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A note on estimating the bent line quantile regression model

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Abstract

This paper considers a new estimating method for the bent line quantile regression model. By a simple linearization technique, the proposed method can simultaneously obtain the estimates of the regression coefficients and the change-point location. Moreover, it can be readily implemented by current software. Simulation studies demonstrate that the proposed method has good finite sample performance. Two empirical applications are also presented to illustrate the method.

Keywords

Quantile regression; Change-point; Linearization technique

1 Introduction

Bent line regression, introduced by Sprent (1961), models a linear relationship which is continuous but has differing slopes before or after a change-point. For example, the log of maximal running speed (MRS) of land mammals shows a sharp increase at low log mass, but a dramatic decrease after reaching a certain amount of log mass. To capture this distinct feature, many authors devoted to the segmented regression based on the mean structure, for example, Quandt (1958, 1960), Robison (1964), Feder (1975), Chappell (1989), Muggeo (2003) and among others. However, there are limiting results for segmented regression based on the quantile structure. It is worthwhile to note that the quantile regression is more flexible for capturing the heteroscedasticity and more robust to outliers in response variables (Koenker and Bassett 1978). Li et al. (2011) was among the first to study the segmented quantile regression with a single changepoint. To be specific, they considered the slope of one scale covariate changing at an unknown change-point, and developed a profile estimation procedure based on the grid-search method.

The main idea of the profile estimation method is that one can separately estimate the parameter vector conditional on the change-point location fixed beforehand, and then estimate the change-point location by a grid-search algorithm. Nevertheless, this is not direct to handle the unknown parameter vector as a completely true parameter. These disadvantages motivate our interest to seek an alternative method to simultaneously estimate the parameter vector and the change-point location, which can also readily implemented. The goal of this paper is to describe the details of an alternative estimating method. The

method we proposed is designed to outperform the grid-search procedure that is conventionally used in the bent line quantile regression. The proposed method is flexible enough to estimate all parameters including the location of change point simultaneously.

The remainder of the paper is organized as follows. Section 2 first reviews the gridsearch method for the bent line quantile regression model, and then develops a new estimating method based on a simple linearization technique by Muggeo (2003). The asymptotic properties are quite direct from the standard theory of quantile regression. Section 3 investigates the good performance of the proposed estimate through simulation studies. Two applications of real data analysis are presented to illustrate the proposed method in Sect.4. Section 5 concludes the paper.

2 Methodology

2.1 The grid-search method

Given $0 < \tau < 1$, Li et al. (2011) considered a continuous bent line quantile regression model for the τ th quantile of Y given x and z :

$$Q_\tau(Y|x, z) = \begin{cases} \alpha_0 + \alpha_1 x + z^\top \gamma, & x \leq t \\ \alpha_0 + \alpha_1 t + \alpha_2(x - t) + z^\top \gamma, & x \geq t, \end{cases} \quad (1)$$

where x is a scalar covariate whose slope changes at the change-point t , z is a $p \times 1$ vector of linear covariates, and $(\alpha_0, \alpha_1, \alpha_2, \gamma^\top)^\top$ are unknown parameters of interest. According to model (1), the linear quantile regression $Q_\tau(Y|x, z)$ is continuous on x at t , but has different straight lines on either side of change-point t . That is, α_1 is the slope of left line segment for $x \leq t$ and α_2 is the slope of right line segment for $x \geq t$. Hence, model (1) captures the non-linear effects between the quantile response Y and the covariate x .

For ease of notation, we re-express the model (1) as follows,

$$Q_\tau(Y|x, z) = \beta_0 + \beta_1 x + \beta_2(x - t)_+ + z^\top \gamma, \quad (2)$$

where $\beta_0 = \alpha_0$, $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_2 - \alpha_1$ is the difference-in-slopes parameter. Here, $(x - t)_+ = (x - t)I(x > t)$, where $I(\cdot)$ is the indicator function. Assume that the observed data $\{(Y_i, x_i, z_i) : i = 1, \dots, n\}$ are n identical and independent samples from (Y, x, z) . In this paper, we focus on the model (2), and aim to estimate the unknown parameters

$$\theta_\tau = (\beta_{0,\tau}, \beta_{1,\tau}, \beta_{2,\tau}, \gamma_\tau^\top, t_\tau)^\top \text{ for a given quantile } \tau.$$

We first briefly review the estimating procedure proposed by Li et al.(2011). Following their notations, let $\eta_\tau = (\beta_{0,\tau}, \beta_{1,\tau}, \beta_{2,\tau}, \gamma_\tau^\top)^\top$, $w_{i,t} = (1, x_i, (x_i - t)_+, z_i^\top)^\top$. To estimate the true parameter vector $\theta_\tau = (\beta_{0,\tau}, \beta_{1,\tau}, \beta_{2,\tau}, \gamma_\tau^\top, t_\tau)^\top$, a common estimator can be expressed by

$$\hat{\theta}_t = \arg \min_{\theta} \sum_{i=1}^n \rho_{\tau} \left\{ Y_i - \beta_0 - \beta_1 x_i - \beta_2 (x_i - t)_+ - z_i^{\top} \gamma \right\}, \quad (3)$$

where $\rho_{\tau}(u) = u[\tau - I(u < 0)]$ is the quantile loss function for $0 < \tau < 1$. However, the estimator $\hat{\theta}_{\tau}$ could not be obtained directly, because the objective function is neither convex nor differentiable. To circumvent this problem, Li et al. (2011) proposed a grid-search estimating procedure, based on the idea of the profile estimation for mean regression with a change-point proposed by Quandt (1958) and Chappell (1989). Specifically, for a fixed t , the parameter vector η_t can be estimated by

$$\hat{\eta}_t(t) = \arg \min_{\eta} \sum_{i=1}^n \rho_{\tau} \left(Y_i - w_{i,t}^{\top} \eta \right).$$

Then, the change-point location t_{τ} can be estimated by

$$\hat{t}_{\tau} = \arg \min_t \sum_{i=1}^n \rho_{\tau} \left\{ Y_i - w_{i,t}^{\top} \hat{\eta}(t) \right\}.$$

Therefore, the estimator of θ_{τ} is $\hat{\theta}_{\tau} = (\hat{\eta}_{\tau}^{\top}(\hat{t}_{\tau}), \hat{t}_{\tau})^{\top}$.

Furthermore, under some regularity conditions and the condition that $\beta_2 \neq 0$, Li et al. (2011) derived the Bahadur representation of $\hat{\theta}_{\tau}$, and proved its asymptotic properties, see more details in their paper.

2.2 The proposed method

As elaborated above, we aim to propose an alternative estimate, which can simultaneously estimate the parameter vector η_{τ} and the change-point location t_{τ} . From the technical point of view, the objective function in (3) is neither linear nor differentiable, and many optimization methods typically used in quantile regression are not directly applicable. To circumvent this issue, we may adopt the linearization technique proposed by Muggeo (2003) for segmented regression models. The main idea is to approximate the non-linear term by a first-order Taylor expansion around an initial value $t^{(0)}$,

$$(x_i - t)_+ = (x_i - t^{(0)})_+ + (-1)I(x_i > t^{(0)})(t - t^{(0)}).$$

Then the parentheses term in the right side of the objective function (3) can be approximated by

$$Y_i - \beta_0 - \beta_1 x_i - \beta_2 u_i^{(0)} - \beta_3 v_i^{(0)} - z_i^{\top} \gamma,$$

where $\beta_3 = \beta_2 \cdot (t - t^{(0)})$, $u_i^{(0)} = (x_i - t^{(0)})_+$ and $v_i^{(0)} = -I(x_i > t^{(0)})$ are two new covariates.

