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Minimal n -noids in hyperbolic and anti-de Sitter 3-space

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We construct minimal surfaces in hyperbolic and anti-de Sitter 3-space with the topology of a n -punctured sphere by loop group factorization methods. The end behaviour of the surfaces is based on the asymptotics of Delaunay-type surfaces, i.e. rotational symmetric minimal cylinders. The minimal surfaces in H^3 extend to Willmore surfaces in the conformal 3-sphere $S^3 = H^3 \cup S^2 \cup H^3$.

1. Introduction

The AdS/CFT correspondence predicts that the physics of the gravitational theory of anti-de Sitter (AdS) space-time is equivalent to the physics of conformal field theory (CFT) on the boundary of that space-time [1]. A particularly important instance is the computation of the Wilson loop expectation value, which by work of Maldacena is given by the (regularized) area of a space-like minimal surface in AdS space-time with the loop as boundary [2–5]. The aim of this paper is to provide new examples of minimal surfaces in anti-de Sitter spaces of non-trivial topological type and with several boundary components.

It is well known that the equations for minimal surfaces in anti-de Sitter space are related to Hitchin self-duality equations [3,6]. In particular, minimal surfaces in AdS_3 are given by solutions corresponding to points in the Hitchin component for rank 2, and minimal surfaces in a totally geodesic $H^3 \subset AdS_4$ are given by rank 2 solutions of the self-duality equations with nilpotent Higgs field. As those, they fall into the class of integrable PDEs, and there are powerful tools for computing large classes of examples [7–9]. On the other hand, it is hard to construct surfaces with non-trivial finitely

Figure 1. Cutaway views of equilateral minimal noids in $\mathbb{H}^3 \cup \mathbb{S}^2 \cup \mathbb{H}^3$ with end counts and cyclic symmetry orders 3 and 5. In all figures, the surface is stereographically projected to $\mathbb{R}^3$, the wireframe designates the ideal boundary and the lines on the surface are curvature lines.

generated topology which are complete, i.e. the surface can be continued to the boundary at infinity of the anti-de Sitter space, and the intersection is a finite number of topological circles. It is worth remarking that surfaces given by global solutions of the self-duality equations are not of that much interest in the AdS/CFT correspondence as the extrinsic monodromy is always non-trivial and the *intersection* with the boundary at infinity gets very complicated. By finite gap integration, cylindrical solutions have been constructed; see for example [10]. In [11] a detailed numerical study for surfaces bounded by a finite number of special curves (including circles, (super)ellipses and boundaries of spherocylinders) is carried out, and the holographic entanglement entropy and the holographic mutual information for those entangling curves have been numerically computed.

For the construction of meaningful examples via loop group factorization methods, two main problems need to be solved: the first is the proof of existence of potentials depending on a loop parameter on a surface with non-trivial monodromy which satisfy a certain reality condition (see remark 2.8). This reality condition is given explicitly only by solving ODEs along non-trivial curves. We solve this problem for a large class of examples. The second problem is concerned with the construction of the minimal surfaces from potentials satisfying the reality conditions: this problem is based on the fact that the generalized Iwasawa factorization is not global, i.e. there exist loops which do not admit an Iwasawa factorization (remark 2.7). It is hard to determine the curves on the surface along which the factorization breaks down, and to characterize the behaviour of the minimal surface along those curves in general. On the other hand, under a mild assumption on the degeneracy of the Iwasawa decomposition, the minimal surface intersects the boundary at infinity transversally; see for example [12, section 5] for technical details and figures 2*a* and 1 for visualizations. It is this failure of the global Iwasawa decomposition which allows us to produce minimal surfaces in anti-de Sitter spaces with non-trivial (finitely generated) topology and predicted topological intersections with the boundary at infinity, at least numerically. We plan to investigate the remaining theoretical questions concerning the intersections at infinity in forthcoming work.

The structure of the paper is as follows: in §2, we briefly describe a unified loop group approach for constant mean curvature (CMC) surfaces in the symmetric spaces $\mathbb{S}^3$, AdS_3 , $\mathbb{H}^3 \cup \mathbb{H}^3$ and dS_3 , including the case of minimal surfaces. In §3, we first recall the conformal surface geometry in the lightcone model, with special emphasis on minimal surfaces in hyperbolic space. We discuss some simple examples of minimal surfaces in $\mathbb{H}^3$: the hyperbolic disc as

the counterpart of the round sphere, and minimal Delaunay cylinders. We then define n -noids and open n -noids, motivated by the behaviour of Delaunay cylinders at their ends. We prove the existence of open n -noids, and conjecture that those surfaces actually give rise to n -noids in the strict sense. In §4, we explain how to modify the techniques of §3 in order to obtain minimal surfaces in AdS_3 . In §5, we study surfaces with three ends more explicitly via the generalized Weierstrass representation (GWR); these investigations have enabled us to perform computer experiments. Among others we construct a family of equilateral trinoids in $\mathbb{H}^3$ with mean curvature $|H| < 1$. The paper is supplemented by figures visualizing global properties of minimal surfaces in $\mathbb{H}^3$, dS_3 and AdS_3 .

2. The loop group method for constant mean curvature surfaces in symmetric spaces

(a) Unitary frames

It is a classical fact in differential geometry that immersed surfaces in space can be described by their first and second fundamental forms. These forms are subject to integrability conditions, the Gauss–Codazzi equations. The Gauss–Codazzi equations for special surface classes, e.g. CMC surfaces, have certain symmetries which gives rise to families of associated surfaces. A systematic investigation of this phenomena has led to the integrable systems approach to CMC surfaces. We refer to [13,14] and references therein for more details.

To construct CMC surfaces in the symmetric spaces $\mathbb{S}^3$, AdS_3 , $\mathbb{H}^3 \cup \mathbb{H}^3$ and dS_3 , we use the following matrix models in $\text{SL}_2(\mathbb{C})$:

space	matrix model	$\langle x, y \rangle$
$\mathbb{S}^3$	SU_2	$\frac{1}{2} \text{tr } x\hat{y}$
AdS_3	SU_{11}	$-\frac{1}{2} \text{tr } x\hat{y}$
$\mathbb{H}^3 \cup \mathbb{H}^3$	$\{X \in \text{SL}_2(\mathbb{C}) \mid \bar{X}^t = X\}$	$-\frac{1}{2} \text{tr } x\hat{y}$
dS_3	$\{X \in \text{SL}_2(\mathbb{C}) \mid \bar{X}^t = e_0 X e_0^{-1}\}$	$\frac{1}{2} \text{tr } x\hat{y},$

(2.1)

where $e_0 = \text{diag}(i, -i)$. The third column of the table specifies the inner product on $M_{2 \times 2}(\mathbb{C})$ extending the metric on the symmetric space, with sign chosen so that the signature of the tangent space is $(\pm, +)$. Here $\hat{y} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ for $y = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

To use integrable systems methods, we introduce the loop group $\Lambda \text{SL}_2(\mathbb{C})$ of real analytic maps $\mathbb{S}^1 \rightarrow \text{SL}_2(\mathbb{C})$, and the subgroup $\Lambda_+ \text{SL}_2(\mathbb{C})$ of loops which extend holomorphically to the interior of the unit disc. The four involutions of $\Lambda \text{SL}_2(\mathbb{C})$

$$\begin{array}{l|l}
 \mathbb{S}^3 & X^*(\lambda) = \overline{X(1/\bar{\lambda})}^{t-1} \\
 \text{AdS}_3 & X^*(\lambda) = e_0 \overline{X(1/\bar{\lambda})}^{t-1} e_0^{-1} \\
 \mathbb{H}^3 & X^*(\lambda) = \overline{X(-1/\bar{\lambda})}^{t-1} \\
 \text{dS}_3 & X^*(\lambda) = e_0 \overline{X(-1/\bar{\lambda})}^{t-1} e_0^{-1}
 \end{array}
 \quad (2.2)$$

determine four real forms $\{X \in \Lambda \text{SL}_2(\mathbb{C}) \mid X^* = X\} \subset \Lambda \text{SL}_2(\mathbb{C})$. By an abuse of terminology, we call elements of such a subgroup *unitary*, or emphasizing the involution, e.g. $\mathbb{H}^3$ -unitary. We will also denote by asterisk (*) the corresponding involutions of the Lie algebra $\Lambda \text{sl}_2(\mathbb{C})$.

Given a choice of one of the real forms induced by (2.2), a *unitary connection* is a $\Lambda \text{sl}_2(\mathbb{C})$ -valued 1-form η on a Riemann surface Σ with the following properties:

- η is flat for all λ .
- $\eta = \eta^*$.

- η has a simple pole at $\lambda = 0$, $\eta_{-1} := \text{res}_{\lambda=0} \eta$ has no $(0, 1)$ part, $\det \eta_{-1} = 0$, and $\langle \eta_{-1}, \eta_{-1}^* \rangle \neq 0$ does not vanish everywhere.

A unitary frame $F: \Sigma \rightarrow \text{ASL}_2(\mathbb{C})$ is a unitary solution to the ODE $dF = F\eta$.

The evaluation formula maps a unitary frame to the symmetric space:

$$f = F|_{\lambda_0} F^{-1}|_{\lambda_1} \quad (2.3)$$

where $\lambda_0, \lambda_1 \in \mathbb{C}^*$ are the evaluation points as follows:

space	evaluation points	mean curvature H
S^3 and AdS_3	$\lambda_0, \lambda_1 \in S^1$ distinct	$\frac{i(\lambda_1 + \lambda_0)}{\lambda_1 - \lambda_0}$
H^3 and dS_3	$\lambda_0, \lambda_1 \in \mathbb{C}^*$ with $\lambda_0 \overline{\lambda_1} = -1$	$\frac{\lambda_1 + \lambda_0}{\lambda_1 - \lambda_0}$

The third column of the table lists the mean curvature H of the induced immersion f up to sign, derived in theorem 2.1. Note that H satisfies $|H| < 1$ for the symmetric spaces H^3 and dS_3 . Formula (2.3) was first derived in [13] for CMC surfaces in H^3 and S^3 .

