Bell's theorem and the problem of decidability between the views of Einstein and Bohr

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Communicated by Federico Capasso, Bell Laboratories, Lucent Technologies, Murray Hill, NJ, October 3, 2001 (received for review June 15, 2001)

Einstein, Podolsky, and Rosen (EPR) have designed a gedanken experiment that suggested a theory that was more complete than quantum mechanics. The EPR design was later realized in various forms, with experimental results close to the quantum mechanical prediction. The experimental results by themselves have no bearing on the EPR claim that quantum mechanics must be incomplete nor on the existence of hidden parameters. However, the well known inequalities of Bell are based on the assumption that local hidden parameters exist and, when combined with conflicting experimental results, do appear to prove that local hidden parameters cannot exist. This fact leaves only instantaneous actions at a distance (called "spooky" by Einstein) to explain the experiments. The Bell inequalities are based on a mathematical model of the EPR experiments. They have no experimental confirmation, because they contradict the results of all EPR experiments. In addition to the assumption that hidden parameters exist, Bell tacitly makes a variety of other assumptions; for instance, he assumes that the hidden parameters are governed by a single probability measure independent of the analyzer settings. We argue that the mathematical model of Bell excludes a large set of local hidden variables and a large variety of probability densities. Our set of local hidden variables includes time-like correlated parameters and a generalized probability density. We prove that our extended space of local hidden variables does permit derivation of the quantum result and is consistent with all known experiments.

We address the question whether the quantum result for the spin pair correlation in Einstein, Podolsky, and Rosen (EPR)-type experiments can be obtained by a hidden parameter theory. We show that this goal can indeed be achieved in spite of the serious objections given in the work of Bell (1).

The work of Bell (1) attempts to show that a mathematical description of EPR-type experiments (2) by a statistical (hidden) parameter theory (3, 4) is not possible. In EPR experiments, two particles having their spins in a singlet state are emitted from a source and are sent to spin analyzers at two spatially separated stations, S_1 and S_2 . The spin analyzers are described by Bell by using unit vectors $\mathbf{a}$, $\mathbf{b}$, etc., of three-dimensional Euclidean space and functions $A = \pm 1$ (operating at station S_1) and $B = \pm 1$ (operating at station S_2); furthermore, neither does A depend on the settings $\mathbf{b}$ of station S_2 nor B on the settings $\mathbf{a}$ of station S_1 . Bell permits particles emitted from the source to carry arbitrary hidden parameters λ of a set Ω that fully characterize the spins and are "attached" to the particles with a probability density ρ (we denote the corresponding probability measure by μ) that does not depend on the settings at the stations. The parameters λ are not permitted to depend on settings $\mathbf{a}$ and $\mathbf{b}$. The assumption that A is independent of $\mathbf{b}$, B of $\mathbf{a}$, and λ of both settings is derived from the experimental procedure. The settings are changed so rapidly that the finite velocity of light does not permit such a dependence. Bell then assumes that the values of functions A and B are determined by the spin analyzer settings and parameters such that:

$$A = A(\mathbf{a}, \lambda) = A_a(\lambda) = \pm 1 \text{ and } B = B(\mathbf{b}, \lambda) = B_b(\lambda) = \pm 1. \quad [1]$$

Thus $A_a(\lambda)$ and $B_b(\lambda)$ can be considered as stochastic processes on Ω , indexed by the unit vectors $\mathbf{a}$ and $\mathbf{b}$, respectively. Quantum theory and experiments show that, for a given time of measurement for which the settings are equal in both stations, we have for singlet state spins

$$A_a(\lambda) = -B_b(\lambda) \quad [2]$$

with probability one. Bell further defines the spin pair expectation value $P(\mathbf{a}, \mathbf{b})$ by

$$P(\mathbf{a}, \mathbf{b}) = \int_{\Omega} A_a(\lambda) B_b(\lambda) \rho(\lambda) d\lambda = - \int_{\Omega} A_a(\lambda) A_b(\lambda) \mu(d\lambda). \quad [3]$$

From Eqs. 1–3, Bell derives his celebrated inequality (1)

$$1 + P(\mathbf{b}, \mathbf{c}) \geq |P(\mathbf{a}, \mathbf{b}) - P(\mathbf{a}, \mathbf{c})| \quad [4]$$

and observes that this inequality is in contradiction with the result of quantum mechanics:

$$P(\mathbf{a}, \mathbf{b}) = -\mathbf{a} \cdot \mathbf{b}. \quad [5]$$

Here $\mathbf{a} \cdot \mathbf{b}$ is the scalar product of $\mathbf{a}$ and $\mathbf{b}$.

