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Guided electromagnetic waves for periodic arrays of thin metallic wires near an interface between two dielectric media

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Guided localized electromagnetic waves propagating along one-dimensional (1D) arrays of thin metallic parallel wires, finite and infinite, are studied. The arrays are embedded into the upper dielectric half-space close to the interface separating it from the lower dielectric medium with different permittivity and the same permeability. Firstly, a dependence of resonance frequencies of excited wave modes for finite array with respect to the array height above the interface is studied. The array is excited by a normally incident plane wave. It is important that the order of the resonance modes changes if the distance between the array and the interface becomes small. An analysis, based on the Pocklington system of integral equations to evaluate resonance frequencies and compute the fields of excited modes above the array, was applied. This approach is based on the longwave approximation of thin wires. Secondly, the waves propagating along infinite 1D array of thin metallic wires that is close to the interface are studied. Dispersion curves are presented for the lowest case of half-wave resonance for different heights of the array over the interface. When the array approaches very close to the interface an anomalous dispersion is observed. The results of the numerical analysis were tested against computations obtained by means of other independent CST Studio Suite simulations.

1. Introduction

The subject of localized guided modes propagating along periodic structures has been very popular for the last two

or more decades in the theory of water waves, acoustics of waveguides, elasticity theory, photonic crystal optics and microwave physics.

In water waves and acoustics, one could cite well-known papers [1–4], for example, in which trapped modes, localized oscillations in unbounded media, are referred to as acoustic resonances, Rayleigh–Bloch waves, array-guided surface waves and bound states. In these papers, one can find a review of quite recent results on these so-called embedded trapped modes in waveguides as well embedded Rayleigh–Bloch modes, travelling waves on one-dimensional (1D) and two-dimensional (2D) periodic arrays of spherical scatterers, below and above the cut-off frequency, and governed by the Helmholtz equation. It is worth mentioning the new approximation method for scattering by long finite arrays of infinite cylinders published in [4] that is relevant to the subject of the paper. Similar effects of embedded trapped modes and travelling waves have also been studied in elasticity (e.g. [5,6]).

Until recently, there has been a strong interest in many research works on metamaterial physics and subwavelength optics (e.g. [7–9]). In some of these studies of the electromagnetic localized waves guided by periodic arrays, scatterers were modelled by some combination of electric and magnetic dipoles; see, for example, research on travelling waves on 1D, 2D and three-dimensional (3D) periodic arrays of lossless scatterers [10,11]. It is worth mentioning another example of the localized guided electromagnetic waves propagating along 1D arrays of dielectric spheres that was studied in [12] on the basis of a rigorous analysis of a wave field superposition of vector spherical wave functions. In [13], electromagnetic energy transport along an array of closely spaced metal rods was investigated experimentally and with numerical simulations. It was discovered that the guided wave of electromagnetic energy was highly confined to the arrays and propagation exists for the rotation angle around 90° with very low power losses.

In another example, the guided localized electromagnetic waves propagating along a 1D finite array of thin metallic (perfectly conducting; PEC) parallel wires of finite length in a free space were studied quite recently in [14]. In this problem, the length of cylindrical wires is assumed to be of the order of the wavelength, whereas the radius of the wires is much smaller than the wavelength. This is a so-called longwave approximation of thin metallic wires that is well known in antenna theory (e.g. [15]). The array was excited by a single thin active vibrator. An analysis based on the Pocklington system of integral equations [15] allowed the resonance frequencies to be computed and the excited wave modes localized near to the array to be constructed. The description of the wave modes, localized above the finite array and given in [14], is directly connected to another problem of guided localized electromagnetic waves, propagating along a 1D periodic infinite array of thin metallic wires. For the second problem, the quasi-periodic wave field is constructed as a superposition of vector wave fields irradiated by linear electric currents. The approach is also based on the method of integral equations of the Pocklington type. Dispersion curves were presented for the first lowest half-wave resonance of a single thin vibrator well known in radio-physics of antenna theory. It is remarkable to note that this is a purely 3D problem in regard to the wide experience of 2D study of Rayleigh–Bloch waves, guided surface waves in the theory of water waves, acoustics and elasticity.

