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Self-consistent triple decomposition of the turbulent flow over a backward-facing step under finite amplitude harmonic forcing

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We investigate the saturation of harmonically forced disturbances in the turbulent flow over a backward-facing step subjected to a finite amplitude forcing. The analysis relies on a triple decomposition of the unsteady flow into mean, coherent and incoherent components. The coherent–incoherent interaction is lumped into a Reynolds averaged Navier–Stokes (RANS) eddy viscosity model, and the mean–coherent interaction is analysed via a semi-linear resolvent analysis building on the laminar approach by Mantić-Lugo & Gallaire (2016 *J. Fluid Mech.* **793**, 777–797. (doi:10.1017/jfm.2016.109)). This provides a self-consistent modelling of the interaction between all three components, in the sense that the coherent perturbation structures selected by the resolvent analysis are those whose Reynolds stresses force the mean flow in such a way that the mean flow generates exactly the aforementioned perturbations, while also accounting for the effect of the incoherent scale. The model does not require any input from numerical or experimental data, and accurately predicts the saturation of the forced coherent disturbances, as established from comparison to time-averages of unsteady RANS simulation data.

1. Introduction

The linear stability theory provides a rigorous foundation to explain the initial growth of disturbances of a certain scale and frequency, and the leading-order nonlinear effects associated with small-amplitude disturbances. Nonetheless, the approach is bound to fail when the system is strongly linearly unstable and the disturbances reach an amplitude large enough for nonlinearities to set in [1–5]. It is the interaction between the finite amplitude fluctuation and the mean flow (not the base flow, i.e. the steady state whose stability is the subject of the linear theory), that is key to understand the growth of disturbances while encompassing the nonlinearity of the system [3,4,6–8]. This has led to consideration of a variety of Reynolds-stresses models in which perturbation equations linearized around a mean flow are coupled to a mean flow equation forced by the Reynolds stresses of the perturbations. For instance, such an approach is used has been used to unravel how the Reynolds stresses of transiently unstable linear and weakly nonlinear disturbances trigger the development of coherent structures in otherwise stable, parallel shear flows [9,10].

This research is carried out in the frame on the self-consistent theory, whose focus has been essentially on capturing the finite amplitude saturation of prototypical exponential instabilities leading to limit-cycle oscillations in laminar separated flows. The approach intends to set up closed, self-consistent models of the mean flow-fluctuation interaction, in the sense that no input from numerical or experimental data is needed. In particular, the mean flow is not taken as a given, but comes instead as an output of a semi-linear approximation of the Navier–Stokes equations. The general picture, consistent with early work in the field of the nonlinear stability theory [11,12], is that perturbations feed on the unstable base flow and grow first according to the linear theory. When they reach a size large enough for nonlinearities to set in, they feed back via their Reynolds stresses. The base flow is distorted into an increasingly stable mean flow, up to the point where the growth rate of the first harmonic becomes zero, and perturbations stop growing and saturate.

The seminal work of Mantić-Lugo *et al.* [13] addresses intrinsic (i.e. self-sustained) monochromatic oscillations with a simple model coupling a quasi-static approximation of the mean flow to the leading eigenmode of the mean flow propagator, assumed to represent accurately the first harmonic. Meliga [14] later tackled the effect of higher harmonics production with an augmented model involving multiple linear harmonic disturbances interacting nonlinearly with one another and feeding back on the mean via their Reynolds stresses, in which case the leading eigenmode is distorted by an additional component generated by the higher harmonics. Comparisons with phase and time-averaged DNS data have assessed the ability of these models to accurately capture the main features of transient and saturated oscillations in strongly nonlinear wakes [13,15,16] and open-cavity flows [14]; see also [17] for an application to thermosolutal convection flows. Mantić-Lugo *et al.* [18,19] also dealt with the saturation of extrinsic, harmonically and stochastically forced oscillations with a model coupling the mean flow to the leading modes of the mean flow resolvent. For this case, the nonlinearity is triggered by the ability of the system to amplify the non-small forcing amplitude. The authors report a good agreement between the self-consistent and nonlinear amplification gain computed in a backward-facing step flow with 1:2 expansion ratio, which completes the proof that saturation mechanics via Reynolds stress coupling is relevant to many flow situations of intrinsic and extrinsic oscillations.

What follows should be regarded as a first step in addressing whether the self-consistent theory can tackle similarly turbulent regimes. The focus is on the saturation of harmonically forced disturbances in a two-dimensional turbulent backward-facing step flow, a test case that has received substantial attention for validation purposes of turbulence models (NASA 2DBFS benchmark [20], also ERCOFTAC C-30) following the seminal experimental measurements of Driver & Seegmiller [21]. The retained approach requires invoking a separation of scales formalized by the triple decomposition [22,23], and decomposing the flow field into a mean component, large-scale coherent disturbances and small-scale incoherent fluctuations. This is very

much in line with recent studies combining Reynolds-stresses models with a statistical description of the small scale to assess its nonlinear effects as a stochastic forcing on the linearized fluctuation equations [24,25] or to analyse the redistribution of turbulent energy among small-scale modes via scattering off the large scale [26], but the line of thought here is different, as we rather lump the incoherent Reynolds stresses into an eddy viscosity model, and set-up a self-consistent description of the mean flow-coherent interaction in the frame of unsteady Reynolds averaged Navier–Stokes (uRANS) modelling. Since the study is intended as a proof of feasibility, we follow the blueprint of several references relying on simple RANS turbulence models to analyse the linear stability properties of turbulent mean flows [27–30] and settle for the one-equation Spalart–Allmaras (SA) model [31].

The key feature of the self-consistent model is that the coherent structures selected by the semi-linear resolvent analysis are those whose Reynolds stresses force the mean flow by their Reynolds stresses in a manner such that the mean flow generates exactly the aforementioned perturbation structures at asymptotically large times. We shall see that the obtained results allow formalizing the correlation between the large-scale dynamics of canonical turbulent shear flows and the leading resolvent modes of the mean flow. This line of thought has been pursued in several simpler linear analyses, that either dismiss the Reynolds stresses of the linear optimal disturbances [32,33], or avoid the problem of closure by assuming *a priori* knowledge of the mean flow (from numerical or experimental data) and then treat the Reynolds stresses nonlinearity in the perturbation equation as an unknown forcing [32–35].

The paper is organized as follows. The flow configuration and governing RANS equations are described in §2, where we also discuss the nature of the steady solutions. In §3, we perform a classical linear resolvent analysis to identify relevant external forcing distributions and frequencies. The semi-linear, self-consistent resolvent analysis is derived in §4. The ability of the model to describe the mechanics of saturation is assessed from exhaustive comparisons with nonlinear data obtained by uRANS simulation. Selected concluding remarks are finally provided in §5.

