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Impact of the sensitivity factor on the signal-to-noise ratio in grating-based phase contrast imaging

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Abstract

The sensitivity factor of a grating-based x-ray differential phase contrast (DPC) imaging system determines how much fringe shift can be observed for a given refraction angle. It is commonly believed that increasing the sensitivity factor will improve the signal-to-noise ratio (SNR) of the phase signal. However, this may not always be the case if the intrinsic phase wrapping effect is taken into consideration. In this work, a theoretical derivation is provided to quantify relationship between the sensitivity and SNR for a given refraction angle, exposure level, and grating based x-ray DPC system. The theoretical derivation shows that the expected phase signal is not always proportional to the sensitivity factor and may even decrease when the sensitivity factor becomes too large. The noise variance of the signal is not always solely dependent on the exposure level and fringe visibility but may become signal-dependent under certain circumstances. As a result, SNR of the phase signal does not always increase with higher sensitivity. Numerical simulation studies were performed to validate the theoretical models. Results show that when the fringe visibility and exposure level are fixed, there exists an optimal sensitivity factor which maximizes the SNR for a given refraction angle; further increase of the sensitivity factor may decrease the SNR.

1. INTRODUCTION

In grating-based phase contrast imaging, the signal, which is the moiré fringe shift ϕ for a given refraction angle θ is given by

$$\phi = S\theta. \quad (1)$$

Here, S is the system-dependent sensitivity factor defined as $\frac{2\pi d}{p}$, where d is the propagation distance and p is the grating period. For a Talbot-Lau interferometer, d is the distance between the phase grating and the analyzer grating, p is the period of the analyzer grating. The noise of the extracted phase shift signal can be calculated as:

$$\sigma = \sqrt{\frac{2}{MN_0 e^2}}, \quad (2)$$

which is related to the total number of phase steps M , average number of photons in each phase step N_0 and the fringe visibility ϵ .¹⁻³ In the past several years, many efforts have been made to increase the sensitivity factor of the system by using higher Talbot orders (increasing d) or by using gratings with submicron period (to decrease p).⁴ As can be seen from Eqs. (1-2), increasing S will always improve the signal-to-noise ratio (SNR) of the differential phase contrast images.

However, this conclusion is based on the assumptions that the fringe shift ϕ is always linearly proportional to the sensitivity factor S and the measured image noise is always independent of the signal. Due to the cyclic nature of the signal and the quantum nature of the x-ray photons, these assumptions may no longer be valid. As a result, the SNR of the phase contrast image is signal-dependent. In other words, there may exist an optimal sensitivity factor for a given signal level that maximizes the SNR. The purpose of this work was to analyze how the SNR depends on both the sensitivity and phase contrast signal level. From the obtained results, it will be shown that increasing sensitivity of the grating interferometer does NOT always lead to an increased SNR and thus there is an optimal sensitivity for a given experimentally accessible phase contrast signal level. Numerical simulation studies were performed to verify the theoretical results.

2. METHODS

2.1 Theoretical analysis of the SNR in grating-based phase contrast imaging

For the widely used phase stepping procedure, assume that the number of photons detected in each phase step is denoted as $N^{(k)}$, $k = 1, 2, 3, \dots, M$, where M is the number of phase steps. In the absence of noise, $N^{(k)}$ is related to the fringe visibility ϵ and the average number of photons in each phase step N_0 is as follows

$$N^{(k)} = N_0 \left[1 + \epsilon \cos\left(\frac{2\pi k}{M} + \phi\right) \right], \quad (3)$$

where ϕ is the phase shift of the moiré fringe induced by the object. For a single measurement of $N^{(k)}$, the signal is always contaminated by noise, which is denoted as

$$\hat{N}^{(k)} = N^{(k)} + \text{noise}. \quad (4)$$

From one set of measured data at each phase step, the widely-used phase retrieval equation was applied to estimate the phase signal

$$\hat{\phi} = \text{Arg} \left(\sum_{k=1}^M \hat{N}^{(k)} \cos \frac{2\pi k}{M} - i \sum_{k=1}^M \hat{N}^{(k)} \sin \frac{2\pi k}{M} \right), \quad (5)$$

where Arg takes the principle value of the argument of a complex number. To calculate the signal-to-noise ratio, repeated experiments are needed to get an ensemble of $\hat{\phi}$. Assume that

the experiment is repeated L times. The expected signal value is given as and the standard deviation of the signal (noise) is given as

$$E(\hat{\phi}) = \lim_{L \rightarrow \infty} \frac{\sum_{l=1}^L \hat{\phi}_l}{L}, \quad (6)$$

and the standard deviation of the signal (noise) is given as

$$\sigma(\hat{\phi}) = \sqrt{\lim_{L \rightarrow \infty} \frac{\sum_{l=1}^L [\hat{\phi}_l - E(\hat{\phi})]^2}{L-1}}. \quad (7)$$

