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Analytic Expressions for Radial Diffusion

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Abstract

I briefly review, compare and contrast two theoretical works that have significantly influenced radial diffusion research thus far, namely, the works of Fälthammar (1965, <https://doi.org/10.1029/JZ070i011p02503>) and the works of Fei et al. (2006, <https://doi.org/10.1029/2005JA011211>). Leveraging Fälthammar's model for magnetic field disturbances, I demonstrate that Fei et al.'s formulas are incorrect: they underestimate radial diffusion by a factor two in the presence of magnetic field disturbances. This underestimation comes from the erroneous assumption that radial displacements driven by magnetic field disturbances are statistically independent from radial displacements driven by induced electric fields while in fact both displacements are proportional to each other. Fei et al.'s approach is similar to Fälthammar's approach in that they both analyze radial diffusion by pieces, depending on the nature of the driver. Yet, the Fokker-Planck equation requires only one radial diffusion coefficient to characterize statistically a trapped radiation belt population cross drift shell motion. Thus, it is worth questioning the practice that consists of defining the coefficient as a sum of independent contributions. In addition, both theoretical models rely on the assumption that the background magnetic field is primarily dipolar, leading to flawed estimates. To overcome these limitations and to improve radial diffusion quantification, I use a general formulation for the variation of the third adiabatic invariant (1) to describe how to compute a radial diffusion coefficient in the most general way and (2) to highlight the assumptions that need to be questioned.

Plain language summary:

Radiation belt particles are constantly moving radially, towards or away from the planet, due to ambient electric and magnetic field perturbations. The individual path of a particle is like a random walk, and the net movement of the radiation belt population is described by a diffusion equation. There are two main theoretical works that relate the diffusion coefficient to statistical properties of the electric and magnetic field fluctuations. Because of apparent similarities between the two resulting sets of formulas, one could think that the most recent set is no more than a generalized version of the original set. However, this is not the case! When applied to the same situation, both sets of formulas provide very different answers for the same radial diffusion coefficient. That is why it is important to reassess the theoretical picture underlying radial diffusion. This article explains how to relate the diffusion coefficient to the general properties of fields, in the most general way. By restarting from scratch, it highlights the assumptions that need to be questioned in order to advance radial diffusion research. This will constitute a major

challenge to tackle to guarantee further progress in our abilities to understand and model radiation belt dynamics.

1. Introduction

Radial diffusion is a statistical description of cross drift shell motion for a trapped radiation belt population that conserves the first two adiabatic invariants. Although there exists some processes of “anomalous” and “neoclassical” radial diffusion that require the violation of one or two of the first two adiabatic invariants (e.g., Cunningham et al., 2018; O’Brien, 2014; Roederer et al., 1973), they are out of the scope of this study.

Quantifying radial diffusion is a scientific challenge of practical interest. From a scientific standpoint, radial diffusion is frequently opposed to local acceleration when it comes to assessing the most important acceleration mechanism for the Earth’s radiation belts (e.g. Thorne, 2010). From a practical standpoint, the radial diffusion coefficient is one of the core inputs of the many numerical models that consist of solving a Fokker-Planck equation to characterize radiation belt dynamics (Beutier & Boscher, 1995; Glauert et al., 2014; Su et al., 2010; Subbotin & Shprits, 2009, Tu et al., 2013).

Two main sets of theoretical expressions are available to quantify radial diffusion in the Earth’s radiation belts: (1) the formulas provided by Fälthammar (1965, 1968) and (2) the formulas provided by Fei et al. (2006).

What motivated the derivation of these two distinct sets of formulas?

Fälthammar (1965) derived expressions for radial diffusion in the early age of radiation belt science. At that time, Parker (1960) had already described how a sudden magnetic field compression would result in the broadening of an initially thin shell of trapped radiation belt particles. Fälthammar (1965) extended the theoretical descriptions available at the time (e.g. Davis & Chang, 1962) by making “somewhat less special assumptions” to describe the field fluctuations. He also showed that fluctuations of the large-scale electric potential field could represent a potentially significant additional source of radial diffusion. The theoretical work presented relied on the assumption that idealized, small, stationary stochastic perturbations with a zero mean were superimposed to a background magnetic dipole field. Schulz and Evitar (1969) generalized the analysis by studying radial diffusion driven by magnetic disturbances in the case of a slightly asymmetric background magnetic field (i.e., they assumed that the small magnetic field fluctuations superimposed to the magnetic dipole field had a non-zero mean). They found that the value of the radial diffusion coefficient is proportional to the power spectrum of the field fluctuations at all harmonics of the drift frequency, even though the first harmonic is the main contributor. In a background dipole field, only the first harmonic of the power spectrum of the magnetic field fluctuations contributes to radial diffusion. Consequently, most subsequent works left out the quantification of the small effect of a slight magnetic field asymmetry. Instead, they analyzed radial diffusion assuming a background magnetic dipole field (e.g. Schulz & Lanzerotti, 1974, p.90).

The development of these early theoretical formulas led to experimental analyses of electric (e.g. Holzworth & Mozer, 1979) and magnetic field disturbances (e.g. Lanzerotti & Morgan, 1973; Lanzerotti et al., 1978) to directly quantify radial diffusion. From the discrete experimental values obtained at $L = 4$ (Lanzerotti & Morgan, 1973) and $L = 6.6$ (Lanzerotti, 1978), Brautigam and Albert (2000) determined a parameterization for radial diffusion driven by magnetic disturbances, as a function of L and K_p index. Following Fälthammar's conclusion that radial diffusion driven by sudden magnetic impulses increases proportionally to L^{10} , the authors assumed a L^{10} dependence for the radial diffusion coefficient - even though the experimental data points at $L=4$ and $L = 6.6$ did not display such dependence. A least squares fitting technique was then implemented to determine a parameterization for the radial diffusion coefficient. Despite an apparent lack of representativeness, radiation belt simulations that rely on Brautigam and Albert's parameterization for magnetically driven radial diffusion yield plausible results when solving the Fokker-Planck equation. Thus, the parameterization became a well-accepted reference quantification for radial diffusion in the Earth's radiation belts. On the other hand, early estimates of radial diffusion driven by electric potential disturbances suffered from a lack of in-situ measurements. Useful at first (e.g., Cornwall, 1968; Lyons & Thorne, 1973), tentative estimates yielded unrealistic outputs when included in modern radiation belt simulations (e.g. Kim et al., 2011). Consequently, the process of radial diffusion driven by electric potential disturbances was left out.