Theorem 2.1. Let η be a unitary connection on a Riemann surface Σ with respect to one of the four real forms (2.2). Let F be a unitary frame satisfying $dF = F\eta$. Then the evaluation formula (2.3) evaluated at evaluation points (2.4) yields a conformal CMC surface f into the corresponding symmetric space with metric

$$v^2 dz \otimes d\bar{z} \quad \text{and} \quad v^2 = 2(\lambda_0^{-1} - \lambda_1^{-1})(\lambda_0 - \lambda_1)\langle \alpha, \alpha^* \rangle, \quad (2.5)$$

and CMC as in (2.4), where $\eta_{-1} = \alpha dz$ and $\eta_1 = \alpha^* d\bar{z}$. The function v^2 is non-negative (and positive for surfaces in S^3 and H^3), and f is a space-like immersion where $v^2 > 0$.

Remark 2.2. The unitary connection η is usually referred to as the associated family of flat connections of the surface f .

Proof. We prove the theorem for AdS_3 ; the proof for the other symmetric spaces is similar with sign changes. The unitary connection decomposes as

$$\eta = (\alpha \lambda^{-1} + \beta) dz + (\beta^* + \alpha^* \lambda) d\bar{z}. \quad (2.6)$$

The flatness of η is equivalent to

$$[\alpha, \beta^*] = \alpha_{\bar{z}}, \quad [\alpha, \alpha^*] + [\beta, \beta^*] = \beta_{\bar{z}} - (\beta^*)_{\bar{z}}, \quad [\beta, \alpha^*] = (\alpha^*)_{\bar{z}}. \quad (2.7)$$

Let $F_0 = F|_{\lambda_0}$ and $F_1 = F|_{\lambda_1}$. To compute the metric

$$f_z = (\lambda_0^{-1} - \lambda_1^{-1})F_0 \alpha F_1^{-1} \quad \text{and} \quad f_{\bar{z}} = (\lambda_0 - \lambda_1)F_0 \alpha^* F_1^{-1}. \quad (2.8)$$

Since $\langle f_z, f_z \rangle = \langle f_{\bar{z}}, f_{\bar{z}} \rangle = 0$ the metric is $v^2 dz \otimes d\bar{z}$ with

$$v^2 = 2\langle f_z, f_{\bar{z}} \rangle = 2(\lambda_0^{-1} - \lambda_1^{-1})(\lambda_0 - \lambda_1)\langle \alpha, \alpha^* \rangle. \quad (2.9)$$

Since $\lambda_0 \neq \lambda_1$ and $\langle \alpha, \alpha^* \rangle \geq 0$ is not constantly vanishing, the metric is generically non-zero, so the evaluation formula induces a conformal immersion.

The normal is $N = F_0 \gamma F_1^{-1}$ where

$$\gamma = -\frac{i}{2} \frac{[\alpha, \alpha^*]}{\langle \alpha, \alpha^* \rangle}. \quad (2.10)$$

Using flatness,

$$f_{z\bar{z}} = (\lambda_0^{-1} - \lambda_1^{-1})F_0(\lambda_0 \alpha^* \alpha - \lambda_1 \alpha \alpha^*)F_1^{-1}. \quad (2.11)$$

Using that $\langle \alpha \alpha^* + \alpha^* \alpha, \gamma \rangle = 0$, the mean curvature of f is

$$H = 2v^{-2} \langle f_{z\bar{z}}, N \rangle = -\frac{1}{2} v^2 (\lambda_0^{-1} - \lambda_1^{-1})(\lambda_0 + \lambda_1) \langle [\alpha, \alpha^*], \gamma \rangle = i \frac{\lambda_1 + \lambda_0}{\lambda_1 - \lambda_0}. \quad (2.12)$$

Remark 2.3. With other choices of evaluation formulae and evaluation points, CMC surfaces can also be constructed from unitary connections in the symmetric spaces related by the Lawson correspondence:

$$\text{corresponding to } S^3: \quad H^3 (|H| > 1) \text{ and } \mathbb{R}^3 \quad (2.13)$$

and

$$\text{corresponding to } \text{AdS}_3: \quad dS_3 (|H| > 1) \text{ and } \mathbb{R}^{2,1}. \quad (2.14)$$

Remark 2.4. In the case the Hopf differential is dz^2 , after a coordinate change the flatness of the unitary connection is Gauss equation on the metric $v^2 = e^{2u}$:

$$S^3 \Delta u + 2 \sinh 2u = 0 \quad H^3 (|H| < 1): \Delta u - 2 \cosh 2u = 0 \quad (2.15)$$

and

$$\text{AdS}_3 \Delta u - 2 \sinh 2u = 0 \quad dS_3 (|H| < 1): \Delta u + 2 \cosh 2u = 0. \quad (2.16)$$

(b) Holomorphic frames

To construct CMC immersions by theorem 2.1, one is required to produce a flat unitary connection. The flatness is the Gauss equation, a partial differential equation on the metric. This section introduces the generalized Weierstrass representation (GWR) [15]. In this construction, the PDE is replaced by an ordinary differential equation together with a Iwasawa loop group factorization. For a more comprehensive treatment of real forms of loop groups, see [14]. The case $\mathbb{R}^{2,1}$ was considered in [16].

A GWR *potential* ξ is a $\text{Asl}_2(\mathbb{C})$ -valued $(1, 0)$ -form on a Riemann surface Σ satisfying the following condition: ξ has a simple pole at $\lambda = 0$, and $\xi_{-1} := \text{res}_{\lambda=0} \xi$ satisfies $\det \xi_{-1} = 0$ and is nowhere zero. A GWR *frame* Φ for ξ is a holomorphic map from the domain to $\text{ASL}_2(\mathbb{C})$ satisfying $d\Phi = \Phi \xi$. Choosing a real form, an *Iwasawa factorization* of Φ is

$$\Phi = FB, \quad F^* = F \quad \text{and} \quad B \in \Lambda_+ \text{SL}_2(\mathbb{C}). \quad (2.17)$$

The Iwasawa factorization can be computed via the Birkhoff factorization as follows. When in the big cell, $\Phi^{*-1} \Phi$ has a Birkhoff factorization

$$\Phi^{*-1} \Phi = X_+^{*-1} X_+ \quad \text{and} \quad X_+ \in \Lambda_+ \text{SL}_2(\mathbb{C}). \quad (2.18)$$

Then the desired Iwasawa factorization of Φ is

$$\Phi = FB, \quad F := \Phi X_+^{-1}, \quad B := X_+, \quad (2.19)$$

because $F^* = F$.

Theorem 2.5. Let ξ be a GWR potential, and Φ a corresponding GWR frame. If Φ has an Iwasawa factorization, then F is a unitary frame. If the symmetric space is S^3 or H^3 , or if the λ^{-1} -term of $\det \xi$ is non-vanishing, F induces a conformal CMC immersion by theorem 2.1.

Proof. Since $d\Phi = \Phi \xi$, $dF = F\eta$ and $\Phi = FB$, then

$$\eta = \xi \cdot (B^{-1}), \quad (2.20)$$

where the dot denotes the gauge action $\xi \cdot g := g^{-1} \xi g + g^{-1} dg$. In particular, flatness of η follows from flatness of ξ . Since $B \in \Lambda_+ \text{SL}_2(\mathbb{C})$ and ξ has a simple pole in λ , then η has a simple pole in λ . Since ξ has no $(0, 1)$ part, then $\eta_{-1} := \text{res}_{\lambda=0} \eta$ has no $(0, 1)$ part. Since $F^* = F$, then $\eta^* = \eta$, hence η is a unitary connection. Due to the positive definite inner product $\langle \eta_{-1}, \eta_{-1}^* \rangle$ vanishes for surfaces in S^3 and H^3 exactly where η_{-1} vanishes. This does not happen, as η_{-1} and ξ_{-1} are conjugated by $B|_{\lambda=0}$, and ξ_{-1} is nowhere vanishing by assumption. In the case of Lorentzian signature, $\langle \eta_{-1}, \eta_{-1}^* \rangle$ vanishes exactly when the kernels of the nilpotent 1-forms η_{-1} and η_1 coincide. If $\langle \eta_{-1}, \eta_{-1}^* \rangle$ vanishes identically, the connection η is upper triangular with respect to a decomposition into the kernel bundle of η_{-1} and a complementary bundle. Because B is positive the leading part, i.e. the -1 term, of the λ -expansion of $\det \xi$ is the same as the leading part of the λ -expansion of

$\det \eta$. The later vanishes identically if η is upper triangular with respect to a decomposition into the kernel bundle of η_{-1} and a complementary bundle. ■

Remark 2.6. The Hopf differential $Q dz^2$ of a CMC immersion induced by a GWR potential ξ is the leading term (coefficient of λ^{-1}) of $\det \xi$.

Remark 2.7. In the case of the spaceform S^3 , the GWR frame always has an Iwasawa factorization. For the other three real forms, the GWR frame may fail to have an Iwasawa factorization, generally on some real analytic subset of the domain. On this set, the surface is singular, in many cases going to the ideal boundary.

If the domain is not simply connected, Φ has monodromy and the induced CMC immersion on the universal cover does not generally *close*, that is, descend to an immersion of the domain. A sufficient condition for closing is the following:

Remark 2.8. The induced CMC immersion closes if the monodromy M of Φ is unitary (intrinsic closing), and at the evaluation points $M(\lambda_0) = M(\lambda_1) \in \{\pm 1\}$ (extrinsic closing).

3. Minimal n -noids in hyperbolic 3-space

We study minimal surfaces in hyperbolic 3-space H^3 which intersect the boundary at infinity perpendicularly. A convenient set-up from a geometric point of view is conformal surface geometry in the lightcone model of the 3-sphere.