One may estimate all parameters $\xi = (\beta_0, \beta_1, \beta_2, \beta_3, \gamma^\top)^\top$ by minimizing the new objective function

$$\sum_{i=1}^n \rho_\tau(Y_i - \beta_0 - \beta_1 x_i - \beta_2 u_i^{(0)} - \beta_3 v_i^{(0)} - z_i^\top \gamma),$$

which is denoted by $\hat{\xi}^{(1)} = (\hat{\beta}_0^{(1)}, \hat{\beta}_1^{(1)}, \hat{\beta}_2^{(1)}, \hat{\beta}_3^{(1)}, \hat{\gamma}^{(1)\top})^\top$. Moreover, a natural iterated estimate for the change-point t is given by

$$t^{(1)} = t^{(0)} + \frac{\beta_3^{(0)}}{\beta_2^{(0)}}.$$

Clearly, one can repeat this process until possible convergence. The iterative algorithm is summarized as follows,

- i. Initialized parameters: $\hat{\xi}^{(0)} = (\hat{\beta}_0^{(0)}, \hat{\beta}_1^{(0)}, \hat{\beta}_2^{(0)}, \hat{\beta}_3^{(0)}, \hat{\gamma}^{(0)})$ and $\hat{t}^{(0)}$, setting $\hat{\beta}_3^{(0)}$ with a small value, such as 0.01.
- ii. Fix $\hat{t}^{(k)}$ at each step k , update $\hat{\xi}^{(k)} = (\hat{\beta}_0^{(k)}, \hat{\beta}_1^{(k)}, \hat{\beta}_2^{(k)}, \hat{\beta}_3^{(k)}, \hat{\gamma}^{(k)\top})^\top$ by

$$\hat{\xi}^{(k)} = \arg \min_{\xi} \sum_{i=1}^n \rho_\tau(Y_i - \beta_0 - \beta_1 x_i - \beta_2 u_i^{(k)} - \beta_3 v_i^{(k)} - z_i^\top \gamma), \quad (4)$$

where $u_i^{(k)} = (x_i - t^{(k)})_+$ and $v_i^{(k)} = -I(x_i > t^{(k)})$.

- iii. Update the change-point estimate $\hat{t}^{(k+1)}$ by

$$\hat{t}^{(k+1)} = \hat{t}^{(k)} + \frac{\hat{\beta}_3^{(k+1)}}{\hat{\beta}_2^{(k+1)}}.$$

- iv. Repeat steps (ii)–(iii) until convergence.

Remark 1 This algorithm inherits most of the advantages of that proposed by Muggeo (2003). One important feature is that one can optimize the non-linear and nondifferentiable objective function in (3) via iteratively minimizing the loss function of the linear quantile regression models (4). Moreover, similar to that of Muggeo (2003), the algorithm always converges in a deterministic model, provided that the change-point exists.

Remark 2 Let $\hat{\xi} = (\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3, \hat{\gamma}^T)^T$ be the limiting estimator of iterative estimators $\hat{\xi}^{(k)} = (\hat{\beta}_0^{(k)}, \hat{\beta}_1^{(k)}, \hat{\beta}_2^{(k)}, \hat{\beta}_3^{(k)}, \hat{\gamma}^{(k)T})^T$. From (4), the statistical inference for estimator $\hat{\xi}$ is quite standard for linear quantile regression. Thus, the relevant statistics such as standard errors, confidence intervals, p values are can readily obtained by the **R** package *quantreg*. Furthermore, the coefficient $\hat{\beta}_3$ measures the difference between the two fitted lines, and is expected to be approximately zero, or rather nonstatistically different from zero, provided the algorithm converges. Meanwhile, the standard error of $\hat{t}$ can be obtained by the delta method,

$$SE(\hat{t}) = \left[\text{Var}(\hat{\beta}_3) + \text{Var}(\hat{\beta}_3)(\hat{\beta}_3/\hat{\beta}_2)^2 - 2(\hat{\beta}_3/\hat{\beta}_2) \text{Cov}(\hat{\beta}_2, \hat{\beta}_3) \right]^{1/2} / |\hat{\beta}_2|.$$

In particular, if $\hat{\beta}_3 \approx 0$, it becomes the simplest version $SE(\hat{t}) = SE(\hat{\beta}_3)/|\hat{\beta}_2|$. Moreover, by McCullagh and Nelder (1989), a $100(1 - \alpha)$ Wald-based confidence interval is given by

$$[\hat{t} - z_{\alpha/2} SE(\hat{t}), \hat{t} + z_{\alpha/2} SE(\hat{t})],$$

where $z_{\alpha/2}$ is the $(1 - \alpha/2)$ th percentile of the standard normal distribution.

Remark 3 Note that the algorithm depends on the existence of a change-point. Hence, it is important to test the change-point existence. From the practical point of view, one can apply one of three tests proposed in Li et al. (2011), or the testing procedure proposed in Lee et al. (2011) and Zhang et al. (2014). However, these are beyond the topics of this paper.

Remark 4 This estimating method can be easily extended to multiple change points problem, along the line of the work of Muggeo (2003). However, testing the existence of multiple change points may require some additional work.

3 Numerical studies

All the R functions for estimation and inference are collected in the R package on github: <https://github.com/FPZhang2015/bentQR>. Two simulations are presented in this section for evaluating the finite sample performance of the proposed estimators. The first simulation is to evaluate the robust properties of the proposed estimator with four different types of error. Another simulation is similar to that in Li et al. (2011) for evaluating the performance in the symmetric and asymmetric bent lines.

3.1 Simulation 1

We consider the following two scenarios:

- I. Independent and identically distributed (IID): $Y = \beta_0 + \beta_1 x + \beta_2(x - t)_+ + \gamma z + e$,
- II. Heteroscedasticity: $Y = \beta_0 + \beta_1 x + \beta_2(x - t)_+ + \gamma z + (1 + 0.3x)e$,

where x is generated from a uniform distribution $U(-2, 5)$, z is generated from a binomial distribution $B(1, 0.5)$. The τ th quantile of the random error term e is 0, that is, $e = \tilde{e} - Q_\tau(\tilde{e})$, where $Q_\tau(\tilde{e})$ is the τ th quantile of $\tilde{e}$. For each scenario, we consider four error cases:

1. a standard normal distribution, $\tilde{e} \sim N(0, 1)$;
2. a t-distribution with three degrees of freedom, $\tilde{e} \sim t_3$;
3. a contaminated standard normal distribution, with 10% observations from a t-distribution with three degrees of freedom, $\tilde{e} \sim 0.9N(0, 1) + 0.1t_3$;
4. a contaminated standard normal distribution, with 10% observations from a standard Cauchy distribution, $\tilde{e} \sim 0.9N(0, 1) + 0.1Cauchy(0, 1)$.

The regression coefficients are set as $(\beta_0, \beta_1, \beta_2, \gamma) = (1, 3, -2, 1)$ and the change point is $t = 1.5$. For each case, we conduct 1000 repetitions with sample sizes $n = 200$. As suggested by a referee, we consider different quantile levels $\tau = 0.1, 0.3, 0.5, 0.7, 0.9$. Tables 1, 2, 3, 4, 5, 6, 7 and 8 report the results by two estimating methods for comparison: the grid-search estimate (“grid”) and the proposed estimate (“proposed”). From these tables, both the two estimators are consistent, since their biases (“Bias”) are small, and the estimated standard errors (“ESE”) are close to standard deviations (“SD”). The 95% coverage probabilities (“CP”) at the tail quantile levels (e.g. $\tau = 0.1, 0.9$) are smaller than the nominal level 95%. This may be because there not enough data at the tail quantile levels. However, this phenomenon disappears for the moderate quantile levels, such as $\tau = 0.3, 0.5, 0.7$. Moreover, both the mean square errors (“MSE”) and the average width of asymptotic confidence intervals (“AW”) of the proposed estimators are similar to those of the grid-search estimators. In summary, the proposed estimator is comparable to the grid-search estimator.

