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EEG for current with two dimensional support.

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Abstract

Objective: The inverse problem of computing the neuronal current density from scalp EEG is highly ill-posed. In part, this is due to the nonuniqueness of the mapping between current sources and scalp potentials. We develop an explicit formula for the scalp EEG for sources constrained to the cortical surface in terms only of the components of the current that affect the EEG signal.

Methods: Starting from the quasistatic form of Maxwell's equations, we develop a formula that involves only the "visible" part of the current (i.e. the part of the current that affects the EEG measurements), as well as certain auxiliary functions which depend on the topology and

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conductivity of the three dimensional domains Ω_c , Ω_f and Ω_b and Ω_s , that model the spaces occupied by the cerebrum, cerebrospinal fluid, bone and scalp respectively.

Results: we derive expressions for the scalp potential for a general nested topology, as well as for the special case of spherical and ellipsoidal surfaces. We verify that the resulting scalp potential, in the case that the current resides in a spherical shell in the cerebrum of thickness 2δ , agrees with the potential obtained via the three-dimensional, formulation, for $\delta = 10^{-8}m$.

Conclusion: the “visible” part of the current can be explicitly characterized and consists of a combination of its component normal to the surface and of a certain function generating the remaining tangential components of the current.

Significance: The resulting ability to restrict the source space greatly reduces the degree of ambiguity in the inverse solutions, offering the potential for more stable inverse solutions, since the auxiliary functions that define the mapping can be computed efficiently using standard numerical methods.

Keywords

Electroencephalography; Brain modeling; Inverse Problems; Biomedical Imaging; Partial differential equations; Computational modeling

1. INTRODUCTION

The EEG inverse problem involves the estimation of the primary intracranial current field that gives rise to measured scalp surface electric potentials. This current field reflects spontaneous or evoked neural activity in the brain. The high bandwidth of the EEG signal, relative to alternative functional imaging modalities such as PET and fMRI, offers the ability to study brain dynamics noninvasively and at a temporal resolution that offers unique insights into the behavior of large scale brain networks.

The primary limitation in mapping brain activity with EEG is the highly ill-posed nature of the spatial inverse problem. Selection of a unique solution requires use of a combination of various constraints, as well as regularization of the inversion procedure. A key constraint follows from the observation that the primary source of extracranial EEG is the transmembrane currents in the apical dendrites of the pyramidal cells in cerebral cortex [1], [2]. As a result, it can be assumed that the current sources lie in the cerebral cortex. An approximate solution can be obtained by assuming that the current sources are constrained to lie on a closed continuous surface representing the midcortex between inner gray/white and pial surfaces. These surfaces can be found by employing structural MRI images of the subject using computation tools such as BrainSuite [3] or FreeSurfer [4]. Alternatively, if individual MRIs are unavailable, an atlas-based cortical surface can be warped to the individual subject by non-rigid coregistration to measurements of the scalp coordinates of their EEG electrodes [5].

Even after restriction to a closed 2D surface, the inverse problem remains ill-posed. The standard approach is then to use regularized inverse methods to solve either for a vector current source directly on the cortical surface or to constrain the orientation of current

sources to be normal to the surface (parallel to the local orientation of the apical dendrites of the pyramidal cells). The regularization term is chosen either to minimize the total energy of the signal (the L_2 norm) or some measure of sparseness of the estimated field (the L_1 or L_p , $p < 1$ measure) [1]–[6].

In [8], the question of non-uniqueness for both EEG and MEG for multiple nested conducting domains modeling the cerebrum, cerebrospinal fluid, skull and scalp was completely resolved. Namely, in the general case that the current is supported in the three dimensional cerebral cortex, the precise part of the current that affects EEG, as well as the part that affects MEG were determined in [7] and [8]. Here, we extend the above result for EEG for the case that the current is supported on a closed surface. In [9] and [10] the direct and inverse problems for both EEG and MEG are solved when the current is constrained on a small disc.

Our main result is the following: let J_3 be the component of the current normal to the given surface, and let J_1 and J_2 be the two components of the current in the tangent plane of this surface. Let $\mathbf{r}(u, v)$ be the equation describing the given surface, where the associated parametric curves are orthogonal. Let ζ be the mean curvature of the surface and let $H = \left\| \frac{\partial \mathbf{r}}{\partial u} \right\| \left\| \frac{\partial \mathbf{r}}{\partial v} \right\|$. Without loss of generality, we can represent J_1 and J_2 in the form

$$J_1 = \frac{1}{H} \left[-\frac{\partial F}{\partial v} + \frac{\partial \Phi}{\partial u} \right], \quad J_2 = \frac{1}{H} \left[\frac{\partial F}{\partial u} + \frac{\partial \Phi}{\partial v} \right], \quad (1.1)$$

where $F(u, v)$ and $\Phi(u, v)$ are two arbitrary functions. We will show that only Φ and J_3 affect the EEG measurements, namely

$$u_s(\mathbf{r}) = -\frac{1}{4\pi} \int_S \left[\left(\frac{1}{H} \left(\frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2} \right) - \zeta J_3 \right) v_s(\mathbf{r}, \tau) \right] dS(\tau), \quad \forall \mathbf{r} \in \partial \Omega_s, \quad (1.2)$$

where u_s is the measured electric potential, S is the given surface, Ω_s denotes the boundary of the scalp where the measurements are made and the auxiliary function v_s depends on the conductivities and on the geometry of the different conducting regions but not on the current.

A numerical solution is presented for a particular example where the current resides in a spherical shell in the cerebrum of thickness 2δ . In this case the solution for $u_s(\mathbf{r})$ obtained via the three-dimensional formulation of [8] is compared with the solution obtained via (1.2). It is shown that for $\delta = 10^{-8}$ meters there is excellent agreement.