(a) The lightcone model for H^3

The lightcone approach to conformal surface geometry is classical; for details we refer to [17,18] and references therein. We consider Minkowski space $V = \mathbb{R}^{4,1}$ with its standard inner product $(\cdot, \cdot)$ inducing the quadratic form

$$q(x_0, \dots, x_4) = -x_0^2 + x_1^2 + \dots + x_4^2. \quad (3.1)$$

The lightcone

$$\mathcal{L} = \{\mathbb{R}x \in PV \mid x \neq 0, q(x) = 0\}, \quad (3.2)$$

is diffeomorphic to the 3-sphere S^3 via

$$(x_1, \dots, x_4) \in S^3 \mapsto \mathbb{R}(1, x_1, \dots, x_4) \in \mathcal{L}. \quad (3.3)$$

Thus, $\mathcal{L}$ inherits a conformal structure from the round metric on S^3 . It is well known that the group $SO(4, 1)$ acts on $\mathcal{L}$ by conformal transformations. In fact,

$$SO_+(4, 1) = \{g \in SO(4, 1) \mid (g(e_0), e_0) > 0\} \quad (3.4)$$

is the group of (orientation preserving) conformal diffeomorphisms of S^3 (equipped with the round conformal structure).

(i) Hyperbolic 3-space

Taking the space-like vector $v_\infty = e_4$, we obtain two copies of hyperbolic 3-space as

$$\mathcal{L} \setminus (\mathcal{L} \cap P(e_4^\perp)) = \{(x_0, x_1, x_2, x_3, -1) \mid -x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1\}. \quad (3.5)$$

This space is naturally equipped with the metric of constant curvature -1 . The subgroup $SO_+(3, 1) \subset SO_+(4, 1)$ (defined by fixing the vector e_4) realizes the isometry group of hyperbolic 3-space.

(ii) Surfaces in the lightcone model

We consider conformal immersions

$$f: \Sigma \rightarrow S^3, \quad (3.6)$$

from a Riemann surfaces Σ into the conformal 3-sphere. The map f is equivalent (in conformal geometry) to the map

$$\sigma = \mathbb{R}(1, f): \Sigma \rightarrow \mathcal{L}. \quad (3.7)$$

The fact that f is an immersion means that for any (local) lift $\tilde{\sigma} = g(1, f)$ of σ (where $g: U \subset \Sigma \rightarrow \mathbb{R}$) and (local) pointwise independent vector fields X, Y on Σ

$$\text{span}(\tilde{\sigma}, X \cdot \tilde{\sigma}, Y \cdot \tilde{\sigma}), \quad (3.8)$$

is a (real) three-dimensional bundle (where $\cdot$ denotes the derivative). Conformality of f means that for a local holomorphic chart z on Σ we have

$$(\tilde{\sigma}_z, \tilde{\sigma}_z) = 0 = (\tilde{\sigma}_{\bar{z}}, \tilde{\sigma}_{\bar{z}}), \quad (3.9)$$

where $g_z := (\partial/\partial z) \cdot g$ and $g_{\bar{z}} := (\partial/\partial \bar{z}) \cdot g$ for any (vector-valued) function g .

A fundamental object in conformal surface theory is the mean curvature sphere congruence. The mean curvature sphere is defined locally by

$$S = \text{span}(\tilde{\sigma}, \tilde{\sigma}_z, \tilde{\sigma}_{\bar{z}}, \tilde{\sigma}_{z\bar{z}}). \quad (3.10)$$

Proposition 3.1. *The surface f considered as a surface in hyperbolic 3-space defined by $v_\infty = e_4$ is of CMC H if and only if*

$$H = (e_4^\perp, e_4^\perp) \quad (3.11)$$

is constant, where $(e_4^\perp)_p$ is the projection of e_4 to S_p . Consequently, f is minimal if and only if e_4 is contained (as a constant section) in S .

In the following, we are interested in conformally parametrized surfaces $f: \Sigma \rightarrow S^3$ into the conformal 3-sphere such that its intersection with

$$H^3 \cup H^3 = S^3 \setminus S^2 = \mathcal{L} \setminus (\mathcal{L} \cap P(e_4^\perp)) \quad (3.12)$$

is a minimal surface. In order to apply the GWR approach, we need an explicit isometry

$$\Psi: \mathcal{L} \setminus (\mathcal{L} \cap P(e_4^\perp)) \rightarrow \{X \in \text{SL}_2(\mathbb{C}) \mid \bar{X}^t = X\}. \quad (3.13)$$

This is provided by

$$\Psi(\mathbb{R}(x_0, x_1, x_2, x_3, x_4)) \mapsto \frac{1}{x_4} \begin{bmatrix} x_0 + x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_0 - x_1 \end{bmatrix}. \quad (3.14)$$

Note that the two copies of $\Psi(\mathcal{L} \setminus (\mathcal{L} \cap P(e_4^\perp)))$ are given by the sets of positive definite and negative definite symmetric SL_2 -matrices.

For visualization, we make use of the *stereographic projection*

$$\{X \in \text{SL}_2(\mathbb{C}) \mid \bar{X}^t = X\} \rightarrow \mathbb{R}^3; \begin{bmatrix} x_0 + x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_0 - x_1 \end{bmatrix} \mapsto \frac{1}{1 + x_0}(x_1, x_2, x_3), \quad (3.15)$$

which is a conformal diffeomorphism from the positive definite symmetric SL_2 -matrices to the unit ball.

Remark 3.2. For visualization of surfaces in deSitter space, we use the *stereographic projection*

$$\{X \in \text{SL}_2(\mathbb{C}) \mid \bar{X}^t = e_0 X e_0^{-1}\} \rightarrow \mathbb{R}^3; \begin{bmatrix} x_0 + x_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - x_1 \end{bmatrix} \mapsto \frac{1}{1 + x_1}(x_0, x_2, x_3), \quad (3.16)$$

which is a conformal diffeomorphism between open subsets of the deSitter space and Minkowski space.

(b) Basic examples

We first illustrate the GWR approach for some basic surfaces. Recall that for H^3 , the real involution on $ASL_2(\mathbb{C})$ is given by

$$X^*(\lambda) = \overline{X(-1/\bar{\lambda})}^t, \quad (3.17)$$

and unitary connections are flat connections of the form

$$d + \eta(\lambda) = d + \eta_0 + \lambda^{-1} \eta_{-1} + \lambda \eta_1, \quad (3.18)$$

with $\bar{\eta}_0^t = -\eta_0$ and $\bar{\eta}_{-1}^t = \eta_1$ where η_{-1} is a nowhere vanishing $(1,0)$ -form with values in the nilpotents.

(i) The sphere in H^3

The simplest example of a GWR potential is given by

$$\xi(\lambda) = \begin{bmatrix} 0 & \lambda^{-1} \\ 0 & 0 \end{bmatrix} dz, \quad (3.19)$$

on the complex plane, with GWR frame

$$\Phi = \begin{bmatrix} 1 & \lambda^{-1}z \\ 0 & 1 \end{bmatrix}. \quad (3.20)$$

The H^3 Iwasawa factorization is given by

$$\Phi = FB = \frac{1}{\sqrt{1-z\bar{z}}} \begin{bmatrix} 1 & \lambda^{-1}z \\ \lambda\bar{z} & 1 \end{bmatrix} \frac{1}{\sqrt{1-z\bar{z}}} \begin{bmatrix} 1 & 0 \\ -\lambda\bar{z} & 1-z\bar{z} \end{bmatrix}, \quad (3.21)$$

and taking $\lambda_0 = 1$ and $\lambda_1 = -1$ we obtain

$$f = F|_{\lambda=1} F^{-1}|_{\lambda=-1} = \frac{1}{1-z\bar{z}} \begin{bmatrix} 1+z\bar{z} & 2z \\ 2\bar{z} & 1+z\bar{z} \end{bmatrix}. \quad (3.22)$$

Restricting to the unit disc $D \subset \mathbb{C}$, this is just a conformally parametrized totally geodesic hyperbolic disc inside hyperbolic 3-space, with induced metric

$$\frac{4}{(1-z\bar{z})^2} dz \otimes d\bar{z}. \quad (3.23)$$

Note that this example is rather special as we are able to write down both the GWR frame and its factorization in terms of elementary functions.

Remark 3.3. The surface f has the same GWR potential as the round minimal 2-sphere in S^3 , and serves as the simplest example of a minimal surface in H^3 . On the other hand, as a map to H^3 , f is not well defined on the whole plane $\mathbb{C}$ or projective line, but crosses the ideal boundary at ∞

$$S_\infty^2 = \mathcal{L} \cap P(e_4^\perp), \quad (3.24)$$

along the unit circle $S^1 \subset \mathbb{C}$ where the Iwasawa decomposition breaks down. By (3.22) and (3.14) (or by geometric reasoning) f can be extended as a conformal surface into the conformal 3-sphere

Figure 2. (a) Delaunay surface in $\mathbb{H}^3 \cup S^2 \cup \mathbb{H}^3$. Each of the two ends of this surface of revolution oscillates between the two copies of $\mathbb{H}^3$, crossing the ideal boundary infinitely often. (b) Minimal Delaunay surface in AdS_3 . At the cone point (centre of image), the profile curve of this surface of revolution crosses the revolution axis and the surface fails to be immersed.

$\mathcal{L}$, a phenomena which turns out to be typical in the examples below. In the case at hand, we obtain a conformally parametrized totally umbilic sphere

$$z \in \mathbb{CP}^1 \mapsto [1 + z\bar{z}, 0, z + \bar{z}, i(\bar{z} - z), 1 - z\bar{z}] \in \mathcal{L}. \quad (3.25)$$

(ii) Delaunay cylinders in $\mathbb{H}^3$

A less trivial class of surfaces is given by Delaunay surfaces in $\mathbb{H}^3$, i.e. rotational symmetric minimal and CMC cylinders. Minimal Delaunay cylinders in $\mathbb{H}^3$ (figure 2a) were first described in [19] in terms of their elliptic spectral data. They come in a real one-dimensional family of geometrically distinct surfaces. On $\mathbb{C}/(2\pi i\mathbb{Z})$, the Hopf differential of a Delaunay cylinder is a constant multiple of $(dw)^2$, and the conformal factor is a solution of the cosh-Gordon equation (remark 2.4). The conformal factor can be given explicitly in terms of the Weierstrass $\wp$ -function on a rectangular elliptic curve; see, for example, [10,19]. The surface is rotational symmetric, and the conformal factor only depends on one (real) variable and is periodic—but blows up once each period where the surface intersects the ideal boundary at ∞ (figure 2a).

