Opinion piece

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An Eulerian formulation of inelasticity: from metal plasticity to growth of biological tissues

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The purpose of this paper is to review and contrast the Lagrangian and Eulerian formulations of inelasticity as they apply to metal plasticity and growth of biological tissues. In contrast with the Lagrangian formulation of inelasticity, the Eulerian formulation is unaffected by arbitrary choices of the reference configuration, an intermediate configuration, a total deformation measure and an inelastic deformation measure. Although the Eulerian formulation for growth of biological tissues includes a rate of mass supply and can be used to understand the mechanics of growth, it does not yet model essential mechanobiological processes that control growth. Much research is needed before this theory can help design medical treatments for growth related disease.

This article is part of the theme issue 'Rivlin's legacy in continuum mechanics and applied mathematics'.

1. Continuum modelling of inelasticity

Prof. Ronald S. Rivlin was one of the most influential researchers in finite deformation continuum mechanics in the twentieth century. One of Rivlin's perhaps most important contributions was stimulated by his desire to understand why Scotch tape is sticky. This led him to realize that no mathematical theory existed at the time for understanding large deformations. Rivlin played a central role in developing the theory of finite elasticity and the associated large deformation constitutive equations. He developed the Mooney–Rivlin constitutive equation for hyperelastic solids and the Reiner–Rivlin equation for the rheology of viscous fluids.

His analytical solutions for large deformation inflation and torsion of tubes and for flow of Reiner–Rivlin fluids have been used to understand the biomechanics of tissues and blood flow. Prof. Rivlin is one of the few researchers who had the drive, talent and physical insight to create a new body of theoretical knowledge that permanently influenced our understanding of continuum mechanics. This modest paper is dedicated to his memory.

A homogeneous uniform hyperelastic solid that has a stress-free configuration is an ideal material which when unloaded returns to its stress-free state with the same volume and shape. The response of a hyperelastic material is independent of the rate of loading and the work done to load the material from any specified configuration to another specified configuration is independent of the path of loading. For these reasons, it is natural to specify a reference configuration as stress-free and to define a strain energy function based on a total strain measure from this reference configuration. The resulting formulation is called Lagrangian because the total strain and the functional form of the strain energy depend on the specification of the reference configuration.

In contrast with a hyperelastic solid, an inelastic material does not have a unique unloaded stress-free configuration even if it is a homogeneous uniform material that has been loaded from a stress-free state causing a homogeneous deformation. In this sense, the inelastic material is more like a fluid than a hyperelastic solid since the response of the material to future loading is insensitive to the specification of a reference configuration. These notions are consistent with the statement by Gilman [1, p. 99]

It seems very unfortunate to me that the theory of plasticity was ever cast into the mold of stress-strain relations because ‘strain’ in the plastic case has no physical meaning that is related to the material of the body in question. It is rather like trying to deduce some properties of a liquid from the shape of the container that holds it. The plastic behavior of a body depends on its structure (crystalline and defect) and on the system of stresses that is applied to it.

The classical theoretical models of plasticity have been developed by enriching standard hyperelastic models with measures of plastic deformation which are determined by integrating evolution equations that capture observed history and rate-dependence of inelastic material response. These models are denoted as Lagrangian formulations of plasticity.

The physics of inelasticity in metals is due to the motion of dislocations through the atomic lattice which lead to large macroscopic deformations. Large distortions of the atomic lattice occur as dislocations move through the lattice with new atoms entering the lattice as defects that cause atoms which were previously part of the lattice to be transferred to nearest neighbours of the lattice. This means that plasticity causes a flux of atoms, with atoms moving through a specific atomic lattice from nearest neighbours to nearest neighbours. This causes the nearest neighbours of a specific atom to change with time. In this sense, plasticity theory is foreign to standard macroscopic continuum models which assume that material points in a material region remain in the material region.

Macroscopic continuum theories of plasticity tacitly assume that a material point contains a large number of atoms and that the flux of atoms due to dislocation motion is balanced with the total number of atoms occupying the material point being constant even though the individual atoms that occupy the material point change with time. An important aspect of the dislocation motion to be modelled is that the distortions of the atomic lattice remain small after the dislocations have caused the transfer of atoms between the nearest neighbours of a specific atomic lattice. Also, these small distortions of the atomic lattice are the elastic deformations that cause macroscopic stress.

Eckart [2] seems to be the first to recognize that since elastic deformations cause stress, it is more physical to propose evolution equations directly for elastic deformation measures. This was done for elastically isotropic inelastic materials and was later proposed by Leonov [3] for elastically isotropic polymeric liquids. These equations are Eulerian in nature since the evolution

equations depend on the current values of the velocity gradient and the state variables that characterize the material response. In this regard, it is noted that Onat [4] discussed the notion of state and emphasized that state variables which are determined by evolution equations that capture the main physics of history dependence of the material response must be measurable, in principle. An Eulerian formulation of plasticity theory valid for elastically anisotropic response has been formulated in [5].

Arbitrariness of the choices of a reference configuration, an intermediate configuration, a total deformation measure and a plastic deformation measure have been discussed in a series of papers [6–8] which emphasize the physical importance of an Eulerian formulation of inelasticity. In this regard, it is noted that Rajagopal & Srinivasa [9] discuss a Eulerian formulation of implicit constitutive equations for inelastic response based on a Gibbs potential. This formulation proposes an evolution equation directly for Cauchy stress. By contrast, here evolution equations are proposed for elastic deformation measures which are used in a Helmholtz free energy to determine a hyperelastic algebraic expression for the Cauchy stress.

One objective of the present paper is to emphasize that the Eulerian formulation of inelasticity is a unified formulation that can model viscoelastic fluids, elastic–inelastic solids and growth of biological tissues. An outline of this paper is as follows. Section 2 presents basic equations, §3 reviews formulations for metal plasticity and §4 presents formulations for growth of biological tissues. Section 5 discusses strongly object, robust integration algorithms and §6 presents conclusion.

2. Basic equations

Recall that a material point which in the current configuration is located by the vector $\mathbf{x}$ relative to an inertial point has velocity $\mathbf{v}$ given by

$$\mathbf{v} = \dot{\mathbf{x}}, \quad (2.1)$$

where a superposed ($\dot{}$) denotes material differentiation with respect to time t . Moreover, the velocity gradient $\mathbf{L}$ and the rate of deformation tensor $\mathbf{D}$ are defined by

$$\mathbf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \quad \text{and} \quad \mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T). \quad (2.2)$$

The conservation of mass and the balance of linear momentum are given by

$$\dot{\rho} + \rho \mathbf{D} \cdot \mathbf{I} = 0 \quad \text{and} \quad \rho \dot{\mathbf{b}} = \rho \mathbf{b} + \text{div} \mathbf{T}, \quad (2.3)$$

where ρ is the current mass density, $\mathbf{I}$ is the second-order identity tensor, $\mathbf{A} \cdot \mathbf{B} = \text{tr}(\mathbf{A}\mathbf{B}^T)$ is the inner product between two second-order tensors ($\mathbf{A}, \mathbf{B}$), $\mathbf{b}$ is the body force per unit mass, $\mathbf{T}$ is the Cauchy stress and $\text{div}()$ denotes the divergence operator relative to the current position $\mathbf{x}$. In addition, the reduced form of the balance of angular momentum requires $\mathbf{T}$ to be symmetric

$$\mathbf{T}^T = \mathbf{T}. \quad (2.4)$$

Within the context of the purely mechanical theory, the rate of material dissipation $\mathcal{D}$ is determined by the rate of work done on the body by body forces and surface tractions minus the rates of change of kinetic and strain energy and is given by

$$\mathcal{D} = \mathbf{T} \cdot \mathbf{D} - \rho \dot{\Sigma} \geq 0, \quad (2.5)$$

where Σ is the strain energy function per unit mass.

