An inversion of the conical Radon transform arising in the Compton camera with helical movement

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Abstract

Since the Compton camera was first introduced, various types of conical Radon transforms have been examined. Here, we derive the inversion formula for the conical Radon transform, where the cone of integration moves along a curve in three-dimensional space such as a helix. Along this three-dimensional curve, a detailed inversion formula for helical movement will be treated for Compton imaging in this paper. The inversion formula includes Hilbert transform and Radon transform. For the inversion of Compton imaging with helical movement, it is necessary to invert Hilbert transform with respect to the inner product between the vertex and the central axis of the cone of the Compton camera. However, the inner product function is not monotone. Thus, we should replace the Hilbert transform by the Riemann–Stieltjes integral over a certain monotone function related with the inner product function. We represent the Riemann–Stieltjes integral as a conventional Riemann integral over a countable union of disjoint intervals, whose end points can be computed using the Newton method. For the inversion of Radon transform, three dimensional filtered backprojection is used. For the numerical implementation, we analytically compute the Hilbert transform and Radon transform of the characteristic function of finite balls. Numerical test is given, when the density function is given by a characteristic function of a ball or three overlapping balls.

Keywords Conical Radon transform · Compton camera · Hilbert transform · Radon transform · Riemann–Stieltjes integral

1 Introduction

Ever since the Compton camera, also known as an electronically collimated γ -camera, was first introduced for use in Single Photon Emission Computed Tomography (SPECT), a Radon-type transform that assigns a function to its surface integral over various sets of cones has attracted substantial attention [2, 3, 5, 8, 9, 12, 13, 16, 17, 20, 23, 27, 29–31]. For the generalized Radon transform including toric Radon transform, see [22].

A typical Compton camera consists of two detectors: a scattering detector and an absorption detector. A photon emitted in the direction of the camera first undergoes Compton scattering in the scattering detector, then is absorbed in the absorption detector. In each detector, the position and energy of the photon are recorded [24].

Next, the recorded energies are used to determine the scattering angle ψ at the position of the scattering detector via the Compton scattering formula (see [1]). This information allows us to know that the observed photon must have been emitted on the surface of a cone with the vertex at a position $\mathbf{u} \in \mathbb{R}^3$ of the scattering detector, the central axis $\mathbf{e}_\beta$ pointing from the absorption detector to the scattering detector, and the scattering angle ψ (see Fig. 1). One of the mathematical problems related to the Compton camera is to reconstruct a function f from its conical Radon transform

$$\begin{aligned} Cf(\mathbf{u}, \beta, \psi) \\ = \sin \psi \int_{S^2} \int_0^\infty f(\mathbf{u} + r\mathbf{e}_\alpha) r \delta(\mathbf{e}_\alpha \cdot \mathbf{e}_\beta - \cos \psi) dr dS(\mathbf{e}_\alpha), \end{aligned} \quad (1)$$

where $\beta = (\beta_1, \beta_2) \in [0, 2\pi) \times [0, \pi]$, $\mathbf{e}_\beta = (\cos \beta_1 \sin \beta_2, \sin \beta_1 \sin \beta_2, \cos \beta_2) \in S^2$, dS is the standard measure on the unit sphere, and δ is the 1-dimensional Dirac delta function.

Various types of conical Radon transforms have been previously studied in [2–6, 8, 9, 12, 13, 15–17, 21, 23, 25, 28,

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Fig. 1 Schematic representation of a standard Compton camera ([9])

[32]. Some of these works [3, 6, 8, 16, 21, 32] were devoted to the conical Radon transform with planar vertex positions and a fixed central axis orthogonal to this plane (i.e., $\mathbf{u} = (u_1, u_2, 0) \in \mathbb{R}^2 \times \{0\}$ and $\mathbf{e}_\beta = (0, 0, 1)$). The inversion of the conical Radon transform with a fixed scattering angle, a fixed central axis, and arbitrary vertex location in the whole space, was derived in [4, 5]. In [9], Jung and Moon provided the inversion formula for the n -dimensional conical Radon transform with a 1-dimensional vertex space, a $(n-2)$ -dimensional central axis space, and a 1-dimensional scattering angle space. Terzioğlu derived several inversion formulas and also obtained the relation between the regular Radon transform and the conical Radon transform with the vertices on the whole space in [28]. In a sequential work [13], the authors developed new analytical reconstruction techniques that are valid for an arbitrary geometry of detectors satisfying an admissibility condition.