We consider the Delaunay cylinders parametrized on the two-punctured sphere $\mathbb{C}^*$, whose Hopf differential is a constant multiple of $(dz)^2/z^2$. To construct this one-dimensional family on the domain $\mathbb{C}^*$ via the GWR approach take evaluation points

$$\lambda_0 = -\lambda_1 = 1 \quad (3.26)$$

determined by the mean curvature $H = 0$ in the last column of table (2.4). The GWR potential on the domain $\mathbb{C}^*$ is $\xi = iA dz/z$, where A is a z -independent loop

$$A = \frac{1}{2\sqrt{q^2 - 1}} \begin{bmatrix} 0 & \lambda^{-1} + q \\ \lambda - q & 0 \end{bmatrix}, \quad (3.27)$$

for $q \in \mathbb{R}$ with $|q| > 1$. It follows that the eigenvalues of the frame monodromy around the puncture $z = 0$ are $\exp(\pm 2\pi\nu)$, where

$$\nu = \frac{i}{2} \sqrt{\frac{(\lambda - q)(-\lambda^{-1} - q)}{q^2 - 1}}. \quad (3.28)$$

The GWR potential ξ has the following properties (remark 2.8):

- intrinsic closing condition, as $A = A^*$; see the third row of (2.2);
- extrinsic closing condition, as the eigenvalues of $A(\lambda_0)$ are $\pm i/2$.

In terms of finite gap theory, see [19], $q \in \mathbb{R}^*$ is the branch-point of the spectral curve: $\det A(q) = 0$.

The GWR frame Φ is based at $z = 1$, i.e. $\Phi|_{z=1} = \mathbb{1}$. Hence $\Phi = \exp(iA \log z)$.

Theorem 3.4. *The GWR construction applied to the above data gives Delaunay cylinders.*

Proof. The GWR frame for the potential ξ is $\Phi = \exp(iA \log z)$. The monodromy of Φ around the puncture $z = 0$ is $M = \exp(2\pi A)$, satisfying the intrinsic closing condition $M^* = M$ due to the symmetry of A , and the extrinsic closing condition (on the cylinder) $M(\lambda_0) = M(\lambda_1) = -\mathbb{1}$ due to the fact that the eigenvalues of A are $\pm i/2$ at each evaluation point. Hence the surface closes on $\mathbb{C}^*$.

To show that the induced surface is a surface of revolution, changing coordinates $\tilde{z} = x + iy = i \log z$, then $\Phi = \exp((x + iy)A) = \exp(xA) \exp(iyA)$. Since $\exp(xA)$ is unitary, then the unitary factor in the Iwasawa decomposition of Φ is $F|_{\tilde{z}=x+iy} = \exp(xA)G(y)$, where G is the unitary factor of $\exp(iyA)$. Hence F is equivariant, i.e. $F|_{t+x+iy} = \exp(tA)F|_{x+iy}$ for $t \in \mathbb{R}$. Hence the surface $F|_{\lambda=\lambda_0} F^{-1}|_{\lambda=\lambda_1}$ is equivariant with equivariant action on the profile curve $X = X(y) = G_{\lambda_0} G_{\lambda_1}^{-1}$ given by

$$X \mapsto \exp(xA(\lambda_0))X \exp(-xA(\lambda_1)). \quad (3.29)$$

This action has closed orbits with period $x \in [0, 2\pi]$ because the eigenvalues of $A(\lambda_0)$ and $A(\lambda_1)$ are $\pm i/2$. ■

Remark 3.5. It is possible to compute the unitary factor G of $\exp(iyA)$ explicitly in terms of elliptic functions. It turns out that G is quasi-periodic in y (i.e. it is equivariant, and the period depends on q), and that the Iwasawa decomposition fails twice in each period. On the other hand, using (3.14) it is possible to extend the Delaunay surface to a conformal immersion of the cylinder $\mathbb{C}^*$ into S^3 ; see also [19] (§6) and figure 2a.

Remark 3.6. It is worth noting that the intersection of a Delaunay cylinder with the boundary at infinity is the disjoint union of circles. This follows from the fact that Delaunay cylinders are equivariant. If we restrict to one component of the Delaunay cylinder inside H^3 , we obtain exactly two boundary circles, which define a Riemann surface of annulus type. It would be interesting to work out in detail the relationship between the free parameter q and the modulus of the annulus.

(c) n -noids

An n -noid in $\mathbb{R}^3$ is a minimal immersion of a n punctured Riemann sphere, each of whose end monodromies has Delaunay eigenvalues, that is, the same eigenvalues as those of a Delaunay cylinder.

In the previous example, we have seen that a Delaunay end in H^3 cannot be defined on a punctured disc when we consider the surface lying only in H^3 . We therefore have to modify our definition:

Definition 3.7. An n -noid in H^3 is a conformal immersion

$$f: \mathbb{CP}^1 \setminus \{p_1, \dots, p_n\} \rightarrow \mathcal{L} \cong S^3, \quad (3.30)$$

such that

- (i) the intersection

$$\text{image}(f) \setminus (\mathcal{L} \cap P(e_4)^\perp) = \text{image}(f) \setminus S^2 = \text{image}(f) \cap (H^3 \cup H^3), \quad (3.31)$$

is a (not necessarily connected) minimal surface;

- (ii) the surface has Delaunay eigenvalues around each end p_k .

Figure 3. Trinoid and fournoid in $\mathbb{H}^3 \cup S^2 \cup \mathbb{H}^3$ with respective cyclic symmetries of orders 2 and 3 about the vertical axis. Neither surface is equilateral.

Remark 3.8. It is necessary to explain the second condition in more detail. In general, if the surface passes through the boundary at infinity, the associated family of flat connections (remark 2.2) does not exist on the n -punctured sphere $\mathbb{CP}^1 \setminus \{p_1, \dots, p_n\}$, and it is therefore not obvious in which sense one should test the second condition. For example, one should expect that the intersection of $\text{image}(f) \cap (\mathbb{H}^3 \cup \mathbb{H}^3)$ around an end p_k is defined on a nested union of disjoint topological annuli, and *a priori* it is unclear why the eigenvalues of the monodromy on all annuli are the same. On the other hand, using condition (1), the surface f is a Willmore surface in S^3 and has an associated family of flat $\text{SL}(4, \mathbb{C})$ -connections which reduces (in a λ -dependent way) to the associated family of rank 2 connections of the minimal surface in $\mathbb{H}^3$ on the corresponding subset. In this way, it can be shown that the monodromy representation up to conjugation is well defined; for details, see [20].

On the other hand, for minimal surfaces constructed from GWR potentials the monodromies of the potential and the associated family agree up to conjugation, and the eigenvalue condition can therefore be checked directly on the potential.

Example 3.9. All minimal Delaunay cylinders are 2-noids. In fact, it follows from [19] that these surfaces can be extended through the boundary at infinity to give a (Moebius-)periodic surface into S^3 from the two-punctured sphere.

So far we only have numerical evidence of the existence of n -noids in $\mathbb{H}^3$ (figures 1, 3 and 7). We are therefore forced to give the following weaker definition.

Definition 3.10. An open n -noid in $\mathbb{H}^3$ is a conformal minimal immersion of a n -holed sphere

$$f: \mathbb{CP}^1 \setminus (D_1 \cup \dots \cup D_n) \rightarrow \mathbb{H}^3, \quad (3.32)$$

for non-intersecting topological discs $D_k \subset \mathbb{CP}^1$, such that the monodromy eigenvalues around each hole D_k are the monodromy eigenvalues of a Delaunay cylinder for some $q \in \mathbb{R}$ with $|q| > 1$.

Remark 3.11. Not every n -noid in $\mathbb{H}^3$ is an open n -noid in the above sense. For example, it might be the case that two or more *ends* of the n -noid start in each of the two $\mathbb{H}^3$ copies inside the conformal 3-sphere, while the two pieces are joined by a surface which is close to a part of a round sphere. In fact, such examples can be constructed by the methods below. On the other hand, we will construct open n -noids in theorem 3.14. We conjecture that those surfaces are n -noids in the sense of definition 3.7 (figures 1, 3 and 7).

(i) 3-noids

A potential for trinoids is

$$\begin{bmatrix} 0 & \lambda^{-1} \\ f(\lambda)Q & 0 \end{bmatrix} dz, \quad (3.33)$$

where $Q dz^2$ is a holomorphic quadratic differential with three double poles and real quadratic residues, $\lambda_0, \lambda_0^{-1}$ are the evaluation points, and for the ambient space H^3

$$f = (\lambda - 1)(\lambda + 1). \quad (3.34)$$

Since at each of its poles the potential is gauge equivalent to a perturbation of a Delaunay potential, by the theory of regular singular points, each monodromy around a puncture has Delaunay eigenvalues. This potential constructs CMC trinoids if the closing conditions of remark 2.8 are satisfied, as shown by the following theorem.

Theorem 3.12. *There exists a real 1-parameter family of GWR potentials on the 3-punctured sphere satisfying the intrinsic and extrinsic closing conditions for minimal surfaces in H^3 .*

The technical proof of the theorem is given in §5.

(d) Existence of n -noid potentials with small necksize

We adopt the techniques of Traizet [21] to prove the existence of open n -noids in H^3 . We conjecture that these surfaces are n -noids in the sense of definition 3.7 as well.

In [21], Traizet showed the existence of GWR potentials for CMC n -noids in $\mathbb{R}^3$ by deforming the GWR potential of the round sphere. His method of solving the monodromy problem for the intrinsic and extrinsic closing conditions can be easily translated to our set-up, with basically identical proofs up to minor changes. Additionally, one can deduce that the Iwasawa factorization works on a subset homeomorphic to a n -holed sphere, which yields actually examples of open n -noids in H^3 . A formally similar method of deforming surfaces in S^3 and H^3 has been introduced in [12,22], respectively.

