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General rogue wave solutions of the coupled Fokas–Lenells equations and non-recursive Darboux transformation

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We formulate a non-recursive Darboux transformation technique to obtain the general n th-order rational rogue wave solutions to the coupled Fokas–Lenells system, which is an integrable extension of the noted Manakov system, by considering both the double-root and triple-root situations of the spectral characteristic equation. Based on the explicit fundamental and second-order rogue wave solutions, we demonstrate several interesting rogue wave dynamics, among which are coexisting rogue waves and anomalous Peregrine solitons. Our solutions are generalized to include the complete background-field parameters and therefore helpful for future experimental study.

1. Introduction

The terminology rogue wave originally refers to an extreme oceanic wave event possessing tremendous destructive power, which is much greater in size than the surrounding surface waves and often appears unexpectedly from directions other than the prevailing wind [1]. It looks like a ‘wall of water’, very steep-sided followed by two unusually deep troughs [2]. As a monster of the deep, the rogue wave has existed as maritime folklore for centuries, and only in the past few decades was it identified by scientists as a real and

actually frequently occurring phenomenon [3,4]. Nowadays, with the availability of reliable lasers and real-time monitoring technologies, the optical counterparts of infamous oceanic rogue waves were observed as well [5,6], not only in many practical settings such as optical fibres [7,8], mode-locked lasers [9], microresonators [10] and photorefractive ferroelectrics [11], but also in evolving dynamics such as filamentation [12], caustics [13] and turbulence [14]. This boom in experimental observation, in turn, stimulates a strong theoretical study of rogue waves spanning different physical branches (fluid dynamics, optics, plasma physics, acoustics, etc.), with the aim to reveal their fundamental origins [15–18] or to understand their unique spatio-temporal dynamics [19–21].

It has now been generally accepted that the Peregrine soliton (sometimes called Peregrine breather or Peregrine rogue wave), first proposed in 1983 by Peregrine as a first-order rational solution to the nonlinear Schrödinger (NLS) equation [22], plays a central role in modelling the extreme wave events [23], thanks to its doubly localized wave-packet that matches well the unpredictable nature of rogue waves, i.e. appearing from nowhere and disappearing without a trace [24]. Actually, apart from the celebrated NLS equation, almost all integrable models, either scalar or vector, admit such kind of rational soliton solutions [19], including their dark [25] and chirped [26] cousins. Of course, there are a limited number of integrable fundamental models (e.g. the classic Korteweg–de Vries equation, the sine-Gordon equation) which do not feature, at the moment, Peregrine-type rogue wave solutions. Despite the diversity of integrable models, however, the Peregrine soliton solutions, if available, can be obtained analytically by standard methods such as Darboux transformation (DT) and Hirota bilinearization.

There is also burgeoning interest in search of the higher-order rogue wave solutions to integrable systems, with the help of the DT method, which could solve for the whole solution hierarchy in a straightforward manner [27–29]. In the past years, intriguing rogue wave dynamics of higher hierarchy, such as complementary rogue waves [30], bullet-like rogue waves [31] and dark three sisters [32], have been unveiled, among other complicated rogue waves patterns [33–35]. Meantime, significant efforts were made to realize these higher-order rogue waves in a laboratorial environment; for example, Chabchoub *et al.* observed successively the super hydrodynamical rogue waves [36–38] and their triplet states [39] using a water-wave tank set-up, and lately, Baronio *et al.* observed a group of dark optical rogue waves in a telecommunication optical fibre [40], to name a few. No doubt, all this experimental progress provides pertinent grounds for the quest of higher-order rogue wave solutions.

Recently, Chen *et al.* reported an interesting anomaly of Peregrine soliton creation in a complex coupled Fokas–Lenells (CFL) system [41]. This kind of Peregrine soliton has a peak amplitude more than three times the background level, and is shown to be able to be numerically excited from a chaotic wave field—a preliminary test prior to experimental examination. It is different from the ultrahigh hump formed by the deterministic collisions of multiple solitons [42,43] and also from the Peregrine soliton state occurring in a non-local nonlinear system, which involves a variable peak amplitude independent of any background parameters [44].

In this article, we formulate a non-recursive (or generalized) DT technique [35,45,46] to obtain the whole rogue wave solution hierarchy of the CFL equations, which has included the special fundamental Peregrine soliton solution presented in [41]. Our solutions have considered both the double-root and triple-root situations of the spectral characteristic equation, which can, as a starting point, give rise to two different forms of rogue wave solutions [47]. We show that these solutions can possess plenty of interesting dynamics such as coexisting rogue waves and anomalous Peregrine solitons. We note that there were several nice papers having contributed to finding the rogue wave solutions of the CFL equations, often expressed in terms of determinants [48,49]. However, in those works, only the double-root situation was taken into account and moreover less general plane-wave solutions were used as initial seeds, either of which inevitably leads to the loss of some intriguing rogue wave dynamics. We expect that our results presented here may help understand the rich anomalous rogue wave dynamics seen in the CFL systems or in other multicomponent nonlinear systems, for instance, microwave turbulence [50].

2. Coupled Fokas–Lenells model and non-recursive Darboux transformation

For our study, we write the CFL equations [51] (named for the pioneering works of Fokas [52] and Lenells [53]) as in a normalized form

$$\text{and} \quad \left. \begin{aligned} iD_\tau \frac{\partial u_1}{\partial \xi} + \frac{\sigma}{2} \frac{\partial^2 u_1}{\partial \tau^2} + (2|u_1|^2 + |u_2|^2)D_\tau u_1 + u_1 u_2^* D_\tau u_2 &= 0 \\ iD_\tau \frac{\partial u_2}{\partial \xi} + \frac{\sigma}{2} \frac{\partial^2 u_2}{\partial \tau^2} + (2|u_2|^2 + |u_1|^2)D_\tau u_2 + u_2 u_1^* D_\tau u_1 &= 0, \end{aligned} \right\} \quad (2.1)$$

where $u_{1,2}(\xi, \tau)$ are the complex envelopes of the two field components, and ξ and τ represent the distance and retarded time, respectively. The asterisk over the field variables stands for the complex conjugate. The parameter σ denotes the anomalous dispersion ($\sigma > 0$) or the normal dispersion ($\sigma < 0$). $D_\tau = 1 + i\epsilon(\partial/\partial\tau)$ is the differential operator with respect to τ and ϵ (> 0) scales the perturbation to the Manakov system. As an important generalization, the CFL system includes, besides the group-velocity dispersion and self- and cross-phase modulation terms that constitute the Manakov system, the effects of space–time coupling (through the presence of D_τ in the first term) [54] and self-steepening (through the last two terms) [55]. These additional terms arise as corrections to the slowly varying envelope approximation [53], when considering the propagation of few-cycle pulses, a situation which can be met in ultrafast optics as well as in hydrodynamics. Thus, the CFL system is relevant to model the propagation of ultrashort optical pulses in birefringent optical fibres [40,53] or the crossing sea states in the open ocean [3,15].

