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Author for correspondence:

Gianni Royer-Carfagni

e-mail: gianni.royer@unipr.it

Membrane analogy for multi-material bars under torsion

Laura Galuppi¹ and Gianni Royer-Carfagni^{1,2}

¹Department of Engineering and Architecture, University of Parma, Parco Area delle Scienze 181/A, 43100 Parma, Italy

²Construction Technologies Institute - Italian National Research Council (ITC-CNR), Via Lombardia 49, 20098 San Giuliano Milanese, Milano, Italy

LG, 0000-0002-1886-0979; GR-C, 0000-0003-4879-9846

Prandtl's membrane analogy for the torsion problem of prismatic homogeneous bars is extended to multi-material cross sections. The linear elastic problem is governed by the same equations describing the deformation of an inflated membrane, differently tensioned in regions that correspond to the domains hosting different materials in the bar cross section, in a way proportional to the inverse of the material shear modulus. Multi-connected cross sections correspond to materials with vanishing stiffness inside the holes, implying infinite tension in the corresponding portions of the membrane. To define the interface constraints that allow to apply such a state of prestress to the membrane, a physical apparatus is proposed, which can be numerically modelled with a two-dimensional mesh implementable in commercial finite-element model codes. This approach presents noteworthy advantages with respect to the three-dimensional modelling of the twisted bar.

1. Introduction

The linear elastic problem of a cylindrical bar with compact simply connected cross section under torsion, which was solved by Barré de Saint-Venant [1] in 1856, is governed by the same differential equations and boundary conditions that describe the small deformation of an inflated initially flat membrane shaped as the cross section of the bar, constrained at the borders and stretched by an uniform equibiaxial stress. This is the *membrane* or *soap-film analogy*, ingeniously proposed by Prandtl [2] in 1903. It can be extended [3,4] to the case in which the material of the twisted bar yields in

certain portion of the cross section. Correspondingly, the membrane deformation is bounded by a surface of constant maximum slope constructed over the edges.

There is a growing interest for non-homogeneous [5,6], multi-material [7,8], laminated [9] and functionally graded [10,11] beams and bars, which may be used as shafts or be prone to flexural torsional buckling. Elements of this type are usually designed using three-dimensional finite-element models, which require a consistent effort for meshing and, even more so, particular attention at the interface and boundary conditions at the beam ends. Following de Saint-Venant's principle, a simple solution for the linear elastic torsion problem can be found, but in a three-dimensional finite-element model (FEM) approach one may obtain unrealistic stress concentrations if the end constraints and the interface ties are not correctly applied. Here, the membrane analogy is revisited and extended to the case of composite multi-material bars. In this case, the inflated membrane shall be differently tensioned in regions that correspond to the domains hosting different materials in the bar cross section, in a way proportional to the inverse of the material shear modulus. Since the membrane problem can be readily solved with a two-dimensional FEM, implementable in most commercial structural codes, the analogy represents a convenient and practical method to analyse the state of stress in complex composite bars.

The proposed method covers the widest range of geometries, including the cases of 'open' and 'closed' bimaterial interfaces. One can find the solution for non-simply connected cross sections by letting the shear modulus of the material go to zero where there is a hole. Once the membrane problem has been numerically solved, it is straightforward to import the data of the deformed surface in a computing software and calculate its slopes, which are associated with the stress components in the twisted bar. The torsional stiffness is related to the volume enclosed between the deformed membrane and its flat reference configuration.

In §2, we state the elastic torsion problem and the membrane analogy. A scheme for the physical model is also presented, which clarifies what type of constraints shall be applied to the *multi-tensioned membrane*. This apparatus is intuitive, but rather than using it as a tool for a quantitative investigation, it is much simpler to solve the membrane problem numerically. In §3, we discuss the results obtainable with the membrane analogy in paradigmatic examples.

2. Multi-tensioned membrane analogy

Consider the de Saint-Venant's problem of a prismatic bar, made of a linear elastic homogeneous and isotropic material, subjected to a force distribution at the ends equipollent to a torque T . Introduce the right-handed reference system (x, y, z) , with z coincident with the bar axis and let Ω represent the simply connected two-dimensional domain representative of its cross section, referred to in the (x, y) plane, bounded by the closed curve Γ_0 . On Γ_0 , let $\mathbf{n}$ and $\mathbf{t}$ denote the outward normal and tangential vectors, respectively, such that the triad $(\mathbf{n}, \mathbf{t}, \mathbf{k})$, with $\mathbf{k}$ the unit vector of the z -axis, is right-handed.