We start with a potential

$$\xi_t(z) := \begin{bmatrix} 0 & \lambda^{-1} dz \\ t(\lambda^2 + 1)\omega(z) & \end{bmatrix}, \quad (3.35)$$

where

$$\omega = \sum_{k=1}^n \left(\frac{a_k}{(z - z_k)^2} + \frac{b_k}{z - z_k} \right) dz. \quad (3.36)$$

Note that in (3.35) we take the evaluation points to be $\lambda_0 = i$ and $\lambda_1 = -i$ in order to have more natural reality conditions. As in [21], we need to allow that the coefficients a_k, b_k, z_k are holomorphic functions in λ , i.e. they are holomorphic on an open neighbourhood of the closed unit disc in the λ -plane. They need to be adjusted for small $t > 0$ such that the intrinsic closing condition is satisfied: we want to find for small $t > 0$ holomorphic functions a_k, b_k, z_k , which are close to constant functions (satisfying a constraint related to some balancing formula), such that the monodromy based at $z = 0$ of the potential (3.35) is in the unitary loop group determined by (2.2) corresponding to the symmetric space H^3 . As we suppose that the functions z_k are close enough to constants, the potential is well defined on a n -holed sphere for all $\lambda \in \{\lambda \in \mathbb{C}^* \mid 0 < |\lambda| < 1 + \epsilon\}$ for some $\epsilon > 0$, and the monodromy is computed on this n -holed sphere. We call $\mathbf{x} = (t, a_1, \dots, z_1, \dots)$ the parameter of the potential, even in the case when $a_1, \dots, z_1, \dots$ depend on λ .

For $\epsilon > 0$, we denote by $\mathcal{B}^\epsilon$ the Banach space of holomorphic functions on

$$\{\lambda \in \mathbb{C} \mid |\lambda| < 1 + \epsilon\}, \quad (3.37)$$

equipped with the generalized Wiener norm as in [21, section 4].

Lemma 3.13. For $k = 1, \dots, n$ let $\tau_k \in \mathbb{R} \setminus \{0\}$ and $p_k \in \mathbb{C} \setminus (\{0\} \cup S^1)$ such that

$$0 = \sum_{k=1}^n \frac{2\tau_k \overline{p_k}}{1 - |p_k|^2} = \sum_{k=1}^n \tau_k \frac{1 + |p_k|^2}{1 - |p_k|^2} = \sum_{k=1}^n \frac{2\tau_k p_k}{1 - |p_k|^2}. \quad (3.38)$$

Then there exists $\epsilon, T > 0$ and unique smooth maps

$$b_k, z_k: [0; T] \rightarrow \mathcal{B}^\epsilon, \quad k = 1, \dots, n \quad (3.39)$$

with

$$b_k(0) = \frac{2\tau_k \overline{p_k}}{1 - |p_k|^2}, \quad z_k(0) = p_k, \quad (3.40)$$

for $k = 1, \dots, n$, and $z_k(0)(0) = p_k$ for $k = 1, \dots, n$, and smooth functions $a_k: [0, T] \rightarrow \mathbb{R}$ with $a_k(0) = \tau_k$, $k = 1, \dots, n$ such that for $t \in [0; T]$ the potential (3.35) with parameter

$$x = (t, a_1(t), \dots, a_n(t), b_1(t), \dots, b_n(t), z_1(t), \dots, z_n(t)), \quad (3.41)$$

has H^3 -unitary monodromy at $z = 0$.

Proof. The proof is analogous to the proof of [21, proposition 3], under the extra assumption that $|p_k| \neq 1$ for $k = 1, \dots, n$, and using the adapted $*$ -operator on functions, i.e. $f^*(\lambda) = \overline{f(-1/\bar{\lambda})}$. The proof that the functions τ_k are independent of λ and real-valued can be done similarly to the proof of [21, proposition 4] by looking at the eigenvalues of the local monodromies. ■

As a corollary, we obtain our main theorem:

Theorem 3.14. For every $n > 1$ there exist open minimal n -noids in H^3 .

Proof. For $n = 2$, we have seen the existence of Delaunay cylinders. For $n \neq 3$, it is easy to see the existence of n pairwise distinct points $p_k \in \mathbb{C}^*$ with $0 < |p_l| < 1$ for $l = 1, \dots, n-1$ and $|p_n| > 1$ and n positive real numbers τ_k satisfying the balancing formula (3.38).

To construct an open minimal n -noid, we consider for $T > t > 0$ the potential provided by lemma 3.13, and the solution Φ of

$$d\Phi = \Phi \xi_t \quad \text{and} \quad \Phi|_{z=0} = \mathbb{1}. \quad (3.42)$$

The Iwasawa decomposition (for the real involution corresponding to H^3) does exist on an open subset of the loop group $\Delta SL_2(\mathbb{C})$, containing the constant loop $\mathbb{1}$. As for $\delta > 0$ small enough and t small enough ξ_t is arbitrarily close to ξ_0 on

$$\Sigma = \{z \in \mathbb{C} \mid |z| < 1 - \delta; |z - p_k| > \delta \text{ for } k = 1, \dots, n-1\} \quad (3.43)$$

the loop $\Phi|_z$ admits an Iwasawa decomposition on the n -holed sphere Σ , and theorem 2.5 for the evaluation points $\lambda_0 = i$, $\lambda_1 = -i$ provides a minimal surface

$$f: \Sigma \rightarrow H^3. \quad (3.44)$$

By construction (and for δ small enough), the monodromies around the holes have Delaunay eigenvalues since at each of its poles the potential is gauge equivalent to a perturbation of a Delaunay potential. ■

Figure 4 presents an equilateral minimal 16-noid. This is a surfaces with dihedral symmetry. Our numerical experiments show that such surfaces stay embedded for an arbitrary number of ends.

Conjecture 3.15. There exist embedded minimal n -noids in H^3 for any $n \in \mathbb{N}_{\geq 2}$.

Figure 4. An equilateral minimal 16-noids in $H^3 \cup S^2 \cup H^3$ with cyclic symmetry of order 16.

4. Minimal n -noids in anti-de Sitter space AdS_3

(a) The lightcone model for AdS_3

We set

$$AdS_3 = SU_{11}, \quad (4.1)$$

equipped with the natural biinvariant Lorentzian metric associated with the quadratic form given by the trace. As for hyperbolic space, anti-de Sitter space can be defined as the complement of the boundary at infinity

$$S_\infty = \{\mathbb{R}(x_0, x_1, x_2, x_3, 0) \mid x_0^2 + x_1^2 - x_2^2 - x_3^2 = 0\}, \quad (4.2)$$

of the lightcone

$$\mathcal{L} = \{\mathbb{R}(x_0, x_1, x_2, x_3, x_4) \mid x_0^2 + x_1^2 - x_2^2 - x_3^2 - x_4^2 = 0\}, \quad (4.3)$$

via

$$\mathbb{R}(x_0, x_1, x_2, x_3, x_4) \in \mathcal{L} \setminus S_\infty \mapsto \frac{1}{x_4} \begin{bmatrix} x_0 + ix_1 & x_2 - ix_3 \\ x_2 + ix_3 & x_0 - ix_1 \end{bmatrix} \in SU_{11}. \quad (4.4)$$

For visualization, we make use of the *stereographic projection*

$$\mathbb{R}(x_0, x_1, x_2, x_3, x_4) \in \mathcal{L}^0 \mapsto \frac{1}{x_0 + x_4} (x_1, x_2, x_3) \in \mathbb{R}^{1,2}, \quad (4.5)$$

defined on

$$\mathcal{L}^0 = \mathcal{L} \setminus \{\mathbb{R}(x_0, x_1, x_2, x_3, -x_0) \mid x_1^2 - x_2^2 - x_3^2 = 0\}. \quad (4.6)$$

This map is a conformal diffeomorphism onto an open subset of Minkowski space $\mathbb{R}^{1,2}$.

(b) Basic examples

We first describe some simple surfaces in terms of the GWR approach.

(i) The sphere in AdS_3

As in the case of S^3 and H^3 , the easiest example of a GWR potential is given by

$$\xi(\lambda) = \begin{bmatrix} 0 & \lambda^{-1} \\ 0 & 0 \end{bmatrix} dz, \quad (4.7)$$

on the complex plane, with GWR frame

$$\Phi = \begin{bmatrix} 1 & \lambda^{-1}z \\ 0 & 1 \end{bmatrix}. \quad (4.8)$$

The factorization for AdS_3 is given by

$$\Phi = FB = \frac{1}{\sqrt{1-z\bar{z}}} \begin{bmatrix} 1 & \lambda^{-1}z \\ \lambda\bar{z} & 1 \end{bmatrix} \frac{1}{\sqrt{1-z\bar{z}}} \begin{bmatrix} 1 & 0 \\ -\lambda\bar{z} & 1-z\bar{z} \end{bmatrix}, \quad (4.9)$$

and taking evaluation points $\lambda_0 = 1$ and $\lambda_1 = -1$ we obtain

$$f = F|_{\lambda=1} F^{-1}|_{\lambda=-1} = \frac{1}{1-z\bar{z}} \begin{bmatrix} 1+z\bar{z} & 2z \\ 2\bar{z} & 1+z\bar{z} \end{bmatrix}. \quad (4.10)$$

Restricted to the unit disc $D \subset \mathbb{C}$ this is just a conformally parametrized totally geodesic hyperbolic disc inside $\text{SU}_{11} = \text{AdS}_3$ with induced metric

$$\frac{4}{(1-z\bar{z})^2} dz \otimes d\bar{z}. \quad (4.11)$$

Remark 4.1. The surface f is not well defined on the whole plane $\mathbb{C}$ or projective line, but crosses the ideal boundary at infinity S_∞ along the unit circle $S^1 \subset \mathbb{C}$. By (4.10) f can be continued on the complement of the closed unit disc as a map into SU_{11} . Again, this phenomena turns out to be typical in the examples below (figures 2*b* and 6).

(ii) Minimal planes

Besides the trivial example of the hyperbolic disc, there is an interesting class of minimal surfaces with trivial topology in AdS_3 which have been investigated in detail in [3], and are parametrized by null polygonal boundaries. In the special case of regular polygons, the Hopf differential is given by a homogeneous polynomial on the complex plane. The metric is rotationally invariant and given by a global solution of the Painlevé III equation.

