}})}{\text{var}(e_{t,V_{xyz}}) - \text{cov}(e_{t,V_{xyz}}, w_t, V_{xyz}) \text{var}(w_t, V_{xyz})^{-1} \text{cov}(w_t, V_{xyz}, e_{t,V_{xyz}})} \\ &= \ln \frac{\text{var}(e_{t,V_{xz}})(1 - \text{corr}(e_{t,V_{xz}}, w_t, V_{xz})^2)}{\text{var}(e_{t,V_{xyz}})(1 - \text{corr}(e_{t,V_{xyz}}, w_t, V_{xyz})^2)} \\ &= \ln \frac{\text{var}(e_{t,V_{xz}})}{\text{var}(e_{t,V_{xyz}})} - \ln \left(\frac{1 - \text{corr}(e_{t,V_{xz}}, w_t, V_{xz})^2}{1 - \text{corr}(e_{t,V_{xyz}}, w_t, V_{xyz})^2} \right) \\ &= F_{y \rightarrow x|z} + F_{x \rightarrow z|y} - F_{x \rightarrow z}, \end{aligned} \quad (\text{A12})$$

as required.

APPENDIX B: DERIVATION OF EQUIVALENCE BETWEEN INFERENCE TESTS

Derivation 8: Equivalence Between Statistical Inference Tests for IGC and Partial Correlation [Eq. (29)]

This proof begins by deriving an analytic relationship between the test variates for IGC and partial correlation, which is followed by establishing distributional equivalence.

In the Theoretical Results section, IGC was shown to be a remapping of partial correlation [Eq. (16)]. We approximate this function using a fourth order Taylor series expansion. To aid readability, let $\rho = \rho_{x_t, y_t | \mathbf{x}_{t-1}^{(p)}, \mathbf{y}_{t-1}^{(p)}}$ so that Eq. (16) becomes $F_{x,y} = -\ln(1 - \rho^2)$. The Taylor series expansion of IGC is then

$$\begin{aligned} F_{x,y} &\approx F_{x,y}(\rho_0) + \left. \frac{dF_{x,y}}{d\rho} \right|_{\rho=\rho_0} (\rho - \rho_0) + \left. \frac{d^2F_{x,y}}{d\rho^2} \right|_{\rho=\rho_0} \frac{(\rho - \rho_0)^2}{2} \\ &\quad + \left. \frac{d^3F_{x,y}}{d\rho^3} \right|_{\rho=\rho_0} \frac{(\rho - \rho_0)^3}{3!} + \left. \frac{d^4F_{x,y}}{d\rho^4} \right|_{\rho=\rho_0} \frac{(\rho - \rho_0)^4}{4!}, \end{aligned} \quad (\text{B1})$$

where ρ_0 is the centre point of the Taylor expansion. The derivative functions can easily be determined as

$$\begin{aligned} \frac{dF_{x,y}}{d\rho} &= -\frac{2\rho}{\rho^2 - 1}, \\ \frac{d^2F_{x,y}}{d\rho^2} &= -\frac{4\rho^2}{(\rho^2 - 1)^2} + \frac{2}{1 - \rho^2}, \\ \frac{d^3F_{x,y}}{d\rho^3} &= \frac{12\rho}{(\rho^2 - 1)^2} + \frac{16\rho^3}{(\rho^2 - 1)^3}, \\ \frac{d^4F_{x,y}}{d\rho^4} &= \frac{12\rho}{(\rho^2 - 1)^2} - \frac{96\rho^2}{(\rho^2 - 1)^3} + \frac{96\rho^4}{(\rho^2 - 1)^4}. \end{aligned}$$

Under the null hypothesis of $H_0 : \rho = 0$, the expected value for sample partial correlation is 0, and hence it is intuitive to centre the Taylor expansion at $\rho_0 = 0$, so that Eq. (B1) becomes,

$$F_{x,y} \approx \rho^2. \quad (\text{B2})$$

Following the same process to approximate Fisher's z-transformation from partial correlation results in,

$$r \approx \rho + \frac{\rho^3}{3} + \frac{\rho^4}{6}. \quad (\text{B3})$$

Under the null hypothesis of zero partial correlation, sample partial correlation values are expected to be small, and hence this approximation can be further reduced to,