3.2 Simulation 2

This simulation setting is similar to that in Li et al. (2011), the data are generated from the following two scenarios:

Symmetric scenario: $Y = 0.5x - (x - 5)_+ + v_1(x)e$,

Asymmetric scenario: $Y = 4 + 0.5x - (x - 2)_+ + v_2(x)e$,

where $v_1(x) = 0.5 + 0.1x$, $v_2(x) = 0.5 + 0.02x$ and e is normal distribution with standard deviation 1 and its τ th quantile 0. In the symmetric scenario, we set $(\beta_0, \beta_1, \beta_2) = (0, 0.5, -1)$ to be symmetric. the scalar covariate $x \sim Unif(0, 10)$ and the change point is the median of x which is $t = 5$. In the asymmetric scenario, we set $(\beta_0, \beta_1, \beta_2) = (4, 0.5, -1)$, the scalar covariate x is a mixture distribution that uniformly on $(0, 1)$ with probability 0.1 and uniformly on $(1, 10)$ with probability 0.9, the change point is $t = 2$.

For each scenario, we conduct 1000 repetitions with sample size $n = 200$, by applying the grid search (“grid”) and the proposed (“proposed”) methods. The simulation results at different quantile levels $\tau = 0.1, 0.3, 0.5, 0.7, 0.9$ are recorded and demonstrated in Tables 9 and 10. We can draw a similar conclusion as that in Simulation 1, which indicates that the proposed estimator is comparable as the grid-search estimator.

4 Empirical analysis

In this section, we discuss two applications to illustrate the proposed method.

4.1 MRS data

The first application is the maximal running speed (MRS) example. Garland (1983) collected 107 terrestrial mammal pairs [MRS (km/h), mass (kg)] from many sources. Although the maximal running speed of the land mammals depends on their size, the dependency is obviously not monotonic. It is observed that the fastest mammals are neither the largest nor the smallest mammals. In order to describe the dependency between MRS and mass, Huxley (1932) introduced an allometric equation

$$MRS = \exp(a) \times \text{mass}^b,$$

where constants a and b may vary when variable mass exceeds some change point. As suggested in Garland (1983), the logarithm transformation makes the allometric equation be linear. Thus, the allometric equation can be considered as a linear relationship between $\log(\text{MRS})$ and $\log(\text{mass})$. Moreover, the bent line quantile regression model (2) may be applied to analyze the MRS data (Li et al. 2011).

Figure 1 presents a scatter plot of the dataset on the log scale. It displays that the $\log(\text{MRS})$ is not always increasing as the $\log(\text{mass})$ rising, which requires considering the nonlinearity of the $\log(\text{MRS})$ in the $\log(\text{mass})$. By a formal test, Li et al. (2011) showed the following bent line quantile regression model is well-fitted on the log scale,

$$Q_{\tau}(Y_i | x_i, z_i) = \beta_0 + \beta_1 x_i + \beta_2 (x_i - t)_+ + \gamma z_i,$$

where Y_i is $\log(\text{MRS})$, x_i is $\log(\text{mass})$, and $z_i = \mathbb{I}(\text{the } i\text{th animal is a hopper})$.

As discussed in Chappell (1989) and Li et al. (2011), the scientists have great interest in how fast mammals can run, especially in the determinants of MRS for organisms in niches for which speed is important. Thus, the median or upper quantiles are definitely interested. The estimated regression coefficients are tabulated in Table 11 for three distinct quantiles, $\tau = 0.5, 0.7, 0.8$. By the proposed method, the change point is about 32 kg for the 0.5th quantile, and 33 kg for 0.7th quantile and 32 kg for 0.8th quantile. The results are in accordance with conclusions in Li et al. (2011): the maximal running speed increases first and then gradually drops as the body mass gets bigger, and hopping has a positive effect on running speed. Figure 1 also confirms the conclusion via three bent quantile regression lines excluding hoppers. The proposed estimating curves are almost the same as the segmented estimation curves by the grid-search method.

4.2 Galton's data

Galton's data was originated from the famous study of Galton (1886) and listed the individual observations for 934 adult children with 8 variables, where the female heights were adjusted by 1.08 to compensate for sex differences. In this paper, we explore the

relationship between the heights of parents and their adult children by the mid-parent's heights and children's heights. Figure 2 shows that the relation between parents and children stature is nonlinear. Furthermore, Wachsmuth et al. (2003) suggested that there is a bend in the data somewhere around the average height of approximately 68 in. for both parents and children. Thus, the bent line quantile regression model (2) may be applied to analyze the Galton's data,

$$Q_{\tau}(Y_i|x_i) = \beta_0 + \beta_1 x_i + \beta_2 (x_i - t)_{+},$$

where Y_i is the mid-parent's height and x_i is the child's height. The estimates of change point and other regression parameters are reported in Table 12 for quantiles $\tau = 0.3, 0.5, 0.7, 0.8, 0.9$. The results show that all the slopes are gentle before the change point and then become steep after the change point for all the quantile levels τ 's. By the proposed method, the change point of the bent line quantile regression is about 68.550 in. for the 0.3th quantile, and 72.145, 70.324 and 71.282 in. for the 0.5th, 0.7th and 0.9th quantiles, respectively. The proposed estimates are almost the same as those by the grid search method. This is also confirmed by Fig.2.

5 Conclusion

Compared to the grid search method proposed by Li et al. (2011), the presented paper has proposed an alternative estimating procedure for the bent line quantile regression model. In virtue of the linearization technique of Muggeo (2003), the proposed method is more efficient than the grid search method, and it is easily implemented. Thus, this work can be considered as a first step toward the development of his idea to quantile regression model. However, a careful investigation on the extension of the proposed method to the test for multiple change-point models is also warranted.

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Fig. 1.
MRS data analysis. $\log(\text{MRS})$ versus $\log(\text{mass})$, with the fitted lines by the bent line quantile regression

Fig. 2.
Fitted lines by the bent line quantile regression for Galton's data

Table 1

Simulation results based on 1000 simulated samples of $n = 200$ observations, for the IID case with the error distribution $\tilde{e} \sim N(0, 1)$ in Simulation 1

τ		Grid					Proposed				
		β_0	β_1	β_2	γ	t	β_0	β_1	β_2	γ	t
0.1	Bias	0.027	0.015	-0.022	0.007	-0.006	0.026	0.014	-0.021	0.007	-0.004
	SD	0.233	0.190	0.242	0.252	0.315	0.235	0.193	0.246	0.253	0.316
	ESE	0.208	0.167	0.249	0.241	0.243	0.201	0.160	0.228	0.224	0.227
	MSE	0.055	0.036	0.059	0.063	0.099	0.056	0.037	0.061	0.064	0.100
	CP	0.879	0.860	0.915	0.916	0.818	0.882	0.860	0.889	0.876	0.825
	AW	0.816	0.654	0.977	0.946	0.951	0.790	0.627	0.895	0.878	0.889
0.3	Bias	0.023	0.015	-0.030	-0.008	0.000	0.023	0.015	-0.030	-0.008	-0.001
	SD	0.174	0.142	0.189	0.187	0.219	0.175	0.144	0.190	0.187	0.222
	ESE	0.167	0.132	0.192	0.191	0.191	0.164	0.131	0.186	0.185	0.184
	MSE	0.031	0.020	0.037	0.035	0.048	0.031	0.021	0.037	0.035	0.049
	CP	0.917	0.906	0.935	0.949	0.891	0.928	0.920	0.922	0.924	0.887
	AW	0.655	0.517	0.752	0.748	0.748	0.644	0.512	0.729	0.724	0.723
0.5	Bias	0.011	0.013	-0.009	0.000	-0.016	0.012	0.013	-0.009	0.000	-0.017
	SD	0.168	0.132	0.178	0.185	0.200	0.168	0.133	0.178	0.184	0.200
	ESE	0.159	0.126	0.181	0.181	0.179	0.157	0.125	0.177	0.177	0.178
	MSE	0.028	0.018	0.032	0.034	0.040	0.028	0.018	0.032	0.034	0.040
	CP	0.920	0.912	0.934	0.930	0.903	0.921	0.916	0.925	0.922	0.918
	AW	0.622	0.494	0.708	0.708	0.703	0.617	0.492	0.695	0.692	0.697
0.7	Bias	0.009	0.011	-0.016	0.002	-0.006	0.009	0.009	-0.015	0.000	-0.004
	SD	0.172	0.142	0.184	0.182	0.222	0.173	0.143	0.185	0.183	0.223
	ESE	0.168	0.132	0.191	0.191	0.191	0.164	0.130	0.185	0.184	0.185
	MSE	0.029	0.020	0.034	0.033	0.049	0.030	0.021	0.034	0.034	0.050
	CP	0.918	0.915	0.938	0.961	0.889	0.910	0.923	0.932	0.937	0.879
	AW	0.659	0.519	0.749	0.747	0.750	0.641	0.509	0.724	0.721	0.725
0.9	Bias	0.013	0.022	-0.042	-0.002	0.000	0.008	0.017	-0.036	0.000	0.002
	SD	0.230	0.185	0.239	0.238	0.288	0.232	0.185	0.239	0.240	0.286
	ESE	0.219	0.174	0.254	0.248	0.250	0.207	0.164	0.234	0.231	0.231
	MSE	0.053	0.035	0.059	0.057	0.083	0.054	0.034	0.058	0.057	0.082
	CP	0.903	0.874	0.919	0.927	0.877	0.875	0.879	0.902	0.892	0.856
	AW	0.859	0.682	0.995	0.973	0.981	0.810	0.642	0.919	0.906	0.906