Introducing the stress function $F(x, y)$, such that the shear stresses τ_{xz} and τ_{yz} are

$$\tau_{xz} = \frac{\partial F(x, y)}{\partial y} \quad \text{and} \quad \tau_{yz} = -\frac{\partial F(x, y)}{\partial x}, \quad (2.1)$$

the problem is governed [12,13] by the differential equation and boundary conditions

$$\Delta F(x, y) = -2G\Theta \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial F(x, y)}{\partial \mathbf{t}} = 0 \Rightarrow F(x, y) = c_0 \quad \text{on } \Gamma_0, \quad (2.2)$$

where $\Delta(\cdot) := \partial^2(\cdot)/\partial x^2 + \partial^2(\cdot)/\partial y^2$, G is the shear modulus of the material and Θ is the torsion angle per unit length. The constant c_0 , having no effects on the stress state, can be set equal to zero.

Prandtl's membrane analogy is based on the fact that $F(x, y)$ satisfies the same differential problem for the out-of-plane displacement of an inflated linear elastic membrane, with the same contour of the bar cross section. If s is the membrane tension, i.e. the uniform and equibiaxial in-plane

Figure 1. Scheme of (a) a multi-material cross section, with both open and closed bi-material interfaces and (b) a multiply connected cross section. (Online version in colour.)

stress multiplied by the thickness, and p the applied pressure, under the hypothesis of small deformation the displacement $w(x, y)$ shall satisfy

$$\Delta w(x, y) = -\frac{p}{s} \quad \text{in } \Omega \quad \text{and} \quad w(x, y) = 0 \quad \text{on } \Gamma_0. \quad (2.3)$$

Equations (2.3) and (2.2) are formally identical in the respective functions by setting $p/s = 2G\Theta$. This means that the pressure plays the role of the torsion angle Θ , while the membrane tension is analogue to $1/(2G)$. The torsional stress is related with the slope of the membrane [12]: lines at constant $w(x, y)$ correspond to curves which are everywhere tangent to the shear stress vector in the cross section of the bar. The torque T and the torsional stiffness $k = T/\Theta$ are given by

$$T = 2 \int_{\Omega} F(x, y) \, dA \quad \text{and} \quad k = \frac{2}{\Theta} \int_{\Omega} F(x, y) \, dA. \quad (2.4)$$

In words, the torque corresponds to the double of the volume V contained between the deformed membrane and its projection on the (x, y) plane, while $k = 2V/p$.

(a) Extension to the torsion of multi-material bars

Consider now the case in which the cross section is composed by N different materials, with shear modulus G_i ($i = 1, \dots, N$), connected by smooth interfaces. With reference to figure 1a, let us denote by Ω_i the domain that hosts the i th material, by Γ_{i0} the intersection of the boundary of Ω_i with the external boundary Γ_0 , and by Γ_{ij} the interface between materials i and j ($i \neq j$). We make a distinction between ‘open’ bi-material interfaces, i.e. open curves whose extremal points lay on Γ_0 , and ‘closed’ interfaces, if they are closed curves inside the cross section. To emphasize this distinction, if Γ_{ij} is a closed interface it will be formally denoted by $\mathring{\Gamma}_{ij}$.

Introduce the stress function as in (2.1). Using the same arguments of the classical approach, for the i th material, one obtains

$$\Delta F_i(x, y) = -2G_i\Theta \quad \text{in } \Omega_i \quad \text{and} \quad F_i(x, y) = 0 \quad \text{on } \Gamma_{i0}. \quad (2.5)$$

On the generic interface Γ_{ij} define the normal and tangential unit vectors $\mathbf{n}$ and $\mathbf{t}$ such that $(\mathbf{n}, \mathbf{t}, \mathbf{k})$ is right-handed. Bi-material matching conditions [14] require the continuity of the normal

traction $\tau_{nz} = \partial F(x, y)/\partial t$ and the continuity of the strain $\gamma_{tz} = \tau_{tz}/G$ in tangential direction. In terms of stress function, one finds

$$\text{and} \quad \left. \begin{aligned} \frac{\partial F_i(x, y)}{\partial t} &= \frac{\partial F_j(x, y)}{\partial t} \Rightarrow F_i(x, y) = F_j(x, y) \quad \text{on } \Gamma_{ij} \\ \frac{1}{G_i} \frac{\partial F_i(x, y)}{\partial \mathbf{n}} &= \frac{1}{G_j} \frac{\partial F_j(x, y)}{\partial \mathbf{n}} \quad \text{on } \Gamma_{ij}. \end{aligned} \right\} \quad (2.6)$$

If the j th material domain Ω_j is enclosed by closed curve Γ_{ij}° , as for materials 2 and 4 in figure 1a, recalling (2.5), one obtains

$$\begin{aligned} \frac{1}{G_i} \oint_{\Gamma_{ij}^\circ} \frac{\partial F_i(x, y)}{\partial \mathbf{n}} dl &= \frac{1}{G_j} \oint_{\Gamma_{ij}^\circ} \frac{\partial F_j(x, y)}{\partial \mathbf{n}} dl \\ &= \frac{1}{G_j} \int_{\Omega_j} \Delta F_j(x, y) dA = -2A_j \Theta, \end{aligned} \quad (2.7)$$

where A_j is the area of Ω_j . Once the stress function has been determined, the torsional stiffness is again obtained through (2.4).