Under superposed rigid body motions (SRBM), the position $\mathbf{x}$ of a material point is transformed to $\mathbf{x}^+$ and time t is transformed to t^+ , such that

$$\mathbf{x}^+ = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{x}, \quad t^+ = t + c, \quad (2.6)$$

where $\mathbf{c}(t)$ is an arbitrary vector function of time that characterizes superposed translation, $\mathbf{Q}(t)$ is a proper orthogonal tensor that characterizes superposed rotations

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{I}, \quad \det(\mathbf{Q}) = +1, \quad \dot{\mathbf{Q}} = \boldsymbol{\Omega}\mathbf{Q} \quad \text{and} \quad \boldsymbol{\Omega}^T = -\boldsymbol{\Omega}, \quad (2.7)$$

with $\boldsymbol{\Omega}$ being a skew-symmetric tensor function of time only and c is a constant time shift. Under SRBM, it can be shown that $(\mathbf{L}, \mathbf{D})$ transform to $(\mathbf{L}^+, \mathbf{D}^+)$, such that

$$\mathbf{L}^+ = \mathbf{Q}\mathbf{L}\mathbf{Q}^T + \boldsymbol{\Omega} \quad \text{and} \quad \mathbf{D}^+ = \mathbf{Q}\mathbf{D}\mathbf{Q}^T, \quad (2.8)$$

and that the balance laws remain unaffected by SRBM when $(\rho, \mathbf{b}, \mathbf{T})$ transform to $(\rho^+, \mathbf{b}^+, \mathbf{T}^+)$, such that

$$\rho^+ = \rho, \quad \mathbf{b}^+ = \dot{\mathbf{v}}^+ + \mathbf{Q}(\mathbf{b} - \dot{\mathbf{v}}) \quad \text{and} \quad \mathbf{T}^+ = \mathbf{Q}\mathbf{T}\mathbf{Q}^T. \quad (2.9)$$

3. Formulations of metal plasticity

(a) Lagrangian formulations

Lagrangian formulations of plasticity enrich the theory of hyperelastic solids with a plastic deformation measure that captures observed effects of history and rate-dependence of material response. A summary of the small deformation theory within the context of thermodynamics can be found in the classical paper by Naghdi [10]. Unfortunately, the large deformation theory of plasticity still is plagued with controversies, some of which have been discussed in the critical review [11]. This section discusses three prominent formulations of large deformation theory: one by Green & Naghdi [12], another attributed to Bilby *et al.* [13], Kröner [14] and Lee [15], and another attributed to Besseling [16].

(b) Green and Naghdi formulation

Green & Naghdi [12] developed a large deformation themomechanical theory of plasticity. Confining attention to the purely mechanical response, this theory introduces the total deformation gradient $\mathbf{F}$ and the total dilatation J , which satisfy the equations

$$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}, \quad J = \frac{\rho_0}{\rho} \quad \text{and} \quad \dot{J} = J\mathbf{D} \cdot \mathbf{I}, \quad (3.1)$$

where ρ_0 is the mass density in the reference configuration. Also, the right Cauchy–Green deformation tensor $\mathbf{C}$ satisfies the equations

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} \quad \text{and} \quad \dot{\mathbf{C}} = 2\mathbf{F}^T\mathbf{D}\mathbf{F}. \quad (3.2)$$

The theory of hyperelasticity is enriched by introducing a symmetric plastic deformation tensor $\mathbf{C}_p$ (similar to $\mathbf{C}$) and a scalar measure of hardening κ by the evolution equations

$$\dot{\mathbf{C}}_p = \Gamma \bar{\mathbf{A}}_p \quad \text{and} \quad \dot{\kappa} = \Gamma H, \quad \Gamma \geq 0, \quad (3.3)$$

where $\bar{\mathbf{A}}_p$ controls the direction of plastic deformation rate, Γ is a non-negative function that controls the magnitude of plastic deformation rate and H controls the rate of hardening. Examples of functional forms of Γ and its dependence on elastic deformation and κ can be found in references cited in §1. For metals, plastic deformation rate is isochoric so $\mathbf{C}_p$ remains unimodular, which imposes the following restrictions:

$$\det(\mathbf{C}_p) = 1 \quad \text{and} \quad \bar{\mathbf{A}}_p \cdot \mathbf{C}_p^{-1} = 0. \quad (3.4)$$

Under SRBM, the total deformation tensor $\mathbf{F}$, the right Cauchy–Green tensor $\mathbf{C}$, the plastic deformation $\mathbf{C}_p$ and the hardening variable κ transform to $(\mathbf{F}^+, \mathbf{C}^+, \mathbf{C}_p^+, \kappa^+)$, such that

$$\mathbf{F}^+ = \mathbf{Q}\mathbf{F}, \quad \mathbf{C}^+ = \mathbf{C}, \quad \mathbf{C}_p^+ = \mathbf{C}_p \quad \text{and} \quad \kappa^+ = \kappa, \quad (3.5)$$

which place restrictions on the functional forms of $(\Gamma, \bar{\mathbf{A}}_p, H)$.

In this theory, the strain energy Σ is assumed to be a function of $\mathbf{F}$ but since Σ is uninfluenced by SRBM it must depend on $\mathbf{F}$ only through the deformation tensor $\mathbf{C}$ so that

$$\Sigma = \Sigma(\mathcal{V}) \quad \text{and} \quad \mathcal{V} = (\mathbf{C}, \mathbf{C}_p, \kappa). \quad (3.6)$$

For both the rate-independent theory with a yield function (e.g. [17]) and the rate-dependent theory, the constitutive equation for stress is taken in the form

$$\mathbf{T} = 2\rho \mathbf{F} \frac{\partial \Sigma}{\partial \mathbf{C}} \mathbf{F}^T. \quad (3.7)$$

Moreover, the rate of material dissipation (2.5) requires

$$\mathcal{D} = -\Gamma \rho \left(\frac{\partial \Sigma}{\partial \mathbf{C}_p} \cdot \bar{\mathbf{A}}_p + \frac{\partial \Sigma}{\partial \kappa} H \right) \geq 0, \quad (3.8)$$

which places a restriction on the functional form of the strain energy Σ .

In addition to solving the conservation of mass and the balance of linear momentum (2.3), this theory requires solution of the evolution equations (3.1) and (3.3) with initial conditions

$$[\mathbf{F}(0), \mathbf{C}_p(0), \kappa(0)]. \quad (3.9)$$

(c) Bilby, Kröner, Lee formulation

Bilby *et al.* [13], Kröner [14] and Lee [15] introduced a formulation that depends on a second-order non-symmetric plastic deformation tensor $\mathbf{F}_p$ (similar to $\mathbf{F}$) determined by an evolution equation of the form

$$\dot{\mathbf{F}}_p = \Gamma \bar{\mathbf{A}}_p \mathbf{F}_p, \quad (3.10)$$

where Γ controls the magnitude of the plastic deformation rate and $\bar{\mathbf{A}}_p$ controls its direction, both of which require a constitutive equation. Again, for metal plasticity the plastic deformation rate is isochoric so $\mathbf{F}_p$ is unimodular and $\bar{\mathbf{A}}_p$ is restricted, such that

$$\det(\mathbf{F}_p) = 1 \quad \text{and} \quad \bar{\mathbf{A}}_p \cdot \mathbf{I} = 0. \quad (3.11)$$

Moreover, an elastic deformation tensor $\mathbf{F}_e$ is defined by the multiplicative form

$$\mathbf{F}_e = \mathbf{F} \mathbf{F}_p^{-1}, \quad (3.12)$$

and a hardening variable κ is introduced which satisfies the evolution equation (3.3). Usually this equation is written in the form $\mathbf{F} = \mathbf{F}_e \mathbf{F}_p$, which suggests that $\mathbf{F}_p$ transforms the reference configuration into an intermediate stress-free configuration and $\mathbf{F}_e$ transforms the intermediate configuration into the present configuration. For general inhomogeneous deformations, $\mathbf{F}$ describes a compatible field with the position $\mathbf{x}$ of a material point in the present configuration being a differentiable function of the position $\mathbf{X}$ of the same material point in the reference configuration. However, both $\mathbf{F}_p$ and $\mathbf{F}_e$ are incompatible tensors which are not determined by differentiation of deformation fields so unloading the material yields a configuration which has residual stresses. In other words, in general, it is not possible to unload the material to a stress-free intermediate configuration.