The work presented in this paper builds on and extends the investigations of localized modes and travelling waves on a linear periodic array of parallel thin metallic wires described in [14]. Now, the finite and infinite arrays are embedded into a medium made of two dielectric half-spaces with different permittivities. The arrays are located close and parallel to the interface separating the half-spaces. It is assumed that the permeability is the same for both half-spaces. In the first part of the work, the dependence of the resonance frequencies of the localized modes excited by a finite array of thin wires with respect to the array height over the interface is studied. The array is illuminated by a normally incident plane electromagnetic wave. It was discovered that, when the height of the array becomes sufficiently small, the order of the mode resonances changes. The numerical analysis based on the system of Pocklington-type integral equations [14,15] to compute the resonance frequencies of the excited localized modes is applied. Then, the 2D portraits of the magnetic component distribution of the modes plotted above the array within a rectangle that is parallel to the XY (plane of the array) are demonstrated. The results of the numerical analysis

are compared with computations obtained by means of another independent numerical model—simulations done by means of the CST Studio Suite (method of finite elements). The agreement between both types of numerical results has proved to be perfect.

It is apparent that the subject of the paper is very important for many applications of metamaterials physics and, in particular, for the metasurfaces. One can easily imagine an artificial surface made of a 2D periodic array of thin metallic wires considered in the longwave approximation. Also, it is worth remarking that structures of periodic resonant wires have been recently proposed as passive multi-mode resonators for magnetic component enhancement to improve magnetic resonance imaging (MRI) [16]. The formulation of the problem described in the previous section was first suggested by the authors of [16]. The subject is also vital in antenna theory as it represents a quite important generalization of the famous canonical Sommerfeld problem of radiation of a horizontal point dipole antenna located above the flat Earth [17].

In the second part of the work, the localized guided electromagnetic waves propagating along an infinite 1D array of thin PEC parallel wires embedded into the upper dielectric half-space of the medium that was mentioned above are studied. The quasi-periodic wave field is constructed as a superposition of the vector wave fields irradiated by linear electric currents. Then, applying the Pocklington integral equation formalism and piece-wise linear discrete approximation for the current, we obtain an infinite system of real linear algebraic equations. Vanishing of the determinant of the associated infinite real matrix provides the conditions for the guided waves to exist, and these are determined numerically after truncation of the infinite system. Dispersion curves are presented for the lowest case of the half-wave resonance for different heights of the array over the interface below the cut-off frequency (below the light line). When the infinite array comes very close to the interface we observe an anomalous dispersion. As before, the results of numerical tests were compared with computations obtained by means of simulations carried out with the CST Studio Suite (method of finite elements). The agreement between both types of numerical results is very good.

The paper is organized as follows. Firstly, in §2a, as an introductory part, a description of the Pocklington integral equation formalism for a single thin wire located over an interface between two dielectric media is given. Subsequently, in §2b, a system of Pocklington integral equations for a finite 1D periodic array of thin wires over the interface between two dielectric media is described. In §3, we present numerical results for two typical modes, excited by a normally incident plane electromagnetic wave for a set of six thin wires. In §4a, guided electromagnetic waves propagating along infinite 1D arrays of thin metallic parallel wires near to the interface between two dielectric media are described analytically. Finally, the numerical results for the dispersion curves of the localized guided wave modes, propagating along the infinite 1D array of thin metallic wires, are discussed in §4b.