The proof of Bell's inequality is based on the obvious fact that for $x, y, z = \pm 1$, we have $|xz - yz| = |x - y| = 1 - xy$. Substituting $x = A_b(\lambda)$, $y = A_c(\lambda)$, $z = A_a(\lambda)$ and integrating with respect to the measure μ , one obtains Eq. 4 in view of Eq. 3. Thus, from the vantage point of mathematics, the Bell inequality is a straightforward consequence of the set of hypotheses and assumptions that are imposed. Note that, unlike other theorems used in physical arguments, the Bell inequality has no experimental basis and actually contradicts all known experiments. It stands on its correctness as a mathematical theorem alone. Extensive discussions of this fact have been given in the literature, mostly concluding that spooky action at a distance was a logical necessity (5, 6). However, very significant analysis suggests that avenues exist that can explain the above facts without recourse to action at a distance (7). We confirm this suggestion by a broad mathematical proof and show that the mathematical model of Bell is not general enough to cover all the physics that may be involved in EPR experiments.

The starting point of our analysis of these hypotheses and assumptions is an examination of the role of time in the characterization of the set of measured data that are represented by functions A_a and B_b in Eq. 1. In a complete EPR experiment, settings $(\mathbf{a}, \mathbf{b})$ are randomly changed. However, to evaluate $P(\mathbf{a}, \mathbf{b})$, etc., for the purpose of checking the Bell inequality,

This paper was submitted directly (Track II) to the PNAS office.

Abbreviations: EPR, Einstein, Podolsky, and Rosen; TLCPs, time-like correlated parameters.

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three settings $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ will have to be selected and considered fixed. The times at which the measurements are taken for the three relevant pairs $(\mathbf{a}, \mathbf{b})$, $(\mathbf{a}, \mathbf{c})$, and $(\mathbf{b}, \mathbf{c})$ will now have to be considered random. Thus, if a time dependence of the functions A and B exists, it is certainly reasonable to allow for an additional stochastic parameter ω related to time that drives the random processes taking place in the two stations. This parameter is not correlated to λ , which can, for example, derive its randomness from current and voltage fluctuations at the source. We would like to emphasize that Bell has introduced a number of assumptions on time dependencies, such as the use of Eq. 2, without regard to the time-disjoint measurements. He also introduced a significant asymmetry in describing the spin properties of the particles and the properties of the measurement equipment. The spins are described by arbitrarily large sets Λ of parameters. On the other hand the measurement apparatus is described by a vector of Euclidean space (the settings), true to Bohr's (4) postulate that the measurement must be classical. Yet, the measurement apparatus must itself in some form contain particles with spins that then, if one wants to be self consistent, also need to be described by large sets of parameters that are related to the settings $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. . .

Time-Like Correlated Parameters (TLCPs)

We state in advance what we believe to be the basis for obtaining the quantum result with the use of hidden parameters: the functions A and B and the densities ρ of hidden parameters may both relate to time. Time correlations, particularly setting-dependent ones, may exist in both stations with no suspicion or hint of spooky action. The introduction of these correlations through time leads, then, to a probability measure that can depend on the settings at both stations, although the functions A , B depend only on the settings of the respective stations, and the parameters, now considered as random variables, are independent when averages are taken over long time periods. A well known fact of probability theory is at the foundation of this assertion: random variables may be conditionally dependent (e.g., for certain time periods), whereas they are independent when no conditions are imposed.

Generalized Bell-type proofs (8) permit any number and form of parameters, so long as separate integrations can be performed over the respective densities, i.e., if the joint conditional densities equal the product of the individual conditional densities. The introduction of TLCPs also presents a critical problem for these proofs, because more than one parameter in the argument of the functions A and B may depend on time. Then, in general, the joint conditional density does not equal the product of individual conditional densities.

We propose the following definition: TLCPs are parameters that exhibit correlations because they are related to periodic processes. They may depend on the setting of the station in which the periodic process occurs. The correlations are caused not by any information transfer over distance but by the fact that the stations are subject to the same physical law.

The essence of our approach is the introduction of such setting- and station-specific time-like correlated parameters $\lambda_{\mathbf{a}}^*$ on one side and $\lambda_{\mathbf{b}}^*$ on the other, which codetermine the functions A , B in addition to the correlated source parameters λ . We show in detail below and in the next sections that TLCPs cannot be fully covered by Bell's or any of the generalized proofs. We also show that these parameters lead in a natural way to setting-dependent probability measures for the parameters without spooky action at a distance. Setting-dependent probability measures were not considered a possibility for three major reasons. First, a single product measure $\mu = \mu_{\mathbf{a}} \times \mu_{\mathbf{b}}$ (where $\mu_{\mathbf{a}}$ depends only on $\mathbf{a}$, and $\mu_{\mathbf{b}}$ depends only on $\mathbf{b}$, which would guarantee Einstein separability of the stations) cannot lead to the quantum result of Eq. 5. Second, stochastic parameters $\lambda_{\mathbf{a}}^*$