Under SRBM $(\Gamma, \bar{\mathbf{A}}_p, \mathbf{F}_p)$ transform to $(\Gamma^+, \bar{\mathbf{A}}_p^+, \mathbf{F}_p^+)$, such that

$$\Gamma^+ = \Gamma, \quad \bar{\mathbf{A}}_p^+ = \bar{\mathbf{A}}_p \quad \text{and} \quad \mathbf{F}_p^+ = \mathbf{F}_p. \quad (3.13)$$

It then follows from (3.5) and (3.12) that under SRBM the elastic deformation tensors $(\mathbf{F}_e, \mathbf{C}_e)$ transforms to $(\mathbf{F}_e^+, \mathbf{C}_e^+)$, such that

$$\mathbf{F}_e^+ = \mathbf{Q} \mathbf{F}_e, \quad \mathbf{C}_e = \mathbf{F}_e^T \mathbf{F}_e \quad \text{and} \quad \mathbf{C}_e^+ = \mathbf{C}_e. \quad (3.14)$$

In this theory, the strain energy Σ is assumed to be a function of $\mathbf{F}_e$ but since Σ is uninfluenced by SRBM it must depend on $\mathbf{F}_e$ only through the deformation tensor $\mathbf{C}_e$ so that

$$\Sigma = \Sigma(\mathbf{C}_e, \kappa). \quad (3.15)$$

For both rate-independent and rate-dependent response, the stress is specified by

$$\mathbf{T} = 2\rho \mathbf{F}_e \frac{\partial \Sigma}{\partial \mathbf{C}_e} \mathbf{F}_e^T, \quad (3.16)$$

and the rate of material dissipation (2.5) requires

$$\mathcal{D} = \Gamma \rho \left(2\mathbf{C}_e \frac{\partial \Sigma}{\partial \mathbf{C}_e} \cdot \bar{\mathbf{A}}_p - \frac{\partial \Sigma}{\partial \kappa} H \right) \geq 0, \quad (3.17)$$

which places a restriction on the functional form of the strain energy Σ .

In addition to solving the conservation of mass and the balance of linear momentum (2.3), this theory requires solution of the evolution equations (3.1), (3.3) for κ and (3.10) with initial conditions

$$[\mathbf{F}(0), \mathbf{F}_p(0), \kappa(0)]. \quad (3.18)$$

(d) Besseling formulation

The formulation discussed by Besseling [16] (see also Besseling & van der Giessen [18]) was motivated by the work of Eckart [2] and Mandel [19] and can be interpreted as proposing an evolution equation for a non-singular second-order non-symmetric tensor $\mathbf{F}_e$ directly by the evolution equation

$$\dot{\mathbf{F}}_e = \mathbf{L}_e \mathbf{F}_e \quad \text{and} \quad \mathbf{L}_e = \mathbf{L} - \mathbf{L}_p, \quad (3.19)$$

where $\mathbf{L}_e$ is the elastic deformation rate, $\mathbf{L}_p$ is the plastic deformation rate and $\mathbf{F}_e$ measures elastic deformations from a stress-free intermediate configuration. Since $\mathbf{L}_p$ is a general second-order tensor, this formulation includes the notion of plastic spin $\mathbf{W}_p$

$$\mathbf{W}_p = \frac{1}{2}(\mathbf{L}_p + \mathbf{L}_p^T), \quad (3.20)$$

which can be used to model the changing orientation of the atomic lattice due to plastic deformation. Moreover, the evolution equation (3.19) will be identical to the evolution equation for $\mathbf{F}_e$ in (3.12) if $\mathbf{L}_p$ is specified by

$$\mathbf{L}_p = \Gamma \mathbf{F}_e \bar{\mathbf{A}}_p \mathbf{F}_e^{-1}, \quad (3.21)$$

which under SRBM satisfies the transformation relation

$$\mathbf{L}_p^+ = \mathbf{Q} \mathbf{L}_p \mathbf{Q}^T. \quad (3.22)$$

(e) Eulerian formulation of elastically isotropic elastic–inelastic materials

Following the work of Eckart [2] and Leonov [3] for elastically isotropic elastic–inelastic materials and using the work of Flory [20] to obtain a pure measure of distortional deformation, it is convenient to introduce an evolution equation for a symmetric positive-definite unimodular tensor $\mathbf{B}'_e$

$$\dot{\mathbf{B}}'_e = \mathbf{L} \mathbf{B}'_e + \mathbf{B}'_e \mathbf{L}^T - \frac{2}{3}(\mathbf{D} \cdot \mathbf{I}) \mathbf{B}'_e - \Gamma \mathbf{A}_p, \quad (3.23)$$

where Γ controls the magnitude of the rate of inelasticity and the direction of inelastic deformation rate $\mathbf{A}_p$ must satisfy the restriction

$$\mathbf{A}_p \cdot \mathbf{B}'_e{}^{-1} = 0, \quad (3.24)$$

to ensure that $\mathbf{B}'_e$ remains unimodular [$\det(\mathbf{B}'_e) = 1$]. Also, the dilatation J is defined by (3.1) with ρ_0 being the mass density in a stress-free state (that need not be associated with any reference configuration).

In this model, the strain energy function for elastically isotropic response is taken to be a function of the total dilatation J , the elastic distortional deformation $\mathbf{B}'_e$ and the hardening κ .

However, under SRBM $(J, \mathbf{B}'_e, \mathbf{A}_p)$ transform to $(J^+, \mathbf{B}'_e^+, \mathbf{A}_p^+)$, such that

$$J^+ = J, \quad \mathbf{B}'_e^+ = \mathbf{Q}\mathbf{B}'_e\mathbf{Q}^T \quad \text{and} \quad \mathbf{A}_p^+ = \mathbf{Q}\mathbf{A}_p\mathbf{Q}^T, \quad (3.25)$$

so the strain energy function can depend on $\mathbf{B}'_e$ only through its two independent invariants (α_1, α_2) , defined by

$$\alpha_1 = \mathbf{B}'_e \cdot \mathbf{I} \quad \text{and} \quad \alpha_2 = \mathbf{B}'_e \cdot \mathbf{B}'_e. \quad (3.26)$$

Thus, the strain energy function Σ is taken in the form

$$\Sigma = \Sigma(J, \alpha_1, \alpha_2, \kappa), \quad (3.27)$$

and for both rate-independent and rate-dependent response the stress is specified by

$$\left. \begin{aligned} \mathbf{T} &= -p\mathbf{I} + \mathbf{T}'', \quad p = -\rho_0 \frac{\partial \Sigma}{\partial J} \\ \text{and} \quad \mathbf{T}'' &= 2\rho \left[\frac{\partial \Sigma}{\partial \alpha_1} (\mathbf{B}'_e - \frac{1}{3}\alpha_1\mathbf{I}) + 2 \frac{\partial \Sigma}{\partial \alpha_2} \left(\mathbf{B}'_e^2 - \frac{1}{3}\alpha_2\mathbf{I} \right) \right], \end{aligned} \right\} \quad (3.28)$$

where $\mathbf{T}''$ is the deviatoric part of $\mathbf{T}$. Also, the rate of material dissipation $\mathcal{D}$ in (2.5) requires

$$\mathcal{D} = \Gamma\rho \left(\frac{\partial \Sigma}{\partial \alpha_1} \mathbf{A}_p \cdot \mathbf{I} + 2 \frac{\partial \Sigma}{\partial \alpha_2} \mathbf{A}_p \cdot \mathbf{B}'_e - \frac{\partial \Sigma}{\partial \kappa} H \right) \geq 0. \quad (3.29)$$

Moreover, the strain energy function is restricted so that the material is stress-free when

$$\mathbf{T} = 0, \quad \frac{\partial \Sigma}{\partial J} = 0 \quad \text{for } J = 1 \text{ and } \mathbf{B}'_e = \mathbf{I}. \quad (3.30)$$

The evolution equations (3.1) for J , (3.23) for $\mathbf{B}'_e$ and (3.3) for κ require initial conditions

$$[J(0), \mathbf{B}'_e(0), \kappa(0)]. \quad (3.31)$$

In this regard, it is assumed that the constitutive equation (3.28) for stress is invertible and that experiments can be performed to determine the value of hardening κ at any state of the material. In particular, the values of $(J, \mathbf{B}'_e)$ in any stress-free state are given by (3.30). Also, when Γ vanishes, the theory represents an Eulerian formulation of a general elastically isotropic hyperelastic material.