The smaller values of MSE are highlighted in bold

Bias the empirical bias, *SD* the empirical standard error, *ESE* the average estimated standard error, *MSE* the mean square error, *CP* 95% coverage probability, *AW* the average width of asymptotic confidence intervals

Table 2

Simulation results based on 1000 simulated samples of $n = 200$ observations, for the heteroscedasticity case with $\tilde{e} \sim N(0, 1)$ in Simulation 1

τ		Grid					Proposed				
		β_0	β_1	β_2	γ	t	β_0	β_1	β_2	γ	t
0.1	Bias	0.021	0.027	-0.112	0.008	0.019	0.019	0.025	-0.106	0.009	0.016
	SD	0.270	0.180	0.668	0.297	0.492	0.273	0.182	0.669	0.297	0.493
	ESE	0.233	0.149	0.394	0.288	0.346	0.243	0.153	0.425	0.273	0.352
	MSE	0.074	0.033	0.458	0.088	0.242	0.075	0.034	0.459	0.088	0.243
	CP	0.891	0.868	0.901	0.939	0.798	0.875	0.873	0.933	0.901	0.827
	AW	0.911	0.586	1.545	1.129	1.357	0.954	0.599	1.664	1.071	1.380
0.3	Bias	0.015	0.016	-0.058	0.004	0.013	0.017	0.016	-0.058	0.004	0.011
	SD	0.198	0.130	0.299	0.220	0.342	0.199	0.131	0.298	0.220	0.342
	ESE	0.187	0.122	0.289	0.228	0.267	0.188	0.117	0.298	0.216	0.272
	MSE	0.039	0.017	0.093	0.048	0.117	0.040	0.018	0.092	0.049	0.117
	CP	0.929	0.911	0.915	0.955	0.842	0.926	0.901	0.948	0.938	0.852
	AW	0.732	0.479	1.131	0.895	1.046	0.738	0.459	1.168	0.845	1.066
0.5	Bias	0.019	0.013	-0.060	-0.005	0.011	0.018	0.012	-0.054	-0.004	0.007
	SD	0.177	0.118	0.288	0.210	0.318	0.179	0.120	0.291	0.210	0.319
	ESE	0.181	0.118	0.280	0.219	0.260	0.176	0.111	0.280	0.203	0.257
	MSE	0.032	0.014	0.087	0.044	0.101	0.032	0.015	0.087	0.044	0.102
	CP	0.935	0.933	0.926	0.954	0.877	0.928	0.921	0.947	0.938	0.883
	AW	0.708	0.461	1.098	0.859	1.018	0.690	0.435	1.096	0.797	1.007
0.7	Bias	0.008	0.007	-0.066	0.002	0.022	0.007	0.007	-0.060	0.004	0.016
	SD	0.190	0.130	0.316	0.217	0.345	0.191	0.132	0.316	0.218	0.344
	ESE	0.190	0.124	0.294	0.229	0.271	0.189	0.118	0.298	0.215	0.272
	MSE	0.036	0.017	0.104	0.047	0.120	0.037	0.018	0.104	0.047	0.119
	CP	0.942	0.927	0.921	0.948	0.858	0.927	0.906	0.934	0.940	0.880
	AW	0.747	0.485	1.153	0.898	1.060	0.739	0.462	1.168	0.842	1.065
0.9	Bias	0.009	0.007	-0.085	0.010	0.020	0.004	0.002	-0.067	0.010	0.010
	SD	0.275	0.174	0.444	0.284	0.480	0.280	0.179	0.446	0.285	0.476
	ESE	0.241	0.154	0.384	0.287	0.355	0.251	0.157	0.402	0.276	0.363
	MSE	0.076	0.030	0.204	0.081	0.230	0.078	0.032	0.203	0.081	0.226
	CP	0.883	0.902	0.900	0.939	0.851	0.893	0.893	0.933	0.916	0.838
	AW	0.947	0.603	1.507	1.127	1.391	0.982	0.615	1.576	1.084	1.422

The smaller values of MSE are highlighted in bold

Bias the empirical bias, *SD* the empirical standard error, *ESE* the average estimated standard error, *MSE* the mean square error, *CP* 95% coverage probability, *AW* the average width of asymptotic confidence intervals

Table 3

Simulation results based on 1000 simulated samples of $n = 200$ observations, for the IID case with $\tilde{e} \sim t_3$ in Simulation 1

τ		Grid					Proposed				
		β_0	β_1	β_2	γ	t	β_0	β_1	β_2	γ	t
0.1	Bias	0.054	0.068	-0.129	0.008	-0.014	0.054	0.067	-0.125	0.004	-0.019
	SD	0.416	0.358	0.431	0.420	0.566	0.419	0.361	0.437	0.422	0.571
	ESE	0.348	0.286	0.448	0.401	0.402	0.339	0.268	0.386	0.356	0.358
	MSE	0.176	0.133	0.202	0.176	0.320	0.179	0.135	0.206	0.178	0.326
	CP	0.851	0.840	0.910	0.926	0.746	0.842	0.812	0.855	0.846	0.755
	AW	1.364	1.121	1.756	1.572	1.577	1.329	1.051	1.514	1.394	1.405
0.3	Bias	0.016	0.016	-0.025	-0.011	-0.007	0.016	0.015	-0.025	-0.013	-0.007
	SD	0.205	0.175	0.227	0.221	0.265	0.208	0.177	0.229	0.221	0.267
	ESE	0.199	0.162	0.232	0.229	0.229	0.193	0.154	0.219	0.216	0.217
	MSE	0.042	0.031	0.052	0.049	0.070	0.043	0.032	0.053	0.049	0.071
	CP	0.940	0.912	0.946	0.946	0.895	0.920	0.880	0.917	0.927	0.880
	AW	0.781	0.634	0.909	0.896	0.896	0.757	0.604	0.857	0.846	0.852
0.5	Bias	0.004	0.012	-0.022	0.011	-0.002	0.003	0.011	-0.020	0.012	-0.003
	SD	0.184	0.152	0.197	0.203	0.241	0.186	0.154	0.198	0.204	0.242
	ESE	0.182	0.146	0.210	0.206	0.207	0.172	0.137	0.195	0.193	0.194
	MSE	0.034	0.023	0.039	0.041	0.058	0.035	0.024	0.039	0.042	0.059
	CP	0.942	0.930	0.953	0.947	0.889	0.923	0.905	0.932	0.930	0.870
	AW	0.715	0.571	0.821	0.808	0.811	0.675	0.537	0.764	0.757	0.761
0.7	Bias	0.015	0.010	-0.026	-0.004	0.011	0.013	0.008	-0.024	-0.003	0.013
	SD	0.205	0.170	0.222	0.223	0.271	0.204	0.171	0.221	0.224	0.269
	ESE	0.205	0.160	0.233	0.229	0.231	0.192	0.152	0.218	0.216	0.217
	MSE	0.042	0.029	0.050	0.050	0.074	0.042	0.029	0.049	0.050	0.072
	CP	0.943	0.924	0.946	0.953	0.887	0.917	0.901	0.925	0.924	0.861
	AW	0.802	0.627	0.912	0.897	0.904	0.754	0.595	0.856	0.846	0.851
0.9	Bias	0.098	0.052	-0.132	-0.012	0.014	0.088	0.040	-0.114	-0.008	0.021
	SD	0.429	0.347	0.447	0.420	0.515	0.438	0.358	0.474	0.424	0.519
	ESE	0.413	0.304	0.470	0.411	0.445	0.336	0.265	0.388	0.364	0.369
	MSE	0.194	0.123	0.218	0.177	0.266	0.200	0.130	0.238	0.180	0.270
	CP	0.887	0.844	0.909	0.932	0.857	0.832	0.817	0.857	0.866	0.783
	AW	1.621	1.193	1.840	1.612	1.743	1.317	1.039	1.521	1.429	1.446