2. Problem formulation

(a) Flow configuration

The two-dimensional backward-facing step flow configuration of Driver & Seegmiller [21] is presented in figure 1. We use a Cartesian coordinate system whose origin is at the foot of the step, with x horizontally along the channel direction, and y vertically upwards. This case has several advantages: the expansion ratio 8:9 is small enough to ensure that pressure effects do not dominate over turbulent effects [36], in particular the upper boundary layer to remain attached despite the separation-induced pressure drop at the lower wall. Also, the experimental inflow conditions are well defined, and there is a substantial amount of reference data, as the case has received substantial attention for validation purposes of turbulence models (NASA 2DBFS and ERCOFTAC C-30 benchmarks; see [20] and references therein). In the following, we set the Reynolds number (built from the step height and the inflow centreline velocity) to the value $Re = 50\,000$ that has long been recommended by NASA [20] and remains used in many turbulence modelling workshops [37]. Because the recommended value has been recently updated to 36 000 to straighten out the comparison with the experimental data, it is worth emphasizing that the reference SA data used in the following for validation purpose are those documented by NASA prior to the update, that pertain to the same Reynolds number $Re = 50\,000$. Only the experimental data are for the updated value 36 000, which is ultimately expected to trigger only minor differences (which has been checked from the numerical standpoint by comparing the SA reference data documented by NASA before and after the update; not shown here for conciseness).

Figure 1. Sketch of two-dimensional backward-facing step considered in this study. The expansion ratio is 8 : 9 and the no-slip walls are indicated with hatching. This configuration is the same as NASA 2DBFS and ERCOFTAC C-30.

(b) Governing equations

The governing equations to be solved are the incompressible Navier–Stokes equations

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - 2\nabla \cdot (\nu \boldsymbol{\varepsilon}(\mathbf{u})) = \mathbf{f}, \quad (2.1)$$

with $\mathbf{u} = (u_x, u_y)$ the velocity field of component u_x (respectively, u_y) in the x (respectively, y) direction, p the pressure to density ratio, $\boldsymbol{\varepsilon}(\mathbf{u})$ the rate of deformation tensors, $\nu = 2 \times 10^{-5}$ the non-dimensional kinematic viscosity, and $\mathbf{f}$ a source term that will be specified as time harmonic in §§3 and 4. Because of incompressibility, it is understood that all velocity fields are divergence-free; therefore this condition is not written explicitly. Before proceeding further, we introduce $\mathcal{N}(\mathbf{u}) = (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - 2\nabla \cdot (\nu \boldsymbol{\varepsilon}(\mathbf{u}))$ as the nonlinear Navier–Stokes operator, and conveniently rewrite (2.1) in compact form as

$$\partial_t \mathbf{u} + \mathcal{N}(\mathbf{u}) = \mathbf{f}. \quad (2.2)$$

While turbulent fluctuations arise in the most general case at various scales and with varying degrees of spatial and temporal coherence, we assume here a clear separation between the large-scale coherent structures that emerge from the externally applied forcing, and the small-scale instability that develops in the shear layers and sustains the production of turbulence by selective amplification of the background noise. This is formalized by the triple decomposition [22,23], and allows decomposing the dynamics into a stochastic, incoherent part, and a coherent part assembled by phase-averaging, i.e.

$$\mathbf{u} = \langle \mathbf{u} \rangle + \overline{\mathbf{u}'} = \bar{\mathbf{u}} + \mathbf{u}' + \mathbf{u}'', \quad (2.3)$$

where the brackets and the overline denotes phase-averaged and time-averaged quantities, respectively, the single prime is for the zero-mean ($\bar{\mathbf{u}'} = \mathbf{0}$) coherent fluctuation, and the double prime is for the zero-mean ($\langle \mathbf{u}'' \rangle = \mathbf{0}$) incoherent fluctuation. Substituting $\langle \mathbf{u} \rangle + \mathbf{u}''$ for $\mathbf{u}$ into (2.2) and taking the phase average yields the coherent equations of motion

$$\partial_t \langle \mathbf{u} \rangle + \mathcal{N}(\langle \mathbf{u} \rangle) = -\langle \mathbf{u}'' \cdot \nabla \mathbf{u}'' \rangle + \mathbf{f}, \quad (2.4)$$

forced by the Reynolds stress of the incoherent fluctuation and the phase-averaged component of the source term $\langle \mathbf{f} \rangle$ (only we have anticipated that $\langle \mathbf{f} \rangle = \mathbf{f}$ on behalf of the forcing being here time

harmonic). Provided the coherent–incoherent interaction is lumped into an eddy viscosity ν_T , the equations of motion within the frame of the Boussinesq approximation are unsteady Reynolds averaged Navier–Stokes (uRANS) equations

$$\partial_t \langle \mathbf{u} \rangle + \tilde{\mathcal{N}}(\langle \mathbf{u} \rangle, \nu_T) = \mathbf{f}, \quad (2.5)$$

formally identical to the unsteady Navier–Stokes equations, only the viscous term of the turbulent evolution operator $\tilde{\mathcal{N}}$ encompasses the total (molecular + turbulent) viscosity $\nu + \nu_T$.

The eddy viscosity is computed with the one-equation SA model [31], written in compact form as

$$\partial_t \tilde{\nu} + \mathcal{E}\mathcal{V}(\langle \mathbf{u} \rangle, \tilde{\nu}) = 0, \quad (2.6)$$

where $\mathcal{E}\mathcal{V}$ is the nonlinear SA operator whose detailed expression is omitted for the sake of concision (the interested reader is referred to [20] and references therein), and $\tilde{\nu}$ is a turbulent working variable that relates to the eddy viscosity via a fourth-order polynomial equation.¹ We use here the so-called SA-R modification [38] intended to reduce the eddy viscosity in regions where the vorticity exceeds the strain rate (here in the vortex core regions developing immediately after the step, where pure rotation should not produce turbulence) while passive in thin shear layers where vorticity and strain are very close [20]. We have slightly modified the model on the term related to the rotation correction to be compatible with the Newton method as detailed in §3b. When marching in time (2.6), we follow [39] and systematically filter out the lowest values of $\tilde{\nu}$ by substituting

$$0.05 \log(1 + \exp(20\chi)), \quad (2.7)$$

for $\tilde{\nu}$ for all degrees of freedom such that $\tilde{\nu}/\nu < 30$, which ensures that the turbulent variable remains strictly positive without resorting to a harsh clipping strategy.