2. Finite array of thin wires above the interface between two dielectric media

In this section, we discuss excitation of the modes localized with respect to the magnetic component $\mathbf{H}$ in a neighbourhood of a finite array of parallel thin PEC wires. These modes are excited by a normally incident plane electromagnetic wave. The total length of cylindrical wires $2L$ is of the order of the wavelength. The radius of the wire a is much smaller than the wavelength (the longwave approximation). The array is embedded into the upper dielectric half-space with ϵ_1 and $\mu = 1$ close to the interface separating the upper dielectric from the lower medium with ϵ_2 and $\mu = 1$. The time-harmonic dependence $e^{-i\omega t}$ of the total electromagnetic fields is omitted everywhere below for the sake of simplicity. The rigorous mathematical formulation of the scattering problem is as follows. The electric $\mathbf{E}(x, y, z)$ and magnetic $\mathbf{H}(x, y, z)$ components of the total field satisfy Maxwell equations

$$\nabla \times \mathbf{E} = ik\mathbf{H}, \quad \nabla \times \mathbf{H} = -ik\mathbf{E}, \quad k = \frac{\omega}{c} \sqrt{\epsilon\mu}, \quad (2.1)$$

where c is the speed of light, k is the wavenumber and ω is the angular frequency. The total field $\mathbf{E}$ and $\mathbf{H}$ components above the interface ($z > 0$) are defined as the sums of the incident, the reflected

and the scattered fields

$$\mathbf{E} = \mathbf{E}_{\text{in}} + \mathbf{E}_{\text{ref}} + \mathbf{E}_{\text{sc}}, \quad \mathbf{H} = \mathbf{H}_{\text{in}} + \mathbf{H}_{\text{ref}} + \mathbf{H}_{\text{sc}}, \quad k_1 = \frac{\omega}{c} \sqrt{\epsilon_1 \mu}. \quad (2.2)$$

Below the interface the $\mathbf{E}$ and $\mathbf{H}$ components are the sums of the transmitted and the scattered fields

$$\mathbf{E} = \mathbf{E}_{\text{tr}} + \mathbf{E}_{\text{sc}}, \quad \mathbf{H} = \mathbf{H}_{\text{tr}} + \mathbf{H}_{\text{sc}}, \quad k_2 = \frac{\omega}{c} \sqrt{\epsilon_2 \mu}. \quad (2.3)$$

It is worth remarking that the reflected fields represent plane waves resulting from the incident plane-wave reflection from the interface, whereas below the interface the transmitted field is the plane wave of transmission of the incident plane wave through the interface. The scattered fields are generated by the induced electric currents distributed along the thin wires.

Further to the formulation of the problem, the total field should satisfy the following boundary conditions: the tangential electric component of the total field vanishes at any points of cylindrical surface for any wire $\mathbf{E} \times \mathbf{n}|_{S_w} = 0$, where $\mathbf{n}$ is the vector of the external normal with respect to the boundary S_w of arbitrary wire. Continuity of $\mathbf{E}$ and $\mathbf{H}$ components while crossing the interface S_i means that the following jumps are identical to zero: $[\mathbf{E} \times \mathbf{n}]_{S_i} = 0$ and $[\mathbf{H} \times \mathbf{n}]_{S_i} = 0$. Far away from the array of wires at infinity, the electric $\mathbf{E}_{\text{sc}}$ and the magnetic $\mathbf{H}_{\text{sc}}$ components of the scattering field must satisfy the Sommerfeld radiation condition. This means that for the finite array both components of the scattered field at infinity should be represented as expanding spherical waves. In the case of an infinite array both components of the scattered field at infinity above the array and below the interface should be represented in the form of the Rayleigh expansion of diffracted orders (outgoing plane waves, propagating and evanescent; e.g. [18]). In the close neighbourhood of the circular edges of a cylindrical surface for any wire, both components of the total field should satisfy the Meixner conditions.

(a) Pocklington integral equation for a single thin wire

Let us assume initially that a single thin PEC cylindrical wire is illuminated by a plane electromagnetic wave located inside an infinite homogeneous dielectric medium with permittivity ϵ and permeability μ . It is directed along the X -axis and its end points are located at $x = -L$ and $x = L$.