acting in station S_1 and $\lambda_{\mathbf{b}}^*$ in station S_2 , although investigated (9), were not included as integration variables in the probability measure because of Eq. 2. It seemed impossible to reconcile the fact

$$A_{\mathbf{a}}(\text{parameters}) = -B_{\mathbf{a}}(\text{parameters}) \quad [6]$$

with that of setting-dependent parameter random variables in the spatially separated stations. The main reasoning was that the settings are changed rapidly between the measurements, and the parameters λ can therefore carry no information about the actual settings at the time period of measurement. Which information could then possibly lead to Eq. 6 without invoking spooky action at a distance between the stations? We show that time-like correlated parameters (derived from a global clock time for both stations) can provide this information. The third reason is the widely held (but mistaken; see ref. 10) belief that if A depends on $\mathbf{a}$, λ and $\lambda_{\mathbf{a}}^*$, then by enlarging the parameter space, one can rewrite A as a function of $\mathbf{a}$ and λ with an enlarged set of parameters λ , and similarly for B .

The following example is designed to show in a more pragmatic way what we understand by TLCPs. TLCPs may actually include space-like labels, such as the settings. However, with respect to their correlations, they are time-like, just as two clocks in two stations show time-like correlations even if some space like settings (e.g., the length of the pendulum) are adjusted separately in the stations. To be definite, assume that two stations have synchronized clocks with the pointer of each clock symbolized by a vector of Euclidean space and denoted by $\mathbf{s}_1$ in station S_1 and by $\mathbf{s}_2$ in station S_2 . Adding the setting vectors in the respective stations, one obtains setting-dependent time-related and correlated parameters, i.e., TLCPs $\mathbf{s}_1 + \mathbf{a}$ in station S_1 and $\mathbf{s}_2 + \mathbf{b}$ in station S_2 . One can find a natural implementation of this example by using gyroscopes in the two stations located on the rotating earth. If such parameters affect functions A and B , then integration over time cannot be factorized, and consequently time cannot be introduced in Bell-type proofs without difficulty. We note in passing that the rotation of the earth also poses the following problem. The quantum result for the spin-pair correlation $P(\mathbf{a}, \mathbf{b}) = -\mathbf{a} \cdot \mathbf{b}$ is invariant to (time-dependent) rotations, whereas the mathematical operations performed in the proof of Bell's theorem are not. Thus rotational symmetry is violated in Bell-type proofs through the factorization process without assessment of its consequences.

To demonstrate the existence of hidden parameters in principle, we may permit any parameter set that can be generated involving t and local setting-dependent operators $O_{\mathbf{a},t}^1$ in station S_1 and $O_{\mathbf{b},t}^2$ in station S_2 . These local operators may act on any parameters (or information) in the respective stations to create new parameters. For example, if a particle that carries the parameter λ^1 arrives from the source within a time period characterized by ω in station S_1 , then the time operator can transform this parameter into a new "mixed" parameter $\Lambda_{\mathbf{a},t}^1(\lambda^1, \omega)$. Recall from the Introduction that the actual time of measurement that determines ω must be random, because the settings are randomly switched (8). We distinguish this random time period ω from the time index in the time operator, because that dependence on time may or may not be random. A more specific way of thinking about these operators is by imagining in the two stations two computers that have synchronized internal clocks. These computers can run any program to create new parameters out of the locally available input. Of equal importance, they can also be used to evaluate these parameters, i.e., assign them a value of ± 1 . Both processes, creation and evaluation, may depend on the respective setting and may be correlated in time.

In the following, we formalize the concepts of TLCPs. The starting point is a set of source parameters $\lambda = (\lambda^1, \lambda^2)$ where the

superscripts indicate information carried to stations S_1 and S_2 , respectively. Random internal parameters λ_a^* operate at station S_1 and λ_b^* at station S_2 . In other words, there is a layer of parameters below the mere settings that will affect the values of functions A, B . Although the observer might imagine that A, B depend on settings only, the values of functions A, B are determined by stochastic processes, indexed by unit vectors $\mathbf{a}$ and $\mathbf{b}$, respectively. For given vectors $\mathbf{a}$ and $\mathbf{b}$, we denote the joint distribution of the resulting random variables $\lambda_a^*(\cdot)$ and $\lambda_b^*(\cdot)$ by $\gamma = \gamma_{ab}$, which we allow to depend on $\mathbf{a}$ and $\mathbf{b}$ to accommodate as broad a situation as possible. A reasonable, though not necessary, assumption on γ is the following continuity condition: for fixed $\mathbf{a}$,

$$\lim_{\mathbf{b} \rightarrow \mathbf{a}} \gamma_{ab}\{\omega: \lambda_b^*(\omega) = \lambda_a^*(\omega)\} \rightarrow 1 = \gamma_{aa}\{\omega: \lambda_a^*(\omega) = \lambda_a^*(\omega)\}. \quad [7]$$

Intuitively speaking, Eq. 7 says that if the vector $\mathbf{b}$ at station S_2 is parallel or close to parallel to the vector $\mathbf{a}$ characterizing the analyzer setting in station S_1 , then for an “overwhelming majority of cases ω ,” the corresponding parameters λ^* and λ^{**} are equal, which then permits Eq. 6 to hold.