(c) Numerical method

The numerical approach used to solve the unsteady RANS equations is adapted from [28]. A mesh composed of about 400 000 triangular elements is generated using the Delaunay–Voronoi algorithm, with strong clustering at the walls and in the near-wake region to accurately represent the small scales typical of the turbulent boundary and shear layers. A uniform cell size 3.5×10^{-3} is prescribed at the walls, which corresponds to a y^+ value of approximately 2 at the inlet, and 3 at the step corner. For this latest position, 230 grid points are distributed across the boundary layer, two in the viscous sublayer, five in the buffer layer and nine in the logarithmic layer, hence 16 points in the inner layer. The computational domain, denoted by Θ , is equipped with the L2 inner product and related norm denoted by $\|\cdot\|_\Theta$. We use a first-order characteristics Galerkin scheme with time step $\Delta t = 5 \times 10^{-2}$, combined with MINI elements, i.e. three degrees of freedom (at each triangle edge) for the pressure, and four degrees of freedom (at each triangle edge and at the barycentre of each triangle) for each velocity component and the turbulent variable, hence about 2 100 000 degrees of freedom in two dimensions. Equations (2.5) and (2.6) are solved together with no-slip conditions at the walls and a stress-free condition at the outlet boundary, located at $x = 40$. At the inlet boundary, located at $x = -4$, we use two symmetrical analytical turbulent boundary layer profiles taken from [40], and set the momentum thickness $\theta = 0.15$ to match the experimental measurements of [21] at this position (figure 2a). The corresponding eddy viscosity profile shown in figure 2c is determined by numerical integration of the momentum equation in the wall-normal direction, which requires going through various steps carefully documented in [41]. The large-scale, sparse matrices obtained after space discretization are built with FreeFEM++ [42], and the matrix inverses are computed using the parallelized MUMPS library. In the following, comparisons are made with reference SA.

¹For consistency, the turbulent evolution operator is thus rewritten in what follows as $\tilde{\mathcal{N}}(\langle \mathbf{u} \rangle, \tilde{\nu})$.

Figure 2. Inlet (a) streamwise velocity $\langle u \rangle$ and (c) eddy viscosity $\tilde{\nu}$ and the steady RANS fields: (b) velocity U and (d) eddy viscosity for $Re = 50\,000$. (Online version in colour.)

3. Linear analysis

(a) Unforced solution

In the absence of forcing ($\mathbf{f} = \mathbf{0}$), the solution to (2.5) and (2.6) converges readily to a steady state, seen in figure 2 to feature a small corner eddy followed by a recirculation extending over the lower wall up to $x = 6$. This value, that proceeds from the zero isocontour of the x -velocity (an almost identical value $x = 5.97$ has been obtained from the lower skin friction distribution) is in close agreement with the reference numerical value of 6.1 reported for the SA model [37]² and the experimental value of 6.0 [21]. The boundary layer thickness measured immediately prior to the step is 1.58, also in agreement with the experimental value of approximately 1.5. In figure 3, we further compare with the reference SA data in terms of vertical cuts of the x -velocity (the same comparisons have been carried out in terms of the lower/top skin friction distributions). Despite little discrepancies that we ascribe to the fact that the reference calculations use a uniform freestream value for the turbulence field variable [37] (we apply a non-uniform profile derived consistently in the frame of the boundary layer theory), a close agreement between the present results and the reference SA data is found regardless of the position, either upstream of the reattachment point at $x = 1, 4$ (figure 3*a,b*), or further downstream at $x = 6, 10$ (figure 3*c,d*). We also report in figure 3 reference data obtained with more sophisticated RANS models, for which the agreement with SA is very satisfactory.

It should be noted that the experimental solution of Driver & Seegmiller [21] underlines a low-frequency behaviour, with a higher frequency $St \sim 0.1$ associated with vortex shedding, and a lower frequency $St \sim 0.03$ characteristic of a flapping of the recirculation region (see also [43] for further analysis of the time-dependent dynamics of this flow by the same authors). Conversely, it is generally acknowledged that the grid-converged uRANS numerical solution for this case is a steady RANS solution [44], whose interpretation can be ambiguous. Namely, it should be understood, not as the average over time of the unsteady solution (which is what is reported in experiments, and also what a steady RANS solution should be if the system was stationary ergodic, but explains in practice the discrepancy between the SA and experimental results in

²Data available from the Langley Research Center [20].

Figure 3. Comparisons between the steady SA RANS solutions ($\langle u \rangle$, in black solid lines) and the reference SA solution provided from NASA (dashed line) [20]. The experimental (grey crosses) [21] streamwise velocities are also shown. (Online version in colour.)

figure 3 at the most upstream position $x = 1$), but as a base solution in the sense of bifurcation theory, evolving into a time-dependent state by some linear instability mechanism. The latter is likely extrinsic, as no intrinsic instability has been found by linear stability analysis (not reported here for conciseness). The line of thought is that basic RANS models prevent the bifurcation from being observed because the important numerical diffusion acts as a low-pass filter whose cut-off scale turns to be (for this case) above that of the shed vortices (in the absence of vortex shedding, the vorticity does not accumulate in the recirculation, hence no flapping either). With more sophisticated models, the outcome is driven by the sharpened competition between numerical diffusion and dispersion. Although caution should be exercised in generalizing from the present results, the premise here is that the converged solution should be steady-state, but that a dubious unsteady dynamics may be recovered under certain conditions.³ This should not, however, call into question the ability of RANS modelling to accurately compute the steady state itself, nor the amplitude of harmonically forced coherent structures, provided their scale is above the cut-off. Such structures emerge naturally from the resolvent analysis as long as the same model is used at all stages of the calculations, which we will show suffices to assess the relevance of the self-consistent theory.

(b) Linear resolvent analysis

We venture now into the field of receptivity analysis to assess the linear response behaviour of the steady RANS solution, denoted for the sake of clarity by $(\bar{\mathbf{u}}_0, \bar{\mathbf{v}}_0)$, to an external harmonic forcing. With no loss of generality, we consider applying a zero-mean (in the sense of time average) forcing at the frequency ω , made non-dimensional using the step height and the inflow centreline velocity, i.e.

$$\mathbf{f}(\mathbf{x}, t) = \mathbf{f}'(\mathbf{x}, t) = 2\Re\{\hat{\mathbf{f}}_1(\mathbf{x}) \exp(i\omega t)\}, \quad (3.1)$$

with unit-norm $\|\hat{\mathbf{f}}_1\|_{\Theta} = 1$. When (3.1) is applied to a stable flow, the asymptotic response is harmonic at the same frequency, hence

$$\mathbf{u}'(\mathbf{x}, t) = 2\epsilon\Re\{\hat{\mathbf{u}}_1(\mathbf{x}) \exp(i\omega t)\}, \quad (3.2)$$