The problem of scattering of a plane electromagnetic wave by a PEC cylinder of finite length is well known in the diffraction theory of classical electrodynamics [15,19]. A traditional way to solve the problem is by reducing it to a 2D Fredholm second-type boundary integral equation for the unknown electric surface current. However, in the case of the longwave approximation, the 2D integral equation with the integral operator over the cylindrical surface of the wire could be reduced to a linear 1D integral equation as the leading term of the approximation. For the longwave approximation, the radius a and the total length $2L$ of a thin wire satisfy the following conditions (see, for example, §62 regarding thin wire diffraction in [19]): $a \ll L$, $a \ll \lambda$, $\lambda = 2\pi/k$, whereas $L \sim \lambda$. As a first step in the longwave approximation one can neglect a contribution of the discs of the cylindrical surface to the total electric surface current as $ka \ll 1$. The second key point is that the transversal component of the lateral electric current is being neglected. It is assumed that only the longitudinal component of the lateral electric current contributes to the total surface current. Thus, in the longwave approximation, using the cylindrical coordinate system related to a cylindrical wire, the electric surface current does not depend on φ , and it has only the x component in the first approximation that depends on x .

As a result, for the leading term of the discussed perturbation scheme in the longwave approximation, one could obtain a linear integral equation of Pocklington type that has been very popular in radio-location and antenna theory for a long period that started, perhaps, in the Second World War (e.g. [15], ch. 2, and [19], §62). Derivation of the Pocklington integral equation for an unknown electric current $I(x)$ in a single thin wire in free space can be found in various books on antenna theory, including [15]. In this case, the total field is a sum of the incident and the scattered waves. The scattered wave is induced by generated longitudinal electric current

$I(x)$, $-L < x < L$. Satisfying the boundary condition along the wire (the total x -component of the electric field is zero for $-L < x < L$), the Pocklington integral equation can be written in the form

$$\frac{i}{ck\sqrt{\epsilon\mu}} \int_{-L}^L dx' I(x') \left(k^2 + \frac{d^2}{dx'^2} \right) \frac{e^{ikR}}{R} = -\mathbf{E}_{\text{in}}(x) \cdot \mathbf{e}_x, \quad -L < x < L, \quad (2.4)$$

for which the boundary conditions should be taken into account $I(-L) = I(L) = 0$. Here on the right-hand side of (2.4) a scalar product of two vectors was introduced where $\mathbf{e}_x$ is the unit vector in the direction of the X -axis, and $\mathbf{E}_{\text{in}}(x)$ is the electric component of the incident plane electromagnetic wave, evaluated at points on the thin wire. The scalar product represents the inhomogeneous term for the integral equation. Here, we use the following notation for the corresponding distance involved: $R = \sqrt{(x - x')^2 + a^2}$.

The Pocklington integral equation (2.4) is to be solved numerically using different versions of the method of moments (e.g. [15]). As soon as a discrete set of values of the electric current $I(x)$ is computed, one can easily evaluate the vector potential and the electric and the magnetic components of the scattered field,

$$A_x = \frac{1}{c} \int_{-L}^L I(x') \frac{e^{ikR}}{R} dx', \quad R = \sqrt{(x - x')^2 + y^2 + z^2} \quad (2.5)$$

and

$$\mathbf{A}_{\text{sc}} = A_x \mathbf{e}_x, \quad \mathbf{E}_{\text{sc}} = \frac{i}{k\sqrt{\epsilon\mu}} (k^2 \mathbf{A}_{\text{sc}} + \nabla(\nabla \cdot \mathbf{A}_{\text{sc}})), \quad \mathbf{H}_{\text{sc}} = \frac{1}{\mu} \nabla \times \mathbf{A}_{\text{sc}}. \quad (2.6)$$