In this paper, we consider the following helical movement Compton imaging ($\mathbf{u} = \gamma(u)$) as in Fig. 2:

$$Cf(u, \beta, \psi) = \sin \psi \int_{S^2} \int_0^\infty f(\gamma(u)) + r \mathbf{e}_\alpha r \delta(\mathbf{e}_\alpha \cdot \mathbf{e}_\beta - \cos \psi) dr dS(\mathbf{e}_\alpha), \quad (2)$$

where $\gamma : (a, b) \rightarrow \mathbb{R}^3$ satisfies $\{k_\beta(u) = \gamma(u) \cdot \mathbf{e}_\beta : u \in (a, b)\} = \mathbb{R}$ for almost all $\mathbf{e}_\beta \in S^2$. $(\cos u, \sin u, u)$ is an example of $\gamma(u)$.

Since $\mathbf{u} = \gamma(u)$ is one-dimensional, less data is required for Compton imaging when the vertices of the cones moves along a curve $\gamma(u)$, as compared to planar geometry. $\gamma(u)$ is a curve contained in three dimensional space, which can be a line $\gamma(u) = \mathbf{a} + u\mathbf{b}$ where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, or a helix $\gamma(u) = (\cos u, \sin u, ku)$, $k \in \mathbb{R}$. If we use line geometry, the part parallel to the line of vertices appears blurred. In order to obtain the conic Radon transform data of the object's hidden backside, care must be taken not to cut off the line of vertices too short, which requires a large data set (see [8, 9]).

A helical geometry is expected to involve continuous data acquisition throughout the volume of interest by simultaneously moving the object through the gantry while the camera rotates like a spiral, which provides rapid imaging with high resolution [10, 11, 33]. Therefore, a helical geometry is the most promising setup for Compton detector.

Section 2 is devoted to the inversion formula of Theorem 1 and a detailed description of the inversion formula in Theorem 2, and based on these theorems, the numerical algorithm and its numerical test for $\gamma(u) = (\cos u, \sin u, u)$ are given in Sect. 3.

2 Inversion formula

In this section, we derive an inversion formula from Cf to f for general $\gamma(u)$ and describe in detail the inversion formula for a helix geometry $\gamma(u) = (\cos u, \sin u, u)$.

The inversion formula for the conical Radon transform with vertices on a curve $\gamma(u)$ is derived in Theorem 1. It is based on Eq. (5) such that a weighted integration of the conical Radon transform with respect to the scattering angle ψ is equal to the Radon transform of f followed by Hilbert

Fig. 2 A helical Compton camera: **a** the detectors of the Compton camera and **b** the helical movement of the Compton camera. Note that, since the detectors are small compared to the object of interest, the small detectors are considered to be the center of a disk

transform with respect to $(\beta, k_\beta(u))$, where $k_\beta(u) = \gamma(u) \cdot e_\beta$ is the norm of the projection of $\gamma(u)$ to the direction e_β . In order to invert the Hilbert transform, the projection norm $k_\beta(u)$ should be one to one. However, generally, this is not the case. Therefore, by introducing a monotone function g_β related to k_β , we can make the inversion formula containing Riemann–Stieltjes integral with respect to this monotone function g_β . Since we are interested in helical geometry $\gamma(u) = (\cos u, \sin u, u)$, in order to derive a numerical method for the inversion formula, the Riemann–Stieltjes integral with respect to g_β is replaced by the Riemann integral on a countable disjoint union of finite intervals when k_β is not one to one. The equation for the starting and ending points of the intervals is explained in Theorem 2 when $\gamma(u) = (\cos u, \sin u, u)$, which will be computed by the Newton method. For the inversion of Radon transform, three-dimensional filtered backprojection method [14] will be used.

$\mathbf{R}f$ is the Radon transform defined by

$$\mathbf{R}f(\beta, s) = \int_{e_\beta^\perp} f(s\mathbf{e}_\beta + \tau) d\tau \quad \text{for } (\mathbf{e}_\beta, s) \in S^2 \times \mathbb{R},$$

where $e_\beta^\perp = \{v \in \mathbb{R}^3 | v \cdot e_\beta = 0\}$. Hf is the Hilbert transform defined by

$$Hf(\tau) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(s)}{\tau - s} ds,$$

where the integral is defined using Cauchy principal value. It is known that $HHf = -f$.

