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Evaluation of Imaging Systems Using the Posterior Variance of Emission Counts

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Abstract

We investigate an approach to evaluation of emission-tomography (ET) imaging systems used for region-of-interest (ROI) estimation tasks. In the evaluation we employ the concept of “emission counts” (EC), which are the number of events per voxel emitted during a scan. We use the reduction in posterior variance of ROI EC, compared to the prior ROI EC variance, as the metric of primary interest, which we call the “posterior variance reduction index” (PVRI). Systems that achieve a higher PVRI are considered superior to systems with lower PVRI. The approach is independent of the reconstruction method and is applicable to all photon-limited data types including list-mode data. We analyzed this approach using a model of 2-D tomography, and compared our results to the classical theory of tomographic sampling. We found that performance evaluations using the PVRI index were consistent with the classical theory. System evaluation based on EC posterior variance is an intuitively appealing and physically meaningful method that is useful for evaluation of system performance in ROI quantitation tasks.

Keywords

Bayes methods; design optimization; Monte Carlo methods; positron emission tomography; single-photon emission computed tomography

I. INTRODUCTION

IMAGING system evaluation is a process that rates the performance of an imaging system for a specific clinically relevant task. The task has to be specified because imaging systems optimized for a given task may not perform optimally for another task. In this work, we derive an approach to imaging-system evaluation for the task of region-of-interest (ROI) quantitation.

In this paper, we use the origin ensemble (OE) algorithm. When using this algorithm the quantity of interest is the number of emissions per voxel rather than the activity, which is

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used in classical approaches. In the first two papers published on the topic [1], [2] the convergence (stability) of the method was proven, and remarkable similarities between the resulting reconstruction and those based on the standard maximum-likelihood principle were found. However, in both of these publications the algorithm was derived *ad hoc* and presented without solid statistical interpretation. The breakthrough in interpretation of the OE algorithm came in 2012 in a published paper [3], in which the algorithm was derived starting from basic principles of statistics and tomography. Owing to the solid theoretical groundwork laid out in that paper, the method constitutes a completely novel statistical approach to analysis (including reconstruction) of photon-limited data. The research to uncover other possible uses of the approach has just begun. In terms of image reconstruction, we do not expect that the method will provide as big a difference in overall image quality as was seen, for example, when iterative methods replaced filtered back projection. The adequacy of the reconstructed image (a “point estimate”) is fundamentally limited by the amount of information provided by the data, and current iterative algorithms already push the limits in utilizing the available information, thereby making room for improvement small. However, as outlined in [3] and in the current paper, the new method developed by us provides, with no additional computing cost, not only the point estimate but also the range (*interval estimation* using statistical terminology) of highly probable solutions. This capability provides a great opportunity for exploring other uses of this additional information. One such usage, application of the estimated interval for imaging system evaluation, is presented in this paper.

A considerable body of research has been directed towards ROI quantitation using metrics such as variance, bias, and mean-squared error (MSE) to indicate system performance with both linear estimator [4], [5] and statistical reconstruction methods (e.g., [6], [7]). In these approaches, randomization has generally only been included in the data domain, while the object has typically been assumed to be constant. For this case, all conclusions could only be drawn for the object that was investigated, so one is then left to speculate that methods optimized for the current object may also work for others. Another limitation of the classical methods based on values of bias and ensemble variance in ROI quantitation is their dependence on the reconstruction method. For example, in order to be able to characterize the imaging system, the desired system resolution or desired ROI ensemble variance has to be specified to indicate the working point along the curve of bias versus ensemble-variance corresponding to a given imaging system and reconstruction method. This requirement considerably limits the generality of findings, unless the superiority of the system can be shown for all possible values of bias or variance.

One may also examine the properties of the likelihood function, assuming the statistical model of the data is specified (e.g., Poisson statistics). For example, the inverse of the Fisher information matrix (FIM) provides a measure of the minimum ensemble variance (EV) for any unbiased estimation method. This minimum EV is called the Cramer-Rao bound (CRB). Although the CRB may provide a solution to some problems listed in the previous paragraph, e.g., because it is independent of the reconstruction method, there are nevertheless difficulties associated with computing the CRB, which requires inversion of high-dimensional singular matrices. Therefore, some approximations, simplifications, and regularizations are needed to calculate the CRB [8]–[10].

One of the biggest limitations of the standard methods in ROI quantification is that data randomization is assumed. Therefore the data variances are “propagated” to yield ensemble variances for the estimated parameters (typically voxel activities). This model works well if the data are acquired directly in sinogram space; however, it is difficult to implement for systems where the data are acquired in so-called list mode. In this mode, every detected event has different detection parameters that can be characterized by float-type quantities (e.g., photomultiplier tube currents), in which case it can be difficult to specify the statistical model of the data in order to use the classical approaches to system evaluation. Furthermore, with *list*-mode data, the dimensionality of the FIM can be very high making it impractical to use CRB-type approaches.

In this work, we investigate a new measure to indicate the imaging system’s ability to perform quantitation within some specified ROI that is independent of the reconstruction algorithm. The method requires a description of the class of objects that will be scanned by specifying priors on the voxel activity values. It also requires specification of the ROI that will be used to calculate the performance measure. The system evaluation can also be based on more than a single ROI, and the method is designed to work with either binned or list-mode data, because it is not necessary to specify the data statistics in projection-data space.

The method is based on the theory of Bayesian experimental design [11], [12] and emission-count posteriors [3]. A key concept that differentiates this approach from standard approaches is that a nonstandard quantity of interest is used. In all approaches used to date, voxel or regional activity values have been utilized. Thus, the ability of the system to estimate these activity values has been used as a system-performance indicator. In our approach, we are interested in the number of events that occurred per voxel and was detected. There is a subtle, but fundamental, difference between these two quantities. The activity is the average number of emissions per voxel per unit time, which is a continuous quantity, whereas the actual number of emissions that occurred is an integer. We take a fully Bayesian view and treat all quantities of interest (voxel activity and numbers of emitted and detected counts) as random variables.

Although in Section II we provide the systematic derivation of the theory underlying the method presented in this paper, we will also provide a conceptual explanation of the approach that will hopefully allow the reader to gain an intuitive appreciation of our approach. Fig. 1 illustrates the method conceptually.

The distributions that we are concerned with in this paper are discrete; therefore, for every discrete number of counts in an ROI, a value of probability is assigned. We perform two hypothetical experiments with experimental setups 1 and 2 (Fig. 1) and determine posterior probabilities of the number of counts in the ROI shown in (B) and (C) for experiment 1 and 2, respectively. Looking at the two illustrative figures [Fig. 1 (B) and (C)] it is obvious that experimental setup 2 provides more precise specification of the number of counts that were emitted. Essentially only ~5 possible numbers are nonzero as opposed to ~9 in setup 1, because the rest are close to zero. In other words, system 2 provides a better (more informative) specification of the number of the actual decays that occurred. Somewhat naively, it can thus be concluded that experimental setup 2 is better than setup 1. However, it

is also possible that in just two experiments the superiority of experimental setup 2 over setup 1 occurred by chance. Therefore, it is important to repeat the experiments for both setups and determine if, on average, setup 2 is better than setup 1.