Since the rate of plasticity causes a tendency for the deviatoric stress $\mathbf{T}''$ to approach zero, Rubin & Attia [21] proposed $\mathbf{A}_p$ in the form

$$\mathbf{A}_p = \mathbf{B}'_e - \left(\frac{3}{\mathbf{B}'_e^{-1} \cdot \mathbf{I}} \right) \mathbf{I}. \quad (3.32)$$

As a special case, the strain energy function is given by

$$\Sigma = f(J) + \frac{1}{2}\mu(\alpha_1 - 3), \quad (3.33)$$

the stress from (3.28) is specified by

$$\mathbf{T} = -p\mathbf{I} + \mathbf{T}'', \quad p = -\frac{df}{dJ}, \quad \mathbf{T}'' = J^{-1}\mu \mathbf{B}''_e \quad \text{and} \quad \mathbf{B}''_e = \mathbf{B}'_e - \frac{1}{3}\alpha_1\mathbf{I}, \quad (3.34)$$

and the rate of material dissipation (3.29) requires

$$\mathcal{D} = \frac{1}{2}J^{-1}\mu\Gamma\mathbf{A}_p \cdot \mathbf{I} = \frac{1}{2}J^{-1}\mu\Gamma \left(\alpha_1 - \frac{9}{\mathbf{B}'_e^{-1} \cdot \mathbf{I}} \right) \geq 0, \quad (3.35)$$

which is automatically satisfied since $\mathbf{B}'_e$ is a unimodular positive-definite tensor. In these equations, the function $f(J)$ controls the response to dilatation, μ is the constant zero-stress shear modulus and $\mathbf{B}''_e$ is the deviatoric part of $\mathbf{B}'_e$.

(f) Eulerian formulation of elastically anisotropic elastic–inelastic materials

An Eulerian formulation for elastically anisotropic inelastic material response, which was motivated by the work of Eckart [2] and Leonov [3], was developed in [5]. The main idea is to model the following physical features of plastic flow in metals:

- elastic deformations of the atomic lattice cause stress,
- elastic deformations of the atomic lattice remain small after dislocations have moved through the lattice,
- the atoms in a specific lattice change with time as dislocations move through the lattice and
- edges of the parallelepiped formed by the atomic lattice do not rotate as material line elements.

In this model, the elastic deformations and orientation of the atomic lattice are modelled by the parallelepiped formed by the triad $\mathbf{m}_i$ ($i = 1, 2, 3$) of linearly independent microstructural vectors

$$m^{1/2} = \mathbf{m}_1 \times \mathbf{m}_2 \cdot \mathbf{m}_3 \geq 0. \quad (3.36)$$

These vectors are determined by the evolution equations

$$\dot{\mathbf{m}}_i = (\mathbf{L} - \mathbf{L}_p) \mathbf{m}_i, \quad \mathbf{L}_p = \Gamma \bar{\mathbf{L}}_p, \quad (3.37)$$

where Γ controls the magnitude and $\bar{\mathbf{L}}_p$ controls the direction of the rate of inelastic deformation tensor $\mathbf{L}_p$, both of which require a constitutive equation. If $\mathbf{L}_p$ vanishes, then the solution of (3.37) causes $\mathbf{m}_i$ to evolve as material line elements so these equations characterize an Eulerian formulation of a general anisotropic hyperelastic solid. Otherwise, $\mathbf{m}_i$ characterize the atomic lattice which is not directly connected to material line elements.

Under SRBM, $(\mathbf{L}_p, \mathbf{m}_i)$ transform to $(\mathbf{L}_p^+, \mathbf{m}_i^+)$ such that

$$\mathbf{L}_p^+ = \mathbf{Q} \mathbf{L}_p \mathbf{Q}^T \quad \text{and} \quad \mathbf{m}_i^+ = \mathbf{Q} \mathbf{m}_i. \quad (3.38)$$

The strain energy function Σ is assumed to be a function of $(\mathbf{m}_i, \kappa)$, but since Σ must be unaffected by SRBM it can depend on $\mathbf{m}_i$ only through the metric m_{ij} of elastic deformation, which satisfies the equations

$$m_{ij} = \mathbf{m}_i \cdot \mathbf{m}_j \quad \text{and} \quad m_{ij}^+ = m_{ij}. \quad (3.39)$$

Moreover, using (3.37) and (3.39) it follows that the elastic metric satisfies the evolution equation

$$\dot{m}_{ij} = 2(\mathbf{D} - \mathbf{D}_p) \cdot (\mathbf{m}_i \otimes \mathbf{m}_j), \quad (3.40)$$

where $\mathbf{a} \otimes \mathbf{b}$ denotes the tensor product between two vectors $(\mathbf{a}, \mathbf{b})$ and the plastic deformation rate $\mathbf{D}_p$ is defined by

$$\mathbf{D}_p = \frac{1}{2}(\mathbf{L}_p + \mathbf{L}_p^T) = \Gamma \bar{\mathbf{D}}_p \quad \text{and} \quad \bar{\mathbf{D}}_p = \frac{1}{2}(\bar{\mathbf{L}}_p + \bar{\mathbf{L}}_p^T). \quad (3.41)$$

For this model, the strain energy function is taken in the form

$$\Sigma = \Sigma(m_{ij}, \kappa), \quad (3.42)$$

with the stress given by

$$\mathbf{T} = 2\rho \frac{\partial \Sigma}{\partial m_{ij}} (\mathbf{m}_i \otimes \mathbf{m}_j), \quad (3.43)$$

so the rate of material dissipation (2.5) requires

$$\mathcal{D} = \Gamma \left(\mathbf{T} \cdot \bar{\mathbf{D}}_p - \rho \frac{\partial \Sigma}{\partial \kappa} H \right) \geq 0. \quad (3.44)$$

Also, the constitutive equation for stress is assumed to be restricted so that a stress-free state is characterized by

$$\mathbf{T} = 0, \quad \frac{\partial \Sigma}{\partial m_{ij}} = 0 \quad \text{for } m_{ij} = \delta_{ij}, \quad (3.45)$$

where δ_{ij} is the Kronecker delta symbol. This means that the triad $\mathbf{m}_i$ has been defined so that $\mathbf{m}_i$ are orthonormal vectors in a stress-free state.