2.1 The derivation of inversion formula

Let $\gamma : (a, b) \rightarrow \mathbb{R}^3$ be a sufficiently smooth function satisfying $k_\beta(a, b) = \mathbb{R}$. Let us define a monotonic function g_β related to k_β as follows:

- the smallest nondecreasing function larger than $k_\beta(u)$, if $\lim_{u \rightarrow a} k_\beta(u) = -\infty$ and $\lim_{u \rightarrow b} k_\beta(u) = \infty$,
- the largest nonincreasing function smaller than $k_\beta(u)$, if $\lim_{u \rightarrow a} k_\beta(u) = \infty$ and $\lim_{u \rightarrow b} k_\beta(u) = -\infty$.

The existence and uniqueness of a monotonic function g_β for a given function k_β is guaranteed by the least upper bound property for real numbers, and its graphical representation is given in Fig. 3, when $\gamma(u) = (\cos u, \sin u, u)$. The range of g_β is the same as k_β and equals the real line itself.

Let us define I_β as

$$I_\beta = \{u \in \mathbb{R} | g_\beta(u) = k_\beta(u)\}.$$

We can infer that g_β is continuous and differentiable almost everywhere, and

- $g_\beta = k_\beta$, $g'_\beta = k'_\beta$ on I_β ,
- $g'_\beta = 0$, elsewhere.

We will describe I_β in Theorem 2 in detail when $\gamma(u) = (\cos u, \sin u, u)$, which is the real line itself or a countable union of disjoint intervals. In a different sense, I_β could be interpreted as the region where g_β is strictly monotonic.

Theorem 1 Let $f \in C^\infty(\mathbb{R}^3)$ have compact support in the cylinder $B_1^2(0) \times \mathbb{R} = \{\mathbf{x} = (x_1, x_2, x_3) : |(x_1, x_2)| < 1\}$. Then we have

$$f(\mathbf{x}) = -\frac{1}{8\pi^4} \int_{S^2} \int_{\mathbb{R}} \int_0^\pi Cf(u, \beta, \psi) \frac{d\psi dg_\beta(u) dS(\mathbf{e}_\beta)}{\cos \psi (\mathbf{x} \cdot \mathbf{e}_\beta - g_\beta(u))^3}, \quad (3)$$

where the second Riemann–Stieltjes integral with respect to $dg_\beta(u)$ is defined using the Cauchy principal value.

Proof We have

$$\begin{aligned} & \int_0^\pi Cf(u, \beta, \psi) \frac{d\psi}{\cos \psi} \\ &= \int_0^\pi \sin \psi \int_{S^2} \int_0^\infty f(\gamma(u) + r\mathbf{e}_\alpha) r \delta(\mathbf{e}_\alpha \cdot \mathbf{e}_\beta - \cos \psi) \\ & \quad \frac{dr dS(\mathbf{e}_\alpha) d\psi}{\cos \psi} \\ &= \int_{S^2} \int_0^\pi f(\gamma(u) + r\mathbf{e}_\alpha) \frac{r^2 dr dS(\mathbf{e}_\alpha)}{r\mathbf{e}_\alpha \cdot \mathbf{e}_\beta} \\ &= \int_{\mathbb{R}^3} f(\gamma(u) + \mathbf{x}) \frac{d\mathbf{x}}{\mathbf{x} \cdot \mathbf{e}_\beta} \\ &= \int_{\mathbb{R}^3} \frac{f(\mathbf{x}) d\mathbf{x}}{\mathbf{x} \cdot \mathbf{e}_\beta - \gamma(u) \cdot \mathbf{e}_\beta} = \int_{\mathbb{R}} \frac{\mathbf{R}f(\beta, s) ds}{s - k_\beta(u)}, \end{aligned} \quad (4)$$

where in the second equation of the second line, we changed the variables $r\mathbf{e}_\alpha \rightarrow \mathbf{x}$. We recognized the last expression in (4) as the Hilbert transform of the Radon transform:

$$\int_0^\pi Cf(u, \beta, \psi) \frac{d\psi}{\cos \psi} = -\pi H\mathbf{R}f(\beta, k_\beta(u)). \quad (5)$$

The similar idea is shown in [25], but not explicitly given as in the form of (5). Note that (5) is changed into the following equation using $g_\beta(u)$