Our probability space Ω consists of all pairs (λ, ω) , where λ is a source parameter and ω is the element of randomness related to time and driving the station parameters $\lambda_a^*(\cdot)$ and $\lambda_b^*(\cdot)$, respectively. Furthermore, we assume that the source parameters λ will interact with the station parameters $\lambda_a^*(\omega)$ and $\lambda_b^*(\omega)$ with built-in time t dependence to form the “mixed” parameters $\Lambda_{a,t}^1(\lambda, \omega)$ and $\Lambda_{b,t}^2(\lambda, \omega)$ (one could visualize this by some many-body interactions). These are not free parameters but rather stochastic processes, indexed by the pairs $(\mathbf{a}, t)$ and $(\mathbf{b}, t)$ at stations S_1 and S_2 , respectively, and defined on Ω . The transition from (λ, ω) to $\Lambda_{a,t}^1(\lambda, \omega)$ and $\Lambda_{b,t}^2(\lambda, \omega)$ is thought to be defined by certain rules that can be represented by station-specific operators $O_{a,t}^1$ and $O_{b,t}^2$ that depend on the globally known time t , which is the same at the stations and at the source. Notice also that the time operations and mixing of parameters occur (in quantum mechanical terms) during the collapse of the wave function. The timing in left and right stations and the values of time involved in the measurement process are also quite flexible. It needs to be guaranteed only that one deals with the same correlated pair.

Thus the connection between the time operators O and the mixed parameters Λ is given by

$$O_{a,t}^1(\lambda^1, \omega) = O_{a,t}^1(\lambda^1, \omega; \lambda_a^*(\omega)) = \Lambda_{a,t}^1(\lambda^1, \omega). \quad [8]$$

Furthermore, the stochastic process $A_{a,t}$ satisfies

$$A_{a,t}(\lambda^1, \omega; \lambda_a^*(\omega), \Lambda_{a,t}^1(\lambda^1, \omega)) = \pm 1. \quad [9]$$

Similar equations apply for station S_2 and settings $\mathbf{b}$, etc. If $\mathbf{b} = \mathbf{a}$, then in analogy to Eq. 2, we have with probability 1

$$B_{a,t}(\lambda^2, \omega; \lambda_a^*(\omega), \Lambda_{a,t}^2(\lambda^2, \omega)) = -A_{a,t}(\lambda^1, \omega; \lambda_a^*(\omega), \Lambda_{a,t}^1(\lambda^1, \omega)). \quad [10]$$

This means that the time operations for equal settings need to be synchronized to lead to Eq. 10. The synchronization may be achieved by the selection of which settings are chosen to be equal in the two stations and by the fact that the stations are in the same inertial frame with identical clock time. (Time shifts and asymmetric station distances can easily be accommodated in our model.)

TLCs in Proofs of Bell's Theorem

We first discuss Bell's proof (and the variation described on p 56 of ref. 8) and demonstrate why it cannot go forward when TLCs

are involved. Later, we shall pinpoint the locations where the argument breaks down in other Bell-type proofs. Bell (8) defines the following parameter sets that are in the backward light cone (as defined by relativity). He lets N denote the specification of all entities that are represented by parameters and belong to the overlap of the backward light cones of both space-like separated stations S_1 and S_2 . In addition, he considers sets of parameters L_a (our notation) that are in the remainder of the backward light cone of S_1 and M_b for S_2 , respectively. Bell denotes the conditional probability that the function A_a assumes a certain value with $A_a = \pm 1$ by $\{A_a|L_a, N\}$ and similarly for $B_b = \pm 1$.

Then, to derive one of the celebrated “local inequalities,” Bell considers the expectation E of the product $A_a B_b$:

$$E\{A_a B_b\} = \sum_{AB} A_a B_b \{A_a|L_a, N\} \{B_b|M_b, N\}. \quad [11]$$

Here we have rewritten the summation or integration process of Eq. 3 in a form that illustrates in greater clarity the assumptions made by Bell and by using the language of relativity. Notice that Eq. 11 uses the assumption that A_a and B_b are conditionally independent given L_a, M_b , and N , whereas this is true only for the given instant of time. This fact shows the crux of the problem with Bell-type derivations of the locality inequalities. The parameter sets in the backward light cones are not constant but evolve and are, certainly in principle, different for all the different times at which each single measurement is taken. In addition, these parameters may be time-like correlated. Using a more precise notation, one must therefore label the sets of parameters with indices that represent time t (or time periods ω), e.g., by $N_t, L_{a,t}$, and $M_{b,t}$. Then it is obvious that the summations or integrations that are needed to derive Bell-type inequalities cannot be performed in a straightforward fashion. For example, as the A_a s depend on the time of measurement, they can no longer be factored in equation 10 on p. 56 of Bell's book (8), because M and M' are measurement sets involving different settings, and thus time must have elapsed while changing from M to M' . Similar arguments lead to a large variety of demonstrations that Bell-type proofs do not go forward when time is involved [K. Hess and W. Philipp, quant-ph, <http://xxx.lanl.gov/abs/quant-ph/0103028>, March 7 (2000)]. For example, Greenberger, Horne, Shimony, and Zeilinger (GHSZ) (10) have derived an “impossibility proof” obtaining a contradiction by using arguments for four correlated spins. However, just as in the case of Bell-type proofs for two correlated spins, time-like correlated parameters are not considered. If they are, the proof is invalidated. We refer the reader to equations 12a–d of GHSZ (10). These results cannot be measured at the same time, because they involve different settings. As a consequence, if time-like correlated parameters are invoked, the GHSZ function $A_\lambda(\phi)$ must be written $A_\lambda(\phi, t_1)$ in equation 12b and $A_\lambda(\phi, t_2)$ in equation 12c of GHSZ (10) with $t_1 \neq t_2$, and the chain of the argument in the proof fails right here (and at several other places as well).

Quantum Result Without Spooky Action

We shall show that a properly chosen sum of setting-dependent subspace product measures (SDSPMs) does not violate Einstein separability and does lead to the quantum result of Eq. 5, while still always fulfilling Eq. 2. By SDSPM, we mean the following. The probability space Ω is partitioned into a finite number M of subspaces Ω_m

$$\Omega = \bigcup_{m=1}^M \Omega_m. \quad [12]$$

For given $\mathbf{a}$ and $\mathbf{b}$, a product measure $(\mu_a \times \mu_b)_m$ is defined on each subspace Ω_m . This measure can be extended to the entire space Ω by setting

$$(\mu_{\mathbf{a}} \times \mu_{\mathbf{b}})_m(\Omega_j) = 0 \text{ if } j \neq m, \quad [13]$$

which we denote by the acronym SDSPM. The final measure μ is then defined on the entire space Ω by

$$\mu = \sum_{m=1}^M (\mu_{\mathbf{a}} \times \mu_{\mathbf{b}})_m. \quad [14]$$

First, we formulate a theorem that provides the stepping stone for this procedure. Note that our measure deviates from a probability measure by at most ε , which can be chosen arbitrarily small. We believe that this presents no physical limitation of the theory but include it for reasons of mathematical precision.

THEOREM. *Let $0 < \varepsilon < 1/2$, and let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be unit vectors. Then there exists a finite measure space $(\Omega, \mathbf{F}, \mu = \mu_{\mathbf{a}, \mathbf{b}})$ and two measurable functions A and B defined on it with the following properties:*

$$\Omega \subset \mathbb{R}^2 \text{ and } \mathbf{F} \text{ depend on } \varepsilon \text{ only.} \quad [15]$$

Ω is a compact set. Its elements are denoted by (u, v) .

The measure μ depends only on $\mathbf{a}$, $\mathbf{b}$, and ε , satisfies

$$1 \leq \mu(\Omega) < 1 + \varepsilon, \quad [16]$$

and has a density $\rho_{\mathbf{ab}}$ with respect to Lebesgue measure.

The functions A and B assume the values

$$A, B = \pm 1, \quad [17]$$

and A depends only on $\mathbf{a}$ and u , B only on $\mathbf{b}$ and v .

Further

$$E\{A_{\mathbf{a}}B_{\mathbf{b}}\} = \int_{\Omega} A_{\mathbf{a}}(u)B_{\mathbf{b}}(v)\rho_{\mathbf{ab}}(u, v)dudv = -\mathbf{a} \cdot \mathbf{b}, \quad [18]$$

and for each vector $\mathbf{a}$, the following equation holds for all $(u, v) \in \Omega$ except on a set of μ -measure $< \varepsilon$:

$$B_{\mathbf{a}}(u, v) = -A_{\mathbf{a}}(u, v). \quad [19]$$

The proof of the theorem requires the following fact, which follows from a basic theorem on B-splines (11). We state the fact here in form of a lemma.