Consider now a membrane, defined by the same external contour Γ_0 , whose tension, supposed equibiaxial, is not uniform. In particular, the membrane portions corresponding to the domains Ω_i are subjected to the tension state $s_i = 1/(2G_i)$. The way in which such a particular tensioning could be obtained in practice will be discussed later on, in §2c. Denoting, again, with p the inflating pressure, and with $w_i(x, y)$ the out-of-plane displacement of the portion Ω_i , the governing equations read

$$\Delta w_i(x, y) = -\frac{p}{s_i} \quad \text{in } \Omega_i \quad \text{and} \quad w_i(x, y) = 0 \quad \text{on } \Gamma_{i0}, \quad (2.8)$$

with interface conditions

$$\text{and} \quad \left. \begin{aligned} w_i(x, y) &= w_j(x, y) \quad \text{on } \Gamma_{ij} \\ s_i \frac{\partial w_i(x, y)}{\partial \mathbf{n}} &= s_j \frac{\partial w_j(x, y)}{\partial \mathbf{n}} \quad \text{on } \Gamma_{ij}, \end{aligned} \right\} \quad (2.9)$$

and, if the contour Γ_{ij}° is closed, using (2.8),

$$s_i \oint_{\Gamma_{ij}^\circ} \frac{\partial w_i(x, y)}{\partial \mathbf{n}} dl = s_j \oint_{\Gamma_{ij}^\circ} \frac{\partial w_j(x, y)}{\partial \mathbf{n}} dl = -pA_j. \quad (2.10)$$

The first condition of (2.9) states the continuity of the membrane displacement at the interface, while the second one deserves consideration. Under the hypothesis of small out-of-plane displacements, the quantity $s_j(\partial w_j(x, y)/\partial \mathbf{n})$ represents the out-of-plane force per unit length that is exerted on the interface by the tension of the membrane portion Ω_j . Therefore, (2.9)₂ corresponds to the out-of-plane equilibrium condition for the interface between membrane portions Ω_i and Ω_j , as schematically represented in figure 2. From this observation, (2.10) is the condition of global equilibrium in the out-of-plane direction of the whole membrane portion Ω_j , delimited by the closed curve Γ_{ij}° .

Observe that the in-plane equilibrium of the interface cannot be satisfied except for the trivial case $s_i = s_j$. Hence, the membrane necessitates constraints in correspondence of the interface, which allow out-of-plane movements but refrain in-plane displacements, providing a constraint reaction that re-establishes in-plane equilibrium. How to construct constraints of this type will be discussed in §2c.

Once the stress function, or equivalently, the out-of-plane displacement of the membrane, has been determined, the torsional stiffness of the composite bar, accounting for the contribution of the different material domains, can be evaluated using (2.4)₂.

Figure 2. Out-of-plane forces at a point of the interface between membrane portions Ω_i and Ω_j . (Online version in colour.)

(b) Multiply connected cross sections

A cross section whose representative domain is not simply connected can be treated similarly, by assuming that the holes host a material with vanishing elastic modulus. For a homogeneous body made by material 1 with shear modulus G_1 , occupying the multiply connected domain Ω_1 shown in figure 1b, let again Γ_{10} denote the outer boundary of Ω_1 , and let Γ_{1j} ($j = 2, \dots, M + 1$) be the closed curves defining the M inner boundaries embracing the holes.

From (2.5), the stress function in the j th hole, where the shear modulus is null, shall satisfy $\Delta F_j(x, y) = 0$, together with (2.6)₂: this implies¹ that $F_j(x, y) = \text{const.} = c_j$. From (2.6)₁, the stress function $F_1(x, y)$ on Ω_1 assumes different constant values on each of the inner boundaries and the elastic problem is solved by

$$\Delta F_1(x, y) = -2G_1\Theta \quad \text{in } \Omega_1, \quad F_1(x, y) = 0 \quad \text{on } \Gamma_{10} \quad \text{and} \quad F_1(x, y) = c_j \quad \text{on } \Gamma_{1j}, \quad (2.11)$$

where, from (2.7), c_j is evaluated by requiring

$$\frac{1}{G_1} \oint_{\Gamma_{1j}} \frac{\partial F_1(x, y)}{\partial \mathbf{n}} d\mathbf{l} = -2A_j\Theta. \quad (2.12)$$

These conditions, here found as a limit case of a bi-material cross section for the vanishing shear modulus of one material, are identical to the classical solution for a multiply connected homogeneous cross section [12,13].