Analogous to the GWR description [23] of *Smyth surfaces* in $\mathbb{R}^3$ with rotationally symmetric metric [24] these minimal planes can be constructed via the GWR potential of the form

$$\begin{bmatrix} 0 & \lambda^{-1} \\ cz^n & 0 \end{bmatrix} dz, \quad n \in \mathbb{N}, \quad c \in \mathbb{R}, \quad (4.12)$$

for appropriate $c \in \mathbb{R} \setminus \{0\}$. The surface (with initial condition at $z=0$ given by $\mathbb{1}$) has discrete ambient cyclic symmetry of order $n+2$. The Iwasawa decomposition is global only for one special value $\hat{c}$ (depending on n); see, for example, the detailed investigations in [25]. In figure 5*a*, a surface for $n=1$ with c numerically close to $\hat{c}$ is shown. A systematic investigation of this boundary behaviour of solutions of the underlying Gauss equation, i.e. the generalized sinh-Gordon equation, on punctured Riemann surfaces with Hopf differentials having higher-order poles is carried out in [26], under the name *crowned Riemann surfaces*.

Figure 5. (a) Minimal Smyth surface in AdS_3 with cyclic symmetry of order three (orientation preserving), respectively, order six (including reflections), (b) swallowtail singularity on a minimal surface in AdS_3 .

(iii) Delaunay cylinders in AdS_3

Minimal Delaunay planes in AdS_3 have been described in [10] in terms of their elliptic spectral data. By imposing the extrinsic closing condition along the equivariant direction they form a real one-dimensional family of geometrically distinct surfaces. The conformal factor with respect to the coordinate dw on $\mathbb{C}/(2\pi i\mathbb{Z})$ is a solution of the sinh-Gordon equation with opposite sign as for CMC surfaces in S^3 (2.15), and the Hopf differential is a constant multiple of $(dw)^2$. The conformal factor can be given explicitly in terms of the Weierstrass $\wp$ -function on a rectangular elliptic curve [10]. The surface is rotationally symmetric, and the conformal factor depends only on one (real) variable and is periodic, but blows up once each (intrinsic) period where the surface intersects the ideal boundary at infinity. Moreover, the conformal factor $v^2 = e^{2u}$ vanishes to second order once in each (intrinsic) period. Geometrically, this means that one tangent direction is mapped to a light-like vector, while the orthogonal tangent direction (with respect to the Riemann surface structure) spans the kernel of the differential of the CMC surface. In the equivariant case, at the singular points, the kernel of the differential of the CMC surface coincides with the kernel of the differential of the conformal factor v^2 , giving us a cone point (figure 2b). This phenomena does not happen generally, where a swallowtail like singularity is expected (figure 5b).

In the following, we consider the Delaunay cylinders parametrized on the 2-punctured sphere $\mathbb{C}^*$, where the Hopf differential is a constant multiple of $(dz)^2/z^2$. These can be constructed via theorem 2.5 analogously to Delaunay cylinders in H^3 .

A family of GWR potentials inducing Delaunay surfaces on $\mathbb{C}^*$ with initial condition $\Phi|_{z=1} = 1$ is

$$A \frac{i dz}{z}, \quad A = B + B^*, \quad B = \begin{bmatrix} \frac{ic}{2} & a\lambda^{-1} \\ b & -\frac{ic}{2} \end{bmatrix}, \quad a \in \mathbb{R}^*, \quad b, c \in \mathbb{R}. \quad (4.13)$$

Imposing the extrinsic closing condition with $\lambda_0 = i$, a 1-parameter family of potentials is given by

$$B = \frac{1}{2\sqrt{q^2 + 1}} \begin{bmatrix} i\sqrt{2(q^2 + 1)} & \lambda^{-1} + q \\ \lambda + q & -i\sqrt{2(q^2 + 1)} \end{bmatrix}, \quad (4.14)$$

for $q \in \mathbb{R}$. The meromorphic frame Φ is based at $z = 1$ with $\Phi|_{z=1} = \mathbb{I}$. As in the previous section, we obtain

Theorem 4.2. *The GWR construction applied to the above data gives Delaunay cylinders in AdS_3 .*

An example is shown in figure 2b.

(c) n -noids in AdS_3

We may define n -noids and open n -noids in AdS_3 in an analogous way as for surfaces in H^3 . We can also adopt Traizet's construction [21] as done for the case of minimal surfaces in H^3 above to prove the existence of open n -noids in AdS_3 . It would be nice to prove that these examples are actually n -noids in the strong sense; a proof of this conjecture might build up on the techniques developed in [27].

Theorem 4.3. *For every $n > 1$ there exist open minimal n -noids in AdS_3 .*

The proof is analogous to the proof of theorem 3.14, taking the appropriate $*$ -operator on $\text{SL}_2(\mathbb{C})$ and on holomorphic functions. Rather than repeating the details of the arguments we give in the next section a general proof of existence of 3-noid potentials for surfaces in S^3 , H^3 , AdS_3 and dS_3 .

5. 3-noids in symmetric spaces

In this last section, we extend slightly the class of surfaces and consider CMC surfaces in the symmetric spaces S^3 , H^3 , AdS_3 and dS_3 . Recall from §2 that these can also be described by the GWR approach. We present a general proof of existence for GWR potentials on a 3-punctured sphere, denoted as trinoids in the following.

The main motivation for giving an explicit proof of existence for trinoids is that the so constructed surfaces can be numerically computed and visualized directly (figures 1, 3, 7 and 6). In particular, the proof might help the reader to perform computer experiments via GWR method. Also, in the case of a 3-punctured sphere, the range of end weights q (determined by the quadratic residues at the ends) is more tractable here than in the implicit function theorem proof of theorem 3.14.

The construction of trinoids is particularly simple because the conjugacy class of the monodromy group in this case is determined by the individual conjugacy classes of the generators. In the case of the involution $\lambda \mapsto 1/\bar{\lambda}$ the closing conditions can be read off from the monodromy eigenvalues, which by the theory of regular singular points, can be read off from the potential.

Each pole of our potential will be a Fuchsian singularity which is GWR gauge equivalent to a perturbation of a Delaunay potential. By the theory of regular singular points, each monodromy around a puncture has Delaunay eigenvalues. This potential constructs CMC trinoids if the closing conditions of remark 2.8 are satisfied. The following theorem constructs trinoids in AdS_3 , H^3 with $|H| < 1$, and dS_3 with $|H| < 1$, in analogy to the construction of trinoids in S^3 in [28].

To show that the trinoid closes, it is necessary to compute a unitarizer X of the monodromy. Unlike the case of S^3 , for the other spaceforms the unitarizer could fail to have an Iwasawa factorization. Then the frame with unitary monodromy $X\Phi$ likewise fails to have an Iwasawa factorization, and hence does not construct a trinoid. Hence it is necessary to find a unitarizer which has an Iwasawa factorization, or equivalently, a unitarizer in $\Lambda_+ \text{SL}_2(\mathbb{C})$.

To solve the technical problem of the existence of a monodromy unitarizer in $\Lambda_+ \text{SL}_2(\mathbb{C})$, we make the simplifying restriction to *isosceles* trinoids, for which two of the three end parameters are equal. As seen in theorem 5.3, under this restriction, a diagonal monodromy unitarizer can be computed explicitly via a scalar Birkhoff factorization. In the general case of non-isosceles trinoids, the proofs of existence of the Birkhoff factorization at a base point z_0 is technically much more involved as they do not reduce to the scalar case. This is beyond the scope of this work, and

Figure 6. Minimal trinoids in AdS_3 and dS_3 : three Delaunay half-cylinders glued to a minimal two-sphere (horizontal plane). A swallowtail singularity (figure 5b) appears on each Delaunay end. (a) Minimal trinoid in AdS_3 , (b) minimal trinoid in dS_3 .

we refer to the implicit methods of §§3 and 4 for the proof of existence of (open) non-isosceles trinoids.

(a) Real forms

The four involutions (2.2) on loops $X: S^1 \rightarrow \text{SL}_2\mathbb{C}$ can be indexed by $\delta \in \{\pm 1\}$ and $\epsilon \in \{\pm 1\}$:

$$X^*(\lambda) = \overline{\eta X(\delta/\bar{\lambda}) \eta^{-1}}^{t^{-1}} \quad \text{and} \quad \eta = \begin{cases} \mathbb{1} & \text{if } \epsilon = 1 \\ \text{diag}(\mathbf{i}, -\mathbf{i}) & \text{if } \epsilon = -1. \end{cases} \quad (5.1)$$

The subgroup of loops X satisfying $X^* = X$ is the real form for the symmetric spaces as tabulated

	$\epsilon = 1$	$\epsilon = -1$	
$\delta = 1$	S^3	AdS_3	.
$\delta = -1$	H^3 ($ H < 1$)	dS_3 ($ H < 1$)	

(5.2)

A loop $M: S^1 \rightarrow \text{SL}_2\mathbb{C}$ is *unitary* if $M^* = M$. A monodromy group is *unitarizable* if there exists a map $X: \mathcal{D}_+ \rightarrow \text{SL}_2\mathbb{C}$ such that for each M in the group $XM X^{-1}$ extends holomorphically to S^1 and is unitary.

For scalar loops define

$$f^*(\lambda) = \overline{f(\delta/\bar{\lambda})}. \quad (5.3)$$

(b) Trinoids potentials

A potential for trinoids is

$$\begin{bmatrix} 0 & \lambda^{-1} \\ f(\lambda)Q & 0 \end{bmatrix} dz, \quad (5.4)$$

where $Q dz^2$ is a holomorphic quadratic differential with three double poles and real quadratic residues, f satisfying $(f/\lambda)^* = f/\lambda$ is

$$S^3 \text{ and } \text{AdS}_3: \quad f = (\lambda - \lambda_0)(\lambda - \lambda_0^{-1}), \quad \lambda_0 \in S^1 \setminus \{\pm 1\} \quad (5.5a)$$

and

$$H^3 \text{ and } dS_3 : f = (\lambda - \lambda_0)(\lambda + \lambda_0^{-1}), \quad \lambda_0 \in \mathbb{R} \setminus \{0\}, \quad (5.5b)$$

and $\lambda_0, \lambda_0^{-1}$ are the evaluation points λ_0, λ_1 are determined by

$$S^3 \text{ and } AdS_3 : \lambda_1 = \frac{1}{\lambda_0} \in S^1 \setminus \{\pm 1\}; \quad H^3 \text{ and } dS_3 : \lambda_1 = \frac{-1}{\lambda_0} \in \mathbb{R}^*. \quad (5.6)$$

For isosceles trinoids choose ends $z = 1, -1, \infty$ and

$$Q = \frac{4a + b(z^2 - 1)}{(z^2 - 1)^2} \quad (5.7)$$

satisfying $\text{qres}_{z=\pm 1} Q \, dz^2 = a$ and $\text{qres}_{z=\infty} Q \, dz^2 = b$.