It is easy to prove that this set of integrable coupled equations is equivalent to the compatibility of the following Lax pair of the linear eigenvalue problem:

$$\mathbf{R}_\tau = \mathbf{U}\mathbf{R} \quad \text{and} \quad \mathbf{R}_\xi = \mathbf{V}\mathbf{R}, \quad (2.2)$$

where $\mathbf{R} = [r, s, w]^T$ (T means a matrix transpose),

$$\text{and} \quad \left. \begin{aligned} \mathbf{U} &= -\frac{i\sigma\lambda\sigma_3}{4\epsilon^3} + i\sqrt{\lambda}(\mathbf{K}\mathbf{A})_\tau \\ \mathbf{V} &= \frac{\sigma}{2\epsilon}\mathbf{U} + \left(\frac{\mathbf{K}\mathbf{A}}{\sqrt{\lambda}} + i\mathbf{K}^2\right)\sigma_3 + \frac{i\sigma_3}{2}\left(\frac{\sigma}{\epsilon^2} - \frac{1}{\lambda}\right), \end{aligned} \right\} \quad (2.3)$$

with λ being the free spectral parameter, $\sigma_3 = \text{diag}(1, -1, -1)$, $\mathbf{A} = \exp((i\tau/\epsilon)\sigma_3)$, and

$$\mathbf{K} = \begin{bmatrix} 0 & u_1 & u_2 \\ u_1^* & 0 & 0 \\ u_2^* & 0 & 0 \end{bmatrix}. \quad (2.4)$$

As usual, the subscripted variables in equations (2.2) and (2.3) indicate a partial derivative.

As in [32,35], we use a non-recursive DT scheme that can directly give the high-order rogue wave solutions without any iteration operation. Specifically, let u_{10} and u_{20} be the initial plane-wave solutions of equation (2.1), then the n th-order rogue wave solutions can be expressed as

$$\text{and} \quad \left. \begin{aligned} u_1^{[n]} &= u_{10} \left(1 - \frac{i}{|u_{10}|} \mathbf{Y}_1 \mathbf{M}^{-1} \mathbf{Y}_2^\dagger \right) \\ u_2^{[n]} &= u_{20} \left(1 - \frac{i}{|u_{20}|} \mathbf{Y}_1 \mathbf{M}^{-1} \mathbf{Y}_3^\dagger \right). \end{aligned} \right\} \quad (2.5)$$

Here the dagger ($\dagger$) indicates a complex-conjugate transpose and the plane-wave seeds u_{j0} ($j = 1, 2$) are defined through their respective amplitudes (a_j), wavenumbers (k_j), frequencies (ω_j):

$$u_{j0} = a_j \exp(ik_j \xi + i\omega_j \tau), \quad (2.6)$$

under the dispersion relations (let $v_j = \epsilon \omega_j - 1$ below)

$$\left. \begin{aligned} k_1 &= 2a_1^2 + \frac{a_2^2(v_1 + v_2)}{v_1} + \frac{\sigma(v_1 + 1)^2}{2\epsilon^2 v_1} \\ k_2 &= 2a_2^2 + \frac{a_1^2(v_1 + v_2)}{v_2} + \frac{\sigma(v_2 + 1)^2}{2\epsilon^2 v_2} \end{aligned} \right\} \quad (2.7)$$

and

$\mathbf{Y}_j$ ($j = 1, 2, 3$) are $1 \times n$ row vectors defined through

$$\begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \\ \mathbf{Y}_3 \end{bmatrix} = [\boldsymbol{\Phi}^{(0)}, \boldsymbol{\Phi}^{(1)}, \boldsymbol{\Phi}^{(2)}, \dots, \boldsymbol{\Phi}^{(n-1)}], \quad (2.8)$$

where $\boldsymbol{\Phi}^{(m)}$ are the series coefficients of the Taylor expansion of the factorized column vector $\boldsymbol{\Phi}(\lambda) = \mathbf{G}^{-1}\mathbf{R}(\lambda)$ about the given spectral parameter $\lambda = \lambda_0$, and $\mathbf{M}$ is an $n \times n$ matrix whose entry can be found via the Taylor expansion of $\boldsymbol{\Phi}^\dagger \mathbf{X} \boldsymbol{\Phi} / (\lambda - \lambda^*)$, with

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{u_{10}^*}{a_1} e^{i\tau/\epsilon} & 0 \\ 0 & 0 & \frac{u_{20}^*}{a_2} e^{i\tau/\epsilon} \end{bmatrix} \quad \text{and} \quad \mathbf{X} = \begin{bmatrix} \lambda\sqrt{\lambda^*} & 0 & 0 \\ 0 & \lambda^*\sqrt{\lambda} & 0 \\ 0 & 0 & \lambda^*\sqrt{\lambda} \end{bmatrix}. \quad (2.9)$$

The key point in the above procedures is to find the exact expression of $\mathbf{R}$ so that the Taylor expansion can be performed. As one can prove, by inserting the plane-wave seeds (2.6) into the linear eigenvalue problem (2.2), the eigenfunction $\mathbf{R}$, which is a function of λ , can be obtained as

$$\mathbf{R}(\lambda) = \mathbf{G}\boldsymbol{\Phi}(\lambda) \quad \text{and} \quad \boldsymbol{\Phi}(\lambda) = \Gamma_1 \mathbf{N}_1 + \Gamma_2 \mathbf{N}_2 + \Gamma_3 \mathbf{N}_3, \quad (2.10)$$

where Γ_j ($j = 1, 2, 3$) are arbitrary complex constants,

$$\mathbf{N}_j = \begin{bmatrix} 1 \\ \frac{4i\sqrt{\lambda}a_1 v_1}{4v_1 + \sigma\lambda/\epsilon^2 - 2\mu_j} \\ \frac{4i\sqrt{\lambda}a_2 v_2}{4v_2 + \sigma\lambda/\epsilon^2 - 2\mu_j} \end{bmatrix} \exp(i\varphi_j) \quad \text{and} \quad \varphi_j = \frac{\mu_j \tau}{2\epsilon} + \frac{v_j \xi}{4\epsilon^2}, \quad (2.11)$$

with

$$v_j = 4\epsilon^2(a_1^2 + a_2^2) + \frac{16\epsilon^4 a_2^2 v_2(1 - v_2/v_1)}{2\epsilon^2(\mu_j - 2v_2) - \sigma\lambda} + \frac{2\epsilon^2(\mu_j - v_1) + \sigma\lambda}{v_1 \lambda} + \sigma(\mu_j + 2) \equiv f(\lambda, \mu_j). \quad (2.12)$$

The μ_j in equations (2.11) and (2.12) are the three roots of the cubic spectral characteristic equation

$$\left(\mu - \frac{\sigma\lambda}{6\epsilon^2} - \frac{2\kappa}{3}\right)^3 - 3\varsigma \left(\mu - \frac{\sigma\lambda}{6\epsilon^2} - \frac{2\kappa}{3}\right) + 2\varrho = 0, \quad (2.13)$$

where

$$\left. \begin{aligned} \varsigma &= \frac{4\sigma\lambda A}{3\epsilon^2} + \frac{\delta^2}{3} + \left(\frac{\kappa}{3} + \frac{\sigma\lambda}{3\epsilon^2}\right)^2 \\ \varrho &= \frac{2\sigma^2\lambda^2 A}{3\epsilon^4} + \frac{\sigma\lambda(2\kappa A - 6\delta B - \delta^2)}{3\epsilon^2} - \frac{\delta^2\kappa}{3} + \left(\frac{\kappa}{3} + \frac{\sigma\lambda}{3\epsilon^2}\right)^3 \end{aligned} \right\} \quad (2.14)$$

and

For the sake of brevity, here and in what follows, we define

$$\delta = v_1 - v_2, \quad \kappa = v_1 + v_2, \quad A = \frac{\epsilon^2}{\sigma}(a_1^2 v_1^2 + a_2^2 v_2^2) \quad \text{and} \quad B = \frac{\epsilon^2}{\sigma}(a_1^2 v_1^2 - a_2^2 v_2^2). \quad (2.15)$$