³For instance, Fadai-Ghotbi *et al.* [44] find a vortex shedding at the proper frequency using a coarse mesh on which numerical dispersion errors trigger spurious stochastic disturbances under the form of small-amplitude oscillations at the step corner, but a steady state using a refined mesh. Interestingly, the laminar backward-facing step flow has been the subject of a similar debate on whether or not the numerics can trigger the onset of unsteadiness (e.g. [45,46]).

where $\hat{\mathbf{u}}_1$ is sought as the solution to the linear disturbance equation

$$\hat{\mathbf{u}}_1 = (i\omega\mathcal{I} + \tilde{\mathcal{L}}_{(\bar{\mathbf{u}}_0, \bar{v}_0)})^{-1} \hat{\mathbf{f}}_1 = \tilde{\mathcal{R}}(\bar{\mathbf{u}}_0, \bar{v}_0, \omega) \hat{\mathbf{f}}_1, \quad (3.3)$$

with $\mathcal{I}$ the identity operator, $\tilde{\mathcal{L}}_{(\bar{\mathbf{u}}_0, \bar{v}_0)} \hat{\mathbf{u}} = (\bar{\mathbf{u}}_0 \cdot \nabla) \hat{\mathbf{u}} + (\hat{\mathbf{u}} \cdot \nabla) \bar{\mathbf{u}}_0 + \nabla \hat{p} - 2\nabla \cdot ((\nu + \nu_T(\bar{v}_0)) \boldsymbol{\varepsilon}(\hat{\mathbf{u}}))$ the turbulent linearized Navier–Stokes operator, and $\tilde{\mathcal{R}}(\bar{\mathbf{u}}_0, \bar{v}_0, \omega) = (i\omega\mathcal{I} + \tilde{\mathcal{L}}_{(\bar{\mathbf{u}}_0, \bar{v}_0)})^{-1}$ the turbulent resolvent that maps the forcing into the velocity response (both operators being formally identical to their laminar counterparts, up to the viscous term defined from the total viscosity $\nu + \nu_T$). The forcing and response amplitudes are measured by the square root of their kinetic energies over a forcing period $T = 2\pi/\omega$, to give $A_f = \sqrt{2} \|\hat{\mathbf{u}}_1\|_\Theta$ and $A_r = \sqrt{2} \|\hat{\mathbf{f}}_1\|_\Theta$. Finally, the gain is defined as the ratio of the response to the forcing amplitudes, hence

$$G(\omega) = \frac{A_r}{A_f} = \frac{\|\hat{\mathbf{u}}_1\|_\Theta}{\|\hat{\mathbf{f}}_1\|_\Theta}. \quad (3.4)$$

The present approach relies on two simplifying assumptions: first, the incoherent dynamics has been slaved to the steady component of the solution. This amounts to a one-way coupling between the coherent and the incoherent perturbation dynamics, in the sense that the incoherent scale affects the coherent scale via the eddy viscosity in the resolvent, only the coherent scale does not feed back. Second, the classical small-amplitude assumption has been invoked to leave aside the nonlinearity in $\hat{\mathbf{u}}_1$ (i.e. the self-interaction of the coherent response) in the disturbance equation. This allows computing $(\bar{\mathbf{u}}_0, \bar{v}_0)$ beforehand as the solution to

$$\tilde{\mathcal{N}}(\bar{\mathbf{u}}_0, \bar{v}_0) = 0 \quad (3.5)$$

and

$$\mathcal{EV}(\bar{\mathbf{u}}_0, \bar{v}_0) = 0, \quad (3.6)$$

which can be done by time-marching (2.5) and (2.6), then, performing the resolvent analysis. In this regard, the reader interested in detailed information is referred to [47,48]. Suffice it to say here that we used an implicitly restarted Arnoldi method to perform the singular value decomposition of the resolvent. The linear optimal gain is given by the largest singular value, the structure of the linear optimal forcing (i.e. the forcing fostering the linear optimal gain) is given by that of the associated right singular vector, and the linear optimal response by the left singular vector.

(c) Results

We show in figure 4 that the steady RANS solution exhibits substantial singular values over an extended range of frequencies $\omega < 1$, which suggests that even a small-amplitude harmonic excitation in the appropriate frequency range can trigger a large, time-periodic response at the same frequency. We find a global maximum for $\omega \sim 0.05$, and another local maximum for $\omega \sim 0.2$. Selected spatial distributions of the forcing and associated flow response are presented in figure 4b–g, for the frequencies pointed at by arrows in figure 4a. There is an obvious difference in the spatial supports of the optimal forcing and response (all the more so as the frequency increases), which stems from the non-normality of the linearized evolution operator [49] advecting the optimal response (respectively, the optimal forcing) downstream (respectively, upstream). For $0.01 \leq \omega \leq 0.1$, the forcing and response are essentially concentrated in the recirculation; see figure 4b,c for a forcing frequency $\omega = 0.05$. Identifying the physical mechanism at play is challenging, but it may be related to a centrifugal instability along the recirculation, or to an elliptic instability in the region between the recirculation and the corner eddy. For $0.2 \leq \omega \leq 0.8$, the optimal forcing and response consist of a series of counter-rotating vortices developing at the lower wall. The forcing amplitude (respectively, the response amplitude) peaks at the step corner (respectively, in the downstream channel); see figure 4d,e for $\omega = 0.3$, which is the hallmark of wave packets growing via lift-up and travelling in the shear layer of the recirculation before shrinking further downstream. The results obtained for larger forcing frequencies are somehow similar, only the forcing and response develop at the upper channel wall and the apparent

Figure 4. (a) Linear optimal gain for different forcing frequencies. Selected (b,d,f) forcing vectors and (c,e,g) corresponding responses for (b,c) $\omega = 0.05$, (d,e) $\omega = 0.3$ and (f,g) $\omega = 0.9$ for $Re = 50\,000$. (Online version in colour.)

wavelength decreases as ω increases (which is consistent with most studies of harmonic optimal gain in laminar, convective flows [47,50–52]); see figure 4f,g for $\omega = 0.9$.