The Bayes risk (BR) is another Bayesian measure that can be used to grade imaging system performance. A system with a smaller BR can be considered better than a system with higher BR. In general, the BR indicates how well the point estimator associated with the imaging system is able to recover the truth. There are two major drawbacks of this approach. In order to calculate the BR, the point estimator has to be specified. Therefore, the method based on BR is not imaging-system specific but it depends on the reconstruction method, as well as on the imaging system. If we are to compare two imaging systems, the same reconstruction method may not be optimal for both of them. Therefore, advantages of one imaging system may be offset by a suboptimal reconstruction method. This limitation makes this approach less general than an approach that is independent of the reconstruction method, such as the one investigated in this work. The other limitation of the BR approach is its requirement that the loss function must be specified. The *loss function* describes a penalty incurred given 1) a particular point estimate, and 2) the truth. Under conditions when it is essential to optimize a combination of imaging system and reconstruction method, however, the BR is probably a better indicator of performance than the PVRI.

In Section II, we derive the theory of our approach and specify the figure of merit (FOM) that will be used for system evaluation. In Section III we present the numerical methods that are used to calculate the FOMs, as well as the computer simulations that will be used as examples of the application of the method. The computer simulations were intentionally chosen to be examples from 2-D tomography so that we can compare the findings of the new method with those from the classical theory of optimal 2-D tomography. In Section IV we present the results and discuss our findings for 2-D tomography. We finish with summary and conclusions in Section V.

II. THEORY

A. Preliminaries

We assume that the scanner is defined by detector elements. In general there can be an infinite number of them for some types of list-mode data, however for clarity of the theory presented here, we will treat the data as a finite vector (possibly of high dimensionality). During a scan, in every detector element, a finite nonnegative number of counts g_k is detected. The integer type vector $\mathbf{g}$ of size K represents the acquired data, while the image is represented by a set of voxels, all of the same size which defines the unit volume. The scan duration is equal to τ units of time. We assume that there are I voxels and each voxel has activity (average number of counts emitted per unit of voxel volume per unit of time) described by vector $\mathbf{f}$ with elements $f_1 \dots f_I$. The imaging system (relation between scanner and reconstruction volume) is characterized by the system *matrix* which describes the probability α_{ki} that an event which occurred in voxel i is detected in projection element k . We also define the sensitivity for each voxel as $\epsilon_i = \sum_{k=1}^K \alpha_{ki}$ which expresses the probability that if an event occurred in voxel i it will be detected.

The key concept used in this work is the concept of emission counts (EC) represented by vector $\mathbf{y}$. We indicate the EC vector as $\mathbf{y}$ and the vector space of all possible $\mathbf{y}$'s as $\mathbf{Y}$. The element of the vector y_{ki} represents the number of events that occurred in voxel i and were detected in projection element k . Although two indexes are used, $\mathbf{y}$ is a vector of size $K \times I$. In general, this quantity is unknown and cannot be measured directly. This quantity is also referred to as complete data in the literature [13]–[15]. Fig. 2 presents a simple two-pixel and two-projection bin system illustrating the EC concept. We define $\mathbf{Y}_g$ as a subspace of $\mathbf{Y}$ such that $\mathbf{y} \in \mathbf{Y}_g: \sum_{i=1}^I y_{ki} = g_k$.

B. Object Variability—Prior distribution $\pi(\mathbf{f})$

In the context of imaging system evaluation, specification of the prior distribution of voxel activities $\pi(\mathbf{f})$ is equivalent to specification of our beliefs about the class of objects that will be imaged with the imaging system. Specification of this prior can be used to define the *object variability*. In the classical approach, the object variability is typically ignored and instead just a single representative object is considered and randomization is only performed for the data. In our approach, we explicitly model object variability by specification of the prior distribution on voxel activities.

We limit the functional form of the prior such that the closed form of the final posterior can be obtained (Section II-C). In Sitek 2012 [3] it was shown that flat, truncated-flat, conjugate, and Jeffreys priors can be used in order to obtain the closed-form of the posterior.

In this work, we use the conjugate prior $\pi(\mathbf{f})$ because it is the most subjective prior and a well-defined class of objects can be represented by it. The conjugate prior is characterized by the same functional dependence on $\mathbf{f}$ as the likelihood and for the case of the Poisson distribution it is defined as a gamma distribution, such that

$$\pi(\mathbf{f}) = \prod_{i=1}^I \pi(f_i) = \prod_{i=1}^I \frac{(\beta_i)^{\beta_i \phi_i}}{\Gamma(\beta_i \phi_i)} (f_i)^{\beta_i \phi_i - 1} e^{-\beta_i f_i}. \quad (1)$$

Parameters $\beta_i \in [0, \infty]$ and $\phi_i \in [0, \infty]$ define the conjugate (gamma distribution) prior for voxel i . In order to have a proper distribution we enforce that $\beta_i \phi_i \geq 1$.

The prior specifies a gamma distribution of activity for each voxel. The prior is designed so that the mean of the gamma distribution is at ϕ_i ; therefore ϕ_i has an intuitive interpretation as the mean prior activity per voxel. The value of parameter β_i defines how strongly the prior distribution is concentrated around ϕ_i . For $\beta_i \rightarrow 0$ that prior approaches a flat distribution (least concentrated) and for $\beta_i \rightarrow \infty$ it approaches the Dirac delta function $\delta(\phi_i)$ (most concentrated). If the prior is a Dirac delta it corresponds to the classical case of a constant object (fixed activity). The activities in voxels are assumed to be conditionally independent which allows us to use the product of gamma distributions for each voxel as a prior for vector $\mathbf{f}$.

C. Emission-Count Posterior

Using the prior defined in the previous section and the Poisson likelihood the discrete posterior of $\mathbf{y}$ can be obtained by marginalization (integration out) of $\mathbf{f}$ and is of the form

$$P(\mathbf{y}|\mathbf{g}) = \frac{1}{Z} \prod_{i=1}^I \frac{\Gamma(c_i + \beta_i \phi_i)}{(\epsilon_i \tau + \beta_i)^{c_i}} \prod_{k=1}^K \frac{(\alpha_{ki})^{y_{ki}}}{y_{ki}!} \text{ for } \mathbf{y} \in \mathbf{Y}_{\mathbf{g}} \quad (2)$$

and $p(\mathbf{y}|\mathbf{g}) = 0$ for $\mathbf{y} \notin \mathbf{Y}_{\mathbf{g}}$. The Z in the equation above represents the normalization constant which is independent of $\mathbf{y}$. The $\Gamma(x)$ is the gamma function equal to $\int_0^\infty e^{-t} t^{x-1} dt$ and

$c_i = \sum_{k=1}^K y_{ki}$ is the total number of events that occurred in voxel i and later were detected by the scanner. For the complete derivation of the above posterior refer to [3]. Basics of the derivation of (2) are included in Appendix A.