2. ELEMENTS OF TWO-DIMENSIONAL GEOMETRY

A point τ on a two-dimensional surface S can be parametrised by two-real parameters u and v . In what follows we assume that the parametric curves are orthogonal. Let us introduce the following notations:

$$\tau_1 = \frac{\partial \tau(u, v)}{\partial u}, \quad \tau_2 = \frac{\partial \tau(u, v)}{\partial v}, \quad \hat{\mathbf{n}} = \frac{\tau_1 \times \tau_2}{|\tau_1 \times \tau_2|}, \quad (2.1)$$

$$H = |\tau_1| |\tau_2|.$$

The vectors τ_1 and τ_2 , together with the unit vector $\hat{\mathbf{n}}$, which is normal to S , satisfy the orthogonality conditions

$$\tau_1 \cdot \tau_2 = \tau_1 \cdot \hat{\mathbf{n}} = \tau_2 \cdot \hat{\mathbf{n}} = 0. \quad (2.2)$$

An arbitrary vector $\mathbf{J}^P$ with support on S can be expressed in the form

$$\mathbf{J}^P = J_1 \tau_1 + J_2 \tau_2 + J_3 \hat{\mathbf{n}}. \quad (2.3)$$

The divergence of $\mathbf{J}^P$ is given by the expression

$$\nabla \cdot \mathbf{J}^P = \frac{1}{H} \frac{\partial}{\partial u} (H J_1) + \frac{1}{H} \frac{\partial}{\partial v} (H J_2) - \zeta J_3, \quad (2.4)$$

where ζ denotes the mean curvature of the surface, defined by

$$\nabla \cdot \hat{\mathbf{n}} = -\zeta. \quad (2.5)$$

Let $\hat{\mathbf{m}}$ be the unit vector which is tangent to S and normal to the curve $\mathcal{S}$ which denotes the perimeter of S . Then, the following identity is valid:

$$\int_S (\nabla \cdot \mathbf{J}^P + \zeta \mathbf{J}^P \cdot \hat{\mathbf{n}}) dS = \int_{\partial S} \mathbf{J}^P \cdot \hat{\mathbf{m}} ds. \quad (2.6)$$

3. A SINGLE CONDUCTOR

We first consider the case of a single homogeneous conductor occupying the domain Ω .

Proposition 3.1: Let Ω be the three-dimensional domain bounded by the smooth surface Ω occupied by a homogeneous conductor of electric conductivity σ . Suppose that the current

$\mathbf{J}^P$ is supported on a two-dimensional surface S which is embedded in the interior of Ω . Suppose that the associated parametric curves of S are orthogonal:

$$S: = \left\{ \tau(u, v), \quad u, v \in \mathbb{R}, \tau_1 = \frac{\partial \tau(u, v)}{\partial u}, \quad (3.1) \right. \\ \left. \tau_2 = \frac{\partial \tau(u, v)}{\partial v}, \quad H = \|\tau_1\| \|\tau_2\| \right\}.$$

Let $\hat{\mathbf{n}}$ be the normal to this surface, and let ζ denote its mean curvature. The electric potential generated by the primary current $\mathbf{J}^P$ and measured outside Ω is given by

$$u(\mathbf{r}) = -\frac{1}{4\pi\sigma} \times \int_S \left\{ \left(\frac{1}{H} \frac{\partial}{\partial u} (HJ_1) + \frac{1}{H} \frac{\partial}{\partial v} (HJ_2) - \zeta J_3 \right) \times \right. \\ \left. v_s(\mathbf{r}, \tau) dS(\tau) \right\}, \quad (3.2)$$

where, $v_s(\mathbf{r}, \tau)$ is the Green's function for the Laplacian for the Neumann boundary value problem in the domain Ω , dS is the differential of this surface, i.e.,

$$dS(\tau(u, v)) = H(u, v) du dv, \quad (3.3)$$

and $\{J_j\}_1^3$ are the three scalar components of $\mathbf{J}^P(\tau)$:

$$\mathbf{J}^P(u, v) = J_1(u, v) \tau_1(u, v) + J_2(u, v) \tau_2(u, v) \quad (3.4) \\ + J_3(u, v) \hat{\mathbf{n}}(u, v)$$

Equation (3.2) shows the explicit dependence of $u(\mathbf{r})$ on the surface S , and in particular on H and ζ . Using the representations

$$J_1 = \frac{1}{H} \left[-\frac{\partial F}{\partial v} + \frac{\partial \Phi}{\partial u} \right], \quad (3.5)$$

$$J_2 = \frac{1}{H} \left[\frac{\partial F}{\partial u} + \frac{\partial \Phi}{\partial v} \right], \quad (3.6)$$

equation (3.2) becomes

$$u(\mathbf{r}) = -\frac{1}{4\pi\sigma} \times \int_S \left[\frac{1}{H} \left(\frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2} \right) - \zeta J_3 \right] v_s(\mathbf{r}, \tau) dS(\tau). \quad (3.7)$$

Proof The quasistatic Maxwell equations are given by

$$\begin{aligned} \nabla \times \mathbf{E} &= 0, & \nabla \times \mathbf{B} &= \mu(\mathbf{J}^P + \sigma \mathbf{E}), \\ \nabla \cdot \mathbf{B} &= 0, \end{aligned} \quad (3.8)$$

where $\mathbf{E}$ and $\mathbf{B}$ denote the electric field and magnetic flux density respectively, whereas μ denotes the conductor's magnetic permeability. The first of these equations implies the existence of a function u called the electric potential,

$$\mathbf{E} = -\nabla u. \quad (3.9)$$

The second of equations (3.8) implies the compatibility condition

$$\nabla \cdot \mathbf{J}^P + \sigma \nabla \cdot \mathbf{E} = 0.$$

Then, equation (3.9) yields

$$\sigma \Delta u = \nabla \cdot \mathbf{J}^P, \quad (3.10)$$

where Δ denotes the 3-D Laplacian.

Let $v_s(\mathbf{r}, \tau)$ satisfy

$$\sigma \Delta_{\mathbf{r}} v_s(\mathbf{r}; \tau) = \delta(\mathbf{r} - \tau), \quad \mathbf{r}, \tau \in \Omega, \quad (3.11)$$

$$\partial_n v_s(\mathbf{r}; \tau) = 0, \quad \mathbf{r} \in \partial \Omega. \quad (3.12)$$

Equations (3.10) and (3.12) imply that u is given by

$$u(\mathbf{r}) = \int_S \nabla_{\tau} \cdot \mathbf{J}^P(\tau) v_s(\mathbf{r}, \tau) dS(\tau). \quad (3.13)$$

Employing in equation (3.13) the expression (2.4) for the divergence, where $\mathbf{J}^P$ is given by (3.4), we find equation (3.2). Then, using (3.5) and (3.6), we find (3.7).