4. Semi-linear, self-consistent resolvent analysis

(a) Formulation

We address now the saturation mechanics of the linear disturbances computed in §3 from a Reynolds decomposition (in the sense of the average over time) of the coherent dynamics. Since $\bar{\mathbf{f}} = \mathbf{0}$, substituting $\bar{\mathbf{u}} + \mathbf{u}'$ for the phase-averaged velocity $\langle \mathbf{u} \rangle$ in (2.5), taking the time average, and subtracting from (2.5), gives

$$\tilde{\mathcal{N}}(\bar{\mathbf{u}}, \bar{\mathbf{v}}) = -\overline{(\mathbf{u}' \cdot \nabla) \mathbf{u}'} \quad (4.1)$$

and

$$\frac{\partial \mathbf{u}'}{\partial t} + \tilde{\mathcal{L}}_{(\bar{\mathbf{u}}, \bar{\mathbf{v}})} \mathbf{u}' = -(\mathbf{u}' \cdot \nabla) \mathbf{u}' + \overline{(\mathbf{u}' \cdot \nabla) \mathbf{u}'} + \mathbf{f}. \quad (4.2)$$

which defines the mean flow as a steady solution to the RANS equations, forced by the coherent Reynolds stresses (i.e. the steady part of the coherent self-interaction), and the coherent fluctuation as a solution to the uRANS equations linearized about the mean flow, forced by the unsteady part of its self-interaction. Additionally, both components are forced by the incoherent self-interaction, encompassed via the eddy viscosity in the left-hand side of (4.1) and (4.2). In this approach, the incoherent dynamics has been slaved to the time-averaged component of the solution, for the eddy viscosity to be computed as a solution to

$$\mathcal{E}\mathcal{V}(\bar{\mathbf{u}}, \bar{v}) = 0, \quad (4.3)$$

as has been done in several recent studies analysing the linear stability of turbulent flows [29,35,41]. The assumption is reminiscent of that underlying the linear analysis carried out in §3. Nonetheless, it amounts to a one and a half-way coupling between the coherent and the incoherent perturbation dynamics, in the sense that the incoherent scale affects the coherent scale via the eddy viscosity in (4.2), and the coherent scale feeds back indirectly via the coherent Reynolds stresses in (4.1).

We consider the same harmonic forcing at frequency ω as in §3, now parametrized by its (real) scalar amplitude A , i.e.

$$\mathbf{f}(\mathbf{x}, t) = \mathbf{f}'(\mathbf{x}, t) = 2A\Re\{\hat{\mathbf{f}}_1(\mathbf{x}) \exp(i\omega t)\}, \quad (4.4)$$

with $\|\hat{\mathbf{f}}_1\|_\phi = 1$. We elaborate further on this matter in the following, but the key assumption here is for the nonlinearity to act through mean flow distortions, not the generation of harmonics. By doing so, the asymptotic response is modelled by a first-order Fourier expansion

$$\mathbf{u}'(t) = 2A\Re\{\hat{\mathbf{u}}_1 \exp(i\omega t)\}, \quad (4.5)$$

at the same frequency, in a way such that substituting in (4.1)–(4.3) and neglecting higher-order harmonics yields the semi-linear, self-consistent system

$$\tilde{\mathcal{N}}(\bar{\mathbf{u}}, \bar{v}) = -2A^2\Re\{(\hat{\mathbf{u}}_1 \cdot \nabla)\hat{\mathbf{u}}_1^*\} \quad (4.6)$$

$$\mathcal{E}\mathcal{V}(\bar{\mathbf{u}}, \bar{v}) = 0 \quad (4.7)$$

$$\text{and} \quad \hat{\mathbf{u}}_1 = \mathcal{R}(\bar{\mathbf{u}}, \omega)\hat{\mathbf{f}}_1, \quad (4.8)$$

where the $*$ superscript indicates complex conjugation.

(b) Resolution

Following [18], an iterative scheme is used to solve (4.6)–(4.8) on the understanding that the forcing distribution is a user-defined, input parameter, and that the time-averaged mean flow is not a given, but comes instead as an output (which is key in setting up a self-consistent, closed description of the mean flow/coherent fluctuation interaction). A number of practical steps must be undertaken to maintain the convergence of the iterative solution. First, we compute the time-averaged mean flow with an under-relaxed Newton method, which requires solving at iteration $k+1$

$$\mathcal{J}_{\bar{\mathbf{u}}^k}(\tilde{\mathcal{N}})\delta\bar{\mathbf{u}}^{k+1/2} + \mathcal{J}_{\bar{v}^k}(\tilde{\mathcal{N}})\delta\bar{v}^{k+1/2} = -\Phi(\hat{\mathbf{u}}_1^k) - \tilde{\mathcal{N}}(\bar{\mathbf{u}}^k, \bar{v}^k) \quad (4.9)$$

and

$$\mathcal{J}_{\bar{\mathbf{u}}^k}(\mathcal{E}\mathcal{V})\delta\bar{\mathbf{u}}^{k+1/2} + \mathcal{J}_{\bar{v}^k}(\mathcal{E}\mathcal{V})\delta\bar{v}^{k+1/2} = -\mathcal{E}\mathcal{V}(\bar{\mathbf{u}}^k, \bar{v}^k), \quad (4.10)$$

with $\Phi(\hat{\mathbf{u}}_1^k)$ the current iteration of the coherent Reynolds stresses, and $\mathcal{J}_{\bar{\mathbf{u}}}$ (respectively, $\mathcal{J}_{\bar{v}}$) the formal Jacobian with respect to the laminar (respectively, the turbulent) variables. In practice, we significantly improve the convergence of the Newton algorithm by implementing a slightly modified version of the SA model, in which all harsh limiters (introduced to prevent unrealistically large values of the turbulence quantities) are replaced by continuously differentiable step functions to circumvent linearization errors. For instance, the production term

Figure 5. Effect of the smooth step function on SA-R model. The black solid curves are current uRANS with smooth step function χ_1 as $\chi_1(s - \Omega_0)$ the same result in figure 3, blue dashed curves are strict SA-R uRANS solutions (using $\min(0, s - \Omega_0)$ in the term Ω), green dotted curves are reference SA model from [20] and the red dotted lines are converged Newton steady solution with smooth step function. (Online version in colour.)

features a minimum limiter $\min(0, S - \Omega)$, with Ω the magnitude of vorticity and S a scalar measure of the deformation tensor, that we replace by the step function

$$\frac{1}{1 + \exp(2k(S - \Omega))}, \quad (4.11)$$

with $k = 0.5$. A similar modification is applied for the destruction term, that features a maximum limiter $\max(\bar{v}/S\kappa^2 d^2, 10)$ with κ the von-Kármán constant and d the distance to the closest wall. For proof of relevance, we show in figure 5 that the velocity profiles of the unforced steady RANS solution, computed iteratively with continuous limiters, yield a better agreement with the reference SA results [20] than their counterparts obtained by time-stepping the unsteady solution with harsh limiters. The only difference is in the very near-wall regions, where the modification actually yields a better agreement with the reference SA results, especially close the step, at $x = 1$, although caution should be exercised in interpreting these results since this position is extremely sensitive to the turbulence model [20,21]. The calculation terminates when the relative variation (in L2-norm) between two consecutive iterations is less than 10^{-5} , at which point the new solution is set to

$$(\bar{\mathbf{u}}^{k+1}, \bar{v}^{k+1}) = (\bar{\mathbf{u}}^k, \bar{v}^k) + \gamma(\delta \bar{\mathbf{u}}^{k+1/2}, \delta \bar{v}^{k+1/2}), \quad (4.12)$$

with γ a relaxation factor adjusted to the forcing amplitude.