D. Posterior of the Total Number of Events in ROI

In order to evaluate the imaging systems we first specify the clinical task to be performed by the imaging system. A common task in medical imaging is the quantitative measurement of activity in some ROI. Then, based on these measured activity values, clinically relevant decisions are made. In our approach we will use the total number of events that occurred per ROI during imaging time τ as a quantity used for the system evaluation. Sometimes the clinical task is dependent on more than a single ROI. Although the theory and methods developed in this work are applicable to multiple ROI, for clarity only a single ROI is used. Extensions of the theory to more ROIs is possible (see discussion). We define a random variable ρ indicating the number of counts emitted in the ROI and detected by the imaging system. It is obvious that

$$\rho = \sum_{i \in \text{ROI}} \sum_{k=1}^K y_{ki} \quad (3)$$

where the first sum in above equation is carried out over the voxels that belong to the ROI. Using the posterior [(2)] and the definition of ρ [(3)] the posterior $p(\rho|\mathbf{g})$ can be approximated using Monte Carlo methods described in Section III-A.

E. Figures of Merit of System Performance

We will use two figures of merit. The main figure of merit will be based on the variance of emission counts. We also use Bayes Risk (BR) as a FOM for the purpose of comparison. A description of the calculation of BR is provided in the Appendix C.

As a main FOM we will define a measure that expresses the ability of the system to determine the number of events that occurred in an ROI and were detected during the scan with duration τ . Since ρ (total number of emissions per ROI that were detected) and $\mathbf{f}$ (voxel activities) are statistically dependent, the distribution of $\pi(\rho)$ can be derived analytically

from the prior $\pi(\mathbf{f})$ and shown that it follows the Gamma–Poisson mixture (GP-M) distribution (see Appendix B for more details). The known GP-M distribution $\pi(\rho)$ will be used as the prior in the system evaluation using prior/posterior variance reduction.

In order to define this index, we first define $\bar{\rho}$ which is a posterior mean of the number of events per ROI as

$$\bar{\rho} = \sum_{\rho=0}^{\infty} \rho p(\rho | \mathbf{g}; \xi). \quad (4)$$

On the right side of the above equation we used ξ which indicates the imaging system. For notation clarity we dropped the dependence of $\bar{\rho}$ on ξ and π and $\mathbf{g}$. Using the equation above we define the posterior variance as

$$V(\xi, \pi, \mathbf{g}) = \sum_{\rho=0}^{\infty} (\rho - \bar{\rho})^2 p(\rho | \mathbf{g}; \xi). \quad (5)$$

It follows that the average posterior variance (averaged over noise realizations) is defined as

$$\begin{aligned} V(\xi, \pi) &= \sum_{\mathbf{g}} V(\xi, \pi, \mathbf{g}) p(\mathbf{g}) \\ &= \sum_{\mathbf{g}} \sum_{\rho=0}^{\infty} (\rho - \bar{\rho})^2 p(\rho, \mathbf{g}; \xi). \end{aligned} \quad (6)$$

Similarly we derive the average prior variance. First, we define $\bar{\rho}_{\pi} = \sum_{\rho=0}^{\infty} \rho \pi(\rho)$ and $V_{\pi}(\pi)$ as

$$V_{\pi}(\pi) = \sum_{\rho=0}^{\infty} (\rho - \bar{\rho}_{\pi})^2 \pi(\rho). \quad (7)$$

Note that the prior variance does not have to be calculated numerically as it is known theoretically for the GP-M distribution [see Appendix B].

Finally, we define the posterior variance reduction index that will be used to quantify the system performance as

$$U(\xi, \pi, \mathbf{g}) = \frac{V_{\pi}(\pi) - V(\xi, \pi, \mathbf{g})}{V_{\pi}(\pi)}. \quad (8)$$

For a 1-D vector ρ , this quantity has a straightforward interpretation as the fractional reduction in posterior variance upon observing data $\mathbf{g}$. Obviously the system performance cannot be dependent on the random variable $\mathbf{g}$; therefore we also average over noise realizations

$$U(\xi, \pi) = \sum_{\mathbf{g}} U(\xi, \pi, \mathbf{g}) p(\mathbf{g}). \quad (9)$$

Using (6), (8), and (9) we arrive at the final expression for the variance reduction figure of merit (FOM)

$$U(\xi, \pi) = 1 - \frac{1}{V_{\pi}(\pi)} \sum_{\mathbf{g}} \sum_{\rho=0}^{\infty} (\rho - \bar{\rho})^2 p(\rho, \mathbf{g}; \xi) \quad (10)$$

where $\sum_{\mathbf{g}}$ is a discrete sum over all possible discrete data vectors $\mathbf{g}$ (average over noise realizations). We will refer to $U(\xi, \pi)$ as the posterior variance reduction index (PVRI). Since $V(\xi, \pi, \mathbf{g})$ is typically smaller than $V_{\pi}(\pi)$ for informative systems (systems that reduce the variance) the $U(\xi, \pi)$ is between 0 and 1. $U(\xi, \pi)=0$ indicates a useless system that does not bring any additional information about ρ over the information contained in the prior. The value of $U(\xi, \pi)=1$ indicates a perfect imaging system as the data $\mathbf{g}$ would imply the exact number of emissions per voxel that occurred in the experiment (posterior variance equal to zero). An example of such a system would be an idealized TOF-PET camera with perfect characteristics, including perfect time-of-flight resolution, which would directly measure the number of events per voxel.

III. METHODS

A. Numerical Monte Carlo Methods

The main numerical challenge is to determine the $p(\rho | \mathbf{g})$. In order to determine the $p(\rho | \mathbf{g})$ we build a histogram of the number of posterior samples for each discrete value of ρ using Monte Carlo methods. For example, for 1-D ρ (i.e., only a single ROI is considered) the possible values of ρ are $0, 1, 2, \dots, \infty$ which enumerate bins of the histogram. When a sample from the posterior is acquired, the value of ρ is determined and the histogram bin corresponding to this value is increased by 1. At the end of the Monte Carlo run all histogram bins are divided by the total number of samples and thus become estimates of the posterior probabilities of ρ for each histogram bin number.

To acquire samples from the posterior $p(\mathbf{y} | \mathbf{g})$ we use the Metropolis–Hastings approach. Although in our previous work [1]–[3] we already discussed the algorithm (which is summarized at the end of this section), it required introduction of the *Origin Ensemble* (OE) space. In the current paper, we provide a new derivation of the algorithm for sampling the posterior that does not require OE space.