In previous studies, it has been discovered that the estimated $\hat{\phi}$ follows a wrapped normal distribution as a result of the function $\text{Arg}(x)$.^{5,6} The wrapped normal distribution is given as

$$f(\hat{\phi}; S) = \sum_{k=-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(\hat{\phi} - S\theta + 2k\pi)^2}{2\sigma^2}\right], \hat{\phi} \in [-\pi, \pi), \quad (8)$$

where σ is the standard deviation of ϕ before the phase wrapping procedure given in Eq. (2). When σ is relatively small ($\sigma < 1$), it was found that the mean of the wrapped normal distribution can be approximated as

$$E(\hat{\phi}; S) = \int_{-\pi}^{\pi} \hat{\phi} f(\hat{\phi}) d\hat{\phi} \approx S\theta + \pi [\text{erf}(A_-) - \text{erf}(A_+)], \quad (9)$$

where the variable $A_{\pm}$ is defined as

$$A_{\pm} = \frac{\pi \pm S\theta}{\sqrt{2}\sigma}. \quad (10)$$

The additional item $\pi [\text{erf}(A_-) - \text{erf}(A_+)]$ is referred to as the signal bias and due to its existence, $E(\hat{\phi})$ is no longer proportional to S . The variance of the wrapped normal distribution can be approximated as

$$\begin{aligned}
 V(\hat{\phi}; S) &= \int_{-\pi}^{\pi} (\hat{\phi} - E(\hat{\phi}))^2 f(\hat{\phi}) d\hat{\phi} \\
 &\approx \sigma^2 - 2\sqrt{2\pi}\sigma \left(e^{-A_+^2} + e^{-A_-^2} \right) + 2\pi^2 [2 - \operatorname{erf}(A_-) - \operatorname{erf}(A_+)] - \pi^2 [\operatorname{erf}(A_-) - \operatorname{erf}(A_+)]^2.
 \end{aligned}
 \tag{11}$$

Detailed derivations are provided in the appendix. Despite the relatively complex form of $E(\hat{\phi})$ and $V(\hat{\phi})$, the first conclusion from Eq. (9) and Eq. (11) is that $E(\hat{\phi})$ differs from ϕ and $V(\hat{\phi})$ differs from σ^2 . The mean and variance of the wrapped normal distribution are no longer those of the original normal distribution. The signal to noise ratio is defined as

$$\text{SNR}(S) = \frac{E(\hat{\phi}; S)}{\sqrt{V(\hat{\phi}; S)}} \neq \frac{S\theta}{\sigma}, \quad (12)$$

which means that the signal-to-noise ratio may not be proportional to sensitivity.

2.2 Numerical simulation method

Theoretical derivation will be verified through numerical simulation studies and the dependence of SNR on the sensitivity factor S will be analyzed. Six phase steps were used ($M=6$) and the visibility (ϵ) was set to be 0.2 to represent a typical grating-based phase contrast system. The refraction angle (θ) was set to be $10^{-6}\pi$. The sensitivity factor (S) varied from 2×10^4 to 98×10^4 , resulting a phase shift from 0.02π to 0.98π . The maximum phase shift was selected to be smaller than π . The mean number of photon detected in each pixel per phase step (N_0) was 80 or 240 to represent a low dose setting. The number of photons in each phase step were calculated based on the sinusoidal relationship in Eq. (3). Poisson noise was added to each phase step and the signal retrieval methods were performed based on Eq. (5). For a group of fixed parameters (M, ϵ, S, N_0), the Poisson noise addition and signal retrieval process were repeated so that an ensemble of $\hat{\phi}$ can be achieved. The distribution of $\hat{\phi}$ was statistically analyzed. The mean, variance and SNR of $\hat{\phi}$ were calculated.

Meanwhile, based on parameters (M, ϵ, S, N_0), the mean, variance and SNR of $\hat{\phi}$ can be theoretically calculated based on Eq. (9)–(12). The theoretically calculated values were compared with those from the simulation to validate the theoretical results.