Since $\mathbf{m}_i$ are linearly independent and not necessarily orthonormal in the present configuration, it is convenient to introduce their reciprocal vectors $\mathbf{m}^i$ by

$$\mathbf{m}^1 = m^{-1/2}(\mathbf{m}_2 \times \mathbf{m}_3), \quad \mathbf{m}^2 = m^{-1/2}(\mathbf{m}_3 \times \mathbf{m}_1) \quad \text{and} \quad \mathbf{m}^3 = m^{-1/2}(\mathbf{m}_1 \times \mathbf{m}_2), \quad (3.46)$$

so that

$$\frac{d}{dt}(m^{1/2}) = m^{1/2}(\mathbf{D} - \mathbf{D}_p) \cdot \mathbf{I}. \quad (3.47)$$

Thus, for isochoric plastic deformation

$$\mathbf{D}_p \cdot \mathbf{I} = 0, \quad (3.48)$$

and it follows with the help of (2.3), (3.1) and (3.47) that

$$\frac{d}{dt} \left[\ln \left(\frac{m^{1/2}}{J} \right) \right] = 0. \quad (3.49)$$

Moreover, since ρ_0 is the mass density in a stress-free state, (3.1) and (3.49) require

$$m^{1/2} = J. \quad (3.50)$$

Also, for elastically isotropic response there can be no preferred dependence on individual $\mathbf{m}_i$ so it is convenient to introduce the symmetric, positive definite tensor $\mathbf{B}_e = \mathbf{m}_i \otimes \mathbf{m}_i$. It then follows that $J = \sqrt{\det(\mathbf{B}_e)}$ and $\mathbf{B}'_e = J^{-2/3} \mathbf{B}_e$, which are determined by the evolution equations (3.1) and (3.23), without the need for specifying $\mathbf{L}_p$ in (3.37).

(g) Arbitrariness

From the perspective of the definition of state variables by Onat [4], the total deformation tensor $\mathbf{F}$, the plastic deformation tensors ($\mathbf{C}_p, \mathbf{F}_p$) and the elastic deformation tensors ($\mathbf{F}_e, \mathbf{C}_e$) are not state variables since they cannot be measured, in principle. In particular, they are affected by arbitrariness of the choices of: the reference configuration; an intermediate configuration; a total deformation measure and a plastic deformation measure, which have been discussed in [6–8].

Rubin [8] showed that when this arbitrariness is removed from the Lagrangian multiplicative formulation associated with (3.12), that the formulation must reduce to the Eulerian formulation based on the microstructural vectors $\mathbf{m}_i$. Moreover in [8], it was shown that $\mathbf{m}_i$ are state variables in the sense of Onat [4] because their initial values can be measured, in principle, by experiments on identical samples of the material in its initial state.

Elastic anisotropy of a material with the strain energy function specified by (3.42) is characterized by the dependence of the strain energy on the vectors $\mathbf{m}_i$. It is important to emphasize that the index (i) in $\mathbf{m}_i$ refers to distinct directions of the atomic lattice. If any of these directions cannot be distinguished by experiments, then the strain energy function must satisfy symmetry conditions which ensure that the material response is also insensitive to these indistinguishable directions.

Comparison of the evolution equation (3.19) for $\mathbf{F}_e$ and (3.37) for $\mathbf{m}_i$ suggests that these formulations may be identical. To understand the differences between these formulations,

consider an *arbitrary* right-handed orthonormal set of constant base vectors $\mathbf{M}_i$ and define the elastic deformation tensor $\mathbf{F}_e$ by

$$\mathbf{F}_e = \mathbf{m}_i \otimes \mathbf{M}_i, \quad (3.51)$$

which satisfies the evolution equation and initial condition

$$\dot{\mathbf{F}}_e = (\mathbf{L} - \mathbf{L}_p) \mathbf{F}_e \quad \text{and} \quad \mathbf{F}_e(0) = \mathbf{m}_i(0) \otimes \mathbf{M}_i. \quad (3.52)$$

However, in [8] it was shown that the elastic response of the material depends on $\mathbf{m}_i$ through the evolution equation (3.37) and on the initial values $\mathbf{m}_i(0)$ of $\mathbf{m}_i$. Although $\mathbf{m}_i(0)$ are measurable, the tensor $\mathbf{F}_e$ contains unphysical arbitrariness of the orientation of $\mathbf{M}_i$ which can be removed by considering the Eulerian formulation based on $\mathbf{m}_i$.

(h) Rate-dependent response

For the rate-independent theory, the rate of inelastic deformation Γ is a homogeneous function of order one in the total rate of deformation $\mathbf{D}$. By contrast, if Γ is not a homogeneous function of order one in $\mathbf{D}$ then the material response is rate-dependent. Examples of rate-dependent response can be found in [22–27]. Also, a recent model exhibiting a smooth elastic–inelastic transition which can model both rate-independent and rate-dependent response can be found in [28,29].

4. Formulations for growth of biological tissues

Biological tissues are complicated materials which are mixtures of many components that can flow relative to each other and interact mechanically, chemically and electrically (e.g. [30–33]). A simplified constrained theory of mixtures with only one velocity field was developed by Humphrey & Rajagopal [30]. Also, review articles on growth and remodelling of tissues can be found in (e.g. [32,34,35]).

(a) Lagrangian formulation of growth

When the tissue is considered to be a homogenized solid, the standard approach to modelling growth for finite deformations is Lagrangian. Rodriguez *et al.* [36] developed a formulation based on the multiplicative form (3.12) by replacing the plastic deformation tensor $\mathbf{F}_p$ with a growth tensor $\mathbf{F}_g$, such that

$$\mathbf{F}_e = \mathbf{F}\mathbf{F}_g^{-1} \quad \text{and} \quad \dot{\mathbf{F}}_g = \mathbf{A}_g \mathbf{F}_g, \quad (4.1)$$

with the rate of growth $\mathbf{A}_g$ specified by a constitutive equation. However, this multiplicative formulation has the same arbitrariness as discussed previously for the Lagrangian formulation of plasticity, which can be removed by the Eulerian formulation discussed below.

(b) Eulerian formulation of growth

Rubin *et al.* [37] developed a unified theoretical structure for modelling interstitial growth and muscle activation in soft tissues. Safadi & Rubin [38] used this theory to study significant differences in the mechanical modelling of confined growth predicted by Lagrangian and Eulerian formulations and Safadi & Rubin [39] used the theory to analyse stresses in arteries. This section reviews the generalized formulation developed within the context of the purely mechanical theory in [38].

In contrast with inert materials like metals, biological tissues are living materials that have mass and energy supplies to allow them to grow and be active. The model developed in [37]

considers the material as an open system with a rate of mass supply. In this model, the balance of mass is given by

$$\dot{\rho} + \rho(\mathbf{D} \cdot \mathbf{I}) = \rho r_m, \quad (4.2)$$

where r_m is the rate of mass supply per unit mass. This model uses only a single velocity field, it ignores details of relative motion of fluids and the solid matrix and it assumes that the tissue is highly vascularized so that mass can be supplied at any point inside the tissue. Also, the rate of material dissipation in the purely mechanical form of the model is given by

$$\mathcal{D} = \mathbf{T} \cdot \mathbf{D} - \rho \dot{\Sigma} + \rho b \geq 0, \quad (4.3)$$

where Σ is the strain energy per unit mass and b is the rate of energy supply per unit mass due to mechanobiological processes. Moreover, it is assumed that b is sufficiently large to satisfy the restriction that material dissipation is non-negative. In other words, the term b allows for energy to be supplied to grow and activate the tissue if necessary.

Specifically, in [38] the elastic dilatation J_e is defined by

$$J_e = \frac{\rho_0}{\rho}, \quad (4.4)$$

where ρ_0 is the constant mass density of the tissue in any stress-free state. Moreover, to simplify the numerical integration algorithm, the rate r_m was specified in [39] by

$$r_m = \Gamma_m \ln \left(\frac{J_e}{J_h} \right), \quad (4.5)$$

so that using the balance of mass (4.2), the evolution equation for the elastic dilatation J_e takes the form

$$\frac{\dot{J}_e}{J_e} = \mathbf{D} \cdot \mathbf{I} - \Gamma_m \ln \left(\frac{J_e}{J_h} \right). \quad (4.6)$$

Also, the symmetric, positive definite, unimodular elastic distortional deformation tensor $\mathbf{B}'_e$ is determined by the evolution equation

$$\dot{\mathbf{B}}'_e = \mathbf{L}\mathbf{B}'_e + \mathbf{B}'_e\mathbf{L}^T - \frac{2}{3}(\mathbf{D} \cdot \mathbf{I})\mathbf{B}'_e - \Gamma\mathbf{A}_g \quad \text{and} \quad \mathbf{A}_g = \mathbf{B}'_e - \left(\frac{3}{\mathbf{B}'_e{}^{-1} \cdot \mathbf{H}} \right) \mathbf{H}. \quad (4.7)$$