It is known that the cubic equation (2.13) can be readily solved by the Cardano's formulae [56]. For our present purpose, we are only concerned with its multiple roots which are closely related

to the rational rogue wave solutions. It turns out that this cubic equation will have a double root, i.e. $\mu_1 = \mu_2 = \mu_0$, when its discriminant satisfies

$$\varsigma^3 - \varrho^2 = 0, \quad (2.16)$$

and, particularly, a triple root $\mu_1 = \mu_2 = \mu_3 = \mu_0$ when its coefficients fulfil $\varsigma = 0$ and $\varrho = 0$. Therefore, solving either the quartic equation (2.16) for the double-root case or simply the coefficient conditions $\varsigma = \varrho = 0$ for the triple-root case, one can obtain a specific value of λ , denoted by λ_0 , which should appear in complex conjugate pairs.

Once the eigenvector $\mathbf{R}(\lambda)$, hence its factorized form $\Phi(\lambda)$, is known, one can expand it with respect to the given spectral parameter λ_0 . For a 3×3 linear eigenvalue problem, there usually exist two ways of expansion, depending on the multiple root of the spectral characteristic equation (2.13). To be specific, for the double-root case, one can set $\lambda = \lambda_0 + (\lambda_0 - \lambda_0^*)\varepsilon^2$ (ε is a complex perturbation parameter) and expand $\Phi(\lambda)$ into the series in powers of ε^2 :

$$\Phi(\lambda) = \Phi^{(0)} + \Phi^{(1)}\varepsilon^2 + \Phi^{(2)}\varepsilon^4 + \dots + \mathcal{O}(\varepsilon^{2n}). \quad (2.17)$$

As such, the elements M_{ij} of the matrix $\mathbf{M}$ in equation (2.5) can also be determined through

$$\frac{\Phi^\dagger \mathbf{X} \Phi}{\lambda - \lambda^*} = \sum_{ij}^n M_{ij} \varepsilon^{*2(i-1)} \varepsilon^{2(j-1)} + \mathcal{O}(|\varepsilon|^{4n}). \quad (2.18)$$

However, for the triple-root case, one needs to set λ to be dependent on ε^3 , say, $\lambda = \lambda_0 + (\lambda_0 - \lambda_0^*)\varepsilon^3$, and then expand $\Phi(\lambda)$ as

$$\Phi(\lambda) = \Phi^{(0)} + \Phi^{(1)}\varepsilon^3 + \Phi^{(2)}\varepsilon^6 + \dots + \mathcal{O}(\varepsilon^{3n}), \quad (2.19)$$

which should now be in powers of ε^3 . Therefore, the entry of $\mathbf{M}$ can be determined by

$$\frac{\Phi^\dagger \mathbf{X} \Phi}{\lambda - \lambda^*} = \sum_{ij}^n M_{ij} \varepsilon^{*3(i-1)} \varepsilon^{3(j-1)} + \mathcal{O}(|\varepsilon|^{6n}). \quad (2.20)$$

In either case, the three row vectors $\mathbf{Y}_j$ are obtained with the help of equation (2.8) and the matrix $\mathbf{M}$ can be obtained from equation (2.18) or (2.20), resulting in the whole rogue-wave solution hierarchy (2.5) whose non-singularity will be automatically guaranteed.

We should point out that the DT method outlined above for non-singular n th-order rogue wave solutions depends on no more than a spectral parameter λ_0 , having an advantage over the conventional recursive DT process where new different spectral parameters need to be given at each iteration stage, in order to avoid solution singularity [27]. For this reason, it is therefore appropriate to call the current Darboux method a non-recursive DT. In the past years, the non-recursive DT was applied successfully to the scalar NLS equations [57] as well as Manakov systems [32,35,45,46]. Here we show that it can also apply to the more complex CFL systems, by defining a special matrix $\mathbf{M}$ as given by equation (2.18) or (2.20). Below, we only present the first-order (fundamental) and the second-order rogue wave solutions of equation (2.1), for both the double- and triple-root cases, following the above non-recursive DT formulation.

3. Rogue wave solutions: the double-root case

Let us first consider the double-root case of the cubic equation (2.13), which holds true provided that the specific complex value λ_0 satisfies the quartic equation (2.16), i.e.

$$(\lambda_0 + \mathbb{X})^4 + p_0(\lambda_0 + \mathbb{X})^2 + q_0(\lambda_0 + \mathbb{X}) + r_0 = 0, \quad (3.1)$$

where the parameters $\mathbb{X}$, p_0 , q_0 and r_0 are all real and given by

$$\left. \begin{aligned} \mathbb{X} &= \frac{\epsilon^2[2A^2(8A + \kappa) + \delta^2(5A + \kappa) + 3B\delta(6A + \kappa)]}{\sigma(4A^2 + 4B\delta + \delta^2)}, \\ p_0 &= \frac{2\epsilon^4[2A^2(12\delta^2 + \kappa^2) + 6B\delta(6A\kappa - 3\delta^2 + \kappa^2) + \delta^2(20A\kappa - 54B^2 - \delta^2 + 3\kappa^2)]}{\sigma^2(4A^2 + 4B\delta + \delta^2)} - 6\mathbb{X}^2, \\ q_0 &= -4\mathbb{X}^3 - 2p_0\mathbb{X} + \frac{4\delta\epsilon^6[\delta^2(3A\delta - 9B\kappa - \delta\kappa) + \kappa^2(5A\delta + B\kappa + \delta\kappa)]}{\sigma^3(4A^2 + 4B\delta + \delta^2)} \\ \text{and } r_0 &= -\mathbb{X}^4 - p_0\mathbb{X}^2 - q_0\mathbb{X} + \frac{\delta^2\epsilon^8(\delta^2 - \kappa^2)^2}{\sigma^4(4A^2 + 4B\delta + \delta^2)}. \end{aligned} \right\} \quad (3.2)$$

It is known that this real-coefficient quartic equation (3.1) admits at least one pair of complex conjugate roots if and only if the discriminant satisfies $\Delta = 4(p_0^2 + 12r_0)^3/27 - (2p_0^3 - 72p_0r_0 + 27q_0^2)^2/27 < 0$, or alternatively, the discriminant satisfies $\Delta \geq 0$, but meanwhile the coefficients fulfil $p_0 \geq 0$ or $r_0 - p_0^2/4 > 0$ [19]. These complex roots are usually very lengthy in form, and can be readily obtained using some common tools. For this reason, we do not present them here for brevity, but state that the above inequality formulae will constitute the parameter conditions for existence of rogue waves. Once the value of λ_0 is known, the double root μ_0 of the cubic equation (2.13) can be expressed in terms of λ_0 , namely, $\mu_0 = \mu^+$ or $\mu_0 = \mu^-$, where

$$\mu^\pm = \frac{2\kappa}{3} + \frac{\sigma\lambda_0}{6\epsilon^2} \pm \sqrt{\frac{\delta^2}{3} + \frac{4\sigma\lambda_0 A}{3\epsilon^2} + \frac{(\epsilon^2\kappa + \sigma\lambda_0)^2}{9\epsilon^4}}. \quad (3.3)$$