In a second step, $\hat{\mathbf{u}}_1^{k+1}$ is obtained by the resolvent analysis of $\mathcal{R}(\bar{\mathbf{u}}^{k+1}, \omega)\hat{\mathbf{f}}_1$ and we compute a Tikhonov regularization of the associated Reynolds stresses as the solution to

$$\Phi(\hat{\mathbf{u}}_1^{k+1}) = (\mathcal{I} + c\nabla^2)(\Re\{(\hat{\mathbf{u}}_1^{k+1} \cdot \nabla)\hat{\mathbf{u}}_1^{k+1*}\}), \quad (4.13)$$

with $c = 0.004$. This allows smoothing out the undesirable consequences of filtering the turbulent variable, that are unnoticeable in the response itself, but show up in its Reynolds stress under the form of small-amplitude, spurious oscillations that rapidly jeopardizes the convergence of the iterative scheme.⁴ At this stage, we start over and repeat until all solutions converge to 1.5×10^{-5} in L2-norm. Note that the need to regularize the Reynolds stress does not proceed from a lack of accuracy of the spatial discretization, as the same behaviour has been found using a refined mesh composed of about 675 000 elements (total degrees of freedom being 3 409 152), or using the same mesh with piecewise quadratic P2 elements with 6 d.f. for all velocity components and the turbulent variable (total d.f. = 2 723 239), as illustrated in figure 6.

Regarding the choice of a relaxation factor, small forcing amplitudes allow for large values of γ , for instance, we set $\gamma = 0.3$ for values of A up to $A = 0.01$ for $\omega = 0.3$, which allows converging all

⁴This also explains the lower level of convergence of the Newton algorithm, since this approach usually converges to the machine precision for laminar flows.

Figure 6. Effect of the Tikhonov regularization on the Reynolds stresses: cut at $x = 5$ of the regularized Reynolds stresses (blue line) versus non-regularized Reynolds stresses (black line) versus non-regularized Reynolds stresses on a finer mesh (red line) with no. triangles 675 000 for $\omega = 0.3$ and $A = 0.005$. (Online version in colour.)

solutions within 20 iterations. In this range, it takes typically 3 h of CPU time using four processors to converging the self-consistent solution to the desired precision, to be compared to the 5 days needed to converge the nonlinear uRANS results under the same conditions. The relaxation factor however decreases rapidly as A increases, as a result of strong linearities considerably slowing the convergence rate of iterative methods. This is not specific to the turbulent regimes, as the laminar cases exhibits a similar behaviour [18], but the self-consistent approach actually keeps performing extremely well, for instance, for $A = 0.03$, we use $\gamma = 0.05$ and it takes up to 40 iterations to achieve convergence, which represents only 6 h of CPU time.

(c) Results

For a given forcing frequency, the line of thought here is to set the forcing to the linear optimal $\hat{\mathbf{f}}_{01}$ computed by linear resolvent analysis of the steady RANS solution. The saturation of the induced response is analysed by self-consistent resolvent analysis, which requires computing the self-consistent gain

$$G_{sc}(\omega) = \frac{A_{r,sc}}{A_f} = \frac{\|\hat{\mathbf{u}}_1\|_{\Theta}}{\|\hat{\mathbf{f}}_{01}\|_{\Theta}}, \quad (4.14)$$

from the forcing and response amplitudes given by $A_f = A\sqrt{2}\|\hat{\mathbf{f}}_{01}\|_{\Theta}$ and $A_{r,sc} = A\sqrt{2}\|\hat{\mathbf{u}}_1\|_{\Theta}$. For various forcing amplitudes, the semi-linear results are compared to their nonlinear counterparts obtained by time-marching the uRANS flow response up to the asymptotic regime. The related gain

$$G_{nl}(\omega) = \frac{A_{r,nl}}{A\sqrt{2}\|\hat{\mathbf{f}}_{01}\|_{\Theta}}, \quad (4.15)$$

proceeds from the response amplitude

$$A_{r,nl} = \sqrt{\frac{1}{T} \int_{t_0}^{t_0+T} \|\mathbf{u}'\|_{\Theta}^2 dt}, \quad (4.16)$$

where t_0 is an arbitrary time large enough for all transient disturbances to die out, set in practice after the mean flow residual has become less than 10^{-6} in L2-norm. When relevant, we also compare to the linear gain and response amplitude obtained by linear resolvent analysis of the steady (unforced) RANS solution, denoted by G_{lin0} (inherently independent of A) and $A_{r,lin0}$ (proportional to A), respectively, and those computed by linear resolvent analysis of the time-averaged (forced) uRANS solution, denoted by G_{lin} and $A_{r,lin}$.

Figure 7. (a) Gain and (b) amplitude of the response as a functions of forcing amplitude: linear gain (dashed red line), self-consistent prediction (triangles), linear response on the mean flow of uRANS (plus symbols) and nonlinear uRANS (filled circles) results for $\omega = 0.3$ and $Re = 50\,000$. (Online version in colour.)

First, we set the forcing frequency to $\omega = 0.3$, for which the linear optimal forcing and response distributions computed in §3 consist of wave packets travelling in the shear layer of the recirculation. The semi-linear gain and response amplitude are reported in figure 7 for increasing values of the forcing amplitude, together with the structure of the various self-consistent solution components provided in figure 8. For $A = 0$, the mean flow reduces to the steady, unforced RANS solution, and the self-consistent results fall onto those predicted by the linear theory. The semi-linear and linear results remain in close agreement for $A \leq 0.003$, after which the nonlinearity comes into play and the coherent Reynolds stresses increasingly distort the steady solution into a mean. This triggers a significant reduction of the optimal gain, by 8% for $A = 0.01$, up to 40% for $A = 0.03$. Figure 8 shows that the nonlinear effects are essentially on the amplitude of the response, not so much on its structure, although the latter *is* distorted in the upstream boundary layer and in the near-wake, where the counter-rotating vortices are slightly squeezed towards the wall.