The smaller values of MSE are highlighted in bold

Bias the empirical bias, *SD* the empirical standard error, *ESE* the average estimated standard error, *MSE* the mean square error, *CP* 95% coverage probability, *AW* the average width of asymptotic confidence intervals

Table 4

Simulation results based on 1000 simulated samples of $n = 200$ observations, for the heteroscedasticity case with $\tilde{e} \sim t_3$ in Simulation 1

τ		Grid					Proposed				
		β_0	β_1	β_2	γ	t	β_0	β_1	β_2	γ	t
0.1	Bias	0.078	0.107	-0.260	0.010	-0.014	0.079	0.109	-0.255	0.011	-0.032
	SD	0.440	0.297	0.614	0.503	0.717	0.452	0.304	0.626	0.508	0.738
	ESE	0.366	0.233	0.855	0.452	0.566	0.478	0.304	0.794	0.529	0.640
	MSE	0.200	0.100	0.444	0.253	0.515	0.210	0.104	0.457	0.258	0.545
	CP	0.851	0.841	0.925	0.938	0.743	0.897	0.886	0.950	0.937	0.826
	AW	1.435	0.913	3.351	1.774	2.219	1.874	1.192	3.113	2.072	2.508
0.3	Bias	0.020	0.019	-0.086	-0.011	0.013	0.020	0.020	-0.084	-0.011	0.009
	SD	0.232	0.156	0.397	0.248	0.439	0.235	0.158	0.398	0.249	0.441
	ESE	0.232	0.155	0.364	0.281	0.327	0.228	0.144	0.365	0.261	0.326
	MSE	0.054	0.025	0.165	0.062	0.193	0.055	0.025	0.165	0.062	0.194
	CP	0.940	0.929	0.935	0.975	0.851	0.928	0.904	0.933	0.954	0.850
	AW	0.911	0.608	1.427	1.102	1.283	0.894	0.565	1.429	1.023	1.280
0.5	Bias	0.008	0.008	-0.072	-0.002	0.019	0.009	0.008	-0.071	-0.002	0.014
	SD	0.211	0.141	0.473	0.224	0.386	0.212	0.144	0.473	0.224	0.387
	ESE	0.216	0.146	0.328	0.261	0.301	0.197	0.125	0.321	0.227	0.288
	MSE	0.045	0.020	0.229	0.050	0.149	0.045	0.021	0.228	0.050	0.150
	CP	0.952	0.949	0.935	0.975	0.884	0.918	0.909	0.934	0.948	0.870
	AW	0.846	0.571	1.287	1.023	1.180	0.772	0.489	1.257	0.890	1.128
0.7	Bias	0.028	0.018	-0.089	0.001	0.016	0.022	0.013	-0.080	0.001	0.014
	SD	0.244	0.159	0.392	0.264	0.422	0.244	0.161	0.397	0.264	0.418
	ESE	0.239	0.157	0.363	0.282	0.329	0.237	0.146	0.371	0.264	0.334
	MSE	0.061	0.026	0.162	0.070	0.178	0.060	0.026	0.164	0.070	0.175
	CP	0.943	0.937	0.940	0.962	0.884	0.935	0.917	0.942	0.945	0.874
	AW	0.938	0.615	1.424	1.106	1.292	0.927	0.573	1.455	1.034	1.309
0.9	Bias	0.108	0.046	-0.266	0.001	0.036	0.088	0.036	-0.203	0.008	-0.010
	SD	0.469	0.296	0.622	0.475	0.699	0.471	0.299	0.632	0.473	0.668
	ESE	0.445	0.271	0.751	0.447	0.628	0.532	0.324	0.830	0.536	0.720
	MSE	0.232	0.090	0.458	0.226	0.489	0.230	0.091	0.441	0.224	0.446
	CP	0.906	0.893	0.912	0.945	0.841	0.930	0.916	0.955	0.938	0.891
	AW	1.743	1.063	2.944	1.753	2.460	2.086	1.270	3.254	2.101	2.821

The smaller values of MSE are highlighted in bold

Bias the empirical bias, *SD* the empirical standard error, *ESE* the average estimated standard error, *MSE* the mean square error, *CP* 95% coverage probability, *AW* the average width of asymptotic confidence intervals

Table 5

Simulation results based on 1000 simulated samples of $n = 200$ observations, for the IID case with $\tilde{e} \sim 0.9N(0, 1) + 0.1t_3$ in Simulation 1

τ		Grid					Proposed				
		β_0	β_1	β_2	γ	t	β_0	β_1	β_2	γ	t
0.1	Bias	0.161	0.016	-0.037	0.005	0.009	0.161	0.018	-0.041	0.005	0.009
	SD	0.204	0.169	0.224	0.222	0.263	0.207	0.171	0.225	0.225	0.264
	ESE	0.196	0.156	0.231	0.230	0.224	0.190	0.151	0.216	0.214	0.213
	MSE	0.067	0.029	0.052	0.049	0.069	0.069	0.030	0.052	0.051	0.070
	CP	0.803	0.886	0.909	0.932	0.843	0.788	0.872	0.902	0.893	0.848
	AW	0.767	0.612	0.906	0.901	0.878	0.746	0.591	0.846	0.837	0.833
0.3	Bias	0.064	0.008	-0.019	-0.002	0.002	0.065	0.007	-0.018	-0.001	0.002
	SD	0.165	0.129	0.175	0.176	0.209	0.166	0.129	0.175	0.176	0.209
	ESE	0.154	0.122	0.175	0.174	0.174	0.151	0.120	0.171	0.170	0.170
	MSE	0.031	0.017	0.031	0.031	0.044	0.032	0.017	0.031	0.031	0.044
	CP	0.883	0.913	0.934	0.944	0.877	0.882	0.921	0.927	0.921	0.878
	AW	0.603	0.478	0.686	0.683	0.682	0.591	0.469	0.669	0.665	0.667
0.5	Bias	0.004	0.005	-0.014	0.001	0.004	0.005	0.005	-0.014	0.000	0.004
	SD	0.150	0.124	0.167	0.166	0.187	0.151	0.125	0.168	0.166	0.187
	ESE	0.143	0.113	0.163	0.165	0.164	0.142	0.113	0.160	0.160	0.161
	MSE	0.022	0.015	0.028	0.028	0.035	0.023	0.016	0.028	0.028	0.035
	CP	0.918	0.913	0.926	0.943	0.889	0.926	0.912	0.924	0.931	0.894
	AW	0.562	0.443	0.641	0.648	0.643	0.558	0.442	0.629	0.628	0.630
0.7	Bias	-0.042	0.010	-0.024	-0.002	0.002	-0.044	0.008	-0.023	0.000	0.003
	SD	0.158	0.123	0.168	0.168	0.204	0.157	0.125	0.171	0.169	0.202
	ESE	0.155	0.122	0.175	0.175	0.174	0.151	0.120	0.170	0.170	0.170
	MSE	0.027	0.015	0.029	0.028	0.041	0.027	0.016	0.030	0.029	0.041
	CP	0.901	0.929	0.936	0.949	0.898	0.906	0.927	0.927	0.932	0.881
	AW	0.608	0.477	0.685	0.685	0.680	0.592	0.469	0.668	0.666	0.665
0.9	Bias	-0.142	0.018	-0.030	0.016	-0.010	-0.145	0.016	-0.026	0.016	-0.009
	SD	0.206	0.170	0.228	0.226	0.257	0.206	0.172	0.231	0.225	0.256
	ESE	0.200	0.155	0.230	0.226	0.227	0.187	0.148	0.211	0.209	0.210
	MSE	0.063	0.029	0.053	0.051	0.066	0.064	0.030	0.054	0.051	0.065
	CP	0.803	0.881	0.903	0.923	0.881	0.781	0.868	0.876	0.892	0.848
	AW	0.782	0.606	0.903	0.887	0.889	0.734	0.581	0.827	0.819	0.823