Fig. 3 The comparison of $k_\beta(u)$ (blue line) and $g_\beta(u)$ (red line) when $\gamma(u) = (\cos u, \sin u, u)$ and $\beta_1 = 0.05\pi$ and **a** $\beta_2 = 0.15\pi$, **b** $\beta_2 = 0.4\pi$, **c** $\beta_2 = 0.6\pi$, and **d** $\beta_2 = 0.85\pi$, respectively. In the cases $\beta_2 = 0.15\pi$

and 0.85π , $k_\beta(u)$ and $g_\beta(u)$ coincide since $k_\beta(u)$ is itself monotonic. (Color figure online)

$$\int_0^\pi Cf(u, \beta, \psi) \frac{d\psi}{\cos \psi} = -\pi H\mathbf{R}f(\beta, g_\beta(u)) \text{ on } I_\beta. \quad (6)$$

Recovering $\mathbf{R}f$ from the right hand side of (5) and then applying the inversion formula for the regular Radon transform gives the inversion formula for the conical Radon transform Cf . Using $H(Hh) = -h$, we have

$$\mathbf{R}f(\beta, s) = \frac{1}{\pi} \int_{\mathbb{R}} H\mathbf{R}f(\beta, \tilde{s}) \frac{d\tilde{s}}{s - \tilde{s}}. \quad (7)$$

The right hand side of (5) cannot be used to replace $H\mathbf{R}f(\beta, \tilde{s})$ in (7), since k_β is not monotonic. Instead of k_β , we use the monotone function g_β by changing the variables $\tilde{s} = g_\beta(u)$. Since $g_\beta(u)$ is strictly monotonic on I_β except $\beta_2 = \frac{\pi}{2}$, by inserting (6) into (7) and using $g'_\beta(u) = 0$ on $\mathbb{R} \setminus I_\beta$,

we and obtain the following Riemann–Stieltjes integral with respect to g_β :

$$\begin{aligned} \mathbf{R}f(\beta, s) &= \frac{1}{\pi^2} \int_{\mathbb{R}} \int_0^\pi Cf(u, \beta, \psi) \frac{d\psi dg_\beta(u)}{\cos \psi (s - g_\beta(u))}, \\ &= \frac{1}{\pi^2} \int_{I_\beta} \int_0^\pi Cf(u, \beta, \psi) \frac{d\psi}{\cos \psi} \frac{k'_\beta(u) du}{s - k_\beta(u)}, \quad (8) \\ &\text{when } \beta_2 \neq \frac{\pi}{2}. \end{aligned}$$

Now applying the following well-known inversion formula for the regular Radon transform (for example, see [7, 18, 19]) completes our proof:

$$f(\mathbf{x}) = -\frac{1}{8\pi^2} \int_{S^2} \partial_{ss} \mathbf{R}f(\beta, \mathbf{x} \cdot \mathbf{e}_\beta) dS(\mathbf{e}_\beta). \quad (9)$$

The condition $\beta_2 \neq \frac{\pi}{2}$ in (8) is negligible, since $\{\mathbf{e}_\beta : \beta_1 \in [0, 2\pi), \beta_2 = \pi/2\}$ is a measure zero set in S^2 . $\square$

When $\gamma(u)$ is a straight line such as $(1, 0, u)$, we have the following inversion formula.

Corollary 1 *Let $\gamma(u) = (1, 0, u)$ and $f \in C^\infty(\mathbb{R}^3)$ have compact support in the cylinder $B_1^2(0) \times \mathbb{R} = \{\mathbf{x} = (x_1, x_2, x_3) : |(x_1, x_2)| < 1\}$. Then we have*

$$\begin{aligned} f(\mathbf{x}) &= -\frac{1}{8\pi^4} \int_{S^2} \int_{\mathbb{R}} \int_0^\pi Cf(u, \beta, \psi) \\ &\quad \times \frac{|\cos \beta_2| d\psi du dS(\mathbf{e}_\beta)}{\cos \psi (\mathbf{x} \cdot \mathbf{e}_\beta - (\sin \beta_2 \cos \beta_1 + u \cos \beta_2))^3}, \quad (10) \end{aligned}$$

where the second integral with respect to u is defined using the Cauchy principal value.

2.2 The implementation of the inversion formula

In the rest of the paper, we will focus on the numerical implementation of (10). For simplicity, we will assume the helical movement $\gamma(u) = (\cos u, \sin u, u)$.