Now, for any perturbation that results in a deviation from the specific value λ_0 , one can express the spectral parameter λ as

$$\lambda = \lambda_0 + \chi\epsilon^2, \quad (3.4)$$

where $\chi = \lambda_0 - \lambda_0^*$ and ϵ is a complex perturbation parameter. Direct substitution of equation (3.4) into the cubic equation (2.13) yields (only the preceding four terms are retained)

$$\left. \begin{aligned} \mu_1 &= \mu_0 + \ell_0\epsilon + \ell_1\epsilon^2 + \ell_2\epsilon^3 + \dots \\ \text{and } \mu_2 &= \mu_0 - \ell_0\epsilon + \ell_1\epsilon^2 - \ell_2\epsilon^3 + \dots \end{aligned} \right\} \quad (3.5)$$

where μ_0 is the double-root obtained with $\lambda = \lambda_0$, and $\ell_{0,1,2}$ are the series coefficients given by

$$\left. \begin{aligned} \ell_0 &= \sqrt{\frac{4\sigma\chi(3\delta B + \delta^2 - \kappa A)}{9\epsilon^2\phi_0} + \frac{2\sigma\chi(2A + \Lambda_0 - \phi_0)}{3\epsilon^2}}, \\ \ell_1 &= \frac{\sigma\chi(4A + 2\Lambda_0 + \phi_0)}{6\epsilon^2\phi_0} - \frac{\ell_0^2}{6\phi_0} \\ \text{and } \ell_2 &= \frac{\sigma^2\chi^2(2A + \Lambda_0 - \phi_0)^2}{18\epsilon^4\ell_0\phi_0^2} - \frac{5\ell_0\ell_1}{12\phi_0} - \frac{\sigma\chi\ell_0(4A + 2\Lambda_0 - 5\phi_0)}{72\epsilon^2\phi_0^2}, \end{aligned} \right\} \quad (3.6)$$

with $\Lambda_0 = \kappa/3 + \sigma\lambda_0/3\epsilon^2$ and $\phi_0 = \mu_0 - 2\kappa/3 - \sigma\lambda_0/6\epsilon^2$. As can be verified, if we take

$$\Gamma_1 = \frac{1}{2} \sum_{i=1}^n \left(\gamma_{2i-1} + \frac{\gamma_{2i}}{\epsilon} \right) \epsilon^{2(i-1)}, \quad \Gamma_2 = \frac{1}{2} \sum_{i=1}^n \left(\gamma_{2i-1} - \frac{\gamma_{2i}}{\epsilon} \right) \epsilon^{2(i-1)} \quad \text{and} \quad \Gamma_3 = 0, \quad (3.7)$$

where γ_i ($i = 1, 2, \dots, 2n$) are arbitrary complex constants, followed by substituting equations (3.4), (3.5), (3.7) into equation (2.10), the obtained vector Φ can be exactly expanded into the Taylor series form (2.17), namely a series expressed in powers of ϵ^2 . As our current concern is with the rational solutions up to the second order ($n = 2$), only its preceding two series terms are provided

below:

$$\Phi^{(0)} = \begin{bmatrix} R_0 \\ R_1 \\ R_2 \end{bmatrix} e^{i\varphi_0} \quad \text{and} \quad \Phi^{(1)} = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \end{bmatrix} e^{i\varphi_0}, \quad (3.8)$$

where $\varphi_0 = \mu_0 \tau / 2\epsilon + f(\lambda_0, \mu_0) \xi / 4\epsilon^2$, with $f(\lambda_0, \mu_0)$ being defined by equation (2.12), and

$$\left. \begin{aligned} R_0 &= \gamma_1 + i\gamma_2 \ell_0 \vartheta, \quad R_j = \frac{2i\sqrt{\lambda_0} a_j v_j}{\alpha_j} (\gamma_1 + i\gamma_2 \ell_0 \theta_j), \\ S_0 &= \gamma_1 p + \gamma_2 q + \gamma_3 + i\gamma_4 \ell_0 \vartheta \\ \text{and} \quad S_j &= \frac{2i\sqrt{\lambda_0} a_j v_j}{\alpha_j} \left[\gamma_1 \left(p + \frac{i\ell_0^2 \theta_j}{\alpha_j} + \frac{\eta_j}{\alpha_j} \right) + \gamma_2 (q + g_j) + \gamma_3 + i\gamma_4 \ell_0 \theta_j \right]. \end{aligned} \right\} \quad (3.9)$$

The other parameters and functions in equations (3.9) are given by

$$\left. \begin{aligned} \vartheta &= \frac{\tau}{2\epsilon} + \left(\frac{\sigma}{4\epsilon^2} + \frac{1}{2v_1 \lambda_0} - \frac{2\delta v_2 a_2^2}{v_1 \alpha_2^2} \right) \xi, \quad \theta_j = \vartheta - \frac{i}{\alpha_j}, \\ \alpha_j &= \frac{\sigma \lambda_0}{2\epsilon^2} - \mu_0 + 2v_j, \quad \eta_j = \frac{\chi}{2} \left(\frac{\alpha_j}{\lambda_0} - \frac{\sigma}{\epsilon^2} \right) + \ell_1, \\ \beta &= \frac{\delta a_2^2 v_2 (2\epsilon^2 \ell_0^2 - \alpha_2 \sigma \chi)}{\epsilon^2 \alpha_2^3 v_1} + \frac{\chi (\mu_0 - v_1)}{2v_1 \lambda_0^2}, \\ \gamma &= \frac{\ell_0 \chi}{2v_1 \lambda_0^2} + \frac{2\ell_0 \delta a_2^2 v_2 (\epsilon^2 \ell_0^2 - \alpha_2 \sigma \chi)}{\epsilon^2 \alpha_2^4 v_1} + \frac{4\ell_0 \ell_1 \delta a_2^2 v_2}{\alpha_2^3 v_1}, \\ p &= -i\beta \xi - \frac{\ell_0^2 \vartheta^2}{2} + i\ell_1 \vartheta, \quad q = i\ell_0 \vartheta p - i\gamma \xi + i\ell_2 \vartheta + \frac{i\ell_0^3 \vartheta^3}{3} \\ \text{and} \quad g_j &= \frac{1}{\alpha_j} \left[-\frac{\ell_0^3 \theta_j^2}{2} + i\ell_0 \theta_j (\eta_j + \ell_1) - i\ell_0 \beta \xi + \ell_2 - \frac{\ell_0 \sigma \chi}{2\epsilon^2 \alpha_j} + \frac{\ell_0^3}{2\alpha_j^2} \right]. \end{aligned} \right\} \quad (3.10)$$

The subsequent operations are rather straightforward; substitution of equation (3.8) into equation (2.8) yields the expressions for $\mathbf{Y}_j$, substitution of equation (3.4) into the second equation (2.9) and then into equation (2.18) yields the expressions of $\mathbf{M}$, and lastly, substitution of all these expressions into equation (2.5) gives the resultant rogue wave solutions up to the second order. Making it explicit, the fundamental rogue wave solution of equation (2.1) can be written as

$$u_1^{[1]} = u_{10} \left(1 - \frac{i\chi R_0 R_1^*}{a_1 m_{11}} \right) \quad \text{and} \quad u_2^{[1]} = u_{20} \left(1 - \frac{i\chi R_0 R_2^*}{a_2 m_{11}} \right), \quad (3.11)$$

where

$$m_{11} = \lambda_0 \sqrt{\lambda_0^*} |R_0|^2 + \lambda_0^* \sqrt{\lambda_0} (|R_1|^2 + |R_2|^2). \quad (3.12)$$