In the membrane analogy, the surfaces enclosed by Γ_{1j} are regions where the membrane is infinitely taut, i.e. $s_j \rightarrow \infty$. Here, $\Delta w_j(x, y) = 0$, with boundary condition derived from the second of (2.9). This implies that $w_j(x, y) = \text{const.} = c_j$ on Γ_{1j} , otherwise the interface would not be in equilibrium in the out-of-plane direction. This means that, in the regions occupied by the holes, the membrane can only undergo a rigid translation. On the deformable region Ω_1 of the membrane, the governing equations are

$$\left. \begin{aligned} \Delta w_1(x, y) &= -\frac{p}{s_1} \quad \text{in } \Omega_1, \\ w_1(x, y) &= 0 \quad \text{on } \Gamma_{10}, \quad w_1(x, y) = c_j \quad \text{on } \Gamma_{1j} \\ s_1 \oint_{\Gamma_{1j}} \frac{\partial w_1(x, y)}{\partial \mathbf{n}} d\mathbf{l} &= -pA_j. \end{aligned} \right\} \quad (2.13)$$

and

¹This result can be demonstrated [15,16] by considering that if a harmonic function, defined in a domain with sufficiently smooth boundary, is not constant, it achieves the maximum on the boundary; hence, its derivative in the direction normal to the boundary is strictly positive *as per* Hopf's lemma, but it can be shown that this is in contrast with (2.6)₂. Therefore, the function shall be constant.

Figure 3. Physical model of the membrane in the undeformed (non-inflated) configuration. Layout and sectional view. (Online version in colour.)

The latter condition, which allows to determine c_j , comes from (2.10) and, recalling the discussing of figure 2, corresponds to the out-of-plane equilibrium of those parts of the membrane occupying the domains associated with the holes.

Extending to this case the expression (2.4)₂, by considering that the stress function is defined also in the $j = 2, \dots, M + 1$ holes of the cross section where $F_j(x, y) = c_j$, the torsional stiffness can be evaluated as

$$k = \frac{2}{\Theta} \left[\int_{\Omega_1} F_1(x, y) \, dA + \sum_{j=2}^{M+1} c_j A_j \right], \quad (2.14)$$

which agrees with the classical result [13]. Analogously, the torsional stiffness is proportional to the volume enclosed between the inflated membrane, including its flat parts corresponding to the holes, and its undistorted configuration.

Of course, this construction can be readily extended to the case of multi-material cross sections with holes.

(c) The physical model for the multi-tensioned membrane

The multi-tensioned membrane described in §2a can be physically constructed in the following way. To illustrate, consider the simplest case of a circular cross section composed by material 2, with shear modulus G_2 , occupying an inner circle represented by the domain Ω_2 , surrounded by material 1, with shear modulus G_1 , in the concentric annular domain Ω_1 (this case is analytically treated in §3a). The corresponding membrane is tensioned at s_2 in the circular core, and at s_1 in the surrounding annular region.

The physical model, in its undeformed (not inflated) configuration, is shown in figure 3. The membrane is stretched over a cylindrical vessel, drawn transparent in the representation of the figure, by lug casing and tension rods like a drum head. The central portion, corresponding to the domain Ω_2 , should be tensioned differently from the surrounding annulus Ω_1 .

As discussed in §2a, along the interface $\dot{\Gamma}_{12}$ it is necessary to design a line constraint that prevents the in-plane displacement in the direction normal to the interface itself and allows the out-of-plane movements. This can be approximated by a sequence of pointwise constraints, obtained with vertical rods, supposed to be fixed at the base and sufficiently rigid to prevent their bending, which pass through plastic or metallic eyelets in the membrane, placed along the interface at a small pitch. The eyelets are supposed to allow the frictionless sliding of the passing-through rods, thus permitting (constraining) the out-of-plane (in-plane) movement of the membrane at the corresponding points. Since the membrane needs to be inflated by pumping

Figure 4. Inflated membrane when $s_2 > s_1$, where the tensioning is obtained by an outward radial movement of the base of the vertical rods. Layout and sectional view. (Online version in colour.)

Figure 5. Inflated membrane when $s_1 > s_2$, where the tensioning is obtained by an inward radial movement of the base of the vertical rods. Layout and sectional view. (Online version in colour.)

air inside the vessel through the inlet tube, special gaskets in the eyelets are supposed to render them sufficiently air-tight.