$$r \approx \rho. \quad (\text{B4})$$

Applying this result to Eq. (B2) immediately gives,

$$F_{x,y} \approx r^2. \quad (\text{B5})$$

Recall that the test statistic of IGC is denoted by $S_{x,y} = NF_{x,y}$, so that,

$$S_{x,y} = h(r) \approx Nr^2, \quad (\text{B6})$$

which has derivative,

$$h'(r) = \frac{dS_{x,y}}{dr} = 2Nr. \quad (\text{B7})$$

Under the null hypothesis the distribution of the partial correlation test variate is given by

$$r \sim N\left(0, \frac{1}{N}\right), \quad (\text{B8})$$

so that

$$p_N(r) = \sqrt{\frac{N}{2\pi}} e^{-\frac{Nr^2}{2}}. \quad (\text{B9})$$

Deriving the distribution of $S_{x,y}$ from r utilizing a change of variable from Eqs. (B6) and (B7) gives [Papoulis and Pillai, 2002]

$$\begin{aligned} p(S_{x,y}) &= 2p_N(r) \left| \frac{1}{h'(r)} \right| \\ &\approx 2\sqrt{\frac{N}{2\pi}} e^{-\frac{NS_{x,y}}{2}} \left| \frac{1}{2\sqrt{NS_{x,y}}} \right| \\ &\approx \frac{1}{\sqrt{2\pi S_{x,y}}} e^{-\frac{S_{x,y}}{2}}, \end{aligned} \quad (\text{B10})$$

where the factor of 2 is a consequence of the noninjective mapping from r to $S_{x,y}$, and the approximation is a result of Eq. (B5).

It now remains to compare the distribution of $S_{x,y}$ derived from r to the known distribution of $S_{x,y}$. Under the null hypothesis the distribution of $S_{x,y}$ [Eq. (13)] is

$$p_{\chi^2}(S_{x,y}) = \frac{1}{\sqrt{2\pi S_{y \rightarrow x}}} e^{-\frac{S_{y \rightarrow x}}{2}}. \quad (\text{B11})$$

which coincides with the distribution of $S_{x,y}$ derived from Fisher's z-transformation of partial correlation [Eq. (B10)], and thus the proof is complete.

Derivation 9: Equivalence Between Statistical Inference Tests for DGC and Partial Variance [Eq. (32)]

The null hypothesis for zero directed influence can be expressed as $H_0 : \text{var}(e_V) = \text{var}(e_A)$ to determine if partial variance for the VAR(p) model significantly differs to partial variance for the AR(p) model [Eq. (30)]. The DGC test statistic, $S_{y \rightarrow x}$, is tested using Eq. (14).

Equivalence of the inference tests for zero directed influence via partial variance and DGC is examined by comparison

of the test distributions under a change of variables [Papoulis and Pillai, 2002]. Denote $N = T - 2p - 1$ as the degrees of freedom, and let p_{χ^2} and p_F denote chi-square and F probability density functions, respectively. We introduce

$$k = \frac{\text{var}(e_{t,A})}{\text{var}(e_{t,V})}, \quad (\text{B12})$$

so that $k = e^{\frac{S_{y \rightarrow x}}{N}}$. In the case of nested models with Gaussian noise, the test statistic for partial variance [Eq. (30)] can be expressed using Eq. (B12), as

$$q = (k - 1) \frac{N}{p}.$$

Rearranging and employing the equivalence relationship between DGC and partial variance [Eq. (19)] gives,

$$\begin{aligned} q &= \frac{N}{p} \left(e^{\frac{S_{y \rightarrow x}}{N}} - 1 \right) \\ &= g(S_{y \rightarrow x}), \end{aligned} \quad (\text{B13})$$

from which the derivative is obtained,

$$\frac{dq}{dS_{y \rightarrow x}} = g'(S_{y \rightarrow x}) = \frac{1}{p} e^{\frac{S_{y \rightarrow x}}{N}}. \quad (\text{B14})$$

Under the null hypothesis the distribution of the DGC test variate is given by

$$p_{\chi^2}(S_{y \rightarrow x}) = \frac{1}{2^{\frac{p}{2}} \Gamma(\frac{p}{2})} S_{y \rightarrow x}^{\frac{p}{2}-1} e^{-\frac{S_{y \rightarrow x}}{2}}. \quad (\text{B15})$$