The smaller values of MSE are highlighted in bold

Bias the empirical bias, *SD* the empirical standard error, *ESE* the average estimated standard error, *MSE* the mean square error, *CP* 95% coverage probability, *AW* the average width of asymptotic confidence intervals

Table 6

Simulation results based on 1000 simulated samples of $n = 200$ observations, for the heteroscedasticity case with $\tilde{\varepsilon} \sim 0.9N(0, 1) + 0.1t_3$ in Simulation 1

τ		Grid					Proposed					
		β_0	β_1	β_2	γ	t	β_0	β_1	β_2	γ	t	
0.1	Bias	0.160	0.064	-0.118	0.004	0.042	0.159	0.063	-0.121	0.004	0.044	
	SD	0.242	0.162	0.488	0.257	0.458	0.245	0.164	0.485	0.258	0.459	
	ESE	0.215	0.137	0.355	0.265	0.316	0.230	0.142	0.374	0.259	0.329	
	MSE	0.084	0.030	0.252	0.066	0.212	0.085	0.031	0.250	0.067	0.212	
	CP	0.804	0.861	0.888	0.957	0.798	0.835	0.854	0.927	0.932	0.838	
	AW	0.844	0.537	1.393	1.039	1.237	0.900	0.558	1.464	1.014	1.290	
0.3	Bias	0.061	0.025	-0.054	0.003	0.020	0.061	0.024	-0.053	0.003	0.019	
	SD	0.185	0.117	0.291	0.207	0.325	0.186	0.119	0.292	0.206	0.325	
	ESE	0.173	0.113	0.266	0.209	0.248	0.171	0.107	0.274	0.196	0.250	
	MSE	0.038	0.014	0.087	0.043	0.106	0.038	0.015	0.088	0.043	0.106	
	CP	0.904	0.921	0.917	0.944	0.855	0.897	0.914	0.931	0.924	0.859	
	AW	0.680	0.443	1.044	0.818	0.970	0.671	0.421	1.073	0.769	0.982	
0.5	Bias	0.005	0.003	-0.037	-0.002	0.016	0.004	0.002	-0.034	-0.002	0.016	
	SD	0.166	0.113	0.271	0.188	0.286	0.169	0.115	0.271	0.190	0.286	
	ESE	0.164	0.107	0.253	0.200	0.239	0.161	0.101	0.253	0.185	0.238	
	MSE	0.028	0.013	0.075	0.035	0.082	0.028	0.013	0.075	0.036	0.082	
	CP	0.940	0.924	0.923	0.968	0.873	0.925	0.919	0.937	0.953	0.882	
	AW	0.642	0.420	0.993	0.782	0.937	0.631	0.396	0.994	0.726	0.932	
0.7	Bias	-0.042	-0.010	-0.054	-0.002	0.018	-0.044	-0.013	-0.052	-0.002	0.019	
	SD	0.177	0.113	0.271	0.196	0.303	0.178	0.114	0.272	0.196	0.302	
	ESE	0.176	0.113	0.266	0.210	0.248	0.173	0.107	0.269	0.195	0.249	
	MSE	0.033	0.013	0.076	0.038	0.092	0.034	0.013	0.077	0.038	0.092	
	CP	0.910	0.923	0.907	0.957	0.870	0.915	0.916	0.939	0.929	0.877	
	AW	0.690	0.445	1.042	0.822	0.974	0.677	0.420	1.053	0.766	0.977	
0.9	Bias	-0.144	-0.042	-0.101	0.023	0.030	-0.147	-0.045	-0.074	0.022	0.019	
	SD	0.233	0.160	0.569	0.263	0.438	0.237	0.161	0.425	0.263	0.427	
	ESE	0.222	0.139	0.412	0.264	0.324	0.230	0.142	0.363	0.252	0.326	
	MSE	0.075	0.027	0.334	0.070	0.193	0.078	0.028	0.186	0.070	0.182	
	CP	0.817	0.874	0.890	0.951	0.846	0.840	0.857	0.916	0.912	0.848	
	AW	0.869	0.544	1.614	1.036	1.272	0.900	0.556	1.422	0.990	1.278	

The smaller values of MSE are highlighted in bold

Bias the empirical bias, *SD* the empirical standard error, *ESE* the average estimated standard error, *MSE* the mean square error, *CP* 95% coverage probability, *AW* the average width of asymptotic confidence intervals

Table 7

Simulation results based on 1000 simulated samples of $n = 200$ observations, for the IID case with $\tilde{e} \sim 0.9N(0, 1) + 0.1Cauchy(0, 1)$ in Simulation 1

τ		Grid					Proposed				
		β_0	β_1	β_2	γ	t	β_0	β_1	β_2	γ	t
0.1	Bias	0.183	0.022	-0.041	0.003	-0.002	0.184	0.023	-0.043	0.002	-0.002
	SD	0.266	0.217	0.277	0.269	0.352	0.270	0.218	0.279	0.268	0.355
	ESE	0.403	0.315	0.482	0.475	0.490	0.225	0.178	0.255	0.249	0.250
	MSE	0.104	0.047	0.079	0.072	0.124	0.106	0.048	0.080	0.072	0.126
	CP	0.799	0.898	0.926	0.938	0.852	0.758	0.866	0.890	0.883	0.817
	AW	1.579	1.234	1.890	1.862	1.922	0.882	0.698	1.001	0.976	0.981
0.3	Bias	0.038	0.012	-0.025	0.002	0.002	0.038	0.011	-0.024	0.001	0.003
	SD	0.174	0.142	0.184	0.187	0.222	0.174	0.143	0.184	0.187	0.223
	ESE	0.344	0.275	0.392	0.392	0.400	0.164	0.130	0.185	0.185	0.184
	MSE	0.032	0.020	0.034	0.035	0.049	0.032	0.021	0.034	0.035	0.050
	CP	0.937	0.935	0.964	0.964	0.914	0.909	0.904	0.931	0.934	0.871
	AW	1.349	1.077	1.535	1.536	1.568	0.643	0.511	0.727	0.723	0.723
0.5	Bias	0.007	0.011	-0.024	0.005	0.000	0.006	0.010	-0.023	0.005	0.002
	SD	0.162	0.129	0.175	0.177	0.199	0.163	0.130	0.177	0.177	0.199
	ESE	0.488	0.376	0.562	0.573	0.579	0.155	0.123	0.174	0.174	0.174
	MSE	0.026	0.017	0.031	0.031	0.040	0.027	0.017	0.032	0.031	0.040
	CP	0.953	0.949	0.965	0.976	0.935	0.919	0.926	0.942	0.937	0.905
	AW	1.914	1.475	2.202	2.245	2.268	0.606	0.480	0.683	0.682	0.680
0.7	Bias	-0.027	0.013	-0.030	0.011	0.007	-0.026	0.013	-0.028	0.011	0.005
	SD	0.180	0.142	0.199	0.183	0.225	0.183	0.144	0.200	0.184	0.227
	ESE	0.471	0.365	0.536	0.546	0.557	0.164	0.129	0.185	0.184	0.184
	MSE	0.033	0.020	0.040	0.034	0.051	0.034	0.021	0.041	0.034	0.052
	CP	0.941	0.939	0.942	0.962	0.923	0.915	0.896	0.918	0.928	0.871
	AW	1.847	1.431	2.102	2.139	2.183	0.642	0.507	0.726	0.723	0.722
0.9	Bias	-0.124	0.038	-0.048	0.002	-0.024	-0.127	0.036	-0.040	0.002	-0.026
	SD	0.293	0.239	0.279	0.271	0.362	0.298	0.241	0.278	0.271	0.358
	ESE	0.325	0.251	0.348	0.312	0.327	0.228	0.182	0.257	0.250	0.252
	MSE	0.101	0.059	0.080	0.073	0.132	0.105	0.059	0.079	0.074	0.129
	CP	0.834	0.910	0.949	0.947	0.878	0.778	0.851	0.898	0.880	0.808
	AW	1.275	0.982	1.365	1.225	1.282	0.894	0.713	1.008	0.980	0.987