Let us assume now that a single thin PEC wire, directed along the X -axis, is located above the interface in the XY -plane between the two media with permittivity ϵ_1 and ϵ_2 , correspondingly, and common permeability $\mu = 1$, at a height h . Let it be located in the upper medium. Let the wire be again illuminated by a plane electromagnetic wave. Using well-known diffraction theory, i.e. the Sommerfeld exact solution for a horizontal point electric dipole hanging over the same interface [17], one can develop a generalized Pocklington integral equation for a wire of total length $2L$ similar to the approach described in [15]. The Pocklington integral equation in this case can be written in the following form:

$$\begin{aligned} & \frac{i}{ck_1\sqrt{\epsilon_1}} \int_{-L}^L dx' I(x') \left(k_1^2 + \frac{d^2}{dx'^2} \right) \left(\frac{e^{ik_1 R_0}}{R_0} - \frac{e^{ik_1 R_1}}{R_1} + \int_0^{+\infty} J_0(\lambda r) e^{-2h\chi_1} F_x^{(1)}(\lambda) \frac{\lambda d\lambda}{\chi_1} \right) \\ & + \frac{i}{ck_1\sqrt{\epsilon_1}} \int_{-L}^L dx' I(x') \frac{d^2}{dx'^2} \int_0^{+\infty} J_0(\lambda r) e^{-2h\chi_1} F_z^{(1)}(\lambda) d\lambda = -(\mathbf{E}_{\text{in}}(x) + \mathbf{E}_{\text{ref}}(x)) \cdot \mathbf{e}_x, \\ & -L < x < L, \quad I(-L) = I(L) = 0, \end{aligned} \quad (2.7)$$

where $J_0(\lambda)$ is the Bessel function of zero order,

$$\chi_1 = \sqrt{\lambda^2 - k_1^2}, \quad \chi_2 = \sqrt{\lambda^2 - k_2^2}, \quad k_{1,2} = \frac{\omega}{c\sqrt{\epsilon_{1,2}}}, \quad R_0 = \sqrt{(x - x')^2 + a^2},$$

$$R_1 = \sqrt{(x - x')^2 + 4h^2}, \quad r = |x - x'|,$$

$$F_x^{(1)}(\lambda) = \frac{2\chi_1}{\chi_1 + \chi_2}, \quad F_z^{(1)}(\lambda) = \frac{2\lambda}{\chi_1 + \chi_2} \left[\frac{k_1^2}{k_2^2} - 1 \right] \left[1 + \frac{k_1^2 \chi_2}{k_2^2 \chi_1} \right]^{-1}.$$

Here on the right-hand side of (2.7) the inhomogeneous term includes a contribution from the plane electromagnetic wave $\mathbf{E}_{\text{ref}}$, reflected from the interface, along with the incident wave. It is worth remarking that the integral equation (2.7) is valid for $h > 0$ as the two integrals inside equation (2.7) with respect to λ lose their exponential convergence at $h = 0$. Following the Sommerfeld analysis for the excitation of the horizontal point electric dipole, located over the interface between two dielectric media, it is convenient to use the Hertz vector potential $\Pi = (\Pi_x, \Pi_y, \Pi_z)$ instead of the vector potential $\mathbf{A}$ [17]. Both vectors are connected as $\Pi = i/(k\sqrt{\epsilon})\mathbf{A}$. It is worth noting that the additional condition $a \ll h$ is to be fulfilled in the problem under

consideration as in the opposite case the longwave approximation of thin wires is not applicable. From the numerical analysis data, we may conclude that at least the following approximate inequality must hold: $h > 10a$. After a discrete set of electric current $I(x)$ values has been evaluated, one can easily compute the vector potential, the electric and the magnetic components of the field scattered from the thin wire. Thus, the scattered field in terms of the Hertz vector in the upper half-space for the observation point $M = (x, y, z)$, ($z > 0$) can be represented as