LEMMA. *Let $n \geq 4$ be an integer. Then there exist real-valued functions $N_i(x)$, $\psi_i(y)$, with $1 \leq i \leq n$ depending only on real variables x and y , respectively, such that*

$$0 \leq N_i(x) \leq 1, \quad 0 \leq \psi_i(y) \leq 2 \text{ for } 0 \leq x, y \leq 1 \quad [20]$$

and

$$0 \leq \sum_{i=1}^n \psi_i(y)N_i(x) - (y-x)^2 \leq \frac{1}{4}n^{-2} \text{ for } 0 \leq x, y \leq 1. \quad [21]$$

Proof of Theorem: Choose an even integer $n > 1/\varepsilon$ and for Ω the square $\Omega = [-3, 3n]^2$ with side of length $3 + 3n$. We endow Ω with Lebesgue measurability, symbolized by the σ -field $\mathbf{F}$, and define:

$$A_{\mathbf{a}}(u) = \begin{cases} \text{sign}(a_k) & \text{if } -k \leq u < -k + 1 & k = 1, 2, 3 \\ -1 & \text{if } 2j \leq u < 2j + \frac{1}{2} & j = 0, 1, \dots, \frac{3n}{2} - 1 \\ +1 & \text{if } 2j + \frac{1}{2} \leq u < 2j + 2 & j = 0, 1, \dots, \frac{3n}{2} - 1 \end{cases} \quad [22]$$

Thus A depends on $\mathbf{a}$ and u only. Here and throughout, we set $\text{sign}(0) = 1$. Similarly, we define

$$B_{\mathbf{b}}(v) = \begin{cases} -\text{sign}(b_k) & \text{if } -k \leq v < -k + 1 & k = 1, 2, 3 \\ +1 & \text{if } 2j + \frac{1}{2} \leq v < 2j + \frac{3}{2} & j = 0, 1, \dots, \frac{3n}{2} - 1 \\ -1 & \text{if } 2j - \frac{1}{2} \leq v < 2j + \frac{1}{2} & j = 1, 2, \dots, \frac{3n}{2} - 1 \\ -1 & \text{if } 0 \leq v < \frac{1}{2} \text{ or } 3n - \frac{1}{2} \leq v < 3n \end{cases} \quad [23]$$

Thus, B depends on $\mathbf{b}$ and v only. Notice that on $\cup_{k=1}^3 [-k, -k + 1]^2$, Eq. 19 is satisfied for all values of (u, v) .

Next, we define $\sigma_{\mathbf{a}}(u)$ and $\tau_{\mathbf{b}}(v)$, respectively, by

$$\begin{aligned} |a_k| \cdot 1\{-k \leq u < -k + 1\} & k \in I_1 \\ N_k(a_1) \cdot 1\{k - 1 \leq u < k\} & k \in I_2 \\ N_{k-n}(a_2) \cdot 1\{k - 1 \leq u < k\} & k \in I_3 \\ N_{k-2n}(a_3) \cdot 1\{k - 1 \leq u < k\} & k \in I_4 \end{aligned} \quad [24]$$

$$\begin{aligned} |b_k| \cdot 1\{-k \leq v < -k + 1\} & k \in I_1 \\ \frac{1}{2} \psi_k(b_1) \cdot 1\{k - 1 \leq v < k\} & k \in I_2 \\ \frac{1}{2} \psi_{k-n}(b_2) \cdot 1\{k - 1 \leq v < k\} & k \in I_3 \\ \frac{1}{2} \psi_{k-2n}(b_3) \cdot 1\{k - 1 \leq v < k\} & k \in I_4 \end{aligned} \quad [25]$$

The symbols $I_1..I_4$ stand for: $I_1 = +3, +2, +1$; $I_2 = 1, \dots, n$; $I_3 = n + 1, \dots, 2n$; $I_4 = 2n + 1, \dots, 3n$, and $1\{\cdot\}$ denotes the indicator function. Furthermore, let $\delta_{jk} = 1$ if $j = k$, and zero otherwise, be the Kronecker symbol. We set

$$\begin{aligned} \kappa(u, v) &= \delta_{ij} \cdot 1\{i - 1 \leq u < i\} \cdot 1\{j - 1 \leq v < j\} \\ i, j &= -2, -1, \dots, 3n. \end{aligned} \quad [26]$$

We finally define the density $\rho_{\mathbf{ab}}$ by

$$\rho_{\mathbf{ab}}(u, v) = \sigma_{\mathbf{a}}(u)\tau_{\mathbf{b}}(v)\kappa(u, v) \quad [27]$$

and the measure $\mu_{\mathbf{ab}}$ by having density $\rho_{\mathbf{ab}}$ with respect to Lebesgue measure. This definition, of course, entails that $\mu_{\mathbf{ab}}$ is a sum of setting-dependent subspace product measures. The integrals that we have to perform will then correspond to summations over integrals of such product measures.

From the above definitions, we obtain the following integrals for the spin-pair correlation functions:

$$\begin{aligned} &\int_{[-3, 0]^2} A_{\mathbf{a}}(u)B_{\mathbf{b}}(v)\rho_{\mathbf{ab}}(u, v)dudv \\ &= - \sum_{k=1}^3 |a_k||b_k|\text{sign}(a_k)\text{sign}(b_k) = -\mathbf{a} \cdot \mathbf{b}. \end{aligned} \quad [28]$$

Furthermore, the integral over the complement of the square $[-3, 0]^2$ vanishes, i.e.,

$$\int_{\Omega \setminus [-3, 0]^2} A_{\mathbf{a}}(u)B_{\mathbf{b}}(v)\rho_{\mathbf{ab}}(u, v)dudv = 0, \quad [29]$$

which proves Eq. 18.