The smaller values of MSE are highlighted in bold

Bias the empirical bias, *SD* the empirical standard error, *ESE* the average estimated standard error, *MSE* the mean square error, *CP* 95% coverage probability, *AW* the average width of asymptotic confidence intervals

Table 8

Simulation results based on 1000 simulated samples of $n = 200$ observations, for the heteroscedasticity case with $\tilde{\varepsilon} \sim 0.9N(0, 1) + 0.1Cauchy(0, 1)$ in Simulation 1

τ		Grid					Proposed				
		β_0	β_1	β_2	γ	t	β_0	β_1	β_2	γ	t
0.1	Bias	0.182	0.085	-0.120	0.002	0.012	0.182	0.086	-0.117	0.000	0.006
	SD	0.302	0.198	0.465	0.330	0.529	0.305	0.202	0.467	0.331	0.534
	ESE	0.424	0.281	0.658	0.520	0.555	0.326	0.204	0.553	0.367	0.456
	MSE	0.124	0.046	0.230	0.109	0.280	0.126	0.048	0.232	0.109	0.286
	CP	0.815	0.867	0.918	0.953	0.811	0.844	0.872	0.947	0.945	0.835
	AW	1.660	1.101	2.581	2.037	2.176	1.277	0.800	2.168	1.437	1.787
0.3	Bias	0.042	0.023	-0.054	0.001	0.005	0.041	0.022	-0.054	0.001	0.005
	SD	0.199	0.126	0.297	0.218	0.321	0.199	0.127	0.299	0.218	0.321
	ESE	0.533	0.394	0.659	0.617	0.664	0.190	0.117	0.293	0.215	0.273
	MSE	0.041	0.016	0.091	0.047	0.103	0.041	0.017	0.093	0.047	0.103
	CP	0.950	0.958	0.948	0.974	0.918	0.923	0.899	0.933	0.934	0.882
	AW	2.088	1.544	2.582	2.419	2.602	0.743	0.460	1.150	0.844	1.071
0.5	Bias	0.011	0.012	-0.062	0.004	0.019	0.010	0.011	-0.058	0.004	0.017
	SD	0.183	0.121	0.271	0.208	0.303	0.184	0.122	0.275	0.209	0.305
	ESE	0.486	0.370	0.600	0.559	0.599	0.175	0.110	0.276	0.203	0.255
	MSE	0.033	0.015	0.077	0.043	0.092	0.034	0.015	0.079	0.044	0.093
	CP	0.965	0.956	0.961	0.982	0.922	0.932	0.912	0.943	0.943	0.891
	AW	1.904	1.451	2.352	2.191	2.347	0.686	0.431	1.081	0.795	1.000
0.7	Bias	-0.003	0.004	-0.056	-0.011	0.004	-0.004	0.002	-0.051	-0.012	0.001
	SD	0.196	0.129	0.310	0.217	0.333	0.196	0.130	0.317	0.218	0.334
	ESE	0.330	0.240	0.440	0.385	0.412	0.191	0.120	0.295	0.216	0.272
	MSE	0.038	0.017	0.099	0.047	0.111	0.038	0.017	0.103	0.048	0.111
	CP	0.963	0.957	0.946	0.975	0.915	0.928	0.914	0.933	0.932	0.879
	AW	1.293	0.942	1.725	1.511	1.616	0.748	0.471	1.156	0.847	1.067
0.9	Bias	-0.121	-0.039	-0.130	-0.011	0.050	-0.127	-0.045	-0.109	-0.010	0.036
	SD	0.311	0.209	0.461	0.329	0.502	0.314	0.211	0.462	0.329	0.501
	ESE	0.320	0.218	0.492	0.365	0.448	0.357	0.216	0.540	0.356	0.493
	MSE	0.111	0.045	0.230	0.108	0.254	0.115	0.046	0.226	0.109	0.252
	CP	0.847	0.881	0.921	0.947	0.873	0.865	0.867	0.951	0.929	0.867
	AW	1.253	0.855	1.929	1.431	1.758	1.399	0.848	2.117	1.397	1.934

The smaller values of MSE are highlighted in bold

Bias the empirical bias, *SD* the empirical standard error, *ESE* the average estimated standard error, *MSE* the mean square error, *CP* 95% coverage probability, *AW* the average width of asymptotic confidence intervals

Table 9

Simulation results based on 1000 simulated samples of $n = 200$ observations, for symmetric scenario in Simulation 2

τ		Grid				Proposed			
		β_0	β_1	β_2	t	β_0	β_1	β_2	t
0.1	Bias	-0.004	0.005	-0.078	0.100	-0.005	0.006	-0.077	0.090
	SD	0.140	0.080	0.347	0.704	0.141	0.081	0.344	0.706
	ESE	0.125	0.064	0.216	0.490	0.129	0.070	0.229	0.513
	MSE	0.020	0.006	0.126	0.505	0.020	0.006	0.124	0.506
	CP	0.933	0.873	0.893	0.816	0.853	0.901	0.941	0.833
	AW	0.490	0.252	0.847	1.919	0.508	0.273	0.897	2.011
0.3	Bias	-0.007	0.005	-0.029	0.023	-0.007	0.005	-0.026	0.018
	SD	0.103	0.059	0.164	0.480	0.103	0.060	0.164	0.480
	ESE	0.115	0.056	0.158	0.390	0.096	0.052	0.161	0.390
	MSE	0.011	0.004	0.028	0.231	0.011	0.004	0.028	0.230
	CP	0.967	0.920	0.918	0.866	0.893	0.898	0.948	0.869
	AW	0.450	0.221	0.620	1.526	0.376	0.206	0.630	1.528
0.5	Bias	-0.005	0.004	-0.024	0.013	-0.004	0.004	-0.023	0.014
	SD	0.102	0.057	0.165	0.466	0.102	0.058	0.166	0.468
	ESE	0.111	0.054	0.148	0.370	0.092	0.050	0.152	0.367
	MSE	0.010	0.003	0.028	0.217	0.010	0.003	0.028	0.219
	CP	0.969	0.929	0.914	0.867	0.892	0.898	0.931	0.867
	AW	0.436	0.212	0.581	1.452	0.360	0.195	0.592	1.437
0.7	Bias	-0.003	0.003	-0.027	0.019	-0.002	0.003	-0.025	0.012
	SD	0.104	0.058	0.166	0.494	0.105	0.060	0.168	0.503
	ESE	0.114	0.056	0.159	0.391	0.098	0.053	0.160	0.387
	MSE	0.011	0.003	0.028	0.244	0.011	0.003	0.029	0.253
	CP	0.968	0.928	0.913	0.869	0.887	0.902	0.934	0.871
	AW	0.447	0.221	0.624	1.534	0.3822	0.206	0.629	1.524
0.9	Bias	0.004	0.001	-0.071	0.069	0.008	0.498	-0.066	0.055
	SD	0.146	0.079	0.325	0.707	0.151	0.083	0.365	0.715
	ESE	0.126	0.068	0.220	0.491	0.130	0.072	0.224	0.518
	MSE	0.021	0.006	0.111	0.504	0.023	0.007	0.137	0.513
	CP	0.926	0.897	0.883	0.830	0.839	0.885	0.907	0.843
	AW	0.494	0.265	0.862	1.923	0.511	0.281	0.878	2.029