Meanwhile, the second-order rogue wave solution reads

$$\left. \begin{aligned} u_1^{[2]} &= u_{10} \left\{ 1 - \frac{i\chi [R_1^* (m_{22} R_0 - m_{21} S_0) + S_1^* (m_{11} S_0 - m_{12} R_0)]}{a_1 (m_{11} m_{22} - m_{12} m_{21})} \right\} \\ \text{and} \quad u_2^{[2]} &= u_{20} \left\{ 1 - \frac{i\chi [R_2^* (m_{22} R_0 - m_{21} S_0) + S_2^* (m_{11} S_0 - m_{12} R_0)]}{a_2 (m_{11} m_{22} - m_{12} m_{21})} \right\}, \end{aligned} \right\} \quad (3.13)$$

where m_{11} is given by equation (3.12) and the other m_{ij} are defined by

$$\left. \begin{aligned} m_{12} &= \lambda_0 \sqrt{\lambda_0^*} R_0^* S_0 + \lambda_0^* \sqrt{\lambda_0} (R_1^* S_1 + R_2^* S_2) + \frac{\chi}{2} \sqrt{\lambda_0^*} |R_0|^2 - \frac{\lambda_0 + \lambda_0^*}{2\lambda_0} m_{11}, \\ m_{21} &= \lambda_0 \sqrt{\lambda_0^*} R_0 S_0^* + \lambda_0^* \sqrt{\lambda_0} (R_1 S_1^* + R_2 S_2^*) + \frac{\chi \lambda_0}{2\sqrt{\lambda_0^*}} |R_0|^2 - \frac{\lambda_0}{\lambda_0^*} m_{11} \\ \text{and} \quad m_{22} &= \lambda_0 \sqrt{\lambda_0^*} |S_0|^2 + \lambda_0^* \sqrt{\lambda_0} (|S_1|^2 + |S_2|^2) + \frac{\chi^2}{4\sqrt{\lambda_0^*}} |R_0|^2 \\ &\quad + \frac{\chi}{2\sqrt{\lambda_0^*}} (\lambda_0^* R_0 S_0^* + \lambda_0 R_0^* S_0) - \frac{\lambda_0}{\lambda_0^*} m_{12} - \frac{\lambda_0 + \lambda_0^*}{2\lambda_0} m_{21} - \frac{\chi}{2\lambda_0^*} m_{11}. \end{aligned} \right\} \quad (3.14)$$

As seen, these m_{ij} given above are all complex and much more complicated in formulae than those presented for the Manakov system in which m_{11} and m_{22} are always real [32,35]. It is hence clear that the solutions (3.11) and (3.13) have no singularity problem, as their denominators will never be zero, of course under the parameter condition such that λ_0 is complex.

We further emphasize that these fundamental and second-order rogue wave solutions are general and valid for arbitrary plane-wave parameters; they apply not only to both (anomalous and normal) dispersion situations, but also to a situation where coexisting rogue wave dynamics take place. They are compact and derived without prior knowledge of the exact formulae of λ_0 and μ_0 . Once λ_0 and μ_0 are known for given initial parameters, one can simplify the obtained solutions and demonstrate their intriguing dynamics. Let us take the simplest case $\delta = 0$ (i.e. $\omega_1 = \omega_2$) as an example, where λ_0 and μ_0 are explicitly given by

$$\lambda_0 = -\frac{\epsilon^2(8A + \kappa)}{\sigma} \pm \frac{4i\epsilon^2}{\sigma} \sqrt{-A(4A + \kappa)} \quad \text{and} \quad \mu_0 = \frac{\kappa}{2}. \quad (3.15)$$

In this case, the fundamental rational solution (3.11) boils down to

$$u_1^{[1]} = u_{10} \left(1 + \frac{F}{D} \right) \quad \text{and} \quad u_2^{[1]} = u_{20} \left(1 + \frac{F}{D} \right), \quad (3.16)$$

where

$$\left. \begin{aligned} F &= \frac{4i}{\kappa} \left[\vartheta + \frac{\sigma \xi (8A + \kappa)}{\epsilon^2 \kappa^3} \right] + \frac{1}{4A\kappa}, \\ D &= \theta^2 - \frac{16\sigma^2 \xi^2 A(4A + \kappa)}{\epsilon^4 \kappa^6} - \frac{1}{16A\kappa} - i \left(\frac{\vartheta}{\kappa} + \frac{8A\sigma \xi}{\epsilon^2 \kappa^4} \right) \\ \text{and} \quad \vartheta &= \frac{\tau}{2\epsilon} - \frac{(32A + 4\kappa - \kappa^3)\sigma \xi}{4\epsilon^2 \kappa^3}, \end{aligned} \right\} \quad (3.17)$$

which can represent a rogue wave event when the parameter condition $A(4A + \kappa) < 0$ is met. It follows easily that, as $\delta = 0$ or $\omega_1 = \omega_2$, both rogue wave components are decoupled from each other, each having a threefold peak amplitude relative to their respective background levels, no matter whether $a_1 = a_2$ or $a_1 \neq a_2$. Actually, this Peregrine-type rogue wave solution bears a great resemblance to the solution form (22) presented in [58] for the scalar FL equation, and hence brings no added non-trivial properties relevant to the CFL system. We should point out that the rogue waves demonstrated in [48,49] are indeed such kind of decoupled rogue waves, due to the special choice of initial plane-wave parameters.

When $\delta \neq 0$, the two rogue wave components become coupled and some interesting dynamics have arisen, such as coexisting rogue wave patterns and creation of anomalous Peregrine solitons. Figure 1 shows typically a pair of fundamental rogue waves, defined by equation (3.11), in an anomalous dispersion regime ($\sigma = 1$) for given background parameters $\epsilon = 3/5$, $\omega_1 = 820/567$, $\omega_2 = 320/567$, $a_1 = \sqrt{6}/5$ and $a_2 = \sqrt{1254}/25$, which, according to equations (3.1) and (3.3), can give rise to two sets of valid values: $\lambda_0 = -2/5 + (2/8085)_J \equiv \lambda_0^+$, $\mu_0 = -5/7 + [(40 + 25i\sqrt{19}/713097)]_J \equiv \mu_0^+$ [column (a)] and $\lambda_0 = -2/5 - (2/8085)_J \equiv \lambda_0^-$,

Figure 1. Coexisting fundamental rogue waves formed in the anomalous dispersion regime ($\sigma = 1$), with the same set of background parameters being given in the text: column (a) $\lambda_0 = \lambda_0^+$, $\mu_0 = \mu_0^+$; column (b) $\lambda_0 = \lambda_0^-$, $\mu_0 = \mu_0^-$. (Online version in colour.)