Through simple computation, we have

$$\begin{aligned} k_\beta(u) &= \cos(u - \beta_1) \sin \beta_2 + u \cos \beta_2 = \sin \beta_2 (\cos(u - \beta_1) + u \cot \beta_2), \\ k'_\beta(u) &= -\sin(u - \beta_1) \sin \beta_2 + \cos \beta_2 = -\sin \beta_2 (\sin(u - \beta_1) - \cot \beta_2), \\ k''_\beta(u) &= -\cos(u - \beta_1) \sin \beta_2. \end{aligned} \quad (11)$$

In the following theorems, we will compute I_β in detail and reformulate (10) using the computed I_β . The detailed relations among k_β , g_β , and I_β are given as follows:

Theorem 2 *Depending on the polar angle β_2 , we have the following relations among the function k_β , the monotonic function g_β , and the domain of integration I_β :*

- If $0 \leq \beta_2 \leq \frac{\pi}{4}$, then $g_\beta = k_\beta$, g_β is a nondecreasing function going to ∞ , and $I_\beta = \mathbb{R}$.
- If $\frac{\pi}{4} < \beta_2 < \frac{\pi}{2}$, then $g_\beta \geq k_\beta$, g_β is a nondecreasing function going to ∞ , and there are two numbers a_0 and b_0 such that $0 < b_0 - \beta_1 < \frac{\pi}{2}$, $0 < b_0 - a_0 < 2\pi$, and $I_\beta = \cup_{n=-\infty}^\infty (2n\pi + a_0, 2n\pi + b_0)$.
- If $\frac{\pi}{2} < \beta_2 < \frac{3\pi}{4}$, then $g_\beta \leq k_\beta$, g_β is a nonincreasing function going to $-\infty$, and there are two numbers a_0 and b_0 such that $-\frac{\pi}{2} < b_0 - \beta_1 < 0$, $0 < b_0 - a_0 < 2\pi$, and $I_\beta = \cup_{n=-\infty}^\infty (2n\pi + a_0, 2n\pi + b_0)$.
- If $\frac{3\pi}{4} \leq \beta_2 \leq \pi$, then $g_\beta = k_\beta$, g_β is a nonincreasing function going to $-\infty$, and $I_\beta = \mathbb{R}$.

Proof The cases for $0 \leq \beta_2 \leq \frac{\pi}{4}$ and $\frac{3\pi}{4} \leq \beta_2 \leq \pi$ can be easily obtained by considering the positivity and negativity of k'_β , respectively.

Let us consider the case $\frac{\pi}{4} < \beta_2 < \frac{\pi}{2}$. Since $\cos(\beta_2) > 0$, k_β goes to ∞ and g_β is a nondecreasing function going to ∞ . Let $b_n (n = \dots, -1, 0, 1, 2, \dots)$ be the increasing sequence of local maxima of k_β such that

$$k'_\beta(b_n) = 0, \quad k''_\beta(b_n) < 0. \quad (12)$$

Based on the fact that k'_β is 2π -periodic and (12), we can arrange

$$b_n = 2n\pi + b_0, \quad b_0 - \beta_1 \in \left(0, \frac{\pi}{2}\right), \quad \sin(b_0 - \beta_1) = \cot \beta_2. \quad (13)$$

By considering k'_β in (11), we can obtain that k_β is decreasing at $(2n\pi + b_0, 2n\pi + c_0)$ and increasing at $(2n\pi + c_0, 2n\pi + b_1)$, where $c_0 - \beta_1 = \pi - (b_0 - \beta_1)$ and $c_0 > b_0$. In addition, from the fact that

$$k_\beta(b_{n+1}) - k_\beta(b_n) = 2\pi \cos \beta_2, \quad (14)$$

$k_\beta(b_n)$ is an increasing sequence with respect to n . Therefore, there is a unique real number $a_{n+1} \in (2n\pi + c_0, 2n\pi + b_1) \subset (b_n, b_{n+1})$ such that

$$k_\beta(a_{n+1}) = k_\beta(b_n), \quad (15)$$

by the least upper bound property. If $k_\beta(a_{n+1}) = k_\beta(b_n)$, then $k_\beta(a_{n+1} + 2\pi) = k_\beta(b_n + 2\pi)$, which means

$$a_n = 2n\pi + a_0.$$

Therefore, we have

$$I_\beta = \cup_{n=-\infty}^{\infty} (a_n, b_n), \quad \dots < a_{-1} < b_{-1} < a_0 < b_0 < a_1 < b_1 < \dots,$$

where $0 < b_n - a_n < 2\pi$. The case for $\frac{\pi}{2} < \beta_2 < \frac{3\pi}{4}$ can be proven similarly. $\square$