For $A \leq 0.03$, the semi-linear values in figure 7 are almost identical to their nonlinear uRANS counterparts. The same remarkable agreement is found for the structure of the coherent flow response and of the coherent Reynolds stresses, which is further illustrated in figure 9 for $A = 0.01$ (the uRANS coherent response is simply a snapshot of the coherent fluctuation $\mathbf{u}' = \langle \mathbf{u} \rangle - \bar{\mathbf{u}}$, for which we pay attention in comparing identical phases over a forcing period). Note, the nonlinear distortion of the coherent response is evidenced in figure 9 by comparing to the linear resolvent analysis of the steady RANS solution, whose solution closely resembles at first sight, but actually features a larger apparent wavelength by 14% (from figure 9 $\lambda_L \simeq 7.7$ versus $\lambda_{sc} \simeq 6.6$), also reflected in subtle differences in the Reynolds stresses distribution. Detailed comparisons of the self-consistent and uRANS mean flows are also provided in figure 10 for the different values of the amplitude. A close agreement is found for $A = 0.01$, both inside the mean recirculation region at $x = 1, 4$, and further downstream, at $x = 6, 10$. On the other hand, the agreement slightly deteriorates for larger forcing amplitude $A = 0.03$, which is because the flow response is not exactly monochromatic (as will be discussed in the discussion section). In general, the mean flow does exhibit a little nonlinear distortion, best visible from $\bar{u}_y$, while $\bar{u}_x$ is rather robust to the nonlinearity. Such results clearly assess the ability of the model to capture the main mechanism responsible for the saturation of the harmonically forced disturbances, that involves mean flow modifications driven by the coherent Reynolds stresses, similarly to what has been reported in the laminar case [18].

Additional results pertaining to two other forcing frequencies $\omega = 0.05$ and $\omega = 0.9$ are reported in figure 11, to evidence the robustness of the theoretical model in a broadband frequency range and with respect to various instability mechanisms. For $\omega = 0.9$, the linear optimal forcing and response distributions computed in §3 consist of wave packets travelling in the upper boundary

Figure 8. Selected self-consistent predictions of velocity perturbations $\hat{\mathbf{u}}$ (*a,c,e*) and Reynolds stresses (*b,d,f*) for different forcing amplitudes (*a,b*) $A = 0.005$, (*c,d*) $A = 0.015$ and (*e,f*) $A = 0.03$ for $\omega = 0.3$ and $Re = 50\,000$. (Online version in colour.)

layer. A close agreement on the gain is found for forcing amplitudes up to $A = 0.02$ (figure 11*a*), also the structures of the self-consistent mean flow, coherent response and Reynolds stresses (not shown here for conciseness). The semi-linear values fall onto their linear counterparts for $A \leq 0.003$, after which the nonlinearity triggers a severe reduction of the linear optimal gain, by 11% for $A = 0.01$, up to 24% for $A = 0.02$. For $\omega = 0.05$, an excellent agreement on the gain is reported in figure 11*b* for $A = 0.002$, after which the iterative algorithm used to solve the self-consistent system diverges. This is likely because the forcing and response exhibits strong gradients in the very vicinity of the step (figure 4*b,c*). This yields large amplitudes of the Reynolds stresses in the very region where it is most altered by the Tikhonov regularization, as evidenced in figure 6. This prevents the Newton algorithm from converging when the nonlinearity is steep, which is the case here, since the decrease in the linear gain is already by 30% for $A = 0.002$, an effect accurately captured by the self-consistent model. Again, the nonlinearity acts essentially on the gain rather than on the structure of the coherent response.

In the course of discussing the above results, we have brought up the fact that the nonlinear gain distortions are frequency. For instance, the relative variations for $\omega = 0.9$ are larger than those reported at $\omega = 0.3$ regardless of A but only because the linear gain at this frequency is smaller. The absolute variations are actually larger for $\omega = 0.3$, which suggests that the nonlinearity at play is stronger for this case but still lower than that for $\omega = 0.05$, that triggers a steep decrease. As a result, the nonlinearity can end up selecting linearly suboptimal frequencies, which is best seen from figure 11*c* synthesizing the computed self-consistent gains against the forcing amplitude.

Figure 9. Comparison of the velocity perturbations $\hat{\mathbf{u}}$ among (a,b) self-consistent, (c,d) nonlinear uRANS, (e,f) linear around the mean flow of uRANS and (g,h) linear around the steady flow of RANS for $A = 0.01$ and $\omega = 0.3$. (Online version in colour.)

It turns out that only for the smallest values $A \leq 0.001$ does the analysis select $\omega \sim 0.05$ as the global maximum. For larger amplitudes, the strong nonlinearity at play at this frequency shifts the global maximum close to 0.2–0.3, which suggests that different ranges of frequencies are likely to emerge for various amplitudes of stochastically forcing uRANS simulations.

Finally, since the self-consistent mean flow provides by design an accurate approximation of the time-averaged mean flow, it is suggested here that relevant quantitative predictions can be obtained from a linear mean flow resolvent analysis,⁵ an approach widely accepted and applied in laminar and turbulent regimes despite its lack of a solid theoretical justification (we recall that it requires merely computing the time-averaged uRANS solution beforehand, and performing the singular value decomposition of the resolvent). For this case, this is confirmed by the close agreement reported in figures 7–11 between the linear mean flow results and their self-consistent/nonlinear counterparts. The self-consistent analysis serves as a foundation for properly interpreting the obtained results as applying in the case where the coherent flow response is purely monochromatic, without further assumption on the amplitude, which contributes to formalizing the previously reported correlation between the large-scale dynamics of canonical turbulent shear flows and the leading mean flow resolvent mode [32,34,35,53]. This

⁵Including the amplitude of the forced coherent disturbances, unlike mean flow linear stability analyses, that inherently miss on the amplitude of self-sustained oscillations.

Figure 10. Detailed comparison on the mean velocities ((a–d) for $\bar{u}_x$ and (e,f) for $\bar{u}_y$) between the self-consistent predictions (solid lines) and time-averaged uRANS results (dashed lines) at different positions (a,e) $x = 1$, (b,f) $x = 4$, (c,g) $x = 6$ and (d,h) $x = 10$ for various forcing amplitudes $A = 0$ (blue), $A = 0.01$ (red) and $A = 0.03$ (black): dark solid curves are self-consistent model and light dashed curves are mean flow of uRANS. (Online version in colour.)