In the 90s, the temporal and spatial accuracy of radiation belt observations improved significantly. Complex structures and rapid dynamics were revealed thanks to a growing network of satellites and ground stations providing multipoint measurements. These observations challenged the traditional picture of radiation belt dynamics provided by the Fokker-Planck equation. In particular, observations showed that relativistic electron fluxes at geostationary orbit could increase significantly (by a couple orders of magnitude), faster than expected (on a timescale ranging from a couple of hours to a couple of days). In addition, a good correlation between enhanced relativistic electron flux and ULF wave power was reported near geostationary orbit (e.g. Rostoker et al, 1998). As a result, it was proposed that the outer belt relativistic electron flux enhancements could be due to a drift resonant interaction with a single mode monochromatic ULF oscillation in a distorted magnetic field (Elkington et al., 1999, 2003; Hudson et al., 1999). The idea that (re-)emerged from these considerations was that the asymmetry of the background magnetic field could drive a form of enhanced radial diffusion in the presence of multiple ULF frequencies. Consequently, new radial diffusion formulas were developed in order to characterize radial diffusion in a slightly asymmetric background field (a dipole field to which a small, time-stationary, local-time dependent perturbation is superimposed). Yet, with the new formulas introduced by Fei et al. (2006), the role played by the small distortion of the field was found to be unimportant. Thus, most (but not all, e.g. (Cunningham, 2016)) subsequent works assumed once again a background magnetic dipole field.

These new set of theoretical expressions led to a new series of numerical and/or experimental estimates for the radial diffusion coefficients (e.g., Ali et al., 2015, 2016; Jaynes et al., 2018; Liu et al., 2016; Ozeke et al., 2012, 2014; Tu et al., 2012). In particular, Ozeke et al. (2014) analyzed many years of ground and space based measurements to propose a new radial diffusion parameterization. Yet, the resulting experimental formulation

is surprisingly similar to Brautigam and Albert's formulation for radial diffusion driven by magnetic disturbances. In fact, the difference between radiation belt simulations with either of the two parameterizations for radial diffusion is negligible (Drozdov et al., 2017).

This article is written in an attempt to break the deadlock and to move forward. The objective of the Sections 2 and 3 is to review, compare and contrast the theoretical frameworks underlying the two different sets of theoretical formulas. In particular, it is demonstrated that 1) both theoretical works have limitations and that (2) Fei et al.'s formulas are erroneous. A more general way to quantify radial diffusion is introduced Section 4.

2. Fälthammar's and Fei et al.'s expressions for radial diffusion: a brief review

2.1. Fälthammar's radial diffusion description

Fälthammar (1965) discusses separately the effects of magnetic field disturbances and the effects of electric potential disturbances ($\nabla \times \mathbf{E} = 0$) on a population of equatorial particles trapped in a magnetic dipole field. Idealized electric and magnetic field fluctuations are introduced to describe small drift motion perturbations.

2.1.1. Magnetic disturbances and electromagnetic radial diffusion—The magnetic field model considered consists of a simple, linearized, perturbation $\mathbf{b}$ superimposed to the background magnetic dipole field $\mathbf{B}_d$. In spherical coordinates, the dipole field is:

$$\mathbf{B}_d = -\frac{B_E R_E^3}{r^3} \begin{pmatrix} 2\cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \quad (1)$$

where $B_E \sim 30,000$ is the magnitude of the equatorial magnetic field at one Earth radius $R_E = 6,400$ km, and the small perturbation is:

$$\mathbf{b} = \begin{pmatrix} S(t) \cos \theta + A(t) r \sin 2\theta \cos \varphi \\ -S(t) \sin \theta + A(t) r \cos 2\theta \cos \varphi \\ -A(t) r \cos \theta \sin \varphi \end{pmatrix} \quad (2)$$

where $|\mathbf{b}/\mathbf{B}_d| \ll 1$. The fluctuation is composed of a symmetric part proportional to $S(t)$, i.e., a component that is independent of local time φ and an asymmetric part proportional to $A(t)$ – which depends on local time. It is worth noticing that this magnetic field disturbance model is curl free by design, which is a limit to its use.

The system of coordinates chosen for this article is the classical spherical coordinates, in which the azimuthal angle is counted positive eastward. The azimuthal angle is counted positive westward in Fälthammar's work – which may be confusing.

Assuming frozen-in flux conditions, the induced electric field $\mathbf{E}_{\text{ind}}$ associated with the time variations of the magnetic disturbance $\mathbf{b}$ is (Fälthammar, 1968):

$$\mathbf{E}_{\text{ind}} = \begin{pmatrix} -\frac{r^2}{7} \frac{dA}{dt}(t) \sin\theta \sin\varphi \\ \frac{2r^2}{7} \frac{dA}{dt}(t) \cos\theta \sin\varphi \\ -\frac{r}{2} \frac{dS}{dt}(t) \sin\theta + \frac{2r^2}{21} \frac{dA}{dt}(t) (3 - 7\sin^2\theta) \cos\varphi \end{pmatrix} \quad (3)$$

It is straightforward to check that $\nabla \times \mathbf{E}_{\text{ind}} = -\mathbf{b}/t$.

The radial component of the drift velocity of an equatorial particle is, to first order approximation in $|b/B_d|$:

$$\frac{dr}{dt} = \frac{E_{\text{ind},\varphi}}{B_d} - \frac{M}{q\gamma B_d r_o} \frac{\partial b}{\partial \varphi} \quad (4)$$

where r_o is the unperturbed value of the particle radial location, $B_d = B_E R_E^3 / r_o^3$ is the amplitude of the magnetic dipole field at the equatorial radial distance r_o , M is the relativistic magnetic moment and γ is the Lorentz factor. In this model, the electric and magnetic perturbations are small in the sense that their contributions to the drift motion are much smaller than the contribution of the dipole field gradient.

Noting that the angular drift velocity is $\Omega = 3M/(\gamma q r_o^2)$ in a dipole field, and that the drift phase φ can be re-formulated in terms of angular drift velocity ($\varphi(t) = -\Omega t + \varphi(t=0)$), it results that the radial velocity for an equatorial particle initially located at r_o with a phase α is:

$$\frac{dr}{dt} = -\frac{r_o}{2B_d} \frac{dS}{dt}(t) - \frac{8r_o^2}{21B_d} \frac{dA}{dt}(t) \cos(\Omega t - \alpha) - \frac{r_o^2 \Omega}{3B_d} A(t) \sin(\Omega t - \alpha) \quad (5)$$

where the two first terms on the right side of the equation are due to the electric drift associated with the induced electric fields, and the third term is due to the magnetic drift perturbation. A partial integration of this equation yields:

$$\begin{aligned}
r(t) - r_o = & -\frac{r_o}{2B_d}(S(t) - S(0)) \\
& -\frac{8r_o^2}{21B_d}\left((A(t)\cos(\Omega t - \alpha) - A(0)\cos(\alpha)) + \Omega \int_0^t A(\xi)\sin(\Omega \xi - \alpha)d\xi\right) \\
& -\frac{r_o^2\Omega}{3B_d}\int_0^t A(\xi)\sin(\Omega \xi - \alpha)d\xi
\end{aligned} \tag{6}$$

Thus, the total radial displacement of the equatorial particle after a time t is

$$\begin{aligned}
r(t) - r_o = & -\frac{5}{7}\frac{r_o^2\Omega}{B_d}\int_0^t A(\xi)\sin(\Omega \xi - \alpha)d\xi - \frac{r_o}{2B_d}(S(t) - S(0)) \\
& -\frac{8}{21}\frac{r_o^2}{B_d}(A(t)\cos(\Omega t - \alpha) - A(0)\cos(\alpha))
\end{aligned} \tag{7}$$

With the exception of the integral term

$$X(t) = -\frac{5}{7}\frac{r_o^2\Omega}{B_d}\int_0^t A(\xi)\sin(\Omega \xi + \alpha)d\xi \tag{8}$$

all the other terms on the right hand side of the equal sign equation (7) are bounded, and these terms are of the order of $b/B_d \ll 1$. Only $X(t)$ can potentially lead to large cumulative effects. Therefore, it is important to take a closer look at this integral:

- If the signal A has frequencies in the neighborhood of the angular drift velocity Ω , the integral X can increase with time, and the radial displacement can become significant.
- The integral $X(t)$ only depends on the signal A , i.e., it only depends on the characteristics of the asymmetric perturbations of the magnetic field.
- The integral $X(t)$ consists of the partial integration of two nearly equal contributions: (1) the induced electric field contributes $8/21$ of the $5/7$ factor in the radial displacement (i.e., about 55%), and (2) the magnetic disturbance contributes $1/3$ of the $5/7$ factor in the radial displacement (i.e., about 45%).