From (13) and (15) with $n = 0$, we have the following integration for $\frac{\pi}{4} < \beta_2 < \frac{\pi}{2}$ and $\frac{\pi}{2} < \beta_2 < \frac{3\pi}{4}$:

$$\int_{I_\beta} h(u) du = \sum_{n=-\infty}^{\infty} \int_{2n\pi+a_0}^{2n\pi+b_0} h(u) du,$$

where

$$\begin{aligned} \sin(b_0 - \beta_1) &= \cot \beta_2 \\ \cos(a_0 - \beta_1) + a_0 \cot \beta_2 &= \cos(b_0 - \beta_1) + (b_0 - 2\pi) \cot \beta_2. \end{aligned} \quad (16)$$

a_0 and b_0 in (16) can be computed using the Newton method.

With the aid of (8) and Theorem 2, Theorem 1 can be represented as follows:

Corollary 2 Let $f \in C^\infty(\mathbb{R}^3)$ have compact support in the cylinder $B_1^2(0) \times \mathbb{R} = \{\mathbf{x} = (x_1, x_2, x_3) : |(x_1, x_2)| < 1\}$. Then we have

$$f(\mathbf{x}) = -\frac{1}{8\pi^4} \sum_{n=-\infty}^{\infty} \int_{S^2} \int_{2n\pi+a_0(\beta_2)}^{2n\pi+b_0(\beta_2)} \int_0^\pi \frac{Cf(u, \beta, \psi) k'_\beta(u)}{\cos \psi (\mathbf{x} \cdot \mathbf{e}_\beta - k_\beta(u))^3} d\psi du dS(\mathbf{e}_\beta), \quad (17)$$

where $a_0(\beta_2)$ and $b_0(\beta_2)$ satisfy (16).

Proof Inserting (8) into (10), we have

$$f(\mathbf{x}) = -\frac{1}{8\pi^4} \int_{S^2} \int_{I_\beta} \int_0^\pi \frac{Cf(u, \beta, \psi) k'_\beta(u)}{\cos \psi (\mathbf{x} \cdot \mathbf{e}_\beta - k_\beta(u))^3} d\psi du dS(\mathbf{e}_\beta), \quad (18)$$

Based on Theorem 2, we can compute $a_0(\beta_2)$ and $b_0(\beta_2)$ and obtain

$$I_\beta = \cup_{n=-\infty}^{\infty} (2n\pi + a_0(\beta_2), 2n\pi + b_0(\beta_2)). \quad (19)$$

By inserting (19) into (18), we have (17). $\square$

Note that $a_0(\beta_2) = 0, b_0(\beta_2) = 2\pi$ if $\beta_2 \in [0, \frac{\pi}{4}] \cup [\frac{3\pi}{4}, \pi]$.

3 Numerical test

The problem we consider in this paper is how to reconstruct the density $f(\mathbf{x}), \mathbf{x} \in \mathbb{R}^3$ from the conical Radon transform $Cf(u, \beta, \psi)$. Our inversion formula for a three-dimensional case can be decomposed into two steps as follows:

$$\begin{aligned} \text{1st Step} \quad Cf(u, \beta, \psi) &\longrightarrow \mathbf{HR}f(\beta, s) \\ \text{2nd Step} \quad \mathbf{HR}f(\beta, s) &\longrightarrow \mathbf{R}f(\beta, s) \longrightarrow f(\mathbf{x}). \end{aligned}$$

We do not treat the first step for the following reasons: First, we do not have experimental data for the conical Radon transform. Second, numerically simulating Cf from equation (2) can be difficult due to the treatment of the Dirac delta function in the equation. Third, the numerical stability of the singular integral $\mathbf{HR}f$ from the conical Radon transform Cf is not known for a discontinuous density function such as a characteristic function, which is considered in this section as a numerical phantom.

3.1 Numerical algorithm

Therefore, in this paper, we will concentrate on the second step from $\mathbf{HR}f$ to f for the numerical computation. We will analytically compute $\mathbf{HR}f$ for the density f , which is the characteristic function of the weighted union of finite balls:

Theorem 3 Let f be a finite sum of the characteristic functions of balls such as

$$f(\mathbf{x}) = \sum_{j=1}^K \chi_{B_{r_j}(\mathbf{c}_j)}(\mathbf{x}), \quad (20)$$

where $B_{r_j}(\mathbf{c}_j)$ is a ball with radius r_j centered at $\mathbf{c}_j \in \mathbb{R}^3$ and χ_A is a characteristic function of a set A . Then, we have

$$\mathbf{HRf}(\boldsymbol{\beta}, s) = \pi \sum_{j=1}^K \left[2r_j s_j + (s_j^2 - r_j^2) \log \left(\frac{|s_j - r_j|}{|s_j + r_j|} \right) \right], \quad (21)$$

where $s_j = s - \mathbf{e}_\beta \cdot \mathbf{c}_j$.