In what follows we present an alternative proof of equation (3.2) which can be generalised to the more realistic model involving also the additional domains Ω_f , Ω_b and Ω_s modeling cerebrospinal fluid, bone and scalp, respectively. Let

$$\mathbf{J}^P(\mathbf{r}) = \int_{\Omega_c} \mathbf{J}^P(\tau) \delta(\mathbf{r}-\tau) dV(\tau). \quad (3.14)$$

Thus,

$$u(\mathbf{r}) = \int_{\Omega_c} U(\mathbf{r}, \tau) \delta(\mathbf{r}-\tau) dV(\tau), \quad (3.15)$$

where $U(\mathbf{r}, \tau)$ solves the equation

$$\sigma \Delta_{\mathbf{r}} U(\mathbf{r}, \tau) = \nabla_{\mathbf{r}} \cdot \mathbf{J}^P(\tau) \delta(\mathbf{r}-\tau). \quad (3.16)$$

Using the identities

$$\begin{aligned} \nabla_{\mathbf{r}} \cdot \mathbf{J}^P(\tau) \delta(\mathbf{r}-\tau) &= \mathbf{J}^P(\tau) \cdot \nabla_{\mathbf{r}} \delta(\mathbf{r}-\tau) \\ &= -\mathbf{J}^P(\tau) \cdot \nabla_{\mathbf{r}} \delta(\mathbf{r}-\tau), \end{aligned}$$

equation (3.16) becomes

$$\sigma \Delta_{\mathbf{r}} U(\mathbf{r}, \tau) = -\mathbf{J}^P(\tau) \cdot \nabla_{\mathbf{r}} \delta(\mathbf{r}-\tau).$$

Thus,

$$U(\mathbf{r}, \tau) = \frac{1}{4\pi} \mathbf{J}^P(\tau) \cdot \nabla_{\mathbf{r}} v_s(\mathbf{r}, \tau). \quad (3.17)$$

where $v_s(\mathbf{r}, \tau)$ solves (3.11) and (3.12). Hence, (3.15) yields

$$u(\mathbf{r}) = \frac{1}{4\pi} \int_{\Omega_c} \mathbf{J}^P(\tau) \cdot \nabla_{\mathbf{r}} v_s(\mathbf{r}, \tau) dV(\tau). \quad (3.18)$$

We recall that u satisfies a homogeneous Neumann boundary condition on Ω , namely,

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega, \quad (3.19)$$

since $\mathbf{J}^P$ is zero on Ω_c , equation (3.18) yields

$$u(\mathbf{r}) = -\frac{1}{4\pi} \int_{\Omega_c} (\nabla \cdot \mathbf{J}^P(\tau)) v_s(\mathbf{r}, \tau) dV(\tau). \quad (3.20)$$

In the case that $\mathbf{J}^P$ has 2-D support equation (3.20) becomes equation (3.13).

We note that equations (3.5) and (3.6) correspond to the well known two dimensional differential representation

$$\mathbf{J}^P - J_3 \hat{\mathbf{n}} = \nabla \Phi + \hat{\mathbf{n}} \times \nabla F. \quad (3.21)$$

Remark 3.1 It is important to note that if one replaces (3.14) by

$$\mathbf{J}^P(\mathbf{r}) = \int_S \mathbf{J}^P(\tau(u, v)) \delta(\mathbf{r} - \tau(u, v)) dS(\tau), \quad (3.22)$$

then one has to compute the fundamental solution of the Laplacian associated with $\delta(\mathbf{r} - \tau(u, v))$ and this procedure is more complicated than the one used in the proof of proposition 3.1

4. THE GENERAL CASE

Proposition 4.1: Let Ω_c denote the three dimensional domain modelling the cerebrum, which is surrounded by three shells $\Omega_f, \Omega_b, \Omega_s$ denoting the three dimensional domains modelling the cerebrospinal fluid, the skull, and the scalp. These compartments are distinguished by their different electric conductivities which will be denoted, respectively, by $\sigma_c, \sigma_f, \sigma_b$ and σ_s . The regions $\Omega_c, \Omega_f, \Omega_b$ and Ω_s are bounded by the surfaces $\Omega_c, (\Omega_c, \Omega_b), (\Omega_f, \Omega_b)$ and (Ω_b, Ω_s) , respectively.

Assume that the current $\mathbf{J}^P$ is supported in a two dimensional surface S embedded in Ω_c .

The electric potential on the scalp is given by

$$u_s(\mathbf{r}) = -\frac{1}{4\pi} \int_S \left\{ \left(\frac{1}{H} \frac{\partial(HJ_1)}{\partial u} + \frac{1}{H} \frac{\partial(HJ_2)}{\partial v} - \zeta J_3 \right) v_s(\mathbf{r}, \tau) \right\} dS(\tau), \quad \mathbf{r} \in \Omega_s, \quad (4.1)$$

where ζ is the mean curvature of the surface S defined in (3.1), dS is given in (3.3), $\{J_j\}_1^3$ are the three components of the current $\mathbf{J}^P$ defined in (3.4), and the auxiliary function v_s depends on σ_j and Ω_j , $j = c, f, b, s$, but not on the current. The mathematical definition of v_s , as well as a code for its numerical computation, are given in section 5.

Using the representations (3.5) and (3.6), equation (4.1) becomes

$$u_s(\mathbf{r}) = -\frac{1}{4\pi} \times \int_S \left[\frac{1}{H} \left(\frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2} \right) - \zeta J_3 \right] v_s(\mathbf{r}, \tau) dS(\tau). \quad (4.2)$$

Proof Using the approach employed for the alternative proof of proposition 3.1, the results of [8] imply

$$u_s(\mathbf{r}) = -\frac{1}{4\pi} \int_{\Omega_c} (\nabla_\tau \cdot \mathbf{J}^P)(\tau) v_s(\mathbf{r}, \tau) dV(\tau), \quad (4.3)$$

$$\mathbf{r} \in \Omega_s.$$

In the case that $\mathbf{J}^P$ has 2-D support, employing (2.4), equation (4.3) becomes (4.1). Then (3.5) and (3.6) imply (4.2).