Deriving the distribution of the partial variance test variate, q , from the DGC test variate, $S_{y \rightarrow x}$, utilizing a change of variable from Eqs. (B13) and (B14) gives [Papoulis and Pillai, 2002]

$$\begin{aligned} p(q) &= p_{\chi^2}(S_{y \rightarrow x}) \left| \frac{1}{g'(S_{y \rightarrow x})} \right| \\ &= \frac{(N \ln(\frac{qp}{N} + 1))^{\frac{p}{2}-1}}{2^{\frac{p}{2}} \Gamma(\frac{p}{2})} e^{-\frac{N}{2} \ln(\frac{qp}{N} + 1)} \left| \frac{Np}{qp + N} \right|. \end{aligned} \quad (\text{B16})$$

Under the null hypothesis of zero directed influence it is reasonable to introduce the assumption that $\text{var}(e_{t,V}) \gg \text{var}(e_{t,A}) - \text{var}(e_{t,V})$, so that the $\ln(\frac{qp}{N} + 1)$ term can be approximated by

$$\begin{aligned} \ln\left(\frac{qp}{N} + 1\right) &= \ln\left(\frac{\text{var}(e_{t,A}) - \text{var}(e_{t,V})}{\text{var}(e_{t,V})} + 1\right) \\ &\approx \frac{qp}{N}. \end{aligned}$$

Employing this approximation Eq. (B16) becomes,

$$p(q) \cong \frac{q^{\frac{p}{2}-1} p^{\frac{p}{2}}}{2^{\frac{p}{2}} \Gamma(\frac{p}{2})} \left(\frac{N}{qp + N} \right)^{\frac{N+2}{2}}. \quad (\text{B17})$$

It now remains to compare the distribution of q derived from $S_{y \rightarrow x}$ to the known distribution of q . Under the null hypothesis the distribution of q [Eq. (30)] expands to

$$p_F(q) = \frac{\sqrt{(qp)^p N^N}}{(qp+N)^{p+N}} q B\left(\frac{N}{2}, \frac{2}{2}\right), \quad (\text{B18})$$

where B is the beta function. For large N we can introduce the approximation

$$B\left(\frac{N}{2}, \frac{p}{2}\right) \approx \Gamma\left(\frac{p}{2}\right) \left(\frac{N}{2}\right)^{-\frac{p}{2}},$$

so that

$$p_F(q) \cong \frac{q^{\frac{p}{2}-1} p^{\frac{p}{2}}}{2^{\frac{p}{2}} \Gamma\left(\frac{p}{2}\right)} \left(\frac{N}{qp+N}\right)^{\frac{p+N}{2}}. \quad (\text{B19})$$

Comparing the known distribution of q [Eq. (B19)] with the distribution of q derived from $S_{y \rightarrow x}$ [Eq. (B17)] it can be seen that for a model order of $p = 2$ they are equivalent, and for all other model orders they are asymptotically equivalent with degrees of freedom, N , determined by timeseries length.

Derivation 10: Equivalence Between Statistical Inference Tests for DGC and Multiple Correlation [Eq. (33)]

This derivation proceeds by deriving a mapping between the test statistics for partial variance and multiple

correlation [Eqs. (30) and (31)]. Application of Eq. (32) then establishes the relationship between multiple correlation and DGC. A proof of the equivalence of the statistical tests for multiple correlation and partial variance can be found in Greene (2008, p.90), and is provided here for completeness. Using the multiple correlation coefficient, and denoting true multiple correlation by R , the null hypothesis for zero directed influence is expressed as, $H_0 : R_{x_t, \begin{bmatrix} x_{t-1}^{(p)} \\ y_{t-1}^{(p)} \end{bmatrix}} = R_{x_t, x_{t-1}^{(p)}}$. Equation (31) is employed to determine if multiple correlation in the VAR(p) model is significantly different from that of the AR(p) model. The equivalence of the test statistics for multiple correlation and partial variance is seen by substituting the relationship between directed influence measures [Eqs. (19) and (22)],

$$h = \frac{N}{p} \left(\frac{\text{var}(e_{t,A})}{\text{var}(e_{t,V})} - 1 \right), \quad (\text{B20})$$

and

$$\begin{aligned} q &= \frac{N}{p} \left(\frac{\text{var}(e_{t,A})}{\text{var}(e_{t,V})} - 1 \right) \\ &= h. \end{aligned} \quad (\text{B21})$$

Application of the asymptotic equivalence between the test statistics for DGC and partial variance [Eq. (32)] then completes the derivation.