By displacing the base of the vertical rods, and operating on the tension rods at the outer border, the membrane can be differently tensioned in the regions of interest. In particular, one can design a sequence of movements that tunes the membrane in such a way that it is uniformly tensioned at s_1 in Ω_1 and at s_2 in Ω_2 . To apply with sufficient precision the required displacements, the vertical rods can be fixed to the metallic base of the vessel with electromagnetic devices, so that there is complete freedom from what concerns their placement. If the membrane is moderately inflated, so that the slope of the deformed surface may be considered an infinitesimal quantity of the first order, in-plane displacements can be considered infinitesimal quantities of the second order with respect to out-of-plane displacements and the stress state in the membrane will not vary to the first order.

For the simple configuration here considered, to obtain $s_2 > s_1$ the rods shall be moved radially outwards, as schematically represented in figure 4. Of course, the membrane should be made of a sufficiently stiff material, so that the tensioning can be obtained at the price of moderate displacements of the rods. When the vessel is pressurized by pumping air through the inlet tube, the deformation of the membrane will be of the type represented in the figure. The curvature in the tauter central part will be less than in the looser surrounding annulus.

The complementary case, when $s_2 < s_1$, can be obtained with a inward radial movement of the base of the vertical rods, as shown in figure 5. Consequently, the surface corresponding to the inflated membrane will have higher curvature in the inner circle than in the surrounding annulus.

Of course, there are other technical solutions that could allow to stress the membrane according to the required programme. For the particular case with circular symmetry here illustrated, it could be possible to reinforce the interface with a rigid ring, so that the resulting portions could be stressed in a different way, but this represents a very special condition. More in general, a possibility could be to overstress some parts of the membrane with tendons, connecting points on the interface to the borders of the vessel through rollers that allow the movement in a vertical direction. In this way the tendons can remain horizontal during the deformation of the membrane, otherwise they would transmit to the interface a non-null vertical component that would spoil the static state of the membrane with respect to the sought analogy.

In any case, the physical model only represents a way to practically visualize the static state of the membrane, permitting to intuitively guess its deformed shape under the action of the inflating pressure. The constraint conditions at the interface of the membrane can be easily implemented in commercial numerical codes, which have in their library a particular type of finite element that can reproduce the mechanical response of a equibiaxially stressed membrane. The deformation of membranes composed by differently tensioned portions can be readily calculated, by constructing the corresponding two-dimensional mesh and using elements with different settings. The numerical analysis is straightforward and the output provides the deformed surface of the membrane, which can be used to calculate the stress state in the twisted bar according to the proposed analogy.

3. Examples

A few examples can illustrate how the analogy can be used in cases of practical interest. Closed-form analytical solutions are found for circular and elliptical multi-material cross sections, while more complex cases are solved numerically.

(a) Multi-material cross section with circular symmetry

The case of a multi-material cross section with mechanical and geometrical circular symmetry is the simplest and can be easily solved analytically. For a bi-material cross section, consider the domain shown in figure 6a, formed by the union of the circle Ω_2 of radius R_{21} with boundary Γ_{21}° , hosting a material with shear modulus G_2 , surrounded by the annulus Ω_1 with external boundary Γ_{10} of radius R_{10} , formed with a material with shear modulus G_1 .

Introduce a cylindrical reference system (r, θ, z) , due to the symmetry of the problem, the stress function depends only on r , i.e. $F_i(r, \theta) = F_i(r)$, $i = 1, 2$. Equations (2.5) and (2.6) may be written as

$$\left. \begin{aligned} \Delta F_1(r) &= -2G_1\Theta \quad \text{for } r \in [R_{21}, R_{10}], & \Delta F_2(r) &= -2G_2\Theta \quad \text{for } r \in [0, R_{21}], \\ F_1(r) &= 0 \quad \text{for } r = R_{10}, & F_1(r) &= F_2(r) \quad \text{for } r = R_{21} \\ \text{and} \quad \frac{1}{G_1} \frac{dF_1(r)}{dr} &= \frac{1}{G_2} \frac{dF_2(r)}{dr} \quad \text{for } r = R_{21}, \end{aligned} \right\} \quad (3.1)$$

where, in polar coordinates, $\Delta(\cdot) = (1/r)(d \cdot / dr) + (d^2 \cdot / dr^2)$. By requiring that the stress function is bounded for $r \rightarrow 0$, the solution takes the form

$$F_1(r) = G_1\Theta \frac{R_{10}^2 - r^2}{2} \quad \text{and} \quad F_2(r) = G_1\Theta \frac{R_{10}^2 - R_{21}^2}{2} + G_2\Theta \frac{R_{21}^2 - r^2}{2}, \quad (3.2)$$

which verifies (2.7) with $A_2 = \pi R_{21}^2$. The tangential shear stress $\tau_{\theta z}(r)$ is obtained by differentiation of the stress function. From (2.4), the torsional stiffness reads

$$k = \frac{\pi}{2} [G_1(R_{10}^4 - R_{21}^4) + G_2R_{21}^4]. \quad (3.3)$$

In the limit case $G_2 \rightarrow 0$, $F_2(r)$ is constant, and the classical solution for hollow circular section is obtained.