Since the field perturbations are not well known, Fälthammar assumed that $A(t)$ are realizations of a stationary stochastic process. In other words, it is assumed that the signal A fluctuates randomly around a zero mean, with time-independent statistical properties. In that context, after a time t that is much longer than the autocorrelation time of the signal A , and much longer than the particle drift period, the expected value of the square displacement $(r(t) - r_o)^2$ will grow linearly with time t . Thus, over a long period of time t , the expected value of the square displacement per unit time will be constant. It is that constant rate of change value that determines the radial diffusion coefficient D_{LL} :

$$\frac{d}{dt}[(r(t) - r_0)^2] = \text{cst} = 2R_E^2 D_{LL} \quad (9)$$

where the symbol $[\]$ denotes the expectation value (averaging over all possible scenarios, including all possible initial drift phases α).

With the idealized models chosen, the radial diffusion coefficient is:

$$D_{LL, m, eq} = \frac{1}{2R_E^2} \frac{d}{dt}[X^2(t)] = \frac{1}{2} \left(\frac{5}{7} \right)^2 \left(\frac{r_0^2 \Omega}{R_E B_d} \right)^2 \int_0^\infty [A(t) A(t + \xi)] \cos(\Omega \xi) d\xi \quad (10)$$

where the subscript m indicates that electromagnetic radial diffusion is driven by magnetic disturbances (magnetic and induced electric field fluctuations), and the subscript eq refers to equatorial particles. Because A is a stationary signal, $[A(t) A(t + \xi)]$ is independent of time t. It only depends on the lag ξ . For ξ greater than the autocorrelation time of the signal A, $[A(t) A(t + \xi)]$ is zero, and the integration over ξ can be extended to infinity.

Introducing $P_A(\Omega)$ the power spectrum of the asymmetric field perturbation A evaluated at the angular drift velocity Ω :

$$P_A(\Omega) = 4 \int_0^\infty [A(t) A(t + \xi)] \cos(\Omega \xi) d\xi \quad (11)$$

It results that:

$$D_{LL, m, eq} = \frac{1}{8} \left(\frac{5}{7} \right)^2 \frac{R_E^2 L^{10}}{B_E^2} \Omega^2 P_A(\Omega) \quad (12)$$

In terms of drift frequency ($\nu = \Omega/2\pi$), the diffusion coefficient is

$$D_{LL, m, eq} = \frac{\pi^2}{2} \left(\frac{5}{7} \right)^2 \frac{R_E^2 L^{10}}{B_E^2} \nu^2 P_A(\nu) \quad (13)$$

For a given kinetic energy, the radial diffusion coefficient $D_{LL, m}$ for off-equatorial particles is proportional to the diffusion coefficient in the equatorial case $D_{LL, m, eq}$ (Fälthammar, 1968)

$$D_{LL, m}(\alpha_{eq}) = \Gamma(\alpha_{eq}) D_{LL, m, eq} \quad (14)$$

where $\Gamma(\alpha_{eq})$ is a multiplying factor that strongly depends on the pitch angle at magnetic equator α_{eq} . $\Gamma(\alpha_{eq})$ is equal to 1 in the equatorial case ($\alpha_{eq} = 90^\circ$) and it is close to 0.1 for the most field-aligned particles.

2.1.2. Electric potential disturbances and electrostatic radial diffusion—

Similar calculations were carried out in the case of electric potential disturbances ($\nabla \times \mathbf{E} = \mathbf{0}$), in the absence of magnetic field perturbations, in a background dipole field. The component of the electric field fluctuation that leads to radial motion is the azimuthal component. It is described by a partial Fourier sum around r_0 :

$$E_\varphi(r_0, \varphi, t) = \sum_{n=1}^N E_{\varphi n}(t) \cos(n\varphi + \gamma_n) \quad (15)$$

where the phases γ_n are assumed not to vary with time t .

The quantities $E_{\varphi n}(t)$ are considered be individually and jointly stationary and ergodic processes, with a zero mean. As a result, $[E_{\varphi m}(t - \tau) E_{\varphi n}(t)] = [E_{\varphi m}(t) E_{\varphi n}(t + \tau)]$ and these quantities are independent of t , both when $m = n$ and when $m \neq n$. In that case, the radial diffusion coefficient is:

$$D_{LL,e} = \frac{1}{2} \left(\frac{1}{R_E B_d} \right)^2 \sum_{n=1}^N \int_0^\infty [E_{\varphi n}(t) E_{\varphi n}(t + \xi)] \cos(n\Omega \xi) d\xi \quad (16)$$

where the subscript e in $D_{LL,e}$ stands for electrostatic radial diffusion driven by electric potential disturbances.

With $P_E(n\nu)$ the power spectrum of the n^{th} harmonic of the electric field fluctuations evaluated at the n^{th} harmonic of the drift frequency ν , the radial diffusion coefficient is

$$D_{LL,e} = \frac{L^6}{8R_E^2 B_E^2} \sum_{n=1}^N P_E(n\nu) \quad (17)$$

This expression is valid for all equatorial pitch angles (because of the assumption of equipotential field lines, see also (Fälthammar, 1968)).

2.2. Fei et al.'s description of radial diffusion (2006)

New expressions for the radial diffusion coefficients were proposed by Elkington et al. (2003), and further developed by Fei et al. (2006) to describe the effect of field fluctuations on a population of trapped equatorial particles in the presence of a slightly asymmetric background magnetic field. No theoretical description was proposed for non-equatorial particles.

The background magnetic field model considered is the superposition of a dipole field and a time-stationary asymmetric disturbance in the equatorial plane. The magnitude of the magnetic field B_0 is:

$$B_0 = B_d + \Delta B \cos \varphi \quad (18)$$

where ΔB is a small perturbation: $\Delta B/B_d \ll 1$.