In general, in order to have a well behaved Markov Chain Monte Carlo algorithm for sampling the posterior, the *detailed balance* condition has to be enforced. Let us by $\mathbf{y}$ and $\mathbf{y}'$ denote two EC vectors that differ by the location of the origin of just a single event that was detected in projection element k . Without losing generality, we assume that for this event origin is located in i and for $\mathbf{y}$ it is located in j . It follows that $y_{ki} = y_{kj} + 1$ and $y_{ki} = y_{kj} - 1$. Using the definition of the posterior (2) the ratio of posterior probabilities of $\mathbf{y}$ to $\mathbf{y}'$ is

$$\frac{p(\mathbf{y}'|\mathbf{g})}{p(\mathbf{y}|\mathbf{g})} = \frac{\alpha_{kj}(\epsilon_i \tau + \beta_i)(c_j + \beta_j \phi_j)(y_{ki})}{\alpha_{ki}(\epsilon_j \tau + \beta_j)(c_i + \beta_i \phi_i - 1)(y_{kj} + 1)}. \quad (11)$$

Suppose we construct a Markov chain where a single step in Markov process is defined by a move of one event origin detected in some projection element k from a location in voxel i to a new location in voxel j . The two configurations of origins are denoted as $\mathbf{y}$ and $\mathbf{y}'$. Let us by $P(\mathbf{y} \rightarrow \mathbf{y}')$ indicate that probability of transition from $\mathbf{y}$ to $\mathbf{y}'$ and vice versa. The condition of detailed balanced says that if $P(\mathbf{y} \rightarrow \mathbf{y}') p(\mathbf{y}|\mathbf{g}) = P(\mathbf{y}' \rightarrow \mathbf{y}) p(\mathbf{y}'|\mathbf{g})$ the chain will lead to equilibrium, and once in equilibrium, samples from the posterior $p(\mathbf{y}|\mathbf{g})$ will be obtained.

In order to construct the Markov chain with such P 's we first note that $P(\mathbf{y} \rightarrow \mathbf{y}') = S(\mathbf{y}'|\mathbf{y}) A(\mathbf{y}'|\mathbf{y})$, where $S(\mathbf{y}'|\mathbf{y})$ is the probability of selecting $\mathbf{y}'$ if the current state is $\mathbf{y}$ and $A(\mathbf{y}'|\mathbf{y})$ is the acceptance probability that once selected the $\mathbf{y}'$ is accepted as the new state of the MC. If it is not accepted, the chain remains at the state $\mathbf{y}$. To determine the selection probability for the algorithm in which we randomly select a single event, we note that if we randomly select an event from all possible detected events the probability of selecting $\mathbf{y}'$ if the system is in $\mathbf{y}$ is

$$S(\mathbf{y}'|\mathbf{y}) = \frac{y_{ki}}{N} \frac{1}{I}. \quad (12)$$

The factor (y_{ki}/N) is the probability that an event emitted in i and detected in k is selected and then multiplied by the chance that exactly voxel j (specified voxel) is selected from all voxels $(1/I)$. The symbol N is the total number of detected events. Similarly $S(\mathbf{y}|\mathbf{y}')$ can be specified and the ratio of such selection probabilities is equal to

$$\frac{S(\mathbf{y}'|\mathbf{y})}{S(\mathbf{y}|\mathbf{y}')} = \frac{y_{ki}}{y_{kj} + 1}. \quad (13)$$

Since the random selection of origins provides the above ratio of selection probabilities it follows from (11) that in order for the chain to obey detailed balance the acceptance ratio must be

$$\frac{A(y'|y)}{A(y|y')} = \frac{\alpha_{kj}(\epsilon_i \tau + \beta_i)(c_j + \beta_j \phi_j)}{\alpha_i(\epsilon_{kj} \tau + \beta_j)(c_i + \beta_i \phi_i - 1)} \quad (14)$$

which is identical to the acceptance ratio derived using OE space [3]. To evaluate this acceptance ratio, the actual value of y_{ki} is not necessary and therefore the method does not carry excessive memory requirements.

Using the above we can construct the following algorithm which we refer to as the origin ensemble (OE) algorithm [1]–[3] that will provide samples from the posterior.

- 1) Select starting state $\mathbf{y}_0$. This can be done by randomly selecting for each event detected in projection element k a voxel i (with nonzero α_{ki}) i.e., assign the origin of the event to this voxel.
- 2) Randomly select an event n , and note the voxel i in which the origin of n is located.
- 3) Randomly select a candidate new voxel j for event n .
- 4) Move origin of event n to the candidate voxel with a chance equal to $\min\left(1, \frac{\alpha_{kj}(\epsilon_i \tau + \beta_i)(c_j + \beta_j \phi_j)}{\alpha_i(\epsilon_{kj} \tau + \beta_j)(c_i + \beta_i \phi_i - 1)}\right)$.
- 5) Repeat last three steps

We define an iteration of this algorithm as steps 2)–4) being repeated N times (N is the total number of detected events).

B. Computer Simulations

To test the new methodology, we compared system performance for the case of 2-D tomographic imaging with predictions from the classical theory of tomographic sampling using computer simulations. The reconstruction matrix consisted of 32×32 square pixels with pixel size equal to 1. The class of objects (object variability) was defined by the specification of the vector Φ shown in Fig. 3, and assuming β equal to either 0.5 or 0.1 for all pixels.

We simulated parallel tomographic projection data acquired over 180° . The number of bins per projection was varied to study effects of lateral sampling; we considered the following number of bins: 2, 8, 16, 24, 32, 48, 64, 72, 128, and 256. Projections with equal angular separation were generated, and the number of projections was also varied to test the following numbers of angular views: 1, 2, 4, 6, 10, 16, 24, 36, 48, 72, and 196. Each element of the system matrix α_{ki} was defined as the area of pixel i inside the parallel strip defined by projection bins: 2, 8, 16, 24, 32, 48, 64, 72, 128, and 256. Projections with equal angular separation were generated, and the number of projections was also varied to test the following numbers of angular views: 1, 2, 4, 6, 10, 16, 24, 36, 48, 72, and 196. Each element of the system matrix α_{ki} was defined as the area of pixel i inside the parallel strip defined by projection bin k , divided by the number of projections. This resulted in a total sensitivity

ϵ for each pixel equal to the area of the pixel, which was equal to 1, since the pixel size was 1. To generate a data set, the activity for each voxel f_i was determined as a sample from the prior $\pi(\mathbf{f})$ and then noiseless projection data were generated based on activity and scaled by $20\,000\tau / \sum_{i=1}^I \phi_i$. The scaling enforced the constraint that the total number of emitted counts, on average, should be 20 000 for $\tau=1$. The total counts were then scaled in proportion to the acquisition time τ , regardless of the number of projections, projection bins, and pixels. Since we aimed to investigate statistical properties in this work, an idealized tomographic system was used in which no Compton scatter or finite spatial and energy resolutions of the detectors were simulated. This was done so that effects due to inaccuracies in system modeling could be avoided.