Figure 6. Domains with circular symmetry corresponding to (a) bi-material and (b) multi-material cross sections. (Online version in colour.)

Figure 7. Plot of the membrane out-of-plane displacement $w(x, y)$, coinciding with the stress function, associated with a bi-material cross section with circular symmetry (case $R_{10} = 10$ mm, $R_{21} = 5$ mm, $s_1 = 0.5$ N mm $^{-1}$, $p = 0.1$ MPa and various s_2). (Online version in colour.)

The corresponding membrane is tensioned at $s_1 = 1/(2G_1)$ in Ω_1 and at $s_2 = 1/(2G_2)$ in Ω_2 . The out-of-plane displacement $w(x, y)$, which takes the same form (3.2) of the stress function $F(x, y)$, is plotted in figure 7 for the case $R_{10} = 10$ mm and $R_{21} = 5$ mm, with $s_1 = 0.5$ N mm $^{-1}$ and $p = 0.1$ MPa (corresponding to $G_1 = 1$ MPa and $\Theta = 0.1$ rad mm $^{-1}$), and various values of s_2 (corresponding to different values of G_2).

It is evident that the applied tension strongly affects the deformation, which is with null curvature in the central portion when $s_2 \rightarrow \infty$ ($G_2 \rightarrow 0$).

A similar analysis can be repeated for cross sections with circular symmetry composed by an arbitrary number of annular domains, as shown in figure 6b. Figure 8 plots the stress

Figure 8. Plot of the membrane out-of-plane displacement $w(x, y)$, coinciding with the stress function, for a tri-material cross section with circular symmetry, for $s_1 = 0.5 \text{ N mm}^{-1}$, $p = 0.1 \text{ MPa}$ and different values of s_2 and s_3 . (Online version in colour.)

function, or equivalently the deformed shape of the multi-tensioned membrane, for three different materials, in the case that $R_{10} = 10 \text{ mm}$, $R_{21} = 7 \text{ mm}$, $R_{32} = 4 \text{ mm}$, $s_1 = 0.5 \text{ N mm}^{-1}$, $p = 0.1 \text{ MPa}$ (corresponding to $G_1 = 1 \text{ MPa}$ and $\Theta = 0.1 \text{ rad mm}^{-1}$) and different values of s_2 and s_3 (equivalently, G_2 and G_3).

(b) Elliptic bi-material cross section

A solution in closed form can also be obtained for an elliptical cross section defined by the countour Γ_{10} of equation $x^2/a^2 + y^2/b^2 = 1$, containing a central core Ω_2 defined by the curve Γ_{21} of equation $x^2/a^2 + y^2/b^2 = \zeta^2$, with $\zeta \in (0, 1)$, hosting a material with shear modulus G_2 , and the complementary part Ω_1 , where the material has shear modulus G_1 .

The stress function, or equivalently the out-of-plane displacement of the membrane, has to satisfy equations (2.5) and (2.6). The solution is of the form

$$\left. \begin{aligned} F_1(x, y) &= -G_1 \Theta \frac{a^2 b^2}{a^2 + b^2} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) \\ \text{and} \quad F_2(x, y) &= \Theta \frac{a^2 b^2}{a^2 + b^2} \left[G_1 (\zeta^2 - 1) - G_2 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - \zeta^2 \right) \right] \end{aligned} \right\} \quad (3.4)$$

When $G_2 \rightarrow 0$, it converges to the classical solution for a hollow elliptical section [12]).

The response is qualitatively similar to that obtained for multi-material sections with circular symmetry.

(c) Laminated rectangular cross sections

Laminated structures are manufactured in multiple layers and widely used in civil, aeronautical, aerospace, automotive and naval constructions. They are often composed by two external plies of the same material, coupled by a softer interlayer. The determination of their structural capacity in terms of bending [9,17] and torsion [18–20], is attracting the attention of several researchers.

Consider, as a representative example, the rectangular cross section of width b shown in figure 9, composed by two external layers represented by the rectangular domains Ω_1 and Ω_3 with thickness h_1 and h_3 , respectively, hosting the same material ($G_1 = G_3$), sandwiching an inner layer of thickness h_2 and shear modulus G_2 , referred to as the domain Ω_2 .