Example 1 (spherical surface)

Let S be given by

$$\tau = \alpha(\sin\theta\cos\varphi, \sin\theta, \cos\theta), \quad (4.4)$$

$$\theta_1 \leq \theta \leq \theta_2, \quad \varphi_1 \leq \varphi \leq \varphi_2,$$

where α is a positive constant. In this case, $u = \theta$ and $v = \phi$. Furthermore, the following formulae are valid:

$$\tau_1 = \alpha(\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta),$$

$$\tau_2 = \alpha(-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0),$$

$$\hat{\mathbf{n}} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta),$$

$$H = \alpha^2 \sin \theta, \quad \zeta = -\frac{2}{\alpha}.$$

Hence, equation (4.2) becomes

$$u_s(\mathbf{r}) = -\frac{1}{4\pi} \int_{\theta_1}^{\theta_2} \int_{\varphi_1}^{\varphi_2} \left\{ \left[\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial \varphi^2} + 2\alpha \sin \theta J_3 \right] v_s(\mathbf{r}, \tau) \right\} d\varphi d\theta. \quad (4.5)$$

Example 2 (ellipsoidal surface)

Let S be given by [11]

$$\begin{aligned} \tau(\mu, v) = \frac{1}{\alpha_1 \alpha_2 \alpha_3} \times \quad (4.6) \\ \left(\alpha_1 h_1 \mu v, h_2 \alpha_2 \sqrt{\mu^2 - h_3^2} \sqrt{h_3^2 - v^2} \right. \\ \left. , h_3 \alpha_3 \sqrt{h_2^2 - \mu^2} \sqrt{h_2^2 - v^2} \right), \end{aligned}$$

where

$$\begin{aligned} h_1 = \sqrt{\alpha_2^2 - \alpha_3^2}, \quad h_2 = \sqrt{\alpha_1^2 - \alpha_3^2}, \quad (4.7) \\ h_3 = \sqrt{\alpha_1^2 - \alpha_2^2}, \quad 0 < \alpha_3 < \alpha_2 < \alpha_1 < \infty. \end{aligned}$$

In this case,

$$\begin{aligned} \tau_1 = \frac{1}{h_1 h_2 h_3} \times \quad (4.8) \\ \left(\alpha_1 h_1 \mu, -h_2 \alpha_2 v \frac{\sqrt{\mu^2 - h_3^2}}{\sqrt{h_3^2 - v^2}}, -h_3 \alpha_3 v \frac{\sqrt{h_2^2 - \mu^2}}{\sqrt{h_2^2 - v^2}} \right), \end{aligned}$$

$$\begin{aligned} \tau_2 = \frac{1}{h_1 h_2 h_3} \times \quad (4.9) \\ \left(\alpha_1 h_1 v, h_2 \alpha_2 \mu \frac{\sqrt{h_3^2 - v^2}}{\sqrt{\mu^2 - h_3^2}}, -h_3 \alpha_3 \mu \frac{\sqrt{h_2^2 - v^2}}{\sqrt{h_2^2 - \mu^2}} \right), \end{aligned}$$

$$\hat{\mathbf{n}} = \frac{\boldsymbol{\tau}_1 \times \boldsymbol{\tau}_2}{|\boldsymbol{\tau}_1 \times \boldsymbol{\tau}_2|}, \quad (4.10)$$

$$H = \frac{(\alpha_1^2 - \mu^2)(\alpha_1^2 - v^2)(\mu^2 - v^2)}{\alpha_2 \alpha_3 \sqrt{\mu^2 - h_3^2} \sqrt{h_2^2 - \mu^2} \sqrt{h_3^2 - v^2} \sqrt{h_2^2 - v^2}}, \quad (4.11)$$

$$\zeta = - \frac{\alpha_1 \alpha_2 \alpha_3}{\sqrt{\alpha_1^2 - \mu^2} \sqrt{\alpha_1^2 - v^2}} \times \left(\frac{1}{\alpha_1^2 - \mu^2} + \frac{1}{\alpha_1^2 - v^2} \right). \quad (4.12)$$

5. COMPUTATION OF $v_s(\mathbf{r}, \tau)$

For a given geometry, the function $v_s(\mathbf{r}, \tau)$ can be computed by solving the following boundary value problem:

$$\nabla^2 v_c(\mathbf{r}, \tau) = 0, \quad \mathbf{r} \in \Omega_c, \tau \in \Omega_c, \quad (5.1)$$

$$\frac{\partial}{\partial n} \left[\frac{1}{|\mathbf{r} - \boldsymbol{\tau}|} + v_c(\mathbf{r}, \tau) \right] = \sigma_f \frac{\partial v_f(\mathbf{r}, \tau)}{\partial n}, \quad \mathbf{r} \in \partial \Omega_c;$$

$$\nabla^2 v_f(\mathbf{r}, \tau) = 0, \quad \mathbf{r} \in \Omega_f, \tau \in \Omega_c, \quad (5.2)$$

$$v_f(\mathbf{r}, \tau) = \frac{1}{\sigma_c} \left[\frac{1}{|\mathbf{r} - \boldsymbol{\tau}|} + v_c(\mathbf{r}, \tau) \right], \quad \mathbf{r} \in \partial \Omega_c;$$

$$\sigma_f \frac{\partial v_f(\mathbf{r}, \tau)}{\partial n} = \sigma_b \frac{\partial v_b(\mathbf{r}, \tau)}{\partial n}, \quad \mathbf{r} \in \partial \Omega_f, \tau \in \Omega_c;$$

$$\nabla^2 v_b(\mathbf{r}, \tau) = 0, \quad \mathbf{r} \in \Omega_b, \tau \in \Omega_c, \quad (5.3)$$

$$v_b(\mathbf{r}, \tau) = v_f(\mathbf{r}, \tau), \quad \mathbf{r} \in \partial \Omega_f,$$

$$\sigma_b \frac{\partial v_b(\mathbf{r}, \tau)}{\partial n} = \sigma_s \frac{\partial v_s(\mathbf{r}, \tau)}{\partial n}, \quad \mathbf{r} \in \partial \Omega_b, \tau \in \Omega_c;$$

$$\nabla^2 v_s(\mathbf{r}, \tau) = 0, \quad \mathbf{r} \in \Omega_s, \tau \in \Omega_c, \quad (5.4)$$

$$v_s(\mathbf{r}, \tau) = v_b(\mathbf{r}, \tau), \quad \mathbf{r} \in \partial \Omega_b,$$

$$\frac{\partial v_s(\mathbf{r}, \tau)}{\partial n} = 0, \quad \mathbf{r} \in \partial \Omega_s.$$

Equations (5.1) - (5.4) are *independent* of the current $\mathbf{J}^P(\tau)$ and depend only on the geometry and on the conductivities σ_c , σ_b , σ_f and σ_s .