$\mu_0 = -5/7 - [(40 + 25i\sqrt{19}/713097)]_J \equiv \mu_0^-$ [column (b)], where $J = \sqrt{-340186 + 19250i\sqrt{19}}$. This choice of background parameters is arbitrary, but still accessible to experiment design, as both the two frequencies $\omega_{1,2}$ and the two amplitudes $a_{1,2}$ chosen above are slightly different from each other. It is clearly seen that, first, this pair of Peregrine rogue waves is different in shape, but could coexist on the same background, hence termed coexisting rogue waves, as unveiled in [35,59], and second, while the u_1 rogue wave component in the left column has a normal peak amplitude which is below three times the background, the one in the right column can reach a climax much higher than this characteristic value (around six times here), hence termed anomalous Peregrine soliton, as uncovered in [41]. When $\delta = 0$, these interesting phenomena will be absent totally.

In the same manner, coexisting patterns also occur for the second-order rogue wave solutions (3.13), as seen in figures 2 and 3, which correspond to the anomalous ($\sigma = 1$) and normal ($\sigma = -1$) dispersion situations, respectively. In these plots, for the sake of comparison, we set the background parameters as $\epsilon = 3/5$, $\omega_1 = 1/\epsilon - \sigma/(21\epsilon^3)$, $\omega_2 = 5\omega_1 - 4/\epsilon$, $a_1 = \sqrt{6}/5$ and $a_2 = \sqrt{1254}/25$, and use the same set of structural parameters $\gamma_1 = 2i$, $\gamma_2 = 1$, $\gamma_3 = 0$ and $\gamma_4 = 10$. It is clear that both field components could exhibit a pair of coexisting triplet rogue wave patterns on the same background, for either the anomalous or the normal dispersion case.

We finally point out that, despite that our solutions (3.11) and (3.13) have predicted analytically the possibility of the coexistence of rogue waves for given initial parameters, there are still some open issues concerning them. As examples, one may wonder whether or not these coexisting rogue waves can be both excited numerically from a chaotic wave field, and whether or not they can be elucidated by the current theory of modulation instability (MI) [60,61]. These interesting topics are beyond the scope of the present article and will be left for future studies.

In addition, we should emphasize that the coexistence of rogue waves exhibits a universality in coupled nonlinear systems [19,35,59], but its occurrence is largely subject to a certain parameter condition. For example, it will be absent in the triple-root case discussed below.

Figure 2. Coexisting triplet rogue wave patterns formed in the anomalous dispersion regime ($\sigma = 1$): column (a) $\lambda_0 = \lambda_0^+$, $\mu_0 = \mu_0^+$; column (b) $\lambda_0 = \lambda_0^-$, $\mu_0 = \mu_0^-$. The other parameters have been indicated in the text. (Online version in colour.)

Figure 3. Coexisting triplet rogue waves in the normal dispersion regime ($\sigma = -1$): column (a) $\lambda_0 = \lambda_0^+$, $\mu_0 = -\mu_0^+$; column (b) $\lambda_0 = \lambda_0^-$, $\mu_0 = -\mu_0^-$. The other parameters are specified in the text. (Online version in colour.)

4. Rogue wave solutions: the triple-root case

Let us now proceed to treat with the interesting triple-root case of the CFL system. It is noted that, when $\varsigma = \varrho = 0$, the cubic equation (2.13) admits a triple root μ_0 , which results in the parameter

condition

$$a_1 = \frac{2\sqrt{-2\sigma\delta^2\nu_1}}{\epsilon(3\delta^2 + \kappa^2)} \quad \text{and} \quad a_2 = \frac{2\sqrt{-2\sigma\delta^2\nu_2}}{\epsilon(3\delta^2 + \kappa^2)}, \quad (4.1)$$

implying that the amplitudes of two background fields bear relation to their frequencies. Under this condition, one can easily find that

$$\lambda_0 = \frac{\epsilon^2\kappa(9\delta^2 - \kappa^2)}{\sigma(3\delta^2 + \kappa^2)} + \frac{3i\epsilon^2}{2\sigma\ell} \quad \text{and} \quad \mu_0 = \frac{\kappa(7\delta^2 + \kappa^2)}{2(3\delta^2 + \kappa^2)} + \frac{i}{4\ell}, \quad (4.2)$$

where

$$\ell = \pm \frac{\sqrt{3}(3\delta^2 + \kappa^2)}{6\delta(\delta^2 - \kappa^2)}. \quad (4.3)$$

As this pair of complex λ_0 could actually give rise to the same rogue wave structures (except for some translational displacements), we shall therefore take the '+' sign in equation (4.3) in the following derivations and discussions, without loss of generality.

As outlined in §2, we can now assume the spectral parameter λ in equation (2.10) to be

$$\lambda = \lambda_0 + i\ell^3\chi\epsilon^3, \quad (4.4)$$

where $\chi = \lambda_0 - \lambda_0^*$, as defined before. Insertion of equation (4.4) into equation (2.13) yields the formulae for the corresponding three roots, expressed in power series

$$\left. \begin{aligned} \mu_1 &= \mu_0 + \phi_2\epsilon - i\phi_1\ell\epsilon^2 - \frac{\ell^2}{2}\epsilon^3 + \frac{2i\phi_2\ell^3}{3}\epsilon^4 + \frac{\phi_1\ell^4}{3}\epsilon^5 + \frac{\phi_2\ell^6}{9}\epsilon^7 + \mathcal{O}(\epsilon^8), \\ \mu_2 &= \mu_0 + \phi_1\epsilon - i\phi_2\ell\epsilon^2 - \frac{\ell^2}{2}\epsilon^3 + \frac{2i\phi_1\ell^3}{3}\epsilon^4 + \frac{\phi_2\ell^4}{3}\epsilon^5 + \frac{\phi_1\ell^6}{9}\epsilon^7 + \mathcal{O}(\epsilon^8) \end{aligned} \right\} \quad (4.5)$$

and

$$\mu_3 = \mu_0 - \epsilon + i\ell\epsilon^2 - \frac{\ell^2}{2}\epsilon^3 - \frac{2i\ell^3}{3}\epsilon^4 - \frac{\ell^4}{3}\epsilon^5 - \frac{\ell^6}{9}\epsilon^7 + \mathcal{O}(\epsilon^8),$$

where $\phi_1 = \exp(-i\pi/3) = \phi_2^*$ and μ_0 is the triple root given by the second of equations (4.2). In an analogous manner, if we let the parameters $\Gamma_{1,2,3}$ in equation (2.10) be

$$\left. \begin{aligned} \Gamma_1 &= \sum_{i=1}^n \left(\frac{\gamma_{3i-2}}{3} - \frac{\phi_1\gamma_{3i-1}}{\epsilon} - \frac{\phi_2\gamma_{3i}}{\epsilon^2} \right) \epsilon^{3(i-1)}, \\ \Gamma_2 &= \sum_{i=1}^n \left(\frac{\gamma_{3i-2}}{3} - \frac{\phi_2\gamma_{3i-1}}{\epsilon} - \frac{\phi_1\gamma_{3i}}{\epsilon^2} \right) \epsilon^{3(i-1)} \\ \Gamma_3 &= \sum_{i=1}^n \left(\frac{\gamma_{3i-2}}{3} + \frac{\gamma_{3i-1}}{\epsilon} + \frac{\gamma_{3i}}{\epsilon^2} \right) \epsilon^{3(i-1)}, \end{aligned} \right\} \quad (4.6)$$