To simulate attenuated projections for SPECT, it was assumed that voxels with nonzero Φ (see Fig. 3) have an attenuation coefficient of 0.15 cm^{-1} , and the pixel size was assumed to be 1 cm. This resulted in approximate sensitivities 0.2, 0.1, 0.1, and 0.2 for ROIs 1–4. Those sensitivity values varied slightly, depending on the number of projections. The projections were simulated over 180 (from 3 o'clock to 9 o'clock) (see Fig. 3). The value of τ used for simulation with attenuation was equal to 5, which resulted in approximately 24 000 detected events; this is equivalent to $\tau \approx$ approximately 24 000 detected events; this is equivalent to for the case of no attenuation.

Calculations of the posterior distributions of EC [inner sum in (10)] in the ROIs were done using 5000 iterations of the OE algorithm [3] as the initial run for the system to reach equilibrium, followed by 50 000 iterations to create the estimate of the posterior distribution (Section III-A). To make sure that the system reaches equilibrium we observed the number of iterations beyond which the number of counts in the ROIs did not change significantly (which was around 500 iterations) and then multiplied it by 10 to make sure equilibration was reached. We assumed that correlations between $\mathbf{y}$ vectors in subsequent iterations could be ignored because with similar correlation times they will average out in the long run of the Markov Chain. This assumption was verified by comparing results when samples were obtained every 1, 5, and 1000 iterations (Fig. 4) for a very long run with a total of 20 million iterations. In order to estimate the sum over $\mathbf{g}$ in (10) we used 50 samples of $\mathbf{g}$ drawn from the joint distribution $p(\mathbf{f}, \mathbf{g})$. To generate the samples from the joint distribution, first $\mathbf{f}$ is drawn from the gamma distribution; noiseless projections are then computed and finally Poisson noisy projections are generated.

Since the method used to calculate the variances is based on Monte Carlo sampling, estimation of the standard error of derived quantities is important. In all calculations we used the bootstrap method [16]. In short, if t Monte Carlo samples are available we bootstrap (randomly chosen samples from the set with possible repetitions) a hundred additional sets of t samples and calculate standard error as $\sqrt{q^2 - \bar{q}^2}$ where q is a quantity of interest calculated from t samples, and $\bar{q}$ and $\bar{q}^2$ are the average of this quantity and average of squared values of this quantity over 100 bootstrapped sets of samples.

IV. RESULTS

In Section IV-A, we present the investigation of various sampling configurations for 2-D tomography using the PVRI and Bayes risk.

A. Investigation of PVRI for 2-D Tomography

We investigated the system for the task of estimating the total number of emitted events in four four-pixel ROIs placed on cold and hot lesions as demonstrated in Fig. 3 with and without attenuation.

Fig. 5 shows that increasing the number of bins in the projections always improves the PVRI [Fig. 5(C), (D), (G), (H)] for hot lesion ROIs. However, the improvements are minimal when the number of bins is higher than 64, which corresponds to the Shannon–Nyquist frequency [Fig. 5(C) and (D)]. It is also evident that PVRI is much smaller for cold lesions than for hot lesions which is consistent with the observation that it is more difficult to recover information from tomographic data about the cold regions.

An interesting detail can also be observed by comparing Fig. 5(C) and (D). It is quite obvious that the ROIs placed on smaller lesions [Fig. 5(D)] have much higher PVRI. This is most likely due to the fact that the ROI pixels placed on the larger hot lesions [Fig. 5(C)] are surrounded by high intensity pixels which have higher absolute variance compared to the variance of pixels in the background that surrounds the smaller hot lesion. This higher variance of neighboring pixels increases the posterior variance in the ROI due to correlations, resulting in markedly reduced PVRI.

For pixels that are more attenuated [Fig. 5(f) and (G)] that PVRI is much smaller than for less attenuated pixels [Fig. 5(E) and (H)]. The value of τ was set to 1 or 5, respectively, for experiments without and with attenuation. The total number of detected counts were 20 000 and 24 000, respectively. We note that although slightly more counts were detected from ROI 1 for the experiment with attenuation [Fig. 5(E)] than in the experiment without attenuation due to longer imaging time [Fig. 5(A)] the PVRI with attenuation is smaller, which we attribute to the fact that for the case with attenuation, ROI 1 is overlapped in projections with pixels that have a much higher number of emitted and detected counts and, therefore, higher absolute noise. This is not the case for hot regions [Fig. 5(D) and (H)], where we observed a slightly better PVRI with attenuation for ROIs that emitted more counts.

Fig. 6 presents results obtained when the number of projections was varied over a 180° angular span, for the same expected number of emitted counts. For the hot lesion ROIs [Fig. 6(C) and (D)], about 32 projections were sufficient to obtain the maximum PVRI. Increasing the number of projections above 32 only minimally improved the index. Data for cold lesion ROIs [Fig. 6(A) and (B)] suggest that a smaller number of projections is needed to obtain an adequate PVRI.

Results presented in Fig. 7 show the dependence of PVRI on imaging time τ . If the number of emitted counts is low (small τ), increasing the number of projections does not

significantly improve PVRI which is quite evident in all graphs in Fig. 7. We also note that for the hot lesion ROIs [Fig. 7(C) and (D)] a larger number of counts always results in a higher PVRI index. Interestingly, for the cold lesion ROIs [Fig. 7(A) and (B)], increasing the number of emitted events by increasing the acquisition time from $\tau=0.1$ to 0.25 did not result in significant improvement in PVRI.

B. Comparison to Bayes Risk

We compared the PVRI to the value of Bayes risk (BR) for MAP estimation assuming a gamma prior and quadratic loss function (Appendix C). Unlike the PVRI, a low value of BR indicates a better system. As mentioned in the introduction, the BR not only measures a property of the imaging system, but it also reflects the performance of the reconstruction; therefore, BR is a semi-optimal measure. We provide a summary of the results of BR in Fig. 8

As we noted for the PVR index, the BR value also indicates that 64 bins is approximately optimal for estimation of the activity in the ROI. The BR is much lower for ROI 3, which indicates that the estimator performs better for the case of fewer bins with larger ROIs [Fig. 8(C) versus (D)].

C. Robustness to Prior Selection

There are two parameters that define the prior. The Φ defines the geometric shape of the object (Fig. 3) while β indicates the spread of the possible values of activities around the Φ . We investigated how robust the findings expressed by the value of PVRI are to the selection of parameter β . In order to determine this, we calculated the PVRI for $\beta = 0.1$ and $\beta = 0.5$ for various number of projections and projection bins with same imaging time. The prior was always the same when data were generated and used in estimation of the posterior. Fig. 9 presents results of this comparison. We obtained excellent correlation between PVRI's obtained with different β 's which indicates that method is robust to the selection of β . In other words, the data suggest that the method would yield the same conclusions regardless of the value of β . The slope is significantly higher than zero because the PVRI indicates the reduction of the posterior variance relative to the initial prior variance. Therefore, we expect for larger prior variance the relative gain brought by observing the same data will be larger which is consistent with the finding that the slopes are significantly larger than 1.