The smaller values of MSE are highlighted in bold

Bias the empirical bias, *SD* the empirical standard error, *ESE* the average estimated standard error, *MSE* the mean square error, *CP* 95% coverage probability, *AW* the average width of asymptotic confidence intervals

Table 10

Simulation results based on 1000 simulated samples of $n = 200$ observations, for asymmetric scenario in Simulation 2

τ		Grid				Proposed			
		β_0	β_1	β_2	t	β_0	β_1	β_2	t
0.1	Bias	-0.001	0.009	-0.010	0.010	0.001	0.009	-0.009	0.009
	SD	0.147	0.142	0.141	0.220	0.147	0.144	0.143	0.219
	ESE	0.364	0.534	0.537	0.743	0.143	0.139	0.141	0.196
	MSE	0.022	0.020	0.020	0.049	0.022	0.021	0.021	0.048
	CP	0.833	0.836	0.853	0.832	0.815	0.845	0.871	0.864
	AW	1.425	2.095	2.105	2.912	0.562	0.543	0.555	0.770
0.3	Bias	-0.004	0.007	-0.007	0.001	-0.005	0.008	-0.007	0.001
	SD	0.114	0.107	0.108	0.155	0.116	0.110	0.110	0.155
	ESE	0.112	0.104	0.106	0.145	0.110	0.100	0.102	0.139
	MSE	0.013	0.012	0.012	0.024	0.013	0.012	0.012	0.024
	CP	0.888	0.895	0.900	0.899	0.891	0.916	0.921	0.898
	AW	0.439	0.408	0.415	0.567	0.434	0.393	0.400	0.545
0.5	Bias	-0.002	0.005	-0.005	0.005	-0.003	0.006	-0.006	0.005
	SD	0.107	0.103	0.1034	0.154	0.107	0.106	0.106	0.156
	ESE	0.106	0.096	0.098	0.134	0.105	0.096	0.098	0.133
	MSE	0.011	0.011	0.011	0.024	0.011	0.011	0.011	0.024
	CP	0.913	0.900	0.908	0.881	0.896	0.907	0.913	0.891
	AW	0.414	0.377	0.383	0.524	0.414	0.377	0.383	0.523
0.7	Bias	-0.002	0.003	-0.004	0.011	-0.002	0.002	-0.002	0.013
	SD	0.116	0.108	0.108	0.155	0.118	0.112	0.111	0.156
	ESE	0.109	0.101	0.103	0.142	0.111	0.101	0.103	0.142
	MSE	0.013	0.012	0.012	0.024	0.014	0.012	0.012	0.025
	CP	0.900	0.873	0.885	0.888	0.873	0.887	0.899	0.879
	AW	0.428	0.395	0.403	0.555	0.435	0.396	0.403	0.555
0.9	Bias	-0.003	0.009	-0.010	0.012	-0.003	0.008	-0.009	0.015
	SD	0.148	0.139	0.138	0.201	0.151	0.143	0.142	0.203
	ESE	0.139	0.133	0.135	0.186	0.144	0.132	0.136	0.186
	MSE	0.022	0.019	0.019	0.041	0.023	0.021	0.020	0.041
	CP	0.844	0.839	0.856	0.868	0.815	0.852	0.878	0.881
	AW	0.544	0.520	0.530	0.730	0.564	0.519	0.531	0.729

The smaller values of MSE are highlighted in bold

Bias the empirical bias, *SD* the empirical standard error, *ESE* the average estimated standard error, *MSE* the mean square error, *CP* 95% coverage probability, *AW* the average width of asymptotic confidence intervals

Table 11
Estimating results for MRS data, including the standard errors and 95% asymptotic confidence intervals

Method		β_0	β_1	β_2	γ	t	
$\tau = 0.5$	Grid	Estimate	3.232	0.292	-0.413	0.606	3.515
		SE	0.099	0.031	0.058	0.130	0.458
		95% CI	[3.038, 3.426]	[0.232, 0.352]	[-0.527, -0.299]	[0.352, 0.860]	[2.618, 4.412]
	Proposed	Estimate	3.249	0.297	-0.417	0.586	3.458
		SE	0.072	0.026	0.056	0.159	0.383
$\tau = 0.7$		95% CI	[3.108, 3.390]	[0.246, 0.347]	[-0.527, -0.307]	[0.274, 0.897]	[2.707, 4.209]
	Grid	Estimate	3.430	0.280	-0.398	0.432	3.515
		SE	0.084	0.025	0.078	0.108	0.454
		95% CI	[3.265, 3.596]	[0.231, 0.328]	[-0.550, -0.246]	[0.220, 0.644]	[2.625, 4.405]
	Proposed	Estimate	3.430	0.280	-0.395	0.432	3.503
$\tau = 0.8$		SE	0.077	0.028	0.060	0.171	0.434
		95% CI	[3.279, 3.581]	[0.226, 0.334]	[-0.513, -0.277]	[0.097, 0.766]	[2.652, 4.354]
	Grid	Estimation	3.580	0.272	-0.412	0.320	3.596
		SE	0.069	0.020	0.085	0.094	0.468
		95% CI	[3.445, 3.714]	[0.232, 0.312]	[-0.578, -0.246]	[0.136, 0.504]	[2.680, 4.512]
Proposed		Estimation	3.580	0.272	-0.386	0.320	3.455
		SE	0.072	0.026	0.059	0.163	0.420
		95% CI	[3.438, 3.721]	[0.222, 0.322]	[-0.502, -0.271]	[0.000, 0.640]	[2.632, 4.277]

Table 12
Estimating results for Galton's data, including the standard errors and 95% asymptotic confidence intervals

Method	$\hat{\theta}_0$	$\hat{\theta}_1$	$\hat{\theta}_2$	t
$\tau = 0.3$	Grid	Estimate	34.145	0.504
		se	16.817	0.265
		95% CI	[1.185, 67.106]	[-0.016, 1.024]
Proposed		Estimate	40.048	0.410
		se	12.696	0.196
		95% CI	[15.164, 64.931]	[0.025, 0.794]
$\tau = 0.5$	Grid	Estimate	49.340	0.287
		se	2.400	0.035
		95% CI	[44.637, 54.043]	[0.218, 0.356]
Proposed		Estimate	48.798	0.295
		se	2.096	0.031
		95% CI	[44.690, 52.905]	[0.235, 0.355]
$\tau = 0.7$	Grid	Estimate	55.870	0.201
		se	2.747	0.041
		95% CI	[50.486, 61.254]	[0.121, 0.280]
Proposed		Estimate	53.989	0.229
		se	3.478	0.052
		95% CI	[47.173, 60.805]	[0.128, 0.330]
$\tau = 0.9$	Grid	Estimate	52.107	0.272
		se	3.047	0.044
		95% CI	[46.135, 58.078]	[0.185, 0.359]
Proposed		Estimate	51.608	0.280
		se	4.013	0.059
		95% CI	[43.743, 59.472]	[0.164, 0.395]
		Estimate	66.2	-0.193
		se	4.248	0.269
		95% CI	[57.873, 74.527]	[-0.719, 0.334]
		Estimate	68.550	-0.127
		se	2.941	0.199
		95% CI	[62.786, 74.315]	[-0.517, 0.264]
		Estimate	71.500	0.159
		se	1.328	0.102
		95% CI	[68.896, 74.104]	[-0.041, 0.359]
		Estimate	72.145	0.222
		se	1.071	0.108
		95% CI	[70.045, 74.244]	[0.011, 0.434]
		Estimate	70.000	0.319
		se	0.591	0.117
		95% CI	[68.841, 71.159]	[0.091, 0.548]
		Estimate	70.324	0.308
		se	0.681	0.083
		95% CI	[68.989, 71.659]	[0.146, 0.470]
		Estimate	71.850	0.943
		se	0.942	0.704
		95% CI	[70.004, 73.696]	[-0.438, 2.323]
		Estimate	71.282	0.698
		se	0.557	0.175
		95% CI	[70.190, 72.374]	[0.355, 1.041]