Proof Using the linearity, without loss of generality, it is sufficient to prove the theorem when

$$f(\mathbf{x}) = \chi_{B_{\mathbf{c}_1}}(\mathbf{x}).$$

The Radon transform of f is given by

$$\mathbf{Rf}(\boldsymbol{\beta}, s) = \pi(r_1^2 - s_1^2) \chi_{(-r_1, r_1)}(s_1), \quad (22)$$

where $s_1 = s - \mathbf{e}_\beta \cdot \mathbf{c}_1$. $\mathbf{HRf}(\boldsymbol{\beta}, s)$ is computed as follows:

$$\begin{aligned} \mathbf{HRf}(\boldsymbol{\beta}, s) &= \frac{1}{\pi} \int_{-1}^1 \frac{\mathbf{Rf}(\boldsymbol{\beta}, \tau)}{s - \tau} d\tau = - \int_{-r_1}^{r_1} \frac{r_1^2 - \tau_1^2}{\tau_1 - s_1} d\tau_1 \\ &= - \int_{-r_1-s_1}^{r_1-s_1} \frac{r_1^2 - (\eta + s_1)^2}{\eta} d\eta \\ &= \left[2r_1 s_1 + (s_1^2 - r_1^2) \log \left(\frac{|s_1 - r_1|}{|s_1 + r_1|} \right) \right], \end{aligned} \quad (23)$$

where in the second equality, we changed the variables $\tau - \mathbf{e}_\beta \cdot \mathbf{c}_1 \rightarrow \tau_1$ and all integrals are computed in terms of

Fig. 4 Relative RMS error for Case 1, when $z_1 = z_3 = 0, z_2 = 0.3, N_{\beta_1} = 50, 100, 200, 400, N_{\beta_2} = N_{\beta_1}/2, N_u = N_{\beta_1}, N_v = N_{\beta_1}, N_z = 100$: **a** three dimensional relative RMS error and sliced relative RMS error at $z = 0$ and **b** three dimensional error for filtered backprojection error

the Cauchy principal value. Therefore, we have proven the theorem. $\square$

Note that, using the limits $\lim_{s_1 \rightarrow r_1} (s_1 - r_1) \log |s_1 - r_1| = \lim_{s_1 \rightarrow -r_1} (s_1 + r_1) \log |s_1 + r_1| = 0$, we have $\mathbf{HRf}(\mathbf{e}_\beta, s_1 = \pm r_1) = \pm 2r_1^2$. Therefore, $\mathbf{HRf}$ in the Eq. (21) is a continuous function. In this paper, we only consider the numerical density given in (20).

The suggested numerical algorithm is as follows:

- $\mathbf{HRf}(\boldsymbol{\beta}, k_\beta(u)) := \int_0^\pi C f(u, \boldsymbol{\beta}, \psi) / \cos \psi d\psi$:
This data is computed in (21) for the density function given by (20).
- $\mathbf{Rf}(\boldsymbol{\beta}, s)$:
Compute $\mathbf{Rf}$ using (8), (16), and (19).
- $f(\mathbf{x})$:

Using Eq. (9), we reconstruct f from the Radon transform data $\mathbf{Rf}$. We used three-dimensional filtered backprojection given in [14]. The method is the same as a finite difference method for the second derivative of the Radon transform with respect to s followed by an integration with respect to $\mathbf{e}_\beta \in S^2$.

3.2 Numerical results

$f(\mathbf{x})$ will be reconstructed from $\mathbf{HRf}(\boldsymbol{\beta}, k_\beta(u))$, when $\gamma(u) = (\cos u, \sin u, u)$ and the numerical phantom density function f^p is given by the characteristic function of the following geometry:

from data $\mathbf{Rf}$ and the Hilbert transformation plus the filtered backprojection error from $\mathbf{HRf}$. Both data sets in (b) are computed as presented in Theorem 3

- **Case 1** A ball centered at $(0, 0, 0)$ with radius 0.3.
- **Case 2** Three balls centered at
 - $(0.2, 0, 0)$ with radius 0.2.
 - $(-0.3, 0, 0)$ with radius 0.3.
 - $(-0.3, 0, 0.2)$ with radius 0.2,

which is the special case given in (20).