V. DISCUSSION AND CONCLUSION

Unfortunately, at the moment the classical theory is not sufficiently general to grade imaging performance for complex imaging geometries, different scan patterns and object definition, and various clinical tasks. Therefore in order to validate the methods developed here, we used a simple model of 2-D tomography and found that our method provided answers consistent with the standard theory developed for these simple 2-D models.

From the Shannon–Nyquist sampling theory, we expect that the lateral sampling should be at least twice the number of pixels in the image for the case studied here. We found that larger numbers of projection bins are always beneficial, although when the number of projections bins is greater than twice the number of pixels (64, in Fig. 5) the gains in PVRI are minimal

for high intensity regions. Similar findings may be seen for cold ROIs, but because of the larger errors it is more difficult to determine the optimal number of bins; however, the range of the number of detector sampling bins required to include both cold and hot lesion quantitation appears to agree reasonably well with the predictions of Huesman [17], who suggested that the optimal number of bins should be between 1.4 and 2.5 times the number of pixels, which corresponds to 45–80 for our case. This finding also appears to hold true for different numbers of projections, whether attenuation is present or not. If attenuation is simulated the PVRI is much smaller for more attenuated ROIs [e.g., Fig. 5(G) versus (H)]. We note that the PVRI for the nonattenuated case [Fig. 5(D)] and attenuated case [Fig. 5(G)] are very similar which is because the same number of emission counts were emitted from ROIs, as the sensitivity of ROI 4 was about 0.2 and a five-times longer imaging time was simulated. We found that the BR—which measures performance of both the imaging system and the reconstruction methods—also shows that 64 bins is optimal in the sense that increasing the number of bins does not significantly improve the BR. For cold regions, a similar trend is seen for the BR metric.

In terms of the value of PVRI for predicting the required number of projections, we found that about 32 projections (which corresponds to the number of image pixels or, approximately, to the object size) are needed in order to achieve optimal performance for the case of lateral sampling with 64 bins per projection (Fig. 6). This number is smaller than the value, $1.5 \times 32 = 48$ predicted by Huesman [17] and others [18], [19] but it is consistent with empirical reports [20], [21] and intuitive arguments [22] that suggest that a much smaller number of projections than suggested in [17]–[19] is sufficient for artifact-free recovery of the object. Our results suggest that with a smaller number of bins per projections, optimal performance can be achieved using a smaller number of angles. We observed that for cold ROIs, optimal performance can be achieved using a smaller number of projections. The effect of attenuation was not simulated for this experiment because the number of projections greatly affects the voxel sensitivities which depend strongly on the starting angle. Therefore, the PVRI would be indicative of the combined effect of the number of projections and the sensitivity which depends on the starting angle.

The methods described in this work provide means for grading of imaging systems based on combined effects of tomographic sampling, noise, and specific region of interest (cold or hot region) which are difficult with standard methods. Even the trade-off can be considered. For example, one may consider the number of additional imaging time that is needed to compensate reduced number of projections (incomplete tomography), etc.

If the acquisition time is varied (Fig. 7), we found that the PVRI for cold ROIs are very small, and approximately the same below $\tau = 1$. Significant gains in PVRI could only be seen for imaging times greater than 1 ($\tau = 1$ corresponds to a total of 20 000 counts). This suggests that in order to use the data efficiently for cold ROIs, it is important to have a total number of counts above some threshold. Our results indicate that the use of tomographic imaging (many projections versus a single projection) adds very little additional information below this threshold. At this point, we are uncertain if the existence of a threshold can be shown, or if the value of such a threshold can be determined theoretically, but we will continue to develop the theory in this direction in the future.

As shown in Fig. 9, the new approach is quite robust to the selection of the prior width parameter. The correlation between PVRI obtained by two different priors is approximately linear, which strongly indicates that conclusions reached by evaluation of the system using one prior will be the same as conclusions reached by using other priors that differ in values of β . When β is increased the *a priori* number of emissions is known with increased precision, and the range of possible improvements due to experiment becomes smaller. This can be appreciated in Fig. 9, as the slope is about 1.4 which indicates that the improvement in PV is higher for smaller β (0.1 versus 0.5).

Although in this paper we used the number of emissions per voxel to derive the FOM, it is possible to derive the posterior and the new FOM based on $p(\tilde{f}|g)$ using the following:

$$\frac{p(\tilde{f}|g)}{\pi(\tilde{f})} = \sum_{\rho=0}^{\infty} p(\rho|\tilde{f}) \frac{p(\rho|g)}{\pi(\rho)}. \quad (15)$$

This equation is derived from Bayes theorem and axioms of probability theory. The terms $\pi(\tilde{f})$ and $\pi(\rho)$ are assumed and calculated (Appendix II-B) priors. The $p(\rho|\tilde{f})$ is the Poisson pdf and $p(\rho|g)$ is the posterior determined by Monte Carlo methods described in this work. Therefore, $p(\tilde{f}|g)$ can be readily obtained. Interestingly, $p(\tilde{f}|g)$ is very difficult to calculate directly using Monte Carlo methods from the posterior $p(\mathbf{f}|g) \propto p(\mathbf{g}|\mathbf{f}) \pi(\mathbf{f})$. This is due to the high computational cost required to evaluate $p(\mathbf{g}|\mathbf{f})$. By virtue of the theory presented in this paper, the posterior can be determined indirectly using (15). To verify the theoretical claim that (15) represents the posterior of the ROI activity requires additional research and will be an interesting continuation of this work. If successful, it would provide a method to compute $p(\tilde{f}|g)$ and a FOM for system evaluation based on $p(\tilde{f}|g)$ at a fraction of the computational cost required for the direct Monte Carlo calculation of $p(\tilde{f}|g)$.

In many clinical tasks, we will only consider ρ with one dimension (a single ROI) and therefore a 1-D posterior $p(\rho|g)$ will be considered. The extension of this method to multiple dimensions (multiple ROIs) is straightforward as long as the FOM is defined based on the posterior covariance for the multidimensional case. The determination of the posterior covariance is similar to the approaches described in Section II-A, except that multi-dimensional histograms are used. An example of a FOM derived from the multi-dimensional posterior covariance be the reduction in the weighted sum of multiple ROI posterior variances. Weighting can be designed to take into account the relative importance of the ROIs.

We used the reduction in posterior variance as the FOM. Another possibility for the FOM is the Kulback–Leibler divergence (KLD) between the posterior and the prior [23], which is a well-known measure of the information content in the data. The KLD has the advantage that it is directly applicable to the multidimensional case, but the value of KLD has a more difficult interpretation than the reduction in posterior variance, which is why we did not use it in this work. Our preliminary calculations suggest that the KLD will correlate well with the PVRI.