It remains to be shown that $\rho_{\mathbf{ab}}$ defines a measure μ that is close, within ε , to a probability measure, i.e., fulfills Eq. 16. For this, we consider the mass distribution between the square $[-3, 0]^2$ and its complement. The amount of mass M_1 distributed over $[-3, 0]^2$ is

$$M_1 = \sum_{k=1}^3 |a_k||b_k|. \quad [30]$$

The mass M_2 of $\Omega \setminus [-3, 0]^2$ equals

$$M_2 = \frac{1}{2} \sum_{k=1}^3 \sum_{i=1}^n N_i(|a_k|)\psi_i(|b_k|). \quad [31]$$

Thus the total mass distributed equals in view of Eq. 21

$$M_1 + M_2 = \sum_{k=1}^3 |a_k||b_k| + \frac{1}{2} \sum_{k=1}^3 \sum_{i=1}^n N_i(|a_k|)\psi_i(|b_k|)$$

$$M_1 + M_2 = \sum_{k=1}^3 |a_k||b_k| + \frac{1}{2} \sum_{k=1}^3 (|a_k| - |b_k|)^2 + \theta n^{-2}$$

$$M_1 + M_2 = 1 + \theta n^{-2} < 1 + \varepsilon, \quad [32]$$

where $0 \leq \theta < 1$. For the case $\mathbf{b} = \mathbf{a}$, we have

$$0 \leq M_2 < \theta n^{-2} < \varepsilon \quad [33]$$

As was observed right after the definitions of A and B , Eq. 19 holds for all $(u, v) \in \cup_{k=1}^3 [-k, -k+1]^2$ and thus for all $(u, v) \in \Omega$, except, perhaps, on a set of μ -measure $< \varepsilon$. This completes the proof of the theorem.

The proof clearly shows that for $\mathbf{b} = \mathbf{a}$, we can choose Ω to be a probability space, i.e., $\mu(\Omega) = 1$.

Suppose now that $\Lambda_{\mathbf{a},t}^1, \Lambda_{\mathbf{b},t}^2$ are mixed parameters as defined above. Let f and g be real-valued bounded functions on the space of the $\Lambda_{\mathbf{a},t}^1$ s and $\Lambda_{\mathbf{b},t}^2$ s. We do not assume that these two λ -spaces are identical, nor is it necessary to specify them at this point. However, we need to assume that, for fixed $\mathbf{a}, \mathbf{b}$ and time operators, the mappings $f(\Lambda_{\mathbf{a},t}^1)$ and $g(\Lambda_{\mathbf{b},t}^2)$ from $\Omega \rightarrow R$ are measurable, so that they can be considered as random variables. Because f and g are assumed to be bounded, we may assume without loss of generality that the ranges of $f(\Lambda_{\mathbf{a},t}^1)$ and $g(\Lambda_{\mathbf{b},t}^2)$ equal the interval, $[-3, 3n]$. A mathematical model for EPR experiments can now be obtained by an application of the theorem. For fixed time operators and source parameters λ , we define the joint density of $f(\Lambda_{\mathbf{a},t}^1)$ and $g(\Lambda_{\mathbf{b},t}^2)$ to equal $\rho_{\mathbf{ab}}(u, v)$, as defined in Eq. 27. Then by Eq. 18 and by the standard transformation formula for integrals, we have for fixed λ and time operators $O_{\mathbf{a},t}^1, O_{\mathbf{b},t}^2$:

$$E\{A_{\mathbf{a},t}(\lambda^1, \cdot, f(\Lambda_{\mathbf{a},t}^1(\lambda^1, \cdot)))B_{\mathbf{b},t}(\lambda^2, \cdot, g(\Lambda_{\mathbf{b},t}^2(\lambda^2, \cdot)))\} = -\mathbf{a} \cdot \mathbf{b}. \quad [34]$$

Here the expectation E operates on the space of ω , a subspace of Ω ; the dummy variable ω of the integration is symbolized by $(\cdot)$.