It turns out that it is convenient to construct the functions $v(\mathbf{r}, \tau)$ via first constructing certain functions $u_j(\mathbf{r}, \tau)$ which are defined in terms of a single dipole: let the functions

$$u_j(\mathbf{r}, \tau), \quad \mathbf{r} \in \Omega_j, \quad \tau \in \Omega_c, \quad j \in \{c, f, b, s\}, \quad (5.5)$$

satisfy the following equations:

$$\sigma_c \nabla^2 u_c(\mathbf{r}) = \nabla \cdot Q \delta(\mathbf{r} - \tau), \quad \mathbf{r} \in \Omega_c, \quad (5.6)$$

$$\sigma_c \frac{\partial u_c}{\partial n} = \sigma_f \frac{\partial u_f}{\partial n}, \quad \mathbf{r} \in \partial \Omega_c;$$

$$\nabla^2 u_f(\mathbf{r}) = 0, \quad \mathbf{r} \in \Omega_f, \quad (5.7)$$

$$u_f(\mathbf{r}) = u_c(\mathbf{r}), \quad \mathbf{r} \in \partial \Omega_c,$$

$$\sigma_f \frac{\partial u_f(\mathbf{r})}{\partial n} = \sigma_b \frac{\partial u_b}{\partial n}, \quad \mathbf{r} \in \partial \Omega_f;$$

$$\nabla^2 u_b(\mathbf{r}) = 0, \quad \mathbf{r} \in \Omega_b, \quad (5.8)$$

$$u_b(\mathbf{r}) = u_f(\mathbf{r}), \quad \mathbf{r} \in \partial \Omega_f,$$

$$\sigma_b \frac{\partial u_b(\mathbf{r})}{\partial n} = \sigma_s \frac{\partial u_s}{\partial n}, \quad \mathbf{r} \in \partial \Omega_b;$$

$$\nabla^2 u_s(\mathbf{r}) = 0, \quad \mathbf{r} \in \Omega_s, \quad (5.9)$$

$$u_s(\mathbf{r}) = u_b(\mathbf{r}), \quad \mathbf{r} \in \partial \Omega_b,$$

$$\frac{\partial u_s(\mathbf{r})}{\partial n} = 0, \quad \mathbf{r} \in \partial \Omega_s.$$

Then u_j and v_j are related by the equation

$$u_j(\mathbf{r}, \tau) = \frac{1}{4\pi} Q(\tau) \cdot \nabla_{\tau} v_j(\mathbf{r}, \tau), \quad (5.10)$$

$$j \in \{f, b, s\}, \mathbf{r} \in \Omega_j, \tau \in \Omega_c.$$

Spherical Geometry

The advantage of constructing the function $v_s(\mathbf{r}, \tau)$ via $u_s(\mathbf{r}, \tau)$ becomes clear by considering the particular case of spherical geometry. In this case, an analytical expression for $v_s(\mathbf{r}, \tau)$ is derived in [12]. This expression involves the inversion of a 7×7 matrix. It was observed in [13] that this matrix is ill conditioned and requires regularization. On the other hand, for the case of N spherical layers, with N arbitrary, an analytical expression for $u_s(\mathbf{r}, \tau)$ is obtained in [14], [15]. This approach is straightforward and does *not* require matrix inversion. Indeed, let $r_1 < r_2 < \dots < r_{N-1} < r_N$, and let $\sigma_1, \dots, \sigma_N$ denote the radii and the corresponding conductivities of the domains $\Omega_1, \dots, \Omega_N$. We consider a single dipole source characterized by (Q, τ) . The position vector of the observation point is denoted by $\mathbf{r}$. Let us introduce the following notations:

$$\hat{\tau} = \frac{\tau}{|\tau|}, \quad \hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|}, \quad (5.11)$$

$$\cos(\alpha) = \hat{\tau} \cdot \hat{Q}, \quad \cos(\gamma) = \hat{\tau} \cdot \hat{\mathbf{r}},$$

$$\cos(\beta) = (\hat{\tau} \times \hat{Q}) \cdot (\hat{\tau} \times \hat{\mathbf{r}}).$$

It was shown in [15], that the observed potential at $\mathbf{r}$ is given by the following expression:

$$u_s(\mathbf{r}, \tau) = \quad (5.12)$$

$$\frac{|Q|}{4\pi\sigma_N r^2} \sum_{n=1}^{\infty} \left(\frac{\tau}{r}\right)^{n-1} \left[\frac{2n+1}{nM_{2,2} + (1+n)M_{2,1}} \right] \times$$

$$\left[n \cos(\alpha) P_n[\cos(\gamma)] + \cos(\beta) \sin(\alpha) P_n^1[\cos(\gamma)] \right],$$

where $M_{2,1}$ and $M_{2,2}$ denote the elements of a 2×2 matrix M . The computation of this matrix is discussed in detail in [15]. Recall that $v_s(\mathbf{r}, \tau)$ can be expressed in the form [8], [12]

$$v_s(\mathbf{r}, \tau) = \sum_{n=1}^{\infty} H_n \frac{2n+1}{4\pi} \frac{\tau^n}{r^{n+1}} P_n(\hat{\tau} \cdot \hat{\mathbf{r}}), \quad (5.13)$$

where the unknown coefficients H_n can be computed by inverting the 7×7 matrix mentioned earlier. By using equations (5.10) and (5.13) one can bypass this inversion and obtain directly $v_s(\mathbf{r}, \tau)$:

$$v_s(\mathbf{r}, \tau) = \quad (5.14)$$

$$\sum_{n=1}^{\infty} \left[\frac{2n+1}{nM_{2,2} + (1+n)M_{2,1}} \right] \frac{1}{\sigma_N} \frac{\tau^n}{r^{n+1}} P_n(\hat{\tau} \cdot \hat{\mathbf{r}}).$$