and

where γ_i ($i=1, 2, \dots, 3n$) are arbitrary complex constants, then the vector Φ can be exactly expanded into the series form (2.19), with terms in powers of ϵ^3 . It is easy to find that, as $n=2$, the first two expansion terms $\Phi^{(0)}$ and $\Phi^{(1)}$, the same as seen in equation (3.8), are now defined by

$$\left. \begin{aligned} R_0 &= \gamma_1 - 3i\gamma_2\vartheta - \gamma_3 \left(\frac{3\vartheta^2}{2} + 3\ell\vartheta - \frac{6\sqrt{3}\sigma\delta\ell\xi}{\epsilon^2\ell_2^3} \right), \\ R_j &= \frac{2i\sqrt{\lambda_0}a_jv_j}{\alpha_j} \left\{ R_0 - \frac{3}{\alpha_j}[\gamma_2 - i\gamma_3(\ell + \theta_j)] \right\}, \\ S_0 &= \gamma_1c_0 + \gamma_2d_0 + \gamma_3e_0 + \gamma_4 - 3i\gamma_5\vartheta - \gamma_6 \left(\frac{3\vartheta^2}{2} + 3\ell\vartheta - \frac{6\sqrt{3}\sigma\delta\ell\xi}{\epsilon^2\ell_2^3} \right) \\ S_j &= \frac{2i\sqrt{\lambda_0}a_jv_j}{\alpha_j} \left\{ S_0 + \gamma_1c_j + \gamma_2d_j + \gamma_3e_j - \frac{3}{\alpha_j}[\gamma_5 - i\gamma_6(\ell + \theta_j)] \right\}, \end{aligned} \right\} \quad (4.7)$$

and

where γ_i ($i=1, \dots, 6$) are six arbitrary complex constants, $\vartheta = \tau/2\epsilon + (\iota_2^2 + 4)\sigma\xi/4\epsilon^2\iota_2^2$, $\theta_j = \vartheta - i/\alpha_j$, $\alpha_j = \sigma\lambda_0/2\epsilon^2 - \mu_0 + 2\nu_j$ and $\iota_1 = \sqrt{3}\delta + i\kappa = \iota_2^*$. The other polynomial functions in equations (4.7) read

$$\left. \begin{aligned} c_0 &= \frac{i\vartheta^3}{6} + i\ell\vartheta^2 - \frac{i\ell^2\vartheta}{2} + \frac{2\sqrt{3}\sigma\delta\ell\xi}{\epsilon^2\iota_2^4}(\kappa\ell - i\iota_2\vartheta), \\ c_j &= \frac{1}{\alpha_j} \left[\frac{\theta_j^2}{2} + 2\ell\theta_j - \frac{2\sqrt{3}\sigma\delta\ell\xi}{\epsilon^2\iota_2^3} - \frac{(3\iota_1\phi_j - 2\iota_2)\ell^2}{2\iota_2} \right], \\ d_0 &= \frac{\vartheta^4}{8} + \frac{3\ell\vartheta^3}{2} + 2\ell^3\vartheta + \frac{18\sigma^2\delta^2\ell^2\xi^2}{\epsilon^4\iota_2^6} - \frac{3\sigma\ell\xi}{\epsilon^2\iota_2^4}(\sqrt{3}\delta\iota_2\vartheta^2 + 6\delta^2\ell\vartheta + 3\iota_1\ell), \\ d_j &= -\frac{i}{\alpha_j} \left[\frac{\theta_j^3}{2} + \frac{9\ell\theta_j^2}{2} - \frac{9\ell^2(\iota_1\phi_j - \iota_2)\theta_j}{2\iota_2} - \frac{6\sigma\delta\ell\xi(3\delta\ell + \sqrt{3}\iota_2\theta_j)}{\epsilon^2\iota_2^4} - \frac{i}{2\alpha_j^3} - \frac{(3\iota_1^3 - \iota_2^3)\ell^3}{2\iota_2^3} \right], \\ e_0 &= -\frac{i\vartheta^5}{40} - \frac{i\ell\vartheta^4}{2} - \frac{3i\ell^2\vartheta^3}{4} - \frac{i\ell^3\vartheta^2}{2} - i\ell^4\vartheta + \frac{18\sigma^2\delta^2\ell^2\xi^2(2\ell\kappa - i\iota_2\vartheta)}{\epsilon^4\iota_2^7} \\ &\quad + \frac{i\sqrt{3}\sigma\delta\ell\xi}{\epsilon^2\iota_2^3} \left[\vartheta^3 + \frac{3(\sqrt{3}\delta + \iota_2)\ell\vartheta^2}{\iota_2} + \frac{3(3\iota_1 - 2\sqrt{3}\delta)\iota_1\ell^2\vartheta}{\iota_2^2} + \frac{4(4i\kappa^3 - \iota_1^3)\ell^3}{\iota_2^3} \right] \\ \text{and } e_j &= \frac{1}{\alpha_j} \left\{ -\frac{\theta_j^4}{8} - 2\ell\theta_j^3 + \left[\frac{3\sqrt{3}\sigma\delta\ell\xi}{\epsilon^2\iota_2^3} - \frac{9\ell^2(2\iota_2 - \iota_1\phi_j)}{4\iota_2} \right] \theta_j^2 + \frac{3\sqrt{3}(\iota_1 + 3\iota_2)\sigma\delta\ell^2\xi\theta_j}{\epsilon^2\iota_2^4} \right. \\ &\quad + \left[\frac{\iota_1^2\ell^3(2\iota_1 - 3\iota_2\phi_j^2)}{2\iota_2^3} + \frac{3\ell^3(4\iota_1\phi_j - 3\iota_2)}{2\iota_2} \right] \theta_j + \frac{3\sqrt{3}[2\iota_1^2 + 3\iota_2^2 - \iota_1\iota_2(3\phi_j + 1)]\sigma\delta\ell^3\xi}{\epsilon^2\iota_2^5} \\ &\quad \left. - \frac{18\sigma^2\delta^2\ell^2\xi^2}{\epsilon^4\iota_2^6} - \frac{15\phi_j^2\iota_1^2\ell^4}{4\iota_2^2} + \frac{(2\iota_1^3 - 15\iota_2^3)\ell^4}{8\iota_2^3} + \frac{3\phi_j\iota_1\ell^4(\iota_1^3 + 14\iota_2^3)}{8\iota_2^4} \right\}. \end{aligned} \right\} \quad (4.8)$$

Consequently, we can express the fundamental rogue wave solution as

$$u_1^{[1]} = u_{10} \left(1 - \frac{i\chi R_0 R_1^*}{a_1 m_{11}} \right) \quad \text{and} \quad u_2^{[1]} = u_{20} \left(1 - \frac{i\chi R_0 R_2^*}{a_2 m_{11}} \right), \quad (4.9)$$

and express the second-order rogue wave solution as

$$\left. \begin{aligned} u_1^{[2]} &= u_{10} \left\{ 1 - \frac{i\chi [R_1^*(m_{22}R_0 - m_{21}S_0) + S_1^*(m_{11}S_0 - m_{12}R_0)]}{a_1(m_{11}m_{22} - m_{12}m_{21})} \right\} \\ \text{and } u_2^{[2]} &= u_{20} \left\{ 1 - \frac{i\chi [R_2^*(m_{22}R_0 - m_{21}S_0) + S_2^*(m_{11}S_0 - m_{12}R_0)]}{a_2(m_{11}m_{22} - m_{12}m_{21})} \right\} \end{aligned} \right\} \quad (4.10)$$