Figure 11. Gains for different forcing frequencies (a) $\omega = 0.9$ and (b) $\omega = 0.05$. Amplitude of the response as a functions of forcing amplitude: linear gain (dashed red line), self-consistent prediction (triangles), linear response on the mean flow of uRANS (plus symbols) and nonlinear uRANS (filled circles) results for $Re = 50\,000$. (c) Comparison self-consistent gains for $\omega = 0.05$ (inverted triangles), 0.3 (triangles) and 0.9 (left-facing triangles). (Online version in colour.)

is all the more important because the classical argument that the Reynolds stresses must be unperturbed at the order of the coherent perturbation is somehow inconsistent, as it requires invoking, on the one hand, a small amplitude assumption to dismiss the nonlinearity in (4.2), while taking, on the other hand, the coherent Reynolds stresses in (4.1) to be those of the finite-amplitude uRANS response, not that of the small-amplitude resolvent mode (otherwise, the mean flow turns into the steady RANS solution). Nonetheless, it also warns against the validity of

the approach when the nonlinearity ceases to restrict to mean flow modifications and starts to generate harmonics, as a purely linear approach inherently leaves aside the nonlinear distortions of the resolvent modes (unlike the self-consistent theory, a matter to which we return in §5).

5. Conclusion and discussion

Characterizing accurately the saturation of forced coherent disturbances in a turbulent flow is key for the design of control strategies at practically meaningful Reynolds numbers. The primary contribution of this work is the development of a generalized semi-linear, self-consistent model that relies on the triple decomposition to analyse the nonlinear perturbation dynamics of the harmonically forced, turbulent flow over a backward-facing step flow with 8:9 expansion ratio, at Reynolds number 50 000. The model assumes a purely monochromatic response, couples the mean flow to its leading resolvent mode, and provides a consistent picture for the saturation of the harmonically forced disturbances, found to be driven by Reynolds stresses-induced mean flow distortions.

The key steps for computing the self-consistent solution are summarized as follows: (i) compute the self-consistent mean flow by solving nonlinear, steady forced RANS equations with the Newton method, (ii) compute the coherent response as the mean flow leading resolvent mode, (iii) compute the coherent Reynolds stresses feeding back onto the mean, (iv) repeat until convergence. Two numerical steps are undertaken to ensure the convergence of the iterative solution: (i) a continuous version of the SA is implemented to avoid poor convergence of the Newton algorithm and (ii) we force the steady RANS equations by a Tikhonov regularization of the Reynolds stresses to alleviate the effect of filtering the turbulent variable, meant to avoid negative values of the viscosity. A poor convergence of the iterative algorithm is reported for $\omega = 0.05$ because of the large Reynolds stresses amplitudes prevailing just downstream of the step. This is a very nonlinear case, however, and the model actually remarkably recovers the steep decrease of the gain achieved at this frequency with forcing amplitudes as low as $A = 0.002$. Other than that, the model is shown to perform accurately for several frequencies associated with different instability mechanisms, and to predict quantitatively the gain and amplitude of the large-scale coherent structures, as well as of the spatial distribution of the mean flow, of the coherent response and of the coherent Reynolds stresses.

There are limitations and challenges to this approach, that require further consideration to allow generalization to a wide variety of turbulent flows. From the theoretical standpoint, the agreement between the semi-linear and nonlinear uRANS data are seen to slightly deteriorate for the highest forcing amplitudes (e.g. figure 7 for $\omega = 0.3$ and $A > 0.03$). The model predictions remains quite satisfactory, as we show in figure 12 that the structure of the self-consistent solution components computed for $\omega = 0.3$ and $A = 0.03$ closely resemble that of the nonlinear solution. We ascribe the discrepancy to the fact that the nonlinearity starts to generate higher harmonics, as we show in figure 13 that the spectrum of the streamwise velocity measured in the recirculation at $x = (1, 0.6)$ for a small amplitude $A = 0.0025$ is purely monochromatic, while that for $A = 0.03$ features a substantial second harmonic, whose amplitude is approximately 20% of the fundamental, and even a small-amplitude third harmonic. This supports in return pushing forward the development of higher-order self-consistent models, as has been done in [14], for which we anticipate that the related coherent fluctuation must be sought as the superposition of a linear component, aligned along the leading mode of the mean flow resolvent, and a nonlinear component generated by the higher harmonics, that progressively distorts the amplitude of the resolvent mode.

Beyond the asymptotical response behaviour, that turns to be the natural scope of most resolvent analyses, it is also of interest to address the transient dynamics of the induced disturbances, as has been done in laminar regimes for intrinsic, self-sustained oscillations [14,15]. Assuming a zero initial condition, the instantaneous coherent fluctuation should read

$$\tilde{\mathcal{R}}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}, \omega) \hat{\mathbf{f}}_1 \exp(i\omega t) - \tilde{\mathcal{A}}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}, t) \hat{\mathbf{u}}_1, \quad (5.1)$$

Figure 12. Same as figure 9 but for $A = 0.03$. (Online version in colour.)

Figure 13. Power spectral density of velocity fluctuation normalized for the fundamental at the forcing frequency to peak unity at $(x, y) = (1, 0.6)$ for forcing frequency $\omega = 0.3$ for different forcing amplitudes: $A = 0.0025$ (black) and $A = 0.03$ (blue). (Online version in colour.)

with $\tilde{A} = \exp(\tilde{L}t)$ the propagator that maps an initial condition into a flow response at time t , and the second component in (5.1) therefore reflects the existence of transient disturbances. In the absence of an exponential instability at play (which has been ascertained by systematically analysing the linear stability analysis of the self-consistent mean flows; not shown here), the transient component is expected to be of the same order of magnitude as its periodic counterpart, up to the point where it decays algebraically. Encompassing its effect in the model is extremely challenging, as it would likely require introducing a time-dependent, time-averaged mean flow, invoking a separation of scales between the fast time scale on which the flow response oscillates and that on which the forced disturbances grow/decay in amplitude, and computing the transient coherent Reynolds stresses that feed back onto the mean. While this could be done on paper by performing a singular value decomposition of the propagator, there is a formal difficulty inherent to the fact that the number of propagator singular vectors needed to approximate the leading resolvent singular mode is unknown and varies as the coherent response develops.

Finally, as mentioned in the Introduction, this work relies on a simple RANS turbulence model as it is intended to prove feasibility, but there remains significant room for improvement from the numerical standpoint. Our current efforts revolve around improving the description of the small-scale turbulent motion using $k - \epsilon$ or $k - \omega$ RANS models that can be linearized without tremendous developments [54] (which is an obligatory step to compute the self-consistent mean flow without marching in time the uRANS equations). We also aim at performing such analyses in the frame of large-eddy simulation (LES), which we believe is key to extend the scope of the approach to practically meaningful turbulent regimes, where the computation of high fidelity numerical solution is challenging even for sophisticated RANS models, on behalf of the strong interactions between all temporal and spatial scales.

Data accessibility. This article has no additional data.

Authors' contributions. P.M. and F.G. conceived the project; E.Y. performed the project; E.Y., P.M. and F.G. discussed the models and E.Y. and P.M. wrote the manuscript with inputs from F.G.

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