The unperturbed drift trajectory of energetic particles trapped in a magnetic field is characterized by $B_0 = \text{cst}$. Thus, with the model chosen, the equation of the drift contour is:

$$r(\varphi) = r_o \left(1 + \frac{\Delta B}{3B_d} \cos \varphi \right) \quad (19)$$

and it is considered that the third adiabatic coordinate L^* is a spatial coordinate that merges with the normalized average radius of the drift contour:

$$L^* = r_o/R_E \quad (20)$$

Differentiating the equation (19), the authors obtained that:

$$\frac{dL^*}{dr} = \frac{1}{R_E} \left(1 - \frac{4}{3} \frac{\Delta B}{B_d} \cos \varphi \right) \quad (21)$$

Thus, with Fei et al.'s model, a displacement of an equatorial particle away from the initial drift contour dr/dt leads to a time variation of the L^* parameter:

$$\frac{dL^*}{dt} = \frac{dL^*}{dr} \frac{dr}{dt} \quad (22)$$

In the same vein as Fälthammar's approach, the effects of two different drivers for radial diffusion are discussed separately in Fei et al.'s model: (1) the magnetic field disturbances and (2) the electric field disturbances.

2.2.1. Magnetic disturbances and magnetic diffusion according to Fei et al.'s model—The magnetic field fluctuations considered are in the direction of the background magnetic field (compressional perturbations). They are described by a Fourier sum around r_o :

$$\delta B(r, \varphi, t) = \sum_{n=1} \delta B_n(t) \cos(n\varphi) \quad (23)$$

Like Fälthammar, the authors assume that the perturbations $\delta B_n(t)$ are realizations of stationary stochastic processes. The induced electric fields are omitted and the radial drift motion driven by magnetic field disturbances is equal to

$$\frac{dr}{dt} = -\frac{M}{q\gamma B_d r_o} \frac{\partial(\delta B)}{\partial \varphi} \quad (24)$$

Combining equations (21), (22), (23), and (24), the authors obtained that:

$$\begin{aligned} \frac{dL^*}{dt}(r, \varphi, t) = & + \frac{ML^{*2}}{q\gamma B_E R_E^2} \sum_{n=1} n \delta B_n(t) \sin(n\varphi) + \frac{2}{3} \frac{ML^{*5}}{q\gamma B_E R_E^2} \frac{\Delta B}{B_E} \sum_{n=1} n \delta B_n(t) \sin((n+1)\varphi) \\ & + \frac{2}{3} \frac{ML^{*5}}{q\gamma B_E R_E^2} \frac{\Delta B}{B_E} \sum_{n=1} n \delta B_n(t) \sin((n-1)\varphi) \end{aligned} \quad (25)$$

The resulting diffusion coefficient is obtained with an approach similar to the one proposed by Fälthammar (1965). The equation (25) is integrated between a time $t = 0$ and a time t to obtain the variation of L^* . Then, the variation of L^* is squared.

$$(L^*(t) - L^*(0))^2 = (a+b+c)^2 \quad (26)$$

where a , b and c are the integrals of the three terms on the right hand of the equation (25).

It is then considered that

$$\frac{d}{dt}[(L^*(t) - L^*(0))^2] = \frac{d}{dt}[a^2] + \frac{d}{dt}[b^2] + \frac{d}{dt}[c^2] \quad (27)$$

where the symbol $[]$ denotes the expectation value and d/dt denotes the rate of change.

As a result, Fei et al. (2006) obtained a diffusion coefficient driven by magnetic disturbances equal to:

$$\begin{aligned} D_{LL, b. eq} = & \frac{M^2}{8q^2\gamma^2 B_E^2 R_E^4} L^{*4} \sum_{n=1} n^2 P_n^B(n\Omega) + \frac{2}{9} \frac{M^2}{q^2\gamma^2 B_E^2 R_E^4} \left(\frac{\Delta B}{B_E}\right)^2 L^{*10} \sum_{n=1} n^2 P_n^B((n+1)\Omega) \\ & + \frac{2}{9} \frac{M^2}{q^2\gamma^2 B_E^2 R_E^4} \left(\frac{\Delta B}{B_E}\right)^2 L^{*10} \sum_{n=1} n^2 P_n^B((n-1)\Omega) \end{aligned} \quad (28)$$

where P_n^B is the power spectrum of the n^{th} harmonic of the magnetic field fluctuation δB :

$$P_n^B(\omega) = 4 \int_0^\infty [\delta B_n(t) \delta B_n(t+\xi)] \cos(\omega \xi) d\xi \quad (29)$$

and Ω is the angular drift velocity of the population considered.

The subscript b in $D_{LL,b,eq}$ indicates that the coefficient quantifies magnetic radial diffusion driven by magnetic field disturbances (omitting induced electric field fluctuations) according to Fei et al's model.

The first term on the right hand of equation (28) does not depend on the asymmetry of the magnetic field (B). The second and third terms on the right hand of equation (28) characterize radial diffusion enabled by the asymmetry of the field. Because they are proportional to $(B/B_E)^2$, and $B/B_d \ll 1$ by design, these two terms are small in comparison with the first term (Figure 13 in (Fei et al., 2006)).

2.2.2. Electric disturbances and electric diffusion according to Fei et al.'s model—The electric field disturbance is assumed to be in the azimuthal direction. It is described by a Fourier sum around r_0 :

$$\delta E_\varphi(r, \varphi, t) = \sum_{n=1} \delta E_{\varphi n}(t) \cos(n\varphi) \quad (30)$$

The radial motion driven by electric field fluctuations is:

$$\frac{dr}{dt} = \frac{\delta E_\varphi}{B_d} \quad (31)$$

Combining equations (21), (22), (30) and (31), it results that

$$\begin{aligned} \frac{dL^*}{dt}(r, \varphi, t) &= \frac{1}{B_d} \sum_{n=1} \delta E_{\varphi n}(t) \cos(n\varphi) \quad (32) \\ &- \frac{2}{3} \frac{\Delta B}{B_d^2} \sum_{n=1} \delta E_{\varphi n}(t) \cos((n+1)\varphi) \\ &- \frac{2}{3} \frac{\Delta B}{B_d^2} \sum_{n=1} \delta E_{\varphi n}(t) \cos((n-1)\varphi) \end{aligned}$$

Following an approach similar to the one presented in the case of magnetic disturbances, the authors obtained that:

$$\begin{aligned}
D_{LL, \epsilon, eq} = & \frac{L^{*6}}{8B_E^2 R_E^2} \sum_{n=1} P_n^E(n\Omega) \\
& + \frac{2}{9B_E^2 R_E^2} \left(\frac{\Delta B}{B_E} \right)^2 L^{*12} \sum_{n=1} n^2 P_n^E((n+1)\Omega) \\
& + \frac{2}{9B_E^2 R_E^2} \left(\frac{\Delta B}{B_E} \right)^2 L^{*12} \sum_{n=1} n^2 P_n^E((n-1)\Omega)
\end{aligned} \quad (33)$$

where P_n^E is the power spectrum of the n^{th} harmonic of the electric field fluctuation δE_ϕ . The subscript ϵ in $D_{LL, \epsilon, eq}$ indicates that the coefficient quantifies electric radial diffusion driven by electric disturbances according to Fei et al.'s model (including both electrostatic potential disturbances, and induced electric field disturbances). The first term on the right hand of equation (33) does not depend on the asymmetry of the magnetic field B . The second and third terms on the right hand of equation (33) characterize additional radial diffusion due to the asymmetry of the field. Because they are proportional to $(\Delta B/B_E)^2$, they are small in comparison with the first term (Figure 13 in (Fei et al., 2006)).