(c) Unitarization

The isosceles trinoid potential (5.4)–(5.7) has symmetries

$$\sigma^* \xi(\lambda) = \xi(\lambda).g_1 \quad \sigma(z) = -z, \quad g_1 = \text{diag}(\mathbf{i}, -\mathbf{i}) \quad (5.8a)$$

and

$$\tau^* \xi(\delta/\bar{\lambda}) = \xi(\lambda).g_2, \quad \tau(z) = \bar{z}, \quad g_2 = \text{diag}\left(\frac{\sqrt{\delta}}{\lambda}, \frac{\lambda}{\sqrt{\delta}}\right). \quad (5.8b)$$

Let M_0 and M_1 be the monodromies around $z = 1$ and $z = -1$, respectively, with basepoint $\Phi|_{z=0} = \mathbf{1}$. The symmetries of the potential imply

$$M_0 = \begin{bmatrix} r & p\lambda \\ -q\lambda^{-1} & r^* \end{bmatrix} \quad \text{and} \quad M_1 = \begin{bmatrix} r & -p\lambda \\ q\lambda^{-1} & r^* \end{bmatrix}, \quad (5.9)$$

for some holomorphic functions $p, q, r : \mathbb{C}^* \rightarrow \mathbb{C}$ satisfying $rr^* + pq = 1$ and

$$p^* = \delta p \quad \text{and} \quad q^* = \delta q. \quad (5.10)$$

Lemma 5.1. *If q/p^* in (5.9) has a Birkhoff factorization*

$$\frac{q}{p^*} = \delta \epsilon x_+^* x_+, \quad x_+ : \mathcal{D}_+ \rightarrow \mathbb{C}^*, \quad (5.11)$$

then M_0 and M_1 are unitarizable in the sense of (5.1).

Proof. If $q/p^* = \delta \epsilon x_+^* x_+$, then $x_+ : \mathcal{D}_+ \rightarrow \mathbb{C}^*$ has a single-valued square root $\mathcal{D}_+ \rightarrow \mathbb{C}^*$, so $X := \text{diag}(\sqrt{x_+}, 1/\sqrt{x_+})$ is a single-valued map $\mathcal{D}_+ \rightarrow \text{SL}_2(\mathbb{C})$. Then

$$P_0 := XM_0X^{-1} = \begin{bmatrix} r & s \\ -\epsilon s^* & r^* \end{bmatrix}, \quad P_1 := XM_1X^{-1} = \begin{bmatrix} r & -s \\ \epsilon s^* & r^* \end{bmatrix}, \quad s := px_+\lambda \quad (5.12)$$

extend holomorphically to S^1 and satisfy $P_0 = P_0^*$ and $P_1 = P_1^*$ in the sense of (5.1). ■

(d) Scalar Birkhoff factorization

To unitarize the trinoid monodromy via lemma 5.1, we need a more detailed analysis of the scalar Birkhoff factorization. With $\mathbb{CP}^1 = \mathcal{D}_+ \sqcup S^1 \sqcup \mathcal{D}_-$,

$$\mathcal{D}_+ := \{\lambda \in \mathbb{C} \mid |\lambda| < 1\} \quad \text{and} \quad \mathcal{D}_- := \{\lambda \in \mathbb{C} \mid |\lambda| > 1\} \cup \{\infty\} \quad (5.13)$$

let

$$\Lambda = \{\text{real analytic loops on } S^1 \rightarrow \mathbb{C}^*\} \quad (5.14a)$$

$$\Lambda_+ = \{f \in \Lambda \mid f \text{ is the boundary of a holomorphic map } \mathcal{D}_+ \rightarrow \mathbb{C}^*\} \quad (5.14b)$$

$$\text{and} \quad \Lambda_- = \{f \in \Lambda \mid f \text{ is the boundary of a holomorphic map } \mathcal{D}_- \rightarrow \mathbb{C}^*\}. \quad (5.14c)$$

1. By [29], every loop $f \in \Lambda$ has a unique Birkhoff factorization

$$f = c\lambda^n f_{-} f_{+}, \quad n \in \mathbb{Z}, \quad c \in \mathbb{C}^* \quad (5.15a)$$

and

$$f_{+} \in \Lambda_{+}, \quad f_{-} \in \Lambda_{-}, \quad f_{+}(0) = 1, \quad f_{-}(\infty) = 1, \quad (5.15b)$$

where by abuse of notation $f_{\pm}$ also denotes the holomorphic extensions of $f_{\pm} \in \Lambda$ to $\mathcal{D}_{\pm}$. Let star be as in (5.3) for either choice of $\delta \in \{\pm 1\}$.

2. If $f \in \Lambda$ satisfies $f = f^*$, then f has a unique Birkhoff factorization

$$f = \epsilon f_{+}^* f_{+}, \quad \epsilon \in \{\pm 1\}, \quad f_{+} \in \Lambda_{+}, \quad f_{+}(0) \in \mathbb{R}_{+}. \quad (5.16)$$

This defines a homomorphism

$$\text{sign}: \{f \in \Lambda \mid f = f^*\} \rightarrow \{\pm 1\}, \quad f \mapsto \epsilon. \quad (5.17)$$

A meromorphic loop on S^1 is meromorphic in a neighbourhood of S^1 . Let Λ^* be the subgroup of meromorphic loops on S^1

$$\Lambda^* = \{f \mid f^* = f \text{ and } \text{div}_{S^1} f \text{ even}\}, \quad (5.18)$$

where $\text{div}_{S^1} f$ even means that the order of each zero and pole of f on S^1 is even.

3. Every $f \in \Lambda^*$ has a unique Birkhoff factorization

$$f = \epsilon f_{+}^* f_{+}, \quad \epsilon \in \{\pm 1\}, \quad \text{div}_{S^1} f_{+} = \frac{1}{2} \text{div}_{S^1} f \quad (5.19)$$

where f_{+} is the boundary of a holomorphic map $\mathcal{D}_{+} \rightarrow \mathbb{C}^*$. This extends the homomorphism (5.17) to a homomorphism

$$\text{sign}: \Lambda^* \rightarrow \{\pm 1\}, \quad f \mapsto \epsilon. \quad (5.20)$$

In particular, for every meromorphic loop f on S^1 with $f = f^*$

$$\text{sign}[f^2] = 1. \quad (5.21)$$

(e) The trace polynomial

Given a monodromy representation M_0, M_1, M_2 with $M_0 M_1 M_2 = \mathbb{1}$ on the three-punctured sphere, define the polynomial [30]

$$\varphi = 1 - t_0^2 - t_1^2 - t_2^2 + 2t_0 t_1 t_2, \quad t_k = \frac{1}{2} \text{tr}(M_k), \quad k \in \{0, 1, 2\}. \quad (5.22)$$

The trace polynomial vanishes precisely where the monodromy representation is reducible. In the case of the involution $\lambda \mapsto 1/\bar{\lambda}$, if the halftraces are in $(-1, 1)$, then the monodromy is SU_2 -unitarizable if $\varphi > 0$ and SU_{11} -unitarizable if $\varphi < 0$.

For the trinoid potential (5.4) with quadratic residues $(q_0, q_1, q_2) \in \mathbb{R}^3$ of Q , and f as in <eqref>eq:trinoid-f</eqref>,</p>
</div>
<div class="Equation-Block">
$t_k = \cos(2\pi v_k), \quad v_k = \frac{1}{2} - \frac{1}{2}\sqrt{1 + q_k \kappa}, \quad \kappa = 4\lambda^{-1}f(\lambda), \quad k \in \{0, 1, 2\}.$
</div>
<div class="Text">
<p>The function $\kappa : S^1 \rightarrow \mathbb{C}$ satisfies $\kappa^* = \kappa$.</p>
</div>
<div class="Text">
<p>The trace polynomial $\varphi : S^1 \rightarrow \mathbb{C}$ for the monodromies of the trinoid potential satisfies $\varphi^* = \varphi$. Putting together (5.22) and (5.23), its series expansion in κ at $\kappa = 0$ is</p>
</div>
<div class="Equation-Block">
$\varphi(\kappa) = c\kappa^4 + O(\kappa^5), \quad c = \frac{\pi^4}{64}(q_0 + q_1 + q_2)(-q_0 + q_1 + q_2)(q_0 - q_1 + q_2)(q_0 + q_1 - q_2).$
</div>
<div class="Text">
<p>Lemma 5.2. Choose $(q_0, q_1, q_2) \in \mathbb{R}^3$ with $c \neq 0$. For s in a small enough interval $(0, s_0)$, $\varphi_{sq} \in \Lambda^*$ and $\text{sign}[\varphi_{sq}] = \text{sign}(c)$.</p>
</div>

Proof. From the series expansion (5.24),

$$\left| \frac{\varphi_q}{\kappa^4 - c} \right| < \left| \frac{c}{2} \right| \quad \text{for all } |\kappa| < \kappa_0. \quad (5.25)$$

Let $m = 1 + \max_{\lambda \in S^1} |\kappa(\lambda)|$. Then for all $|\kappa| < m$ and $s \in (0, \kappa_0/m)$, we have $|s\kappa| < \kappa_0$, so

$$\left| \frac{\varphi_{sq}(\kappa)}{\kappa^4 - c} \right| = \left| \frac{\varphi_q(s\kappa)}{\kappa^4 - c} \right| < \left| \frac{c}{2} \right|. \quad (5.26)$$

Hence φ_{sq}/κ^4 maps S^1 into the disc centred at c with radius $|c/2|$. This implies that φ_{sq} has no zeros on S^1 , except at the zeros of κ in the case κ has zeros on S^1 . Since these zeros are of order 4, then $\varphi \in \Lambda^*$.