Figure 4. Surface (top) and contour (bottom) plots of (a) unusual Peregrine solitons and (b) their doublet states in the anomalous dispersion regime ($\sigma = 1$). column (a) $\gamma_1 = \gamma_3 = 0$, $\gamma_2 = 1$; column (b) $\gamma_1 = 0$, $\gamma_2 = 8i$, $\gamma_3 = 1$. The background parameters are given by $\epsilon = 3/5$, $\omega_1 = 820/567$, $\omega_2 = 320/567$, $a_1 = \sqrt{8/21}$ and $a_2 = \sqrt{40/21}$. (Online version in colour.)

which have exactly the same solution forms as equations (3.11) and (3.13), respectively, but with

$$\left. \begin{aligned}
 m_{11} &= \lambda_0 \sqrt{\lambda_0^*} |R_0|^2 + \lambda_0^* \sqrt{\lambda_0} (|R_1|^2 + |R_2|^2), \\
 m_{12} &= \lambda_0 \sqrt{\lambda_0^*} R_0^* S_0 + \lambda_0^* \sqrt{\lambda_0} (R_1^* S_1 + R_2^* S_2) + \frac{i\chi \ell^3}{2} \sqrt{\lambda_0^*} |R_0|^2 - \frac{i\ell^3 (\lambda_0 + \lambda_0^*)}{2\lambda_0} m_{11}, \\
 m_{21} &= \lambda_0 \sqrt{\lambda_0^*} R_0 S_0^* + \lambda_0^* \sqrt{\lambda_0} (R_1 S_1^* + R_2 S_2^*) - \frac{i\chi \ell^3 \lambda_0}{2\sqrt{\lambda_0^*}} |R_0|^2 + \frac{i\ell^3 \lambda_0}{\lambda_0^*} m_{11} \\
 \text{and } m_{22} &= \lambda_0 \sqrt{\lambda_0^*} |S_0|^2 + \lambda_0^* \sqrt{\lambda_0} (|S_1|^2 + |S_2|^2) + \frac{\chi^2 \ell^6}{4\sqrt{\lambda_0^*}} |R_0|^2 \\
 &\quad + \frac{i\chi \ell^3}{2\sqrt{\lambda_0^*}} (\lambda_0^* R_0 S_0^* - \lambda_0 R_0^* S_0) + \frac{i\ell^3 \lambda_0}{\lambda_0^*} m_{12} - \frac{i\ell^3 (\lambda_0 + \lambda_0^*)}{2\lambda_0} m_{21} - \frac{\chi \ell^6}{2\lambda_0^*} m_{11}.
 \end{aligned} \right\} \quad (4.11)$$

We point out that the fundamental solution (4.9) thus obtained can reduce, with considerable simplifications, to that presented in [41]. This solution can represent an anomaly of Peregrine soliton creation by which we mean the peak amplitude can go beyond the threefold amplitude limit, as shown in figure 4, where the fourfold-amplitude Peregrine soliton and its doublet state are exhibited. We have also confirmed numerically that this type of unusual Peregrine soliton is *stable* despite the onset of MI and can even be excited from a chaotic wave field [41]. Here we have generalized the stability notion from the usual soliton context to the transient rogue wave context, and referred to a rogue wave as being *stable* if its structure can unfold without significant distortion over a rather long distance, particularly under situations where small perturbations are exerted, or when the spontaneous MI has been activated by quantum noise. Of course, as

Figure 5. (a) Rogue wave quartets and (b) sextets, given by equation (4.10), in the anomalous dispersion regime, obtained with the same background parameters as in figure 4, but with different sets of structural parameters. Column (a), $\gamma_1 = 40, \gamma_2 = 1, \gamma_6 = 150 + 40i$ and $\gamma_3 = \gamma_4 = \gamma_5 = 0$; column (b) $\gamma_1 = 200, \gamma_3 = 1, \gamma_6 = 2000i$ and $\gamma_2 = \gamma_4 = \gamma_5 = 0$. (Online version in colour.)

there exists only a pair of complex conjugated λ_0 for given initial parameters (see the first of equations (4.2)), no coexisting rogue wave phenomena appear now.

We further provide here the second-order rogue wave solution (4.10) which was not reported before, to our best knowledge. Depending on the specific values of six structural parameters γ_j , there would appear the rogue wave quartets ($\gamma_2 = 1$ and $\gamma_3 = 0$) and rogue wave sextets ($\gamma_3 \neq 0$). Figure 5 demonstrates these two types of rogue wave states using the same background parameters as in figure 4, while with different sets of structural parameters. It is seen that the rogue wave components that constitute the quartets or sextets should not strictly have the same height in peak amplitude and their layout depends strongly on the values of parameters γ_j [35].

5. Conclusion

In conclusion, we obtained with a non-recursive or generalized DT method the general n th-order rational rogue wave solutions to the CFL equations, by taking both the double-root and triple-root situations into account. As illustrative examples, the exact explicit fundamental and second-order rogue wave solutions in both situations are presented. We demonstrated several interesting rogue wave dynamics such as coexisting rogue waves and anomalous Peregrine solitons. By the term *coexisting rogue waves* we mean that on the same continuous background could occur two rogue waves of different patterns, each of which will be the solution of the underlying model equation. Actually, this coexisting rogue wave phenomenon was first demonstrated in a multicomponent long-wave and short-wave resonant system [59] and then in the focusing Manakov system [35]. For this reason, we may conclude that the coexistence of rogue waves is a universal feature inherent to many coupled nonlinear systems, although some of its issues remain open. Likewise, by *anomalous Peregrine soliton* we mean a complex fundamental rational solution which consists of quadratic polynomials featured by the Peregrine solitons, while having a peak amplitude beyond

the threefold amplitude limit of the usual Peregrine solitons [27,29,62]. Such a Peregrine soliton anomaly was first reported in [41] and was shown to be able to be excited numerically from a turbulent field caused by noise. In this article, we showed further, on an analytical level, that the enforcement amplitude factor can be unexpectedly large (e.g. more than 5) on the coexisting Peregrine soliton occasions, depending on the initial parameters used. In particular, all solutions presented above have been generalized to include the complete background-field parameters and therefore are suitable for future experimental study of these interesting rogue wave dynamics.

As a final remark, we stress that the non-recursive DT method developed here can also be applied to (2+1)D CFL equations so that the obtained rational solutions can describe the rogue wave phenomena occurring in more realistic multidimensional nonlinear systems, for instance in crossing seas which usually involve two transversal directions and one temporal dimension. We note that similar multidimensional models such as the (2+1)D Manakov system [63] and the Davey–Stewartson equations [64,65] have been explored previously, yielding multidimensional rogue wave solutions. We anticipate that, whereas the (2+1)D CFL system will admit more complicated rogue wave dynamics than those obtained within the present (1+1)D model, most of its rogue wave states relevant for a weak coupling between two transversal dimensions could still be well described by the solutions of the latter simpler model.

Data accessibility. This article has no additional data.

Authors' contributions. Y.Y., Y.Z. and S.C. participated in deriving the analytic solutions. S.C., F.B. and P.G. contributed to analysing the scientific results and writing the article. All authors gave their final approval for publication.

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