3. Comparing Fälthammar's and Fei et al.'s results in the case of magnetic disturbances in a background magnetic field

Given the apparent similarities between Fei et al.'s formulas and Fälthammar's formulas, one could think that Fei et al.'s formulas are no more than a generalized version of Fälthammar's expressions. The objective of this paragraph is to emphasize that: (1) Fei et al.'s formulas are distinct from Fälthammar's formulas and that (2) Fei et al.'s formulas are erroneous. On one hand, Fälthammar's model (1965) provides (1) an electric diffusion coefficient, due to electric potential fluctuations ($\nabla \times \mathbf{E} = 0$) – electrostatic radial diffusion –, and (2) a magnetic diffusion coefficient, due to the combined action of magnetic field perturbations and induced electric field perturbations – electromagnetic radial diffusion –. On the other hand, Fei et al.'s model (2006) quantifies (1) an electric diffusion coefficient, due to both electric potential perturbations and induced electric field perturbations – electric radial diffusion – and (2) a magnetic diffusion coefficient, due to magnetic perturbations only – magnetic radial diffusion –. In all cases, it is commonly considered that the total radial diffusion coefficient is the sum of the different contributions. A representation of the different perturbations taken into account by the different diffusion coefficients computed by Fälthammar (1965) and Fei et al. (2006) is provided Figure 1.

By separating the action of magnetic field perturbations from the action of the associated induced electric fields, Fei et al. ultimately assume that both perturbations are uncorrelated. The validity of this assumption is often wrongly attributed to Brizard and Chan (2001). Yet, it is inconsistent with Faraday's law ($\nabla \times \mathbf{E} = -\mathbf{B}/t$). Thus, Fei et al.'s formulas for radial diffusion are erroneous in the presence of magnetic field fluctuations.

It is possible to estimate the magnitude of the error by applying Fei et al.'s approach to Fälthammar's description, in the absence of electric potential fluctuations. Since Fei et al.'s

magnetic field model does not include a symmetric perturbation of the background magnetic field, let us consider that the symmetric component of the magnetic field model (S) is zero equation (2) and let us assume that asymmetric part (A) is varying with time. Thus, the magnetic fluctuation $A(t) r \cos \varphi$ corresponds to $\delta B(r, \varphi, t)$ in Fei et al's model, with B set to zero equation (19). Following Fei et al.'s approach, one can track cross drift shell motion due to the magnetic disturbance only (omitting the action of the induced electric fields). In that case, the resulting diffusion coefficient is equal to:

$$D_{LL, b, eq} = \frac{1}{8} \left(\frac{1}{3} \right)^2 \frac{R_E^2 L^{10}}{B_E^2} \Omega^2 P_A(\Omega) \quad (34)$$

Similarly, one can track cross drift shell motion due to the induced electric field disturbance only (omitting the action of the magnetic field fluctuations). The resulting diffusion coefficient is:

$$D_{LL, \epsilon, eq} = \frac{1}{8} \left(\frac{8}{21} \right)^2 \frac{R_E^2 L^{10}}{B_E^2} \Omega^2 P_A(\Omega) \quad (35)$$

(The equivalence can be derived directly from equation (33) by noticing that $P_1^E(\Omega) = (8/21)^2 \Omega^2 R_E^4 L^4 P_A(\Omega)$ when the electric field is described according to equation (3)). Therefore, the total radial diffusion coefficient according to Fei et al's formula is:

$$D_{LL, b, eq} + D_{LL, \epsilon, eq} = \frac{1}{8} \left(\left(\frac{8}{21} \right)^2 + \left(\frac{1}{3} \right)^2 \right) \frac{R_E^2 L^{10}}{B_E^2} \Omega^2 P_A(\Omega) \quad (36)$$

However, when both magnetic and induced electric disturbances are taken into account, the cumulative radial displacement is the sum of both contributions (see also the derivation of equation (8) section 2.1.1). Thus, Fälthammar's radial diffusion coefficient for magnetic field disturbances is

$$D_{LL, m, eq} = \frac{1}{8} \left(\frac{8}{21} + \frac{1}{3} \right)^2 \frac{R_E^2 L^{10}}{B_E^2} \Omega^2 P_A(\Omega) = \frac{1}{8} \left(\frac{5}{7} \right)^2 \frac{R_E^2 L^{10}}{B_E^2} \Omega^2 P_A(\Omega) \quad (37)$$

Because

$$\frac{\left(\frac{8}{21} \right)^2 + \left(\frac{1}{3} \right)^2}{\left(\frac{8}{21} + \frac{1}{3} \right)^2} \cong 0.5 \quad (38)$$

It results that

$$D_{LL, b, eq} + D_{LL, \epsilon, eq} \cong 0.5 D_{LL, m, eq} \quad (39)$$

Thus, by wrongly assuming that the radial displacements due to the magnetic field perturbations are uncorrelated from the radial displacements due to the induced electric fields, Fei et al.'s formulas provide an underestimation of the total radial diffusion coefficient by a factor 2 in the case of magnetic field disturbances. This result relies on the assumptions that (1) the magnetic field disturbances are described by the simple model introduced by Fälthammar (1965) (equations (2) and (3)), and that (2) there is no electric potential disturbance.

The artificial separation between electric potential disturbances and magnetic disturbances in Fälthammar's study was justified by the fact that these disturbances originate from different sources. In practice, the correlation between electric potential disturbances and magnetic disturbances is unknown. A potential correlation between these fluctuations would result in a global radial diffusion coefficient distinct from the sum of the different contributions. That is why it is necessary to extend the theoretical framework underlying radial diffusion.

The Fokker-Planck equation calls for only one global radial diffusion coefficient to characterize the statistical properties of cross drift shell motion for a trapped radiation belt population, regardless the nature of the driver. Thus, cross drift shell motion needs to be characterized thoroughly.

4. A general method to quantify radial diffusion

Radial diffusion has to do with the violation of the third adiabatic invariant of a trapped radiation belt population. Thus, in order to quantify radial diffusion, it is essential to first characterize the violation of the third adiabatic invariant, i.e., to determine dL^*/dt .

4.1. Analytic expressions for the violation of the third adiabatic invariant

The appropriate coordinate to discuss radial diffusion is the third adiabatic invariant, which is proportional to the magnetic flux Φ encompassed by a particle drift contour. Equivalently, one can use the L^* coordinate ($L^* = 2\pi B_E R_E^2 / |\Phi|$). Thus, in order to determine Φ or L^* , it is necessary to first determine the drift contour associated with the particle considered. In a steady state, the total energy of the guiding center ϵ is constant along the drift contour Γ (e.g., Whipple, 1978). In other words, for all bounce-averaged guiding center locations $\mathbf{r}$ elements of Γ :

$$\epsilon(\mathbf{r}) = T(\mathbf{r}) + qU(\mathbf{r}) = \text{const.} \quad (40)$$

where U is the electrostatic potential (measured either at the mirror point or equivalently at the magnetic equator – U is constant along equipotential magnetic field lines), and T is the guiding center kinetic energy:

$$T = E_o \sqrt{1 + \frac{2MB_m}{E_o}} - E_o \quad (41)$$

where E_o is the rest mass energy (511 keV for an electron, 938 MeV for a proton), M is the relativistic magnetic moment, and B_m is the mirror point magnetic field intensity.