3. RESULTS

The distributions of the estimated phase signal $\hat{\phi}$ when ($S=2 \times 10^5, N_0=80$), ($S=9 \times 10^5, N_0=80$) and ($S=9 \times 10^5, N_0=240$) are shown in Fig. 1. When N_0 is fixed, for a relatively small S , little of the original normal distribution fell outside the range $[-\pi, \pi]$, as shown in

Fig. 1 (a). The wrapped normal distribution has little difference from the original distribution. However, for relatively larger S , one can clearly see that distribution was cut abruptly at π without the continuous tail of the original normal distribution, as shown in Fig. 1 (b). Due to the phase wrapping effect, the missing part was wrapped to the left, resulting in the small tail around $-\pi$. For a fixed sensitivity factor, increasing N_0 will decrease the standard deviation of the normal distribution before being wrapped, so that a smaller portion of the normal distribution is wrapped to the other side, as seen in the comparison of Fig. 1 (b) and (c). The theoretically calculated probability density curves of the wrapped normal distribution are also shown in Fig. 1 as a comparison with the distribution from the simulated results.

As a result of the phase wrapping effect, the mean of the distribution is not necessarily the mean of the original normal distribution. Fig. 2 shows the relationship between $E(\hat{\phi})$ and S . When $\phi = S\theta$ is much smaller than π , the phase wrapping effect is negligible so that $E(\hat{\phi}) \approx \phi \propto S$. However, when S becomes very large and ϕ approaches π , the relationship between $E(\hat{\phi})$ and S is no longer linear. As an extreme case, $E(\hat{\phi})$ may decrease when S increases, as indicated in the right part in Fig. 2(a). In this case, the measured mean phase shift severely deviates from the ground truth $\phi = S\theta$, which may impact the signal to noise ratio, as will be analyzed later. Increasing the number of photons per phase step decreases the phase wrapping effect, so that the region where $E(\hat{\phi})$ is proportional to S becomes larger.

The relationship between the standard deviation of the phase shift and the sensitivity factor S is shown in Fig. 2(b). For relatively small S , the standard deviation is approximately a constant independent of S , as the phase wrapping effect did not contribute much to the distortion of the original normal distribution. For a higher sensitivity factor, however, the standard deviation increases when S increases. Increasing N_0 results in a larger region where $\sigma(\hat{\phi})$ is a constant independent of S .

As a result of the dependence of the mean and standard deviation of the $\hat{\phi}$ distribution on S , the relationship between SNR and S is shown in Fig. 2(c). The SNR is approximately proportional to the sensitivity factor for relatively small sensitivity factors, where the phase wrapping contributes less. For larger sensitivity factors, the dependence of SNR on S is non-linear. For a given set of experimental parameters (M , N_0 , ϵ) and image object (θ), there is an optimal sensitivity factor so that SNR can reach its maximum.

To further illustrate the impact of the sensitivity factor on the signal-to-noise ratio, a digital non-attenuating wedge with a constant diffraction angle $10^{-6}\pi$ was simulated as a phantom. The number of photons per phase step N_0 was chosen to be 80 which corresponds to the black curve in Fig. 2(c). S was chosen to be 4×10^5 , 6.4×10^5 (optimal sensitivity factor for such setting) and 9×10^5 . The results are shown in Fig. 3. As one can tell, a relatively low sensitivity factor ($S = 4 \times 10^5$) results in a low phase signal (Fig. 3(a)), and increasing the sensitivity factor results in a better SNR since the signal gets amplified (Fig. 3(b)). Continual increase of the sensitivity factor leads to a more severe phase wrapping effect, as shown in Fig. 3(c). Not only does the image present more severe signal bias, the noise also increases so the overall SNR decreases.

4. DISCUSSION AND CONCLUSION

As shown from both theoretical and numerical simulation results, SNR of the phase signal is not always proportional to the sensitivity factor. For a given system and a refraction angle, there is an optimal sensitivity factor that results in the highest SNR; beyond the optimal sensitivity, increasing the sensitivity factor will decrease the SNR.