In these equations, (Γ_m, J_h, Γ) are non-negative functions and $\mathbf{H}$ is a symmetric, positive definite tensor with unimodular part $\mathbf{H}'$

$$\Gamma_m \geq 0, \quad J_h > 0, \quad \Gamma \geq 0, \quad \mathbf{H}^T = \mathbf{H}, \quad \mathbf{H}' = \det(\mathbf{H})^{-1/3} \mathbf{H} \quad \text{and} \quad \det(\mathbf{H}') = 1. \quad (4.8)$$

Also, under SRBM the quantities $(\Gamma_m, J_h, \Gamma, \mathbf{H})$ transform to $(\Gamma_m^+, J_h^+, \Gamma^+, \mathbf{H}^+)$, such that

$$\Gamma_m^+ = \Gamma_m, \quad J_h^+ = J_h, \quad \Gamma^+ = \Gamma \quad \text{and} \quad \mathbf{H}^+ = \mathbf{Q}\mathbf{H}\mathbf{Q}^T, \quad (4.9)$$

which ensure that under SRBM, $(J_e, \mathbf{B}'_e)$ satisfy the transformation relations

$$J_e^+ = J_e \quad \text{and} \quad \mathbf{B}'_e{}^+ = \mathbf{Q}\mathbf{B}'_e\mathbf{Q}^T. \quad (4.10)$$

The terms associated with (Γ_m, J_h) in (4.6) and $(\Gamma, \mathbf{H})$ in (4.7) model the inelastic process of homeostasis. For positive values of the homeostasis rates (Γ_m, Γ) and zero velocity gradient $(\mathbf{L} = 0)$, these evolution equations reduce to

$$\frac{\dot{J}_e}{J_e} = -\Gamma_m \ln \left(\frac{J_e}{J_h} \right) \quad \text{and} \quad \dot{\mathbf{B}}'_e = -\Gamma \left[\mathbf{B}'_e - \left(\frac{3}{\mathbf{B}'_e{}^{-1} \cdot \mathbf{H}} \right) \mathbf{H} \right], \quad (4.11)$$

which cause $(J_e, \mathbf{B}'_e)$ to approach their homeostatic values $(J_h, \mathbf{H}')$. For general loading with non-zero $\mathbf{L}$, homeostasis is an inelastic process that causes a tendency for $(J_e, \mathbf{B}'_e)$ to approach their homeostatic values $(J_h, \mathbf{H}')$. Moreover, the constitutive equation for Cauchy stress takes the form

$$\mathbf{T} = \mathbf{T}(J_e, \mathbf{B}'_e), \quad (4.12)$$

which can depend on other variables, but in the purely mechanical theory is assumed to be restricted so that stress-free states are characterized by

$$\mathbf{T} = 0 \Rightarrow J_e = 1 \quad \text{and} \quad \mathbf{B}'_e = \mathbf{I}. \quad (4.13)$$

Comparison of the evolution equations (3.1) for J , (3.23) (with the specification (3.32)) for plasticity with (4.6) and (4.7) for growing materials reveals that for plasticity there is no inelastic dilatational rate ($J_e = J$) and that plastic distortional deformation rate causes $\mathbf{B}'_e$ to have a tendency to approach its zero-stress value $\mathbf{I}$. By contrast, homeostasis in the evolution equations (4.6) and (4.7) causes a tendency for $\mathbf{T}$ to approach its homeostatic value

$$\mathbf{T}(J_h, \mathbf{H}') \neq 0, \quad (4.14)$$

which does not vanish when $(J_h \neq 1 \text{ and/or } \mathbf{H}' \neq \mathbf{I})$. This means that the constitutive equations for the homeostatic values $(J_h, \mathbf{H}')$ can be used to model experimental observations about the stresses in biological materials in their homeostatic states (e.g. [39]). Also, the constitutive equations for (Γ_m, Γ) control the rates of homeostasis.

Collagen and elastin fibre bundles that typically are found in biological tissues cause these tissues to be elastically anisotropic. Also, since the resistance to distortional deformations of tissues saturated with fluids is usually much smaller than the resistance to volumetric compression they are modelled as nearly incompressible. However, when the tissue is allowed to grow, the total volume can change significantly. Therefore, growing tissues are often modelled as elastically nearly incompressible. This means that it is convenient to separate the effects of dilatation from distortion.

Kuhl [35] discussed volumetric, fibre and area elastic measures for growth of tissues which were reconsidered in [37] within the context of the Eulerian formulation of growth. The elastic dilatation J_e includes volumetric growth. In [37], the elastic stretch λ_{es} of a material fibre currently in the unit $\mathbf{s}$ direction and the elastic area growth λ_{en} of the material surface currently normal to the unit vector $\mathbf{n}$ were defined by

$$\lambda_{es} = (J_e^{-2/3} \mathbf{B}'_e{}^{-1} \cdot \mathbf{S})^{1/2} \quad \text{and} \quad \lambda_{en} = (J_e^{-4/3} \mathbf{B}'_e \cdot \mathbf{N})^{-1/2}, \quad \mathbf{S} = \mathbf{s} \otimes \mathbf{s}, \quad \mathbf{N} = \mathbf{n} \otimes \mathbf{n}, \quad (4.15)$$

where $(\mathbf{S}, \mathbf{N})$ are structural tensors. Also, to model bending and twisting associated with early cardiac morphogenesis Safadi & Rubin [40] introduced a right-handed orthonormal triad $\mathbf{s}_i$ of vectors and structural tensors $\mathbf{S}_{ij}$

$$\mathbf{S}_{ij} = \mathbf{s}_i \otimes \mathbf{s}_j, \quad (4.16)$$

with $\mathbf{s}_1$ being the current direction of a material fibre, $\mathbf{s}_3$ being the current unit normal to a material surface and $\mathbf{s}_2$ completing the triad of vectors. Specifically, Safadi & Rubin [40] introduced the elastic distortional strains

$$\left. \begin{aligned} E'_{eij} &= \frac{1}{2} \lambda'_{ei} \lambda'_{ej} (\mathbf{I} - \mathbf{B}'_e{}^{-1}) \cdot \mathbf{S}_{ij}, \\ \lambda'_{ei} &= (\mathbf{B}'_e{}^{-1} \cdot \mathbf{S}_{ii})^{-1/2} \quad (\text{no sum on } i, j), \\ E'_{eii} &= \frac{1}{2} (\lambda'^2_{ei} - 1) \quad (\text{no sum on } i), \\ E'_{e12} &= -\frac{1}{2} \lambda'_{e1} \lambda'_{e2} (\mathbf{B}'_e{}^{-1} \cdot \mathbf{S}_{12}), \\ E'_{e13} &= -\frac{1}{3} \lambda'_{e1} \lambda'_{e3} (\mathbf{B}'_e{}^{-1} \cdot \mathbf{S}_{13}) \\ E'_{e23} &= -\frac{1}{3} \lambda'_{e2} \lambda'_{e3} (\mathbf{B}'_e{}^{-1} \cdot \mathbf{S}_{23}). \end{aligned} \right\} \quad (4.17)$$

and

Moreover, it can be shown that

$$\dot{E}'_{eij} = \mathbf{K}_{ij} \cdot \mathbf{D} - \Gamma G_{ij}, \quad \mathbf{K}_{ij}^T = \mathbf{K}_{ij} = \mathbf{K}_{ji} \quad \text{and} \quad G_{ji} = G_{ij}, \quad (4.18)$$

where the functional forms of $(\mathbf{K}_{ij}, G_{ij})$ are given in [40].