For radiation belt populations, it is commonly assumed that the kinetic energy is so high that the effect of electrostatic potentials on trapped particle drift motion can be neglected ($T \geq 100$ keV $\gg |qU|$, thus $e \approx T$). As a result, the drift shell and the corresponding drift contour are usually characterized by the relation:

$$B_m(\mathbf{r}) = \text{const.} \quad (42)$$

Therefore, L^* is not a spatial coordinate, it is the magnetic coordinate of a geomagnetically trapped particle (e.g. Roederer & Lejosne, 2018).

There are general analytic expressions for the violation of the third adiabatic invariant. These formulas have been demonstrated by Northrop (1963) in the (α, β, K) coordinate system, where α and β are coordinates related to the magnetic field topology (Euler potentials) and K identifies with the total energy of particles in the static case. The formulas were also derived by Lejosne et al. (2012) using real geographical coordinates $(\mathbf{r})$. Because radial diffusion implicitly assumes conservation of the first two adiabatic invariants, the characteristic time for the variations of the field must be greater than the bounce period of the population considered. Thus, in the following, all the quantities considered are averages over the bounce period of the population considered.

For an equatorial particle trapped in a magnetic field, in the absence of electrostatic fields, the instantaneous rate of change of L^* is:

$$\frac{dL^*}{dt}(\mathbf{r}_o, t) = \frac{L^{*2}}{2\pi B_E R_E^2} \oint_{\Gamma(r_o)} \frac{B_o(\mathbf{r}, t)}{|\nabla_o B_o(\mathbf{r}, t)|} \left(\frac{dB_o}{dt}(\mathbf{r}, t) - \frac{dB_o}{dt}(\mathbf{r}_o, t) \right) dl \quad (43)$$

where $\mathbf{r}_o$ is the guiding center location along the drift contour $\Gamma(r_o)$ at time t , B_o is the equatorial magnetic field intensity, and $\nabla_o B_o$ is the gradient of B_o .

For non-equatorial trapped particles in the absence of electrostatic fields, the formula becomes

$$\frac{dL^*}{dt}(\mathbf{r}_o, t) = \frac{L^{*2}}{2\pi B_E R_E^2} \oint_{\Gamma(r_o)} \frac{B_o(\mathbf{r}, t)}{|\nabla_o B_m(\mathbf{r}, t)|} \left(\frac{dB_m}{dt}(\mathbf{r}, t) - \frac{dB_m}{dt}(\mathbf{r}_o, t) \right) dl \quad (44)$$

where B_m is the magnetic field intensity at the mirror points.

In the most general case:

$$\frac{dL^*}{dt}(\mathbf{r}_o, t) = \frac{L^{*2}}{2\pi B_E R_E^2} \oint_{\Gamma(r_o)} \frac{B_o(\mathbf{r}, t)}{|\nabla_o \varepsilon(\mathbf{r}, t)|} \left(\frac{d\varepsilon}{dt}(\mathbf{r}, t) - \frac{d\varepsilon}{dt}(\mathbf{r}_o, t) \right) dl \quad (45)$$

where ε is the total energy of the guiding center.

These expressions can be reformulated in terms of deviation from a drift-averaged total time derivative. This is done by introducing the drift-average spatial operator $[]_D$ such that

$$[f]_D(t) = \frac{1}{\tau_D} \oint_{\Gamma} \frac{f(\mathbf{r}, t)}{|\mathbf{v}_D(\mathbf{r}, t)|} dl = \frac{1}{\tau_D} \int_0^{\tau_D} f(\mathbf{r}(\tau), t) d\tau \quad (46)$$

where τ_D indicates the drift period of the population considered, and Γ is the associated drift contour at time t . $[f]_D(t)$ determines the spatial average of the quantity f at a time t , along the associated drift contour Γ . Each drift contour element is weighted in the summation by the time spent drifting through that location if the electromagnetic conditions were time-stationary.

With that operator, the equation (45) is also:

$$\frac{dL^*}{dt}(\mathbf{r}_o, t) = \frac{L^{*2}}{q\Omega B_E R_E^2} \left(\left[\frac{d\varepsilon}{dt} \right]_D(t) - \frac{d\varepsilon}{dt}(\mathbf{r}_o, t) \right) \quad (47)$$

where $\Omega = 2\pi/\tau_D$ is the population angular drift velocity. This is the same formula as the one derived by Northrop (1963, eq (3.80), p.64) and reviewed by Cary and Brizard (2009, p. 717).

In the absence of electrostatic field fluctuations, the equation (47) becomes:

$$\frac{dL^*}{dt}(\mathbf{r}_o, t) = \frac{L^{*2}}{2\pi B_E R_E^2} C_m \left(\left[\frac{dB_m}{dt} \right]_D(t) - \frac{dB_m}{dt}(\mathbf{r}_o, t) \right) \quad (48)$$

where C_m is a constant equal to

$$C_m = \oint_{\Gamma(r_o)} \frac{B_o(\mathbf{r}, t)}{|\nabla_o B_m(\mathbf{r}, t)|} dl \quad (49)$$

For equatorial particles, the expression (48) is:

$$\frac{dL^*}{dt}(\mathbf{r}_o, t) = \frac{L^{*2}}{2\pi B_E R_E^2} C_o \left(\left[\frac{dB_o}{dt} \right]_D(t) - \frac{dB_o}{dt}(\mathbf{r}_o, t) \right) \quad (50)$$

where C_o is a constant equal to

$$C_o = \oint_{\Gamma(r_o)} \frac{B_o(\mathbf{r}, t)}{|\nabla_o B_o(\mathbf{r}, t)|} dl \quad (51)$$

In the traditional case of a background dipole field to which small magnetic field perturbations are superimposed,

$$\frac{dL^*}{dt}(\mathbf{r}_o, t) \cong \frac{L^{*4}}{3B_E} \left(\left[\frac{dB_o}{dt} \right]_D(t) - \frac{dB_o}{dt}(\mathbf{r}_o, t) \right) \quad (52)$$

In the case of the the simple model introduced by Fälthammar (1965) (equations (2) and (3)), with

$$\frac{dB}{dt} = \frac{\partial B}{\partial t} + \frac{\mathbf{E}_{ind} \times \mathbf{B}}{B^2} \cdot \nabla B = \frac{5}{2} \frac{dS}{dt} + \frac{15}{7} \frac{dA}{dt} r \cos \varphi \quad (53)$$

the equation (52) becomes:

$$\frac{dL^*}{dt}(\mathbf{r}_o, t) \cong - \frac{5}{7} \frac{L^{*5} R_E}{3B_E} \frac{dA}{dt} \cos \varphi \quad (54)$$

And it is straightforward to retrieve the expression for the radial diffusion coefficient (equation (12)).