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Appendix

The wrapped normal distribution is given as

$$f(x) = \sum_{k=-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu+2k\pi)^2}{2\sigma^2}}, x \in [-\pi, \pi), \quad (13)$$

where $\mu \in [-\pi, \pi)$ and σ^2 are the mean and variance of the original normal distribution, respectively. The mean of the wrapped normal distribution is calculated as

$$\begin{aligned} \left\langle x \right\rangle &= \int_{-\pi}^{\pi} x \sum_{k=-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu+2k\pi)^2}{2\sigma^2}} dx \\ &= \sum_{k=-\infty}^{+\infty} \int_{(2k-1)\pi-\mu}^{(2k+1)\pi-\mu} \frac{\xi + \mu - 2k\pi}{\sigma\sqrt{2\pi}} e^{-\frac{\xi^2}{2\sigma^2}} d\xi \quad (\xi = x - \mu + 2k\pi) \\ &= \mu - \pi \sum_{k=-\infty}^{+\infty} k \left[\operatorname{erf}\left(\frac{(2k+1)\pi - \mu}{\sqrt{2}\sigma}\right) - \operatorname{erf}\left(\frac{(2k-1)\pi - \mu}{\sqrt{2}\sigma}\right) \right], \end{aligned} \quad (14)$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad (15)$$

is the error function. By changing the dummy variable k and using the property $\operatorname{erf}(x) = -\operatorname{erf}(-x)$, equation (14) can be further simplified as

$$\begin{aligned} \langle x \rangle &= \mu + \pi \sum_{k=0}^{+\infty} \left[\operatorname{erf} \left(\frac{(2k+1)\pi - \mu}{\sqrt{2}\sigma} \right) - \operatorname{erf} \left(\frac{(2k+1)\pi + \mu}{\sqrt{2}\sigma} \right) \right] \quad (16) \\ &\approx \mu + \pi \left[\operatorname{erf} \left(\frac{\pi - \mu}{\sqrt{2}\sigma} \right) - \operatorname{erf} \left(\frac{\pi + \mu}{\sqrt{2}\sigma} \right) \right]. \end{aligned}$$

The approximation in equation (16) is based on the assumption that $\sigma < 1$. The variance of the wrapped normal distribution is given by

$$V(x) = \langle (x - \langle x \rangle)^2 \rangle = \langle (x - \mu)^2 \rangle - (\mu - \langle x \rangle)^2 \quad (17)$$

Mean of $(x - \mu)^2$ is given as

$$\langle (x - \mu)^2 \rangle = \int_{-\pi}^{\pi} (x - \mu)^2 f(x) dx = \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} \frac{(x - \mu)^2 \exp \left[-\frac{(x - \mu + 2k\pi)^2}{2\sigma^2} \right]}{\sigma\sqrt{2\pi}} dx \quad (18)$$

Assuming that $\xi = x - \mu + 2k\pi$, equation (18) becomes

$$\begin{aligned} \langle (x - \mu)^2 \rangle &= \sum_{k=-\infty}^{\infty} \int_{(2k-1)\pi - \mu}^{(2k+1)\pi - \mu} \frac{(\xi - 2k\pi)^2 \exp \left[-\frac{\xi^2}{2\sigma^2} \right]}{\sigma\sqrt{2\pi}} d\xi \quad (19) \\ &= \sum_{k=-\infty}^{\infty} \int_{(2k-1)\pi - \mu}^{(2k+1)\pi - \mu} \frac{\xi^2 \exp \left[-\frac{\xi^2}{2\sigma^2} \right]}{\sigma\sqrt{2\pi}} d\xi \\ &\quad + \sum_{k=-\infty}^{\infty} \int_{(2k-1)\pi - \mu}^{(2k+1)\pi - \mu} \frac{-4k\pi\xi \exp \left[-\frac{\xi^2}{2\sigma^2} \right]}{\sigma\sqrt{2\pi}} d\xi \\ &\quad + \sum_{k=-\infty}^{\infty} \int_{(2k-1)\pi - \mu}^{(2k+1)\pi - \mu} \frac{4k^2\pi^2 \exp \left[-\frac{\xi^2}{2\sigma^2} \right]}{\sigma\sqrt{2\pi}} d\xi \quad (20) \end{aligned}$$

As shown in equation (20), the variance can be split into three items. The first item is

$$\sum_{k=-\infty}^{\infty} \int_{(2k-1)\pi - \mu}^{(2k+1)\pi - \mu} \frac{\xi^2 \exp \left[-\frac{\xi^2}{2\sigma^2} \right]}{\sigma\sqrt{2\pi}} d\xi = \sigma^2 \quad (21)$$

The second item is

$$\sum_{k=-\infty}^{\infty} \int_{(2k-1)\pi-\mu}^{(2k+1)\pi-\mu} \frac{-4k\pi\xi \exp\left[-\frac{\xi^2}{2\sigma^2}\right]}{\sigma\sqrt{2\pi}} d\xi \approx -2\sigma\sqrt{2\pi} \left[e^{\frac{-(\pi+\mu)^2}{2\sigma^2}} + e^{\frac{-(\pi-\mu)^2}{2\sigma^2}} \right] \quad (22)$$