Under SRBM $(\mathbf{s}, \mathbf{S}, \mathbf{n}, \mathbf{N}, \mathbf{s}_i, \mathbf{S}_{ij}, \mathbf{K}_{ij}, G_{ij})$ transform to $(\mathbf{s}^+, \mathbf{S}^+, \mathbf{n}^+, \mathbf{N}^+, \mathbf{s}_i^+, \mathbf{S}_{ij}^+, \mathbf{K}_{ij}^+, G_{ij}^+)$, such that

$$\left. \begin{aligned} \mathbf{s}^+ &= \mathbf{Q}\mathbf{s}, & \mathbf{S}^+ &= \mathbf{Q}\mathbf{S}\mathbf{Q}^T, & \mathbf{n}^+ &= \mathbf{Q}\mathbf{n}, & \mathbf{N}^+ &= \mathbf{Q}\mathbf{N}\mathbf{Q}^T, \\ \text{and} & & \mathbf{s}_i^+ &= \mathbf{Q}\mathbf{s}_i, & \mathbf{S}_{ij}^+ &= \mathbf{Q}\mathbf{S}_{ij}\mathbf{Q}^T, & \mathbf{K}_{ij}^+ &= \mathbf{Q}\mathbf{K}_{ij}\mathbf{Q}^T, & G_{ij}^+ &= G_{ij}, \end{aligned} \right\} \quad (4.19)$$

and the elastic deformation measures $(\lambda_{es}, \lambda_{en}, E'_{eij})$ are unaffected by SRBM.

Now, for an elastically anisotropic material, the strain energy function is taken in the form

$$\Sigma = \Sigma(J_e, \alpha_1, \alpha_2, \lambda_{es}, \lambda_{en}, E'_{eij}), \quad (4.20)$$

where it is noted that any number of fibre stretches λ_{es} and area stretches λ_{en} can be easily included. Then, the Cauchy stress $\mathbf{T}$ is specified by

$$\begin{aligned} \mathbf{T} = & \rho_0 \frac{\partial \Sigma}{\partial J_e} \mathbf{I} + 2J_e^{-1} \rho_0 \frac{\partial \Sigma}{\partial \alpha_1} \left(\mathbf{B}'_e - \frac{1}{3} \alpha_1 \mathbf{I} \right) + 4J_e^{-1} \rho_0 \frac{\partial \Sigma}{\partial \alpha_2} \left(\mathbf{B}_e'^2 - \frac{1}{3} \alpha_2 \mathbf{I} \right) \\ & + J_e^{-1} \lambda_{es} \rho_0 \frac{\partial \Sigma}{\partial \lambda_{es}} \mathbf{S} + J_e^{-1} \lambda_{en} \rho_0 \frac{\partial \Sigma}{\partial \lambda_{en}} (\mathbf{I} - \mathbf{N}) + J_e^{-1} \rho_0 \frac{\partial \Sigma}{\partial E'_{eij}} \mathbf{K}_{ij}, \end{aligned} \quad (4.21)$$

with summation implied over the repeated indices (i, j) and with Σ satisfying the restrictions

$$\frac{\partial \Sigma}{\partial J_e} = 0, \quad \frac{\partial \Sigma}{\partial \lambda_{es}} = 0, \quad \frac{\partial \Sigma}{\partial \lambda_{en}} = 0, \quad \frac{\partial \Sigma}{\partial E'_{eij}} = 0 \quad \text{for } J_e = 1 \text{ and } \mathbf{B}'_e = \mathbf{I}, \quad (4.22)$$

to ensure that (4.13) characterizes stress-free states. Furthermore, using (4.3), (4.4), (4.6), (4.7), (4.18), (4.20) and (4.21), it can be shown that the material dissipation rate $\mathcal{D}$ is given by

$$\mathcal{D} = \rho b + \mathcal{D}_m + \mathcal{D}_d, \quad (4.23)$$

where expressions for the material dissipation due to inelastic dilatational changes $\mathcal{D}_m$ and due to inelastic distortional changes $\mathcal{D}_d$ can be found in [40]. Examples in [40] show that this formulation can be used to model the mechanics of inhomogeneous inflation, extension, bending and torsion exhibited in early cardiac morphogenesis.

5. Strongly objective, robust integration algorithms

The objective of a numerical integration algorithm for evolution equations of history-dependent variables is to use their initial values at the beginning of a typical time step $t = t_1$ and to estimate their values at the end $t = t_2$ of the time step with time increment $\Delta t = t_2 - t_1$. This section discusses aspects of numerical integration algorithms for the Eulerian formulations for elastically isotropic metals and for growth of biological tissues. The algorithms are robust in the sense that they are implicit and they are strongly objective because the tensorial measures estimated at the end of the time step satisfy the same transformation relations under SRBM as the exact tensors (e.g. [41,42]). For example, this means that neglecting inertia, the normal and shearing components (relative to material directions) of the traction vector at a material point on the surface of a structure will be uninfluenced by SRBM. An example of a beam which experiences large superposed rigid body rotation can be found in [43].

For integrating tensorial measures use is made of the work of Simo [44,45] to introduce the relative deformation gradient $\mathbf{F}_r$ from the beginning of the time step. This tensor can be

determined by the value $\mathbf{F}(t_1)$ of the total deformation gradient $\mathbf{F}$ at the beginning of the time step and the estimate of $\mathbf{F}(t)$ at any time by the expression

$$\mathbf{F}_r(t) = \mathbf{F}(t)\mathbf{F}^{-1}(t_1). \quad (5.1)$$

This tensor satisfies the evolution equation and initial condition

$$\dot{\mathbf{F}}_r = \mathbf{L}\mathbf{F}_r \quad \text{and} \quad \mathbf{F}_r(t_1) = \mathbf{I}. \quad (5.2)$$

Next, the relative dilatation J_r is defined by

$$J_r(t) = \det(\mathbf{F}_r), \quad (5.3)$$

which satisfies the evolution equation and initial condition

$$\dot{J}_r = J_r(\mathbf{D} \cdot \mathbf{I}) \quad \text{and} \quad J_r(t_1) = 1. \quad (5.4)$$

Then, using these expressions the unimodular part $\mathbf{F}'_r$ of the relative deformation gradient $\mathbf{F}_r$ is defined by

$$\mathbf{F}'_r = J_r^{-1/3} \mathbf{F}_r, \quad (5.5)$$

which satisfies the evolution equation and initial condition

$$\dot{\mathbf{F}}'_r = [\mathbf{L} - \frac{1}{3}(\mathbf{D} \cdot \mathbf{I})\mathbf{I}]\mathbf{F}'_r, \quad \text{and} \quad \mathbf{F}'_r(t_1) = \mathbf{I}. \quad (5.6)$$

Since $(\mathbf{F}_r, J_r, \mathbf{F}'_r)$ can be determined by the guess $\mathbf{F}(t_2)$ for the total deformation gradient at the end of the time step or directly from the displacements associated with this guess, they solve the evolution equations and initial conditions (5.2), (5.4) and (5.6) exactly. Moreover, under SRBM their values $[\mathbf{F}_r(t_2), J_r(t_2), \mathbf{F}'_r(t_2)]$ at the end of time step transform to $[\mathbf{F}_r^+(t_2), J_r^+(t_2), \mathbf{F}_r^{'+}(t_2)]$, respectively, such that

$$\mathbf{F}_r^+(t_2) = \mathbf{Q}\mathbf{F}_r(t_2), \quad J_r^+(t_2) = J_r(t_2) \quad \text{and} \quad \mathbf{F}_r^{'+}(t_2) = \mathbf{Q}\mathbf{F}'_r(t_2). \quad (5.7)$$

This feature of the relative deformation gradient is used to develop the strongly objective numerical algorithms discussed below.