4.2. Comments on the violation of the third adiabatic invariant

4.2.1. L^* can only be violated if the time variations of the field depend on local time.—If the time variations of the fields are the same all along the drift contour: $d\epsilon/dt(\mathbf{r}_o, t) = [d\epsilon/dt]_D(t)$ for all guiding center locations $\mathbf{r}_o$ along the drift contour. As a result, in the case of symmetric field fluctuations:

$$\frac{dL^*}{dt}(\mathbf{r}_o, t) = 0 \quad (55)$$

4.2.2. dL^*/dt is zero on drift-average along the drift contour—The instantaneous rate of change of L^* for a guiding center located at $(\mathbf{r}_0, t)$ along the drift contour is proportional to $([d\epsilon/dt]_D(t) - d\epsilon/dt(\mathbf{r}_0, t))$. Thus, the drift average of the variations of L^* along $\Gamma(\mathbf{r}_0)$ is zero:

$$\left[\frac{dL^*}{dt} \right]_D(t) = 0 \quad (56)$$

4.2.3. There is a competition between the drift period and the characteristic time for the variation of the fields—The expression of dL^*/dt highlights the competition between the characteristic time for the variation of the field τ_C and the drift period τ_D of the population considered. Since the instantaneous rate of change of L^* is proportional to τ_D/τ_C , L^* remains approximately constant if the characteristic time for the variation of the field is very long in comparison with the drift period : $(\tau_D/\tau_C \ll 1) \Rightarrow (dL^*/dt \ll 1)$. This is in agreement with the fact that L^* is an adiabatic invariant associated with drift motion.

4.2.4. Challenges—In the most general case, the quantification of dL^*/dt requires:

- to define the drift contour of the population considered at a given instance,
- to evaluate the electric and magnetic fields, together with their total time derivatives – i.e., to evaluate the total changes as seen by the particles $(d/dt = \partial/\partial t + \mathbf{V}_D \cdot \nabla)$, over the entire drift shell, at a given instance.

Since no measurement can provide such information, there is ineluctable uncertainty when quantifying dL^*/dt . In addition, it is worth keeping in mind that the proposed framework relies on the frozen field condition (e.g. Parker, 1960). This requires no electric field component parallel to the magnetic field direction and a perfectly conducting Earth's surface. In practice, both assumptions should be examined in the region of interest. There are times and regions where the frozen field condition is not valid. The consequences for the drift motion of a trapped radiation belt population are unknown.

4.3. How to derive a radial diffusion coefficient

Let us derive a general formulation for the radial diffusion coefficient, starting from the expression of the instantaneous rate of change of L^* at a location $\mathbf{r}$ and a time t :

$$V_L(\mathbf{r}, t) = \frac{dL^*}{dt}(\mathbf{r}, t) \quad (57)$$

with dL^*/dt described equation (45) – or an appropriate simplification of it.

4.3.1. Integration over a time interval t —After a time t , the variation in the L^* of a particle is equal to

$$\Delta L^* = L^*(\mathbf{r}(t), t) - L^*(\mathbf{r}(0), 0) = \int_0^t V_L(\mathbf{r}(u), u) du \quad (58)$$

4.3.2. Computation of the expectation value for the mean square

displacement—The expectation value of the square of the displacement is equal to

$$[(\Delta L^*)^2] = \int_0^t \int_0^t [V_L(\mathbf{r}(u), u) V_L(\mathbf{r}(v), v)] du dv \quad (59)$$

where $[]$ denotes the expectation value of the bracketed quantity. Therefore, it is necessary to compute the autocorrelation function of the Lagrangian velocity V_L to determine the radial diffusion coefficient D_{LL} . This velocity is a function of both space and time.

The rate of change of $[(\Delta L^*)^2]$ must become constant after some time t to be able to determine the radial diffusion coefficient ($D_{LL} = \frac{1}{2} \frac{d}{dt} [(\Delta L^*)^2]$). Additional theoretical assumptions have been made to guarantee that this is indeed the case.

4.3.3. Separation of the spatial and temporal dependence for the velocity V_L

—The traditional assumption here is that the spatial and temporal functions are independent (see for instance equations (2), (15), (23) and (30)). In addition, because the particles are drifting along closed drift shells, it is considered that the spatial function is a periodic function in local time, with a periodicity defined by the particle unperturbed drift period. Because the radial diffusion formalism assumes small variations for the coordinate of interest, the radial dependence of the spatial function is often omitted. As a result, the velocity V_L is rewritten as a product of two independent functions: a temporal function λ , and a spatial function which only depends on local time $\varphi = -\Omega t + \alpha$

$$V_L(\mathbf{r}(t), t) = \lambda(t) \cos(\Omega t - \alpha) \quad (60)$$

where Ω and α are respectively the angular drift velocity and the initial drift phase of the particle considered. This formulation could be further elaborated by rewriting $V_L(\mathbf{r}(t), t)$ as a Fourier sum $\sum_n \lambda_n(t) \cos(n\varphi + \alpha_n)$. For the sake of simplicity, we only consider the first harmonic ($n=1$) in the following. The generalization is straightforward.

4.3.4. Drift phase averaging

—We compute the expectation value of $V_L(\mathbf{r}(u), u) V_L(\mathbf{r}(v), v)$ by averaging over multiple scenarios, including all possible initial drift phases.

As a result:

$$[V_L(\mathbf{r}(u), u)V_L(\mathbf{r}(v), v)] = \frac{1}{2}[\lambda(u)\lambda(v)]\cos(\Omega(u-v)) \quad (61)$$

4.3.5. Stationary signals—It is then assumed that the signal λ is stationary. Thus, that the autocorrelation $[\lambda(u)\lambda(v)]$ only depends on the lag between u and v . The integral (59) becomes:

$$[(\Delta L^*)^2] = t \int_0^t [\lambda(t)\lambda(t+\tau)]\cos(\Omega\tau)d\tau \quad (62)$$

where $[\lambda(t)\lambda(t+\tau)]$ does not depend on t . Once the time τ becomes longer than the autocorrelation time of the signal λ , the expectation value of $[\lambda(t)\lambda(t+\tau)]$ becomes zero. Thus, the integral reaches a finite value once t is large enough.