Another expansion of the method would be to use the approach to investigate the reduction in the posterior variance of parameters that are derived from ROI measurements. An example of such a quantity is the ratio of values obtained in two different ROIs. More interestingly, for dynamic imaging, the reduction of the posterior variance of kinetic parameters derived from a multi-frame set of projection data or list-mode data can be studied as well. For dynamic imaging, such an index would indicate how well the system performs for quantitative tasks in dynamic imaging.

Finally, we would like to emphasize that the statistical approach presented in this paper is not directly based on the data-likelihood. This, by itself, constitutes a major theoretical advance. As to the impact of methods based on the OE algorithm, it is uncertain. As mentioned in the introduction, if we consider only image reconstruction, the impact is questionable, because it is unclear how much better images reconstructed using our approach can be in comparison to those reconstructed using iterative methods. However, we do not discard the possibility that our basic algorithm may allow us to include different priors that may be better than standard priors since the approach is so unlike iterative data-likelihood-based methods. We see a possible high impact in utilizing the interval information. This work provides a possible use of that information (another was provided in our original paper with the statistical derivation [3]), but there may be others.

The method is practical as it can be implemented for the most complex imaging geometries, as well as for both sinogram and list-mode acquisition modes. In fact it is quite straightforward to implement the OE algorithm. The imaging-system evaluation is performed in exactly the same way as the image reconstruction, i.e., by sampling the posterior. Therefore, if the method can be implemented for other uses like image reconstruction, it can also easily be used for system evaluations. In our laboratory we have implemented it for 3-D PET data from a time-of-flight (TOF) list-mode lanthanum-bromide system [24] and for the Siemens Biograph 3-D PET [25]. We and other investigators have used it for analysis of data from a Compton camera and solid state detectors [26]–[28]. Other investigators have also used it in applications for radiotherapy [29].

In summary, we have presented a fully Bayesian approach that uses the concept of emission counts for imaging system evaluations. The approach is based on the reduction in the variance of the posterior compared to the prior for the specified ROI. The method was applied to the case of 2-D tomography and the findings were compared to results of known tomographic sampling theory. The system evaluation method presented here assumed that the image is represented by voxels, and requires specification of the ROI. The evaluation approach does not depend on the reconstruction method and it is applicable to both binned and list-mode data.

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APPENDIX

A.: Derivation of the EC Posterior $p(\mathbf{y}|\mathbf{g})$

The detailed derivation is presented in [3]. Here, we provide only the highlights of this derivation. From the definition of random vectors $\mathbf{y}$ and $\mathbf{g}$ we have that $p(\mathbf{g}|\mathbf{y}) = 1$ for $\mathbf{y} \in \mathbf{Y}_{\mathbf{g}}$ and zero otherwise. This is because $\mathbf{y}$ unambiguously implies for $\mathbf{g}$ for $\mathbf{y} \in \mathbf{Y}_{\mathbf{g}}$ and conversely for $\mathbf{y} \notin \mathbf{Y}_{\mathbf{g}}$ the probability of random vector $\mathbf{g}$ is zero from the definition. Using Bayes theorem we obtain

$$p(\mathbf{y}|\mathbf{g}) = p(\mathbf{g}|\mathbf{y}) \frac{p(\mathbf{y})}{p(\mathbf{g})} = \frac{p(\mathbf{g}|\mathbf{y})}{p(\mathbf{g})} \int_{\mathbf{f}} p(\mathbf{y}|\mathbf{f}) \pi(\mathbf{f}) d\mathbf{f}. \quad (16)$$

Since $p(\mathbf{y}|\mathbf{f})$ is described by Poisson distribution $\pi(\mathbf{f})$ and is assumed (1) the integral $\int_{\mathbf{f}} = \int_0^{\infty} df_1 \dots \int_0^{\infty} df_L$ can be readily calculated analytically and the above transforms to (2).

B.: Derivation of the $\pi(\rho)$ and Calculation of Prior Variance

The ROI is a union of some number of voxels. The number of events emitted by each voxel (detected or not) is Poisson distributed assuming the voxel activity is known and therefore the distribution of the number of events that occurred in the ROI and were detected by the scanner conditioned on the expected number of emitted counts $\tilde{f} \in \tau$ is

$$p(\rho | \tilde{f} \in \tau) = \frac{(\tilde{f} \in \tau)^\rho e^{-\tilde{f} \in \tau}}{\rho!} \quad (17)$$

Where $\tilde{f} = \sum_{i \in \text{ROI}} f_i$ We used the fact that the sum of Poisson variates is also a Poisson variate, and assumed that the sensitivity values of voxels constituting the ROI are all the same and equal to $\bar{\epsilon}$. This assumption will typically be violated, but if the ROIs are small (which typically is the case) the variation in sensitivity will be minimal as the sensitivity is, spatially, a very slowly varying function. The prior $\pi(\tilde{f})$ can be obtained from $\pi(\mathbf{f})$ by assuming that the voxels that constitute the ROI have the same β 's in which case the $\pi(\tilde{f})$ will be gamma distributed (same distribution as $\pi(\mathbf{f})$) such that

$$\pi(\tilde{f}) = \frac{(\beta)^{\beta\tilde{\phi}}}{\Gamma(\beta\tilde{\phi})} (\tilde{f})^{\beta\tilde{\phi}-1} e^{-\beta\tilde{f}} \quad (18)$$

under the assumption that the event emissions from different voxels are conditionally statistically independent. We used $\tilde{\phi} = \sum_{i \in \text{ROI}} \phi_i$ Now, $\pi(\rho)$ the prior can be obtained from

$$\pi(\rho) = \int_0^\infty d\tilde{f} p(\rho | \tilde{f} \bar{\Xi} \tau) \pi(\tilde{f}) \quad (19)$$

which leads to

$$\pi(\rho) = \int_0^\infty d\tilde{f} \frac{(\tilde{f} \bar{\Xi} \tau)^\rho e^{-\tilde{f} \bar{\Xi} \tau} (\beta \tilde{\phi})^{\beta \tilde{\phi}}}{\Gamma(\beta \tilde{\phi})} (\tilde{f})^{\beta \tilde{\phi} - 1} e^{-\beta \tilde{f}}. \quad (20)$$

Rearranging the terms in the above yields

$$\pi(\rho) = \frac{(\beta \tilde{\phi})^{\beta \tilde{\phi}} (\bar{\Xi} \tau)^\rho}{\Gamma(\beta \tilde{\phi}) \rho!} \int_0^\infty d\tilde{f} (\tilde{f})^{\rho + \beta \tilde{\phi} - 1} e^{-\tilde{f}(\bar{\Xi} \tau + \beta)}. \quad (21)$$

After calculating the integral and rearranging the terms

$$\pi(\rho) = \binom{\rho + \beta \tilde{\phi} - 1}{\rho} \left(1 - \frac{\bar{\Xi} \tau}{\beta + \bar{\Xi} \tau}\right)^{\beta \tilde{\phi}} \left(\frac{\bar{\Xi} \tau}{\beta + \bar{\Xi} \tau}\right)^\rho. \quad (22)$$

The above equation is from the definition the negative binomial distribution with one real-valued parameter ($\beta \tilde{\phi}$) also known as Poisson-binomial distribution. The negative binomial distribution with real-valued parameters is also known as the *Pascal* or *Polya Distribution* [30]. Fig. 10 graphically presents this distribution for different values of the parameters ϕ and β .