The evolution equation (4.6) for the elastic dilatation J_e and (4.7) for the elastic distortional deformation $\mathbf{B}'_e$ for growth of biological tissues reduce to (3.1) for the total dilatation J and (3.23) (with the specification (3.32)) for $\mathbf{B}'_e$ for metal plasticity when $\Gamma_m = 0$ and $\mathbf{H} = \mathbf{I}$. Therefore, the integration algorithms for growth of biological tissues simplify to those for metal plasticity. Consequently, only the algorithm for growth of biological tissue will be discussed in detail here. Using the expression (3.34) for the deviatoric tensor $\mathbf{B}''_e$, the deviatoric part of the evolution equation (4.7) can be written in the form

$$\dot{\mathbf{B}}''_e = \mathbf{L}\mathbf{B}'_e + \mathbf{B}'_e\mathbf{L}^T - \frac{2}{3}(\mathbf{B}'_e \cdot \mathbf{D})\mathbf{I} - \frac{2}{3}(\mathbf{D} \cdot \mathbf{I})\mathbf{B}''_e - \Gamma \left[\mathbf{B}''_e - \left(\frac{3}{\mathbf{B}_e'^{-1} \cdot \mathbf{H}} \right) \mathbf{H}'' \right], \quad (5.8)$$

with $\mathbf{H}$ represented by

$$\mathbf{H} = \mathbf{I} + \mathbf{H}'' \quad \text{and} \quad \mathbf{H}'' \cdot \mathbf{I} = 0, \quad (5.9)$$

where $\mathbf{H}''$ is the deviatoric part of $\mathbf{H}$.

Next, using the work in [44,45], it is convenient to define the elastic trial values $(J_e^*, \mathbf{B}_e^{'*})$ by the expressions

$$J_e^* = J_r(t)J_e(t_1), \quad \mathbf{B}_e^{'*} = \mathbf{F}'_r(t)\mathbf{B}_e'(t_1)\mathbf{F}_r^T(t) \quad \text{and} \quad \mathbf{B}_e^{''*} = \mathbf{B}_e^{''*} - \frac{1}{3}(\mathbf{B}_e^{'*} \cdot \mathbf{I})\mathbf{I}, \quad (5.10)$$

which satisfy the evolution equations and initial conditions

$$\left. \begin{aligned} \dot{J}_e^* &= J_e^*(\mathbf{D} \cdot \mathbf{I}), \quad J_e^*(t_1) = J_e(t_1), \\ \dot{\mathbf{B}}_e^{''*} &= \mathbf{L}\mathbf{B}_e^{''*} + \mathbf{B}_e^{''*}\mathbf{L}^T - \frac{2}{3}(\mathbf{B}_e^{''*} \cdot \mathbf{D})\mathbf{I} - \frac{2}{3}(\mathbf{D} \cdot \mathbf{I})\mathbf{B}_e^{''*} \\ \text{and} \quad \mathbf{B}_e^{''*}(t_1) &= \mathbf{B}_e^{''*}(t_1). \end{aligned} \right\} \quad (5.11)$$

Thus, the evolution equation (4.6) can be rewritten in the form

$$\frac{\dot{J}_e}{J_e} = \frac{\dot{J}_e^*}{J_e^*} - \Gamma_m \ln \left(\frac{J_e}{J_h} \right), \quad (5.12)$$

and the evolution equation (5.8) can be approximated by

$$\dot{\mathbf{B}}_e'' = \dot{\mathbf{B}}_e''^* - \Gamma \left[\mathbf{B}_e'' - \left(\frac{3}{\mathbf{B}_e'^{-1} \cdot \mathbf{H}} \right) \mathbf{H}'' \right]. \quad (5.13)$$

Then, using a backward Euler approximation of the derivative, the implicit solution of the evolution equation (5.12) can be expressed in the form

$$\ln \left[\frac{J_e(t_2)}{J_e^*(t_2)} \right] = -\Delta t \Gamma_m(t_2) \ln \left[\frac{J_e(t_2)}{J_h(t_2)} \right], \quad (5.14)$$

where $[J_m(t_2), J_h(t_2)]$ are estimates of (J_m, J_h) at the end of the time step. This equation can be solved to obtain

$$J_e(t_2) = \exp \left[\frac{\ln\{J_e^*(t_2)\} + \Delta t \Gamma_m(t_2) \ln\{J_h(t_2)\}}{1 + \Delta t \Gamma_m(t_2)} \right]. \quad (5.15)$$

Also, using a backward Euler approximation of the derivative, the implicit solution of the evolution equation (5.13) can be expressed in the form

$$\mathbf{B}_e''(t_2) = \mathbf{B}_e''^*(t_2) - \Delta t \Gamma(t_2) \left[\mathbf{B}_e''(t_2) - \left(\frac{3}{\mathbf{B}_e'^{-1}(t_2) \cdot \mathbf{H}(t_2)} \right) \mathbf{H}''(t_2) \right], \quad (5.16)$$

where $\Gamma(t_2)$ is an estimate of Γ at the end of the time step and $\mathbf{H}$ is represented in the form

$$\mathbf{H}''(t_2) = H_{ij}(t_2) \mathbf{S}_{ij}(t_2) \quad \text{and} \quad \mathbf{S}_{ij}(t_2) = \mathbf{s}_i(t_2) \otimes \mathbf{s}_j(t_2), \quad (5.17)$$

with $H_{ij}(t_2)$ being estimates of the components of $\mathbf{H}''$ relative to $\mathbf{s}_i(t_2)$ at the end of the time step. Also, $\mathbf{s}_i(t_2)$ are given by

$$\mathbf{s}_1(t_2) = \frac{\mathbf{F}_r(t_2) \mathbf{s}_1(t_2)}{|\mathbf{F}_r(t_2) \mathbf{s}_1(t_2)|}, \quad \mathbf{s}_3(t_2) = \frac{\mathbf{F}_r^{-T}(t_2) \mathbf{s}_3(t_2)}{|\mathbf{F}_r^{-T}(t_2) \mathbf{s}_3(t_2)|} \quad \text{and} \quad \mathbf{s}_2(t_2) = \mathbf{s}_3(t_2) \times \mathbf{s}_1(t_2). \quad (5.18)$$

Furthermore, (5.16) can be rewritten in the implicit form

$$\mathbf{B}_e''(t_2) = \lambda \mathbf{B}_e''^*(t_2) + B(1 - \lambda) \mathbf{H}''(t_2), \quad (5.19)$$

with the auxiliary variables (λ, B) defined by

$$\lambda = \frac{1}{1 + \Delta t \Gamma(t_2)} \quad \text{and} \quad B = \frac{3}{\mathbf{B}_e'^{-1}(t_2) \cdot \mathbf{H}(t_2)}. \quad (5.20)$$

Given a value for $\mathbf{B}_e''(t_2)$, the analytical procedure discussed in [21] can be used to determine $\mathbf{B}_e'(t_2)$. Then, the numerical procedure discussed in [38] can be used to iterate on guesses for $[\Gamma(t_2), B]$ until (5.20) for B and the constitutive equation

$$\Gamma(t_2) = \Gamma[J_e(t_2), \mathbf{B}_e'(t_2), \mathbf{H}''(t_2)], \quad (5.21)$$

are satisfied. For metal plasticity with $\mathbf{H}'' = 0$, it can be seen that (5.19) is similar to the radial return algorithm developed by Wilkins [46]. Details of this solution procedure for a smooth elastic-plastic transition model with hardening and both rate-independent and rate-dependent response can be found in [28,43]. This algorithm is strongly objective since $[J_e(t_2), \mathbf{B}_e'(t_2)]$ satisfy the same transformation relations (4.10) under SRBM as the exact values $(J_e, \mathbf{B}_e')$.

6. Conclusion

This paper has reviewed the Lagrangian and Eulerian formulations of metal plasticity and growth of biological tissues. In contrast with the Lagrangian formulations, the Eulerian formulations are insensitive to arbitrariness of the choices of the reference configuration, the intermediate

configuration, a total deformation measure and an inelastic deformation measure. In particular, the Eulerian formulation of growth models the inelastic process of homeostasis which causes the elastic deformation measures to have a tendency to approach their homeostatic values. Also, robust strongly objective numerical algorithms to integrate the evolution equations in the Eulerian formulation have been reviewed. In addition, reference has been made to a number of papers which discuss these issues in detail.

Data accessibility. This article has no additional data.

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