In that context, the mean square of the displacement grows linearly with time once t is greater than the signal autocorrelation time, and the rate of change of $[(\Delta L^*)^2]$ is constant:

$$\frac{d}{dt}[(\Delta L^*)^2] = \int_0^\infty [\lambda(t)\lambda(t+\tau)]\cos(\Omega\tau)d\tau \quad (63)$$

It is the magnitude of the rate of change of $[(\Delta L^*)^2]$ that determines the radial diffusion coefficient:

$$D_{LL} = \frac{1}{2} \frac{d}{dt}[(\Delta L^*)^2] \quad (64)$$

Introducing the power spectrum P_λ of the fluctuations λ at the angular drift velocity Ω :

$$P_\lambda(\Omega) = 4 \int_0^\infty [\lambda(t)\lambda(t+\tau)]\cos(\Omega\tau)d\tau \quad (65)$$

it results that:

$$D_{LL} = \frac{1}{8} P_\lambda(\Omega) \quad (66)$$

For instance, if the autocorrelation of the signal λ is described by an exponential function:

$$[\lambda(t)\lambda(t+\tau)] = [\lambda_0^2] e^{-\tau/\tau_\lambda} \quad (67)$$

where $[\lambda_0^2]$ is the mean square velocity, and the exponential time constant τ_λ represents the characteristic time over which the signal λ is correlated with its previous values, it results that

$$D_{LL}(\Omega) = \frac{[\lambda_0^2]}{2} \frac{\tau_\lambda}{1 + \Omega^2 \tau_\lambda^2} \quad (68)$$

Thus, if $\tau_\lambda \ll 1/\Omega$, i.e. if the autocorrelation time is very small in comparison with the population drift period, $D_{LL}(\Omega) = [\lambda_0^2]\tau_\lambda/2$ and the diffusion coefficient is independent of energy. It increases when the mean square velocity increases (i.e., when the field fluctuations increases), and when the autocorrelation time increases (because this means that the particles are pushed in the same direction for a longer time). On the other hand, if $\tau_\lambda \gg 1/\Omega$,

$D_{LL}(\Omega) = [\lambda_0^2]/(2\Omega^2\tau_\lambda)$, and the diffusion coefficient decreases with increasing energy. This is consistent with the fact that the characteristic time for the variations of the field must be very small in comparison with the drift period of the population considered for the third adiabatic invariant to potentially vary (see also paragraph 4.2.3). Thus, the variations of the diffusion coefficient with particles' energy can provide information on the autocorrelation time of the signal λ , and vice versa.

In Fälthammar's work, the Langrangian velocity equation (60) is integrated along the drift path before calculation of the power spectrum. So it is not the power spectrum of the signal λ that is computed in Fälthammar's work. It is the power spectrum of the integral of the signal that is computed instead. With Λ an integral of λ , the integration of equation (60) yields:

$$\begin{aligned} \Delta L^* &= \int_0^t \lambda(u) \cos(\Omega u + \alpha) du = \Omega \int_0^t \Lambda(u) \cos(\Omega u + \alpha) du \quad (69) \\ &+ (\Lambda(u) \cos(\Omega u + \alpha) - \Lambda(0) \cos(\alpha)) \end{aligned}$$

As a result,

$$P_\lambda(\Omega) = \Omega^2 P_\Lambda(\Omega) \quad (70)$$

and

$$D_{LL} = \frac{\Omega^2}{8} P_\Lambda(\Omega) \quad (71)$$

In that context, we retrieve the traditional dependence of D_{LL} with Ω^2 , which cancels in the case of sudden impulses (see for instance the discussion in Fälthammar (1965)).

5. Conclusions

In this work, I have briefly reviewed the works of Fälthammar (1965, 1968) and Fei et al. (2006) in order to compare and contrast the different radial diffusion models proposed.

Both approaches rely on the assumption that the background magnetic field is mainly dipolar (Fälthammar assumed a background dipole field. Fei et al. assumed a slightly asymmetric background field, with $B/B_d \ll 1$). Regarding the idealized models for the field fluctuations, both approaches assume that:

- the field fluctuations are small (i.e. they can only drive small perturbations for the drift motion of the trapped population considered);
- the spatial and temporal variations are decoupled, i.e. the field fluctuations are the product of two independent functions: a function that depends of time, and a function that depends of space (more precisely, local time);
- the temporal variations are realizations of a stationary stochastic process.

Regarding the differences between the two models:

- Fei et al.'s model only applies to equatorial particles while Fälthammar's work has been extended to non-equatorial particles (1968).
- Fälthammar studied separately radial diffusion driven by magnetic field disturbances and radial diffusion driven by electric potential disturbances ($\nabla \times \mathbf{E} = 0$). On the other hand, Fei et al. (2006) studied separately radial diffusion driven by magnetic field disturbances (in the absence of electric fields) and radial diffusion driven by electric disturbances (in the absence of magnetic field disturbances).

While some of the model assumptions are germane to diffusion theory (namely, the assumptions that the field fluctuations are small and stationary (e.g. Taylor, 1921)), others represent a serious limitation to the current accuracy with which radial diffusion is quantified. In particular, the background magnetic field may significantly depart from a dipole magnetic. Magnetic field disturbances may not necessarily behave according to the linearized model introduced by Fälthammar (1965) (equations (2) and (3)). Magnetic field disturbances and electric potential disturbances may be correlated. This article describes a new method to compute a radial diffusion coefficient. This method is more general than what has been done before. It does not make any assumption regarding the topology of the background magnetic field, nor does it imply any artificial separation between the different fluctuations driving cross drift shell motion. The method starts with the general formula for the violation of the third adiabatic invariant. This formula could help question the assumptions on which radial diffusion models are currently based. In particular, current theoretical radial diffusion formulas rely on idealized field fluctuation models in which the spatial and temporal variations are decoupled. The extent to which this assumption is valid is

unknown. In that context, MHD simulations (e.g., Ilie et al., 2017) could provide useful information.

Currently, there is a need to improve the spatial and temporal accuracy of the radiation belt simulations. The objective is to introduce local time as a 4th dimension in the radiation belt simulations, and to develop event-specific models. In that case, it is pivotal to realize the limitations of radial diffusion theory, and the limitations of the germane Fokker-Planck equation. Radial diffusion theory is appropriate when the objective is to provide a drift-averaged characterization of a population cross drift shell motion, assuming that particles have experienced many small uncorrelated perturbations (e.g., Riley and Wolf, 1992). Finding a compromise between accuracy (MHD and test particle simulations) and expediency (Fokker-Planck diffusion theory) requires a statistical reformulation of the radiation belt dynamics which should be able to include significant, localized (i.e. non-diffusive) radial transport, drift phase bunching, drift echoes, etc.... Such features are specific to trapped population drift motion. Yet, they cannot be reproduced by the current numerical simulations which consist of solving a 3D Fokker-Planck equation.

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Key points:

- Fálthammar's and Fei et al.'s models for radial diffusion are compared and contrasted.
- Both models are limited. Fei et al.'s model for radial diffusion is erroneous.
- An improved method for quantifying radial diffusion is proposed.

Fälthammar (1965): $D_{LL} = D_{LL,m} + D_{LL,e}$

Fei et al. (2006): $D_{LL} = D_{LL,b} + D_{LL,\epsilon}$

Figure 1:

Fälthammar and Fei et al. made different choices when separating the different radial diffusion drivers. Fälthammar (1965) studied radial diffusion due to magnetic field fluctuations, including the effect of the induced electric fields ($D_{LL,m}$). He also studied radial diffusion due to electric potential fluctuations ($D_{LL,e}$). On the other hand, Fei et al. (2006) studied radial diffusion driven by magnetic field fluctuations, in the absence of any kind of electric field fluctuation ($D_{LL,b}$). They also studied independently radial diffusion driven by electric field fluctuations, regardless of their nature ($D_{LL,\epsilon}$). In all cases, the total radial diffusion coefficient D_{LL} is usually introduced as an aggregate, equal to the sum of the different contributions.