Arbitrary Geometry

In the case of arbitrary geometry there already exist several codes for the numerical evaluation of equations (5.6)-(5.9), see for example [16]. The numerical construction of $v_s(\mathbf{r}, \tau)$ involves the following steps:

- 1) Fix a source point $\tau: = [\tau_x, \tau_y, \tau_z] \in \Omega_c$ and an observation point $\mathbf{r} \in \Omega_s$. Consider three dipoles positioned at the source $\tau \in \Omega_c$ with the following orthogonal moments:

$$\mathbf{Q}_x = [1, 0, 0], \mathbf{Q}_y = [0, 1, 0], \mathbf{Q}_z = [0, 0, 1]. \quad (5.15)$$

- 2) For each of the above three dipoles, solve the boundary value problem described by equations (5.6)-(5.9). Denote the solution for the potential due to the dipole oriented in the direction $\mathbf{Q}_i$ by $u_s(\mathbf{r}, \tau; \mathbf{Q}_i)$. Denote the vector

$$\mathbf{u}_s(\mathbf{r}, \tau) := [u_s(\mathbf{r}, \tau; \mathbf{Q}_1), u_s(\mathbf{r}, \tau; \mathbf{Q}_2), u_s(\mathbf{r}, \tau; \mathbf{Q}_3)]. \quad (5.16)$$

- 3) The gradient of $v_s(\mathbf{r}, \tau)$ is given by

$$\begin{aligned}
\frac{\partial v_s(\mathbf{r}, \tau)}{\partial \tau_x} &= u_s(\mathbf{r}, \tau; Q_x), \\
\frac{\partial v_s(\mathbf{r}, \tau)}{\partial \tau_y} &= u_s(\mathbf{r}, \tau; Q_y), \\
\frac{\partial v_s(\mathbf{r}, \tau)}{\partial \tau_z} &= u_s(\mathbf{r}, \tau; Q_z).
\end{aligned} \quad (5.17)$$

The OpenMEEG open source software can be employed to solve $u_s(\mathbf{r}, \tau)$ for any sensor configuration on the scalp, i.e., $\mathbf{r} \in \Omega_s$ in a single run, see [16].

6. NUMERICAL TESTS

The objective of this section is to compare the accuracy of equation (4.1) with the general expression derived in [8] for the electric potential on the scalp. The auxiliary functions in the spherical head model can be computed analytically, thus we restrict this test to a spherical head model. We assume that the current resides on a 2D spherical surface inside the cerebrum. Using the same notation as in section 4, Ω_c denotes the cerebrum, which is surrounded by three shells Ω_f , Ω_b , Ω_s denoting the cerebrospinal fluid, the skull, and the scalp. Their conductivities are respectively denoted by σ_c , σ_f , σ_b and σ_s . The spaces Ω_c , Ω_f , Ω_b and Ω_s are bounded by the surfaces Ω_c , (Ω_c, Ω_f) , (Ω_f, Ω_b) and (Ω_b, Ω_s) . The assumed parameters of the geometry are given in table I.

We recall that an arbitrary vectorial function with support in Ω_c can be expressed as

$$\mathbf{J}^P(\tau) = \nabla \psi(\tau) + \nabla \times \mathbf{A}(\tau), \quad \tau \in \Omega_c, \quad (6.1)$$

where $\mathbf{A}(\tau)$ satisfies the constraint $\nabla \cdot \mathbf{A}(\tau) = 0$ and $\psi(\tau)$ is the irrotational component of the current. It is shown in [8] that the electric potential on the scalp is given by

$$u_s(\mathbf{r}) = -\frac{1}{4\pi} \int_{\Omega_c} (\nabla^2 \psi(\tau)) v_s(\mathbf{r}, \tau) dV(\tau), \quad \mathbf{r} \in \partial \Omega_s. \quad (6.2)$$

We can use Green's first identity to rewrite equation (6.2) into the equivalent form

$$\begin{aligned}
u_s(\mathbf{r}) &= \\
&\frac{1}{4\pi} \int_{\Omega_c} \nabla \psi(\tau) \cdot \nabla_{\tau} v_s(\mathbf{r}, \tau) dV(\tau), \quad \mathbf{r} \in \partial \Omega_s.
\end{aligned} \quad (6.3)$$

To emulate a 2D surface for the support of the current in equation (6.3), we consider the current to reside in a thin spherical shell inside the cerebrum with a thickness of $2\delta = 2 \times 10^{-8}$, where

$$\alpha - \delta \leq \tau \leq \alpha + \delta, \quad \phi_1 \leq \phi \leq \phi_2, \quad \theta_1 \leq \theta \leq \theta_2.$$

We denote the unit tangential and normal vectors at any point on the surface by $\hat{\mathbf{i}}(\theta, \phi, \tau)$ and $\hat{\mathbf{n}}(\theta, \phi, \tau)$ respectively. We assume that the current is confined only in the radial direction, i.e., $\hat{\mathbf{i}}(\theta, \phi, \tau) \cdot \nabla \psi(\tau) = 0$ and $\hat{\mathbf{n}}(\theta, \phi, \tau) \cdot \nabla \psi(\tau) \neq 0$. Thus, using

$$\begin{aligned} \frac{\partial \psi}{\partial n} &= \hat{\mathbf{n}}(\theta, \phi, \tau) \cdot \nabla \psi(\tau), \quad (6.4) \\ dV(\tau) &= \tau^2 \sin(\theta) d\theta d\phi d\tau, \end{aligned}$$

equation (6.3) becomes

$$\begin{aligned} u_s^{(1)}(\mathbf{r}) &= \frac{1}{4\pi} \times \quad (6.5) \\ &\int_{\alpha-\delta}^{\alpha+\delta} \left[\int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} J_3(\tau) \frac{\partial v_s(\mathbf{r}, \tau)}{\partial n} \tau^2 \sin(\theta) d\theta d\phi \right] d\tau, \\ &\mathbf{r} \in \partial \Omega_s. \end{aligned}$$