Let the division numbers for $\beta_1, \beta_2, u, x, y$, and z be $N_{\beta_1}, N_{\beta_2}, N_u, N_x, N_y$, and N_z , respectively. MATLAB R2018b

Fig. 6 The reconstructed density f for Case 2, where the left figures are reconstructed images and the right figures are true phantom images: **a, b** 3-dimensional image **c, d** sliced image at $z = -0.18$, **e, f** sliced image at $z = 0$, and **g, h** sliced image at $z = 0.2$. $N_{\beta_1} = 400, N_{\beta_2} = 200, N_u = 400, N_x = N_y = N_z = 100$

is used for numerical testing. Let us define the following Root Mean Square(RMS) norms:

Fig. 5 The true and reconstructed images for Case 1 when $N_{\beta_1} = 400, N_{\beta_2} = 200, N_u = 400, N_x = N_y = N_z = 100$; **a** three dimensional image and **b** the sliced image at $z = 0$ for true phantom f^p , **c** three dimensional image and **d** the sliced image at $z = 0$ for the reconstructed density f

(a)

(b)

(c)

(d)

(e)

(f)

(g)

(h)

$$\|g\| = \left(\int_{[-1,1] \times [-1,1] \times [z_1, z_2] \cap B^3} |g(\mathbf{x})|^2 d\mathbf{x} \right)^{1/2} \quad \text{and}$$

$$\|g\|_0 = \left(\int_{\{(x_1, x_2): x_1^2 + x_2^2 \leq 1\}} |g(x_1, x_2, z_3)|^2 dx_1 dx_2 \right)^{1/2},$$

where z_1, z_2 , and z_3 are the real numbers of interest.

In order to compare the reconstructed function f from the real density f^p , we computed the following relative RMS (Root Mean Square) error norms (three dimensional and sliced error norms):

$$\frac{\|f - f^p\|}{\|f^p\|} \quad \text{and} \quad \frac{\|f - f^p\|_0}{\|f^p\|_0}.$$

From Fig. 4a, we can observe that the three dimensional and sliced relative RMS errors decrease as N_u increases, and the three-dimensional error decays faster than the sliced error at $z = 0$. This indicates that the suggested algorithm converges as discretization progresses. As shown in Fig. 4b, three-dimensional relative RMS errors are compared for a filtered backprojection error from data $\mathbf{R}f$ and the added Hilbert transform error from data $\mathbf{HR}f$. The data $\mathbf{HR}f$ and the Radon transform $\mathbf{R}f$ for the density function in (20) are given by (21) and (22), respectively. Therefore, the error of the Hilbert transform is approximated by the difference of these two errors, which is considered to be quite small as compared to the filtered backprojection error alone.

Figure 5 shows the reconstructed density f , which is fairly good. Note that this problem is an inverse problem, generally known as an ill-posed problem. Therefore, it is thought to be dependent on the noise. We did not consider any kinds of stabilization techniques, as they are beyond the scope of this paper, and will thus be considered in future research.

In Case 2, we considered a characteristic function of three overlapping balls as a density function as shown in Fig. 6. In the sliced images, two balls are separated in (c, d), contacted in (e, f), and overlapped in (g, h). Although there are some artifacts in the background, the reconstruction was fairly well done. The three dimensional relative RMS error is about 19%, when $z_1 = -0.3$ and $z_2 = 0.4$.

4 Conclusion

In this paper, we derived the inversion formula for the conical Radon transform, where the cone of integration moves along a curve in three-dimensional space such as a helix.

And a detailed inversion formula for helical movement was treated. For the numerical implementation, we used the filtered backprojection method and Newton method for computing the domain of integral related with the helical movement. Instead of conventional data Cf , we used analytically computed $\mathbf{HR}f$ data which is weighted integral of Cf with respect to opening angle ψ , because it is difficult to obtain experimental data. The experimental setup for this new model was not treated in detail, which is beyond the scope of this paper. For the experimental study about detector setup related with this paper, which is different from conventional two parallel plane detectors, see [26, 27].

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Compliance with ethical standards

Conflict of interest Kwon K. declares that he has no conflict of interest in relation to the work in this article.

Ethical approval This article does not contain any studies with human participants or animals performed by the author.

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