The solution of the governing equations (2.5) is usually obtained in approximate form for narrow rectangular section ($b/h \rightarrow \infty$), or found by separation of variables [12] developing the Prandtl stress function $F(x, y)$ into series [3]. The libraries of most commercial numerical codes do not contain a particular finite element that considers the torsion problem *à la* de Saint-Venant for cases of this type. Consequently, the numerical approach followed by most engineers is to consider a full three-dimensional model, where each material is covered by brick elements.

Figure 9. Bi-material laminated rectangular cross section, with indication of the reference domains and interfaces. (Online version in colour.)

Here, we have considered the case of a bar with $h_1 = 30$ mm, $h_2 = 10$ mm, $h_3 = 50$ mm, $b = 200$ mm and 1 m long, whose cross section is of the type represented in figure 9, constructing a three-dimensional model with the software ABAQUS [21]. A particular care needs to be paid at the bar ends, in order to avoid unrealistic stress concentration. For the case at hand, a relative rigid rotation of 0.1 rad of the extreme cross sections has been prescribed, not constraining their axial displacement (warping of the cross section is permitted). The body has been discretized with a structured mesh, with quadratic solid elements (20 nodes) with reduced integration (C3D20R in the code library). Both materials have been modelled as linear elastic, and linear geometric analyses have been run. In general, models of this type are quite time-consuming in their implementation.

Passing to the analogy, the membrane model has been again implemented in ABAQUS, by using linear quadrilateral membrane elements with reduced integration (referred to as M3D4R in the code library). Null displacements have been prescribed on the external boundary, while at the inner (open) interfaces Γ_{21} and Γ_{32} of figure 9, only the in-plane displacements are prevented. According to [22], tractions are applied in each portion of the membrane as an uniform equibiaxial prestress, using the ‘*INITIAL CONDITIONS, TYPE=STRESS’ option. This scheme numerically implements the constraints applied to the membrane in the physical model of figure 4. To account for the geometric stiffness induced by the prestress, nonlinear calculation procedures are used with iterative equation solver. Considered values for the membrane prestress are $s_1 = 0.3$ N mm $^{-1}$, corresponding to $G_1 = 1.667$ MPa, and s_2 varying in the range $0.3 \div 30$ N mm $^{-1}$. The applied pressure is $p = 10^{-4}$ MPa, associated with the assumed torsion angle per unit length equal to 10^{-4} rad mm $^{-1}$. It has been verified that such a choice does not substantially alter the in-plane tension during the deformation.

Figure 10 shows the numerically evaluated deformed shape of the multi-tensioned membrane, for different values of the traction s_2 . This allows a synoptic evaluation of the effects of the interlayer. When $G_2 = G_1$ the cross section is indeed homogenous, and the deformation of the membrane is that expected in the case of uniform equibiaxial tension (figure 10a). When $G_2 < G_1$, the strip Ω_2 is more tensioned than the remaining parts ($s_2 > s_1$) and therefore results to be the stiffest part (figure 10b). Indeed, the central strip refrains the out-of-plane deformation and when $s_2 \gg s_1$ ($G_2 \ll G_1$), in practice the out-of-plane displacement of the strip is negligible (figure 10c). In this case, the deformation of the membrane is equal to that of two disjoint membranes, corresponding to the domain Ω_1 and Ω_3 . This means the cross section behaves as if it was composed by two uncoupled layers made of material 1, whereas the interlayer plays no role apart from that of impeding the complete separation of the layers.

Comparison with the results from the full three-dimensional model can be made in terms of components of shear stress. In the analogy, these are defined at each point by the slopes in the direction of the reference axes of the deflected membrane, which have been easily calculated after exporting the out-of-plane displacement field in Matlab. Under the hypothesis of small deformation, the slopes are defined by the first partial derivatives of the out-of-plane

Figure 10. Numerically evaluated deformed shape of the multitensioned membrane for (a) $s_2 = s_1$ ($G_2 = G_1$), (b) $s_2 = 10s_1$ ($G_2 = 10^{-1}G_1$), (c) $s_2 = 100s_1$ ($G_2 = 10^{-2}G_1$). (Online version in colour.)

Figure 11. Qualitative comparison between the shear stress components evaluated with the three-dimensional numerical model, and the slope of the multi-tensioned membrane. Case $G_2 = 10 G_1$. (Online version in colour.)

displacement field. Figure 11 shows, for the case $G_2 = 10 G_1$, the comparison between $\partial w(x, y)/\partial x$ and $\partial w(x, y)/\partial y$ and the shear stress components τ_{xz} and τ_{yz} . Recall from §2 that, in the membrane analogy, $\tau_{xz} = \partial F(x, y)/\partial y$ and $\tau_{yz} = -\partial F(x, y)/\partial x$.