Assuming that the current is in the radial direction and has support on the surface

$$\tau = \alpha, \quad \phi_1 \leq \phi \leq \phi_2, \quad \theta_1 \leq \theta \leq \theta_2,$$

we have

$$J_1 = J_2 = 0, \quad \zeta = -2/\alpha, \quad dS = \alpha^2 \sin(\theta) d\theta d\phi.$$

In this setting, equation (4.1) becomes

$$\begin{aligned} u_s^{(2)}(\mathbf{r}) &\quad (6.6) \\ &= \frac{1}{2\pi} \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} J_3(\tau) v_s(\mathbf{r}, \tau) \alpha \sin(\theta) d\theta d\phi. \end{aligned}$$

We introduce the following notations:

$$\begin{aligned}\cos(\tilde{\alpha}) &= \hat{\tau} \cdot \hat{n}(\theta, \phi), & \cos(\gamma) &= \hat{\tau} \cdot \hat{\mathbf{r}}, \\ \cos(\tilde{\beta}) &= (\hat{\tau} \times \hat{n}(\theta, \phi)) \cdot (\hat{\tau} \times \hat{\mathbf{r}}).\end{aligned}\quad (6.7)$$

By employing equation (5.17), we can express the term $\frac{\partial v_s(\mathbf{r}, \tau)}{\partial n}$ in equation (6.5) via the analytic representation below:

$$\begin{aligned}\frac{\partial v_s(\mathbf{r}, \tau)}{\partial n} &= \\ \frac{1}{4\pi\sigma_N r^2} \sum_{n=1}^{\infty} \left\{ \left(\frac{\tau}{r} \right)^{n-1} \left[\frac{2n+1}{nM_{2,2} + (1+n)M_{2,1}} \right] \right. \\ &\quad \left. \left[n\cos(\tilde{\alpha})P_n[\cos(\gamma)] + \cos(\tilde{\beta})\sin(\tilde{\alpha})P_n^1[\cos(\gamma)] \right] \right\},\end{aligned}\quad (6.8)$$

where $M_{2,1}$ and $M_{2,2}$ denote the elements of the 2×2 matrix M derived in [15]. Here, the objective is to compare the two independent equations (6.3) and (4.1) on the scalp. In equation (6.3), the term $\nabla_{\tau} v_s(\mathbf{r}, \tau)$ (gradient of the auxiliary function w.r.t τ) is independent of the current. In the case of the 2D model, equation (4.1) indicates that the current independent term is the function $v_s(\mathbf{r}, \tau)$. For a spherical geometry, the form of this auxiliary function is given by (5.14). Thus, a prerequisite step towards the comparison of equations (6.3) and (4.1) is the numerical verification of the relation given by (5.17). To this end, we employ the finite difference (FD) technique on equation (5.14) to compute the partial derivative of $v_s(\mathbf{r}, \tau)$ with respect to three orthogonal directions, i.e., $\frac{\partial v_s(\mathbf{r}, \tau)}{\partial \tau_x}$, $\frac{\partial v_s(\mathbf{r}, \tau)}{\partial \tau_y}$ and $\frac{\partial v_s(\mathbf{r}, \tau)}{\partial \tau_z}$. These partial derivatives constitute the left hand side of equation (5.17). For example, for a given source point τ and observation point $\mathbf{r}$, the partial derivative of $v_s(\mathbf{r}, \tau)$ with respect to $\{\tau_x, \tau_y, \tau_z\}$ can be computed by the finite difference approximation below:

$$\begin{aligned}\frac{\partial v_s(\mathbf{r}, \tau)}{\partial \tau_x} &\approx \frac{v_s(\mathbf{r}, \tau) - v_s(\mathbf{r}, \tau + h\hat{x})}{h}, & \hat{x} &= [1, 0, 0], \\ \frac{\partial v_s(\mathbf{r}, \tau)}{\partial \tau_y} &\approx \frac{v_s(\mathbf{r}, \tau) - v_s(\mathbf{r}, \tau + h\hat{y})}{h}, & \hat{y} &= [0, 1, 0], \\ \frac{\partial v_s(\mathbf{r}, \tau)}{\partial \tau_z} &\approx \frac{v_s(\mathbf{r}, \tau) - v_s(\mathbf{r}, \tau + h\hat{z})}{h}, & \hat{z} &= [0, 0, 1],\end{aligned}\quad (6.9)$$

where the constant $h = 10^{-4}$ denotes the step size. We employ the analytical solution given by equation (5.12) to compute the corresponding right hand side of (5.17).

Figure 2 shows the comparison between two independent approaches for computing $\nabla_{\tau} v_s(\mathbf{r}, \tau)$. In the first approach the finite difference scheme described by (6.9) is used on the analytical solution (5.14). In the second approach $\nabla_{\tau} v_s(\mathbf{r}, \tau)$ is constructed using the analytical expression of (5.12) and three orthogonal dipoles. In the FD, scheme a step size of $h = 10^{-4}$ resulted in an excellent agreement between the two independent approaches in computing (5.17). This shows that for the spherical geometry, (5.14) is indeed the correct solution of the boundary value problem described by equations (5.1)-(5.4).

The final objective is to compare the potential computed by equation (6.5) with that computed by (6.6). In the case of equation (6.5), a 2D surface has been emulated by assuming that the current resides on a thin shell with thickness $2\delta = 2 \times 10^{-8}$. In these numerical tests, we have arbitrarily chosen the normal component of the source $J_3(\tau)$ as

$$J_3(\tau, \theta, \phi) = \tau \sin(\phi) \cos(\theta),$$

$$\alpha - \delta \leq \tau \leq \alpha + \delta, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq 2\pi,$$

where $\alpha = 0.0835$ is the radius of the support of the source. As shown in figure 3, there is an excellent agreement between equation (6.5) and (6.6). Moreover, the numerical tests imply that as $\delta \rightarrow 0$ the following expected relation exists:

$$u_s^{(1)} \approx 4\delta u_s^{(2)}. \quad (6.10)$$

The Matlab scientific platform has been employed in all numerical computations. The two dimensional integral in equation (6.6) has been computed using the built in function “integral2” and the three dimensional integral of equation (6.5) is computed using the built-in function “integral3”.