The variance of the negative binomial distribution is known theoretically and for $\pi(\rho)$ expressed by (22), it is equal to $\bar{\Xi} \tilde{\phi} ((\bar{\Xi} \tau + \beta)/(\beta))$. Using this result we obtain

$$V_\pi(\pi) = \bar{\Xi} \tilde{\phi} \frac{\bar{\Xi} \tau + \beta}{\beta}. \quad (23)$$

This expression for the prior variance is consistent with the known asymptotic behavior for β . For $\beta \rightarrow 0$ the variance goes to infinity as the prior with $\beta \rightarrow 0$ approaches a flat prior. For $\beta \rightarrow \infty$ the $V_\pi(\pi) \rightarrow \bar{\Xi} \tau \tilde{\phi}$, which is the value of the expected number of detected events per ROI. This asymptotic behavior is anticipated because the emission of events for fixed activity (obtained by assuming $\beta \rightarrow \infty$) is governed by the Poisson distribution, for which the variance is equal to the expected number of events.

C.: Calculation of Bayes Risk

We chose to use the maximum *a posteriori* expectation maximization (MAPEM) algorithm, with a gamma prior [31], as a combined reconstruction and estimation method to obtain the BR. The Bayes risk for the MAPEM $r(\pi, L)$ was calculated for the task of estimation of the activity ($\mathbf{f}$) in the ROIs specified in Fig. 3. This was done by estimation of the BR by averaging values of risks calculated over 10 000 repetitions of the following two steps. Step 1: Draw projection data from the joint distribution $p(\mathbf{f}, \mathbf{g})$ (described at the end of Section III-B). Step 2: Perform 1000 iterations of MAPEM to determine the point estimator of the activity in ROI $\tilde{f}$ and then calculate risk assuming a quadratic loss function. The average risk over 10 000 repetitions was used as an estimate of BR. BR errors were calculated using the bootstrapping approach described in Sections III-A and III-B. In general, the BR can be considered as the standard mean-squared error averaged over the prior distribution.

In the work discussed above, we assumed that after 10 000 iterations of MAPEM, convergence would be reached to double-precision numerical accuracy. This was confirmed by the fact that performing iterations beyond 10 000 did not change the pixel values.

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Fig. 1.

Conceptual representation of the evaluation method used in this paper. A: Before the experiment (imaging scan) is performed, the total number of counts emitted in some ROI is described by a prior distribution (depicted as A). Figures B and C show the posterior probabilities for two different experimental setups 1 and 2, respectively.

Fig. 2.

Two-voxel, two-projection system. Measured data are g_1 and g_2 counts acquired in projection elements 1 and 2. Voxels have activities f_1 and f_2 . Emission-counts y_{11} and y_{21} are the numbers of counts emitted in voxel 1 and detected in projection elements 1 and 2, respectively. A similar definition applies for EC values y_{12} and y_{22} . EC space $\mathbf{Y}$ has more dimensions than $\mathbf{G}$ (four as opposed to two dimensions in the given example). A vector corresponds unambiguously to the vector $\mathbf{g}$ such that $g_k = \sum_{i=1}^I y_{ki}$ but a vector $\mathbf{g}$ corresponds to a subspace $\mathbf{Y}_{\mathbf{g}}$. In other words, there are many vectors from EC that correspond to a single acquired dataset $\mathbf{g}$. It follows that although we cannot determine which $\mathbf{y}$ occurred upon observing $\mathbf{g}$ we can determine probabilities of each $\mathbf{y}$ given $\mathbf{g}$. See [5] and (2).

Fig. 3.

Two-dimensional image of prior parameter. The number on the image on the left indicate the value of Φ for pixels in different structures indicated by arrows. Definitions of four four-pixel ROIs are shown on the right. When attenuation was simulated the attenuator (with attenuation coefficient 0.15 cm^{-1}) had the same shape as the background.

Fig. 4.

Correlation between posterior probabilities of emission counts obtained by sampling the posterior every 1, 5, and 1000 iterations of OE algorithm. The total number of iterations used to estimate the posteriors was 20 000 000 (equivalent to 4×10^{11} Markov steps for 20 000 detected counts) Each point on the graph corresponds to a single value of a possible EC number in ROI 3 (Fig. 3). We observe close to perfect correlations between posterior distributions.

Fig. 5.

Posterior variance reduction index (PVRI y-axes) as a function of the number of projection bins. Images A–D correspond to ROIs 1–4 shown in Fig. 3. Image E–H correspond to simulations with attenuation for ROIs 1–4. The error bars for C, D, G, and H are so small that they are hardly visible. The five curves correspond to 2, 6, 16, 48, and 192 projections and the total acquisition time was equal to 1 for A–D and 5 for E–H. In general, increasing the number of bins improves the PVRI, which reaches a plateau around 64 bins.

Fig. 6. Posterior variance reduction index (PVRI y-axes) as a function of the number of projections. Images A–D correspond to ROIs 1–4 shown in Fig. 3. The error bars for C and D are so small that they are hardly visible. Four curves correspond to 2, 24, 64, and 256 bins per projection and acquisition time τ is equal to 1 for all curves. In general, a larger number of projections improves the PVRI, except when using a small number of bins, for which the benefits are not significant.

Fig. 7. Posterior variance reduction index (PVRI y-axes) as a function of the number of projections for a fixed number (64) of projection bins. Images A–D correspond to ROIs 1–4 shown in Fig. 3. Each curve corresponds to a different acquisition time τ . The benefits of using a larger number of projections are greater for lower levels of noise (larger τ).

Fig. 8.

Bayes risk (y-axes) as a function of the number of projection bins. Images A–D correspond to ROIs 1–4 shown in Fig. 3. The task was the estimation of average activity in the ROIs, a gamma prior, and quadratic loss function were used. The five curves correspond to 2, 6, 16, 48, and 192 projections and the total acquisition time τ was equal to 1. The value of BR decreases and reaches a plateau around 64 bins.

Fig. 9.

Correlations between PVRI indexes obtained using different values (0.1 and 0.5) of parameter β in the prior definition. Images A, B, and C present this comparison for a different numbers of projections (192, 24, and 4, respectively). Each point on the graph corresponds to different number of projection bins (a total of 10 points) and going from left to right they corresponds to 2, 8, 16, 24, 32, 48, 64, 72, 128, and 256. Error bars are smaller than the symbols.

Fig. 10.

Examples of the Poisson binomial distributions that were used as priors for $\tau = \bar{\epsilon} = 1$.

Although the distributions are represented as continuous curves for clarity, they are discrete distributions that assign a probability for each value of ρ .