This direct application does not address the key question whether the introduced probability measure is free of the suspicion of spooky action at a distance. To prove this claim, we need to ensure the following. If setting $\mathbf{b}$ at station S_2 is changed into setting $\mathbf{c}$, the probability distribution governing the parameters $\Lambda_{\mathbf{a},t}^1$ at station S_1 must remain unchanged. This task can be achieved in the following way. Choose any of the $(n+1)^2$ squares Q_{jk} with vertices at points $(3j, 3k), (3(j+1), 3k), (3(j+1), 3(k+1))$, and $(3j, 3(k+1))$ for $j, k = -1, 0, 1, 2, \dots, n-1$. Now repeat the entire construction with Q_{jk} replacing Q_{-1-1} . Define A and B to be equal to $\text{sign}(a_i)$ or $-\text{sign}(b_i)$, respectively, on each of the three vertical and horizontal strips of Q_{jk} with $i = 1, 2, 3$. On the vertical and horizontal strips not

containing parts of Q_{jk} , define A and B equal ± 1 in an obvious modification of the above construction. Next assign mass M_1 to Q_{jk} and mass M_2 to the complement $\Omega \setminus Q_{jk}$ of Q_{jk} . M_2 will be distributed on $3n$ unit squares as follows: Q_{jk} and the vertical and horizontal strips associated with them take a total of $(3n+3) \cdot 3 \cdot 2 - 9 = 18n + 9$ unit squares. From the remaining $9n^2$ unit squares, we choose $3n$ and distribute the mass $(1/2)N_i(|a_k|)\psi_i(|b_k|)$ on them (with $1 \leq i \leq n$ and $1 \leq k \leq 3$). For given Q_{jk} , this yields $N = (n+1)^2 \binom{9n^2}{3n}$ possible measures μ_m with $1 \leq m \leq N$. For each of these measures, Eqs. 15–19 hold. Label the corresponding functions A and B as $A_{(m)}$ and $B_{(m)}$ and consider the index (m) a function of the source parameter $\lambda = (\lambda^1, \lambda^2)$ and the time operators $O_{\mathbf{a},t}^1, O_{\mathbf{b},t}^2$. Then the functions $A_{(m)}$ and $B_{(m)}$ can be considered as functions of $\mathbf{a}, \lambda, \Lambda_{\mathbf{a},t}^1$, and $\mathbf{b}, \lambda, \Lambda_{\mathbf{b},t}^2$, respectively. Finally, define a new measure μ on Ω by setting

$$\mu = \frac{1}{N} \sum_{m=1}^N \mu_m. \quad [35]$$

At this point, we consider Ω as the union of N layers of the above type stacked up in three dimensions, reinterpreting A, B and μ accordingly.

The density $\rho_{\mathbf{ab}}$ is now defined on the domain

$$\rho_{\mathbf{ab}}(u, v, m) \text{ with } -3 \leq u, v < 3n; m = 1, 2, \dots, N. \quad [36]$$

For fixed u and v , the joint density governing the pair of parameters $f(\Lambda_{\mathbf{a},t}^1)$ and $g(\Lambda_{\mathbf{b},t}^2)$ is given by

$$= \frac{1}{N} \sum_{m=1}^N \rho_{\mathbf{ab}}(u, v, m)$$

$$= \frac{1}{(3n+3)^2} \left(\sum_{k=1}^3 |a_k||b_k| + \frac{1}{2} \sum_{k=1}^3 \sum_{i=1}^n N_i(|a_k|)\psi_i(|b_k|) \right)$$

$$= \frac{M_1 + M_2}{(3n+3)^2} = \frac{1 + \theta n^{-2}}{(3n+3)^2},$$

with $0 \leq \theta < 1$ in view of Eq. 32. This shows that the joint density of $f(\Lambda_{\mathbf{a},t}^1), g(\Lambda_{\mathbf{b},t}^2)$ is uniform over the square $[-3, 3n]^2$ and therefore $f(\Lambda_{\mathbf{a},t}^1)$ and $g(\Lambda_{\mathbf{b},t}^2)$ considered as random variables are stochastically independent, and themselves have uniform distribution over the appropriate intervals. Therefore, if the setting $\mathbf{b}$ gets changed to the setting $\mathbf{c}$, the random variables $f(\Lambda_{\mathbf{a},t}^1), g(\Lambda_{\mathbf{c},t}^2)$ are also independent, and there is no change in the distribution of $f(\Lambda_{\mathbf{a},t}^1)$ by changing from $\mathbf{b}$ to $\mathbf{c}$.

Conclusion

We have presented a mathematical framework that can derive the quantum result for the spin-pair correlation in EPR-type experiments by use of hidden parameters. A key element of our approach is contained in the introduction of TLCPs and setting-dependent functions of them, leading in a natural way to a setting-dependent probability measure. The construction of this probability measure is complicated by the fact that spooky action must not be introduced indirectly. This is accomplished by letting the probability measure be a superposition of setting-dependent subspace product measures with two important properties: (i) the factors of the product measure depend only on parameters of the station that they describe, and (ii) the joint density of the pairs of setting-dependent parameters in the two stations is uniform. The mathematical basis for this factorization is the theory of B-splines.

Support of the Office of Naval Research (NOOO14-98-1-0604 and MURI) is gratefully acknowledged.

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