Since $\text{sign}(c)\varphi_{sq}/\kappa^4$ takes values in the open right halfplane, it has a single-valued square root $f: S^1 \rightarrow \mathbb{C}^*$ satisfying $f^* = f$. By (5.21), $\text{sign}[\varphi_{sq}] = \text{sign}(c)\text{sign}[\kappa^4 f^2] = \text{sign}(c)$. ■

In the case of isosceles trinoids, for which $(q_0, q_1, q_2) = (a, a, b)$ with a, b as in (5.7), then $c = \frac{\pi^4}{64}b^2(4a^2 - b^2)$, so $\text{sign}[\varphi] = +1$ in a subregion of $\{|b| < 2|a|\}$, and $\text{sign}[\varphi] = -1$ in a subregion of $\{|b| > 2|a|\}$.

(f) Trinoids

Theorem 5.3. *For each of the four symmetric spaces tabulated in (5.2), there exists a two real parameter family of conformal CMC isosceles trinoid potentials satisfying the closing conditions with prescribed mean curvature H .*

Proof. Choose a symmetric space, $\epsilon \in \{\pm 1\}$ and $\delta \in \{\pm 1\}$ as tabulated in (5.2), and an isosceles trinoid potential (5.4)–(5.7).

By lemma 5.2, there exists a subregion of $\{(a, b) \in \mathbb{R}^2\}$ with (a, b) as in (5.7), in which $\varphi \in \Lambda^*$ and $\text{sign}[\varphi] = \delta\epsilon$.

Let M_0 and M_1 be the monodromies of the meromorphic frame Φ based at $\Phi|_{z=0} = \mathbb{1}$ as in (5.9). The respective halftaces t_0, t_1, t_2 of $M_0, M_1, M_2 = M_1^{-1}M_0^{-1}$ are

$$t_0 = t_1 = \frac{1}{2}(r + r^*) \quad \text{and} \quad t_2 = \frac{1}{2}(r^2 - 2pq + r^{*2}), \quad (5.27)$$

so the trace polynomial (5.22) is

$$\varphi = (\mathfrak{i}(r - r^*))^2 pq. \quad (5.28)$$

Since $\varphi \in \Lambda^*$ and $(\mathfrak{i}(r - r^*))^2 \in \Lambda^*$, then $pq \in \Lambda^*$. Since $p^*p \in \Lambda^*$ then $q/p^* = (pq)/(p^*p) \in \Lambda^*$. By (5.21) $\text{sign}[(\mathfrak{i}(r - r^*))^2] = 1$ and $\text{sign}[p^*p] = 1$, so

$$\text{sign}[q/p^*] = \text{sign}[(pq)/(p^*p)] = \text{sign}[pq] \text{sign}[p^*p] = \text{sign}[pq] \quad (5.29a)$$

$$= \text{sign}[(\mathfrak{i}(r - r^*))^2] \text{sign}[pq] = \text{sign}[\varphi] = \delta\epsilon. \quad (5.29b)$$

By the definition of sign , p/q^* has a factorization $p/q^* = \delta\epsilon x_+^* x_+$, where $x_+ : \mathcal{D}_+ \rightarrow \mathbb{C}^*$. By lemma 5.1 there exists a unitarizer X of the monodromy in the sense of (5.1).

By theorem 2.5, the immersion induced by $X\Phi$ via r -Iwasawa factorization is conformal and CMC, and by remark 2.8 it closes around each of its three Delaunay ends. ■

Remark 5.4.

1. In the case of S^3 theorem 5.3 reconstructs a two-dimensional subfamily of the three-dimensional family of trinoids constructed in [28] without the isosceles constraint.
2. In the other three symmetric spaces, where the Iwasawa factorization can leave the big cell, the domain of the trinoids constructed by theorem 5.3 is not known, but numerical experiments indicate that in general the whole punctured Riemann sphere is mapped into the light cone, intersecting the ideal boundary along curves (figures 1, 3, 7 and 6).
3. Noids with more than three ends can be constructed using a cyclic branched cover of the Riemann sphere (figures 1 and 3).

Figure 7. Two views of an equilateral minimal trinoid in $\mathbb{H}^3 \cup S^2 \cup \mathbb{H}^3$ stereographically projected to $\mathbb{R}^3$ via the Poincaré halfspace model. The cutaway view (right) shows the intersection of the surface with the ideal boundary along nested topological circles.

(g) Dressing

By theorem 5.3, we can construct equilateral trinoids in dS_3 (figure 6*b*), and isosceles non-equilateral trinoids in $\mathbb{H}^3$ (figure 3). In this section, we construct equilateral trinoids in $\mathbb{H}^3$ (figure 1) by dressing equilateral trinoids in dS_3 . The dressing action interchanges the real forms for dS_3 and $\mathbb{H}^3$.

More explicitly, the *dressing action* of a loop g (defined on an a circle of radius $r \in (0, 1]$) on a frame F , written $g \# F$, is by definition the r -unitary factor of gF of its r -Iwasawa factorization. In analogy to [31], we take g to be a diagonal *simple factor*

$$g = \text{diag}(p^{1/2}, p^{-1/2}), \quad p = \frac{\lambda - \mu}{\mu\lambda + 1}, \quad p^* = -\frac{1}{p} \quad (5.30)$$

on an r -circle, $r < |\mu|$.

Lemma 5.5. *Diagonal simple factor dressing $F \mapsto g \# F$ interchanges the real form for dS_3 with the real form for $\mathbb{H}^3$.*

Proof. The dressing action of the simple factor g on a frame F in the real form for dS_3 is computed explicitly and algebraically as

$$g \# F = g F k^{-1} g^{-1}, \quad k = \frac{1}{\sqrt{|u|^2 - |v|^2}} \begin{bmatrix} u & \bar{v} \\ v & \bar{u} \end{bmatrix}, \quad \begin{bmatrix} u \\ v \end{bmatrix} = F^{-1}(\mu)\ell, \quad \ell = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (5.31)$$

We have

$$F k^{-1} = \begin{bmatrix} s & t \\ t^* & s^* \end{bmatrix} \implies g \# F = g F k^{-1} g^{-1} = \begin{bmatrix} s & t p \\ t^* p^* & s^* \end{bmatrix}. \quad (5.32)$$

Hence $g \# F$ is in the real form for $\mathbb{H}^3$. The proof for the other direction is the same except for a sign change. ■

Theorem 5.6. *There exists a real 1-parameter family of equilateral trinoids in $\mathbb{H}^3$ for each mean curvature $|H| < 1$ (figure 1).*

Proof. Let F be a unitary frame for a trinoid in the 1-parameter family of equilateral trinoids in dS_3 with specified mean curvature $|H| < 1$ (theorem 5.3). The dressed frame $g \# F$ is in the real form for $\mathbb{H}^3$ (lemma 5.5) so it satisfies the intrinsic closing condition for $\mathbb{H}^3$.

To satisfy the extrinsic closing condition, we choose the simple factor g as follows. The subset of the λ plane along which the monodromy group of F is reducible is the zero set of the trace polynomial φ (5.22). With t the halftrace of the monodromy of F around each end, we have $\varphi = (1 - t)^2(1 + 2t)$ with zero set

$$\left\{ \mu \in \mathbb{C}^* \mid \frac{1}{2} - \frac{1}{2}\sqrt{1 + 4q\mu^{-1}f(\mu)} \in \frac{\mathbb{Z}}{3} \right\}, \quad f \text{ as in (5.5b)}, \quad (5.33)$$

a discrete subset of $\mathbb{C}^*$ which accumulates at 0 and ∞ . Using (5.9) and (5.28), μ can be found in this zero set away from the evaluation points such that ℓ is a common eigenvalue of the monodromy group. This choice of μ implies that the monodromy of $g \# F$ is gMg^{-1} , where M is a monodromy of F (in analogy to [32] for noids in $\mathbb{R}^3$). Hence $g \# F$ satisfies the extrinsic closing conditions for H^3 , so it is the frame for an equilateral trinoid in H^3 . ■

Remark 5.7. The GWR construction of trinoids (theorem 5.3) is as follows:

- (i) In terms of a potential (5.4) on the three-punctured sphere, compute a dressing C which unitarizes the monodromy group of the corresponding holomorphic frame Φ based at $\Phi|_{z=z_0} = \mathbb{1}$.
- (ii) Show that the dressed holomorphic frame $C\Phi$ is in the big cell of the Iwasawa decomposition in some subregion of the domain.
- (iii) Construct a CMC immersion of that subregion into H^3 via the evaluation formula (2.3) applied to the unitary Iwasawa factor of $C\Phi$.

Following this programme with gauge equivalent trinoid potentials, Kobayashi in [33, theorem 4.1] gives an incomplete construction of equilateral trinoids in H^3 with mean curvature $|H| < 1$, failing to address step (2). Our theorem 5.6 fills this gap by constructing a dressing C in the loop group $\Lambda_+SL_2(\mathbb{C})$. Thus $C\Phi$ is in the big cell of the Iwasawa decomposition at, and hence in a neighbourhood of the basepoint z_0 . Moreover, it follows from theorem 3.14 that for small necksizes $C\Phi$ is in the big cell of the Iwasawa decomposition on the 3-holed sphere obtained by removing three discs around the punctures of the 3-punctured sphere.

Dorfmeister, Inoguchi and Kobayashi further state without proof in [7, section 10.5] that the theorem of Kobayashi mentioned above constructs CMC immersions of the three-punctured sphere into H^3 . On the contrary, in light of the numerical evidence of figures 1, 3, 7 and 6, we conjecture that in analogy to 2-noids (Delaunay surfaces), the incompletely constructed trinoids in [33] map the three-punctured sphere not into H^3 but into $H^3 \cup S^2 \cup H^3$.

Data accessibility. Our data are provided as electronic supplementary material.

Authors' contributions. A.I.B., S.H. and N.S. have made substantial contributions to the conception of the work and to the proofs of the theorems. A.I.B., S.H. and N.S. wrote the paper. The images have been produced by N.S., based on joint theoretical investigations with A.B.I. and S.H.. A.I.B., S.H. and N.S. give their final approval of this version to be published. All authors agree to be accountable for all aspects of the work in ensuring that questions related to the accuracy or integrity of any part of the work are appropriately investigated and resolved.

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