The approximation in equation (22) is based on the assumption that $\sigma < 1$. The third item is

$$\sum_{k=-\infty}^{\infty} \int_{(2k-1)\pi-\mu}^{(2k+1)\pi-\mu} \frac{4k^2\pi^2 \exp\left[-\frac{\xi^2}{2\sigma^2}\right]}{\sigma\sqrt{2\pi}} d\xi \quad (23)$$

$$\approx 2\pi^2 \left[\operatorname{erf}\left(\frac{-\pi-\mu}{\sqrt{2}\sigma}\right) - \operatorname{erf}\left(\frac{-3\pi-\mu}{\sqrt{2}\sigma}\right) + \operatorname{erf}\left(\frac{3\pi-\mu}{\sqrt{2}\sigma}\right) - \operatorname{erf}\left(\frac{\pi-\mu}{\sqrt{2}\sigma}\right) \right] \quad (24)$$

$$\approx 2\pi^2 \left[2 - \operatorname{erf}\left(\frac{\pi+\mu}{\sqrt{2}\sigma}\right) - \operatorname{erf}\left(\frac{\pi-\mu}{\sqrt{2}\sigma}\right) \right] \quad (25)$$

The approximation in equation (24)–(25) is also based on $\sigma < 1$. When the results from equations (16), (17), (22) and (25) are combined, the variance is given as

$$\begin{aligned} \langle (x - \langle x \rangle)^2 \rangle &= \langle (x - \mu)^2 \rangle - (\mu - \langle x \rangle)^2 \quad (26) \\ &\approx \sigma^2 - 2\sigma\sqrt{2\pi} \left[\exp\left[-\frac{(\pi+\mu)^2}{2\sigma^2}\right] + \exp\left[-\frac{(\pi-\mu)^2}{2\sigma^2}\right] \right] \\ &\quad + 2\pi^2 \left[2 - \operatorname{erf}\left(\frac{\pi+\mu}{\sqrt{2}\sigma}\right) - \operatorname{erf}\left(\frac{\pi-\mu}{\sqrt{2}\sigma}\right) \right] \\ &\quad - \pi^2 \left[\operatorname{erf}\left(\frac{\pi+\mu}{\sqrt{2}\sigma}\right) - \operatorname{erf}\left(\frac{\pi-\mu}{\sqrt{2}\sigma}\right) \right]^2 \quad (27) \end{aligned}$$

If we define $A_{\pm} = \frac{\pi \pm \mu}{\sqrt{2}\sigma}$, the mean and variance of the wrapped normal distribution will have a less complex form

$$\langle x \rangle \approx \mu + \pi [\operatorname{erf}(A_-) - \operatorname{erf}(A_+)] \quad (28)$$

$$V(x) \approx \sigma^2 - 2\sqrt{2\pi}\sigma \left(e^{-A_+^2} + e^{-A_-^2} \right) + 2\pi^2 [2 - \operatorname{erf}(A_-) - \operatorname{erf}(A_+)] \quad (29)$$

$$- \pi^2 [\operatorname{erf}(A_-) - \operatorname{erf}(A_+)]^2.$$

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Figure 1.

Distribution of $\hat{\phi}$. (a) $S = 2 \times 10^5$ ($\phi = 0.2\pi$), $N = 80$; (b) $S = 9 \times 10^5$ ($\phi = 0.9\pi$), $N_0 = 80$; (c) $S = 9 \times 10^5$ ($\phi = 0.9\pi$), $N_0 = 240$. The bar plot represents the statistical measurement of $\hat{\phi}$ from the numerical simulation. The solid curve represents the theoretical wrapped normal distribution shown in equation (8).

Figure 2.

Mean, standard deviation and SNR of $\hat{\phi}$ versus $\mathcal{S}$. The discrete data points represent measurements of $\hat{\phi}$ from the numerical simulation. The continuous curves represent the theoretical values of the properties of $\hat{\phi}$ from the wrapped normal distribution.

Figure 3.

Simulated images of a wedge phantom when the sensitivity factor S changes. (a) $S = 4 \times 10^5$ (b) $S = 6.4 \times 10^5$ (c) $S = 9.6 \times 10^5$. The display window for all images is $[-\pi, \pi]$. The left part of the image represents the air scan while the right part represents the phase signal of the wedge.