It has been verified that, by using the same mesh geometry for the bar cross section and for the membrane, the numerical solution of the three-dimensional torsional problem requires

Figure 12. Comparison between the maximum shear stresses evaluated either with the three-dimensional numerical model or the membrane analogy, for different values of G_2/G_1 . (a) External plies with material 1 and (b) internal ply with material 2. (Online version in colour.)

Figure 13. Cross section formed by four materials, with both open and closed interfaces. Quotes expressed in millimetre. (Online version in colour.)

a computational time about 25 times higher than that of the membrane analysis. The solution of the membrane problem does not entail any convergence problem, but, just to correctly capture the maximum values of the membrane slope, it is convenient to refine the mesh in the regions where the maximum shear stresses are expected. For the case at hand, the mesh has been thus refined in proximity of the boundaries of each material domain, by using elements with 0.5 mm width.

Figure 12 compares, for different values of G_2/G_1 , the maximum shear stresses τ_{xz} and τ_{yz} in the cross section, evaluated either from the numerical three-dimensional model or the membrane analogy. In particular, figure 12a refers to the external domain $\Omega_1 \cup \Omega_3$ with material 1, while figure 12b to the inner domain Ω_2 with material 2.

Figure 14. Deformation of the inflated membrane corresponding to a four-material cross section for (a) $s_1 = s_2 = s_3 = s_4 = 0.3 \text{ N mm}^{-1}$; (b) $s_1 = s_2 = s_4 = 0.3 \text{ N mm}^{-1}$, $s_3 = 0.9 \text{ N mm}^{-1}$; (c) $s_1 = s_3 = 0.9 \text{ N mm}^{-1}$, $s_2 = s_4 = 0.3 \text{ N mm}^{-1}$; (d) $s_1 = s_3 = 0.3 \text{ N mm}^{-1}$, $s_2 = 0.6 \text{ N mm}^{-1}$, $s_4 = 0.9 \text{ N mm}^{-1}$. Frontal and axonometric views. (Online version in colour.)

The numerical experiment confirms that the proposed analogy represents an efficient and accurate method to estimate the stress state in composite bars under torsion. The approach is computationally fast since it requires a simple two-dimensional modelling.

(d) Generic four-material section

The analogy can be used for composite cross sections of any type. As a further example, consider a multi-material cross section, composed by four different materials with both closed and open interfaces, of the type represented in figure 13.

The problem is again solved numerically for a cylinder 1 m long using a three-dimensional model implemented in Abaqus. The simulation uses the same elements and the same type of boundary conditions considered for the problem in §3c. Due to the complex shape and the high numbers of materials, the FEM analysis is quite time consuming. On the other hand, the FEM study of the equivalent membrane structure is straightforward.

The membrane is modelled in the same way of the corresponding case in §3c, using an unstructured mesh, with elements of size approximately 5 mm in order to accurately calculate the membrane slope, formed by the same FEM membrane elements and using similar-in-type constraints at the external borders and internal interfaces. For different values the uniform equibiaxial prestress s_i in the various portions Ω_i , the membrane deformation under an applied pressure $p = 10^{-4}$ MPa is represented in figure 14. The state of stress may be obtained by exporting the data corresponding to the out-of-plane displacements in Matlab and calculating the surface slope in the desired directions.

This example confirms the applicability of the proposed analogy in numerically solving very complex cross sections.

4. Discussion and conclusions

The classical Prandtl's membrane analogy for the torsion problem *à la* de Saint-Venant has been extended to bars with multi-material cross section. The equations governing the shear stress distribution in the bar are identical to those describing the deformation of an inflated membrane, differently tensioned in the various regions corresponding to the regions of the bar cross section hosting different materials. Such equibiaxial, but non-uniform, tensile state can be obtained by supposing that the interfaces between the aforementioned portions are constrained in such a way that out-of-plane displacements are permitted, but in-plane displacements are not. A scheme for a physical model that fulfills this condition has been proposed, but the membrane deformation can be readily calculated using a commercial FEM code.

For complex shapes and/or a large number of constituent materials, the alternative to evaluate the torsional response is a three-dimensional FEM analysis of a prismatic bar with the same cross section, but this requires particular attention for the mesh preparation, the definition of the end-constraints and the characterization of the interfaces. Using the membrane analogy, one can consider a simple two-dimensional mesh to evaluate the membrane deformation. Once this is known, the stress distribution in the different material domains and the torsional stiffness can be readily calculated. Reference to different working examples has demonstrated the capability of the proposed analogy to furnish both an intuitive and a quantitative representation of the state of stress.

We believe that the method here proposed may represent a powerful tool to engineers, accustomed to use commercial FEM codes, for the characterization of the torsional response of composite and layered structures, an important example of which is certainly represented by laminated structural glass.

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