7. DISCUSSIONS AND CONCLUSIONS

The inverse problems of the important imaging techniques of Electroencephalography (EEG), and Magnetoencephalography (MEG) involve the estimation of the neuronal current from the knowledge of the electric potential on the scalp, and of the magnetic field outside the head, respectively. However, as it was known to Helmholtz since 1853, both these techniques suffer from non-uniqueness. Following the efforts of several researchers, the complete answer to the non-uniqueness question for both EEG and MEG for arbitrary geometry was presented in [8] (see also [7]): if the current is supported in a 3D space, only the irrotational part of the current, Ψ , affects the EEG measurements. In this case, Ψ is called the “visible via EEG” part of the current, since this is the only part of the current that *can* be reconstructed from the knowledge of the EEG data. Similarly, it is shown by equation (1.2), that if the current is supported in a 2D surface, then only the irrotational part of the tangential component of the current, Φ , as well as the normal component of the current, J_3 ,

affect the EEG measurements. Thus, in this case Φ and J_3 are the only “visible via EEG” parts of the current.

Regarding equation (1.2) we note the following: (a) the analogue of equation (6.1) is now equation (3.21), hence $u_s(\mathbf{r})$, in addition to the scalar function Φ , now it also depends explicitly on the normal component J_3 of the current. Thus, although knowledge of $u_s(\mathbf{r})$ yields information only about the single term

$$\frac{1}{H} \left(\frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2} \right) - \zeta J_3,$$

this term depends on both Φ and J_3 . (b) The above term depends crucially on the geometry of the surface. In particular, the larger the mean curvature ζ and the larger the term H , the larger the contribution of J_3 .

It should be emphasized that equation (6.3), in addition to providing a complete answer to the uniqueness question, also provides an effective way for computing the visible via EEG part of the current, i.e., the component $\psi(\tau)$. Similarly, equation (1.2) provides an effective way for computing the part of the current that contributes to the measured data.

In the 3D case the basic equation for the reconstruction of the cortical current is equation (6.3). The auxiliary function $\nabla_\tau v_s(\mathbf{r}, \tau)$ can be obtained via the OpenMEEG open source software by solving equations (5.6)-(5.9).

In this setting, the inverse problem reduces to estimating the scalar function $\psi(\tau)$. This involves constructing a triangulated mesh for the assumed support of the current (for example a variation of the cortical mesh itself) and computing the relevant inversion matrix. The unknown scalar function $\psi(\tau)$ as well as the auxiliary function $\nabla_\tau v_s(\mathbf{r}, \tau)$ can be expanded using elemental interpolation functions, see [20]. Then, the elements of the relevant inversion matrix can then be computed analytically.

The analysis of the 2D case which is based on equation (4.1) is similar (details will be presented elsewhere). The results presented in this paper provide an analytic characterization of the components of the primary current that give rise to measurable scalp potentials in the case where the current field is constrained to a 2D surface. This allows us to explore, directly in the source space, which current patterns on the cortical surface will produce zero measurements. Previously this would need to be performed indirectly since they would only be made explicit through the null space of the forward model. A second use of these results is that we can parameterize the unknown image explicitly in terms of its visible and invisible components. In this way, when regularizing to compute an inverse solution, the component of the source that visible in the data and that which is determined only by the regularizing function can be explicitly identified. This in turn would allow identification of the degree to which the choice of regularizer impacts an estimated inverse solution. Exploration of these and other applications will be addressed in a future manuscript.

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Fig. 1.

This figure shows the multilayer shell geometry representing head model. The bounded domain Ω_c represent the cerebrum. The shell Ω_f represents the cerebrospinal fluid (CSF). The CSF is surrounded by the skull characterized by the domain Ω_b and the skull is surrounded by the scalp, which is denoted by Ω_s .

Fig. 2.

This figure compares $\nabla_{\tau} v_s(\mathbf{r}, \tau)$ and $u_s(\mathbf{r}, \tau; Q)$ as given by equation (5.17). On the three subplots (a)-(c), the x-axis denotes the indices of randomly selected source position vectors, and the y-axis shows the electric potential in units of Volts. A sample of 20 source points $\tau \in \Omega_c$ have been selected randomly inside the cerebrum. A single sensor has been selected randomly on the outermost layer representing the scalp ($\|\mathbf{r}\| = 0.1$ with $\sigma = 0.33$). The partial derivatives of $v_s(\mathbf{r}, \tau)$ with respect to $\{\tau_x, \tau_y, \tau_z\}$ have been computed using the finite difference approximation given by equation (6.9) with the step size $h = 10^{-4}$. Subplot (a)

shows $\frac{\partial v_s(\mathbf{r}, \tau)}{\partial \tau_x} = u_s(\mathbf{r}, \tau; Q_x)$, subplot (b) shows $\frac{\partial v_s(\mathbf{r}, \tau)}{\partial \tau_y} = u_s(\mathbf{r}, \tau; Q_y)$ and subplot (c) shows $\frac{\partial v_s(\mathbf{r}, \tau)}{\partial \tau_z} = u_s(\mathbf{r}, \tau; Q_z)$.

Fig. 3.

This figure shows the numerical results of comparing relation $u_s^{(1)} \approx 4\delta u_s^{(2)}$. The x-axis denotes the sensor indices randomly selected on the outermost layer ($\text{Scalp } \|\mathbf{r}\| = 0.1$) and the y-axis is the potential in units of Volts. A set of 30 sensors have been randomly selected on the surface of the outermost layer (Scalp, see table I). The current resides on a spherical shell with thickness $2\delta = 2 \times 10^{-8}$. The blue solid line with circle are potentials computed by equation (6.6) and the red solid line with cross are potentials computed at the the scalp using equation (6.5). The auxiliary function term $v_s(\mathbf{r}, \tau)$ in equation (6.6) is computed explicitly by (5.14). The normal component of the gradient $\nabla_{\tau} v_s(\mathbf{r}, \tau)$ in the outward direction, i.e., $\frac{\partial v_s(\mathbf{r}, \tau)}{\partial n}$ that appears in equation (6.5) is explicitly given by (6.8).

TABLE I

THE RADII AND CORRESPONDING CONDUCTIVITIES FOR THE DIFFERENT SHELLS IN THE SPHERICAL HEAD MODEL.

Domain	radius (meter)	Conductivity (S/m)
Cerebrum	0.084	0.33
CSF	0.087	0.0042
Skull	0.095	1.79
Scalp	0.1	0.33