$$\Pi_x(M) = \frac{i}{ck_1\sqrt{\epsilon_1}} \int_{-L}^L dx' I(x') \left(\frac{e^{ik_1 R_0}}{R_0} - \frac{e^{ik_1 R_1}}{R_1} + \int_0^{+\infty} J_0(\lambda r) e^{-(z+h)\chi_1} F_x^{(1)}(\lambda) \frac{\lambda d\lambda}{\chi_1} \right) \quad (2.8)$$

and

$$\Pi_z(M) = \frac{-i}{ck_1\sqrt{\epsilon_1}} \int_{-L}^L dx' I(x') \frac{d}{dx} \int_0^{+\infty} J_0(\lambda r) e^{-(z+h)\chi_1} F_z^{(1)}(\lambda) \frac{\lambda d\lambda}{\chi_1}, \quad (2.9)$$

where

$$R_0 = \sqrt{(x-x')^2 + y^2 + (z-h)^2}, \quad R_1 = \sqrt{(x-x')^2 + y^2 + (z+h)^2}, \quad r = \sqrt{(x-x')^2 + y^2}.$$

The electric and the magnetic components of the field scattered from the thin wire can be found from the following relations:

$$\mathbf{E}_{sc}(M) = k_1^2 \Pi + \nabla(\nabla \cdot \Pi), \quad \mathbf{H}_{sc}(M) = -i\sqrt{\epsilon_1} k_1 \nabla \times \Pi, \quad \Pi = (\Pi_x, 0, \Pi_z).$$

For the scattered field in the lower half-space ($z < 0$), one can derive that

$$\Pi_x(M) = \frac{i}{ck_1\sqrt{\epsilon_1}} \int_{-L}^L dx' I(x') \int_0^{+\infty} J_0(\lambda r) e^{z\chi_2 - h\chi_1} F_x^{(2)}(\lambda) \frac{\lambda d\lambda}{\chi_2} \quad (2.10)$$

and

$$\Pi_z(M) = \frac{-i}{ck_1\sqrt{\epsilon_1}} \int_{-L}^L dx' I(x') \frac{d}{dx} \int_0^{+\infty} J_0(\lambda r) e^{z\chi_2 - h\chi_1} F_z^{(2)}(\lambda) \frac{\lambda d\lambda}{\chi_2}, \quad (2.11)$$

where

$$F_x^{(2)}(\lambda) = \frac{k_1^2}{k_2^2} \frac{2\chi_2}{\chi_1 + \chi_2}, \quad F_z^{(2)}(\lambda) = \frac{2\lambda}{\chi_1 + \chi_2} \left[\frac{k_1^2}{k_2^2} - 1 \right] \left(1 - \left[1 + \frac{k_1^2 \chi_2}{k_2^2 \chi_1} \right]^{-1} \right)$$

and

$$\mathbf{E}_{sc}(M) = k_2^2 \Pi + \nabla(\nabla \cdot \Pi), \quad \mathbf{H}_{sc}(M) = -i\sqrt{\epsilon_2} k_2 \nabla \times \Pi.$$

Formulae (2.8)–(2.11) provide a global representation of the scattered field, and they are suitable for computing the field in the near-field zone. In the far-field zone, one could apply a traditional asymptotic analysis of the saddle point method to construct the spherical wave scattering patterns, the surface wave contributions and the lateral waves, originating from the saddle point, pole and branch points. However, in this paper we omit this analysis.

It should be noted that the representation (2.8)–(2.11) was constructed on the basis of the Sommerfeld solution [17]. The dipole moment p_x for the point source of the horizontal electric dipole in [17] in our case of a finite length thin wire should be replaced by $\int_{-L}^L dx I(x)$. Thus, the scattered field representation (2.8)–(2.11) satisfies the boundary conditions on the interface separating both dielectrics. Namely, this representation provides continuity of the tangent components of the scattered electric and magnetic fields across the interface. Secondly, one should note that the total electric field as the sum of the scattered fields, described by (2.8) and (2.9), the incident and the reflected plane waves has to satisfy the boundary condition over the lateral surface of the thin PEC wire. This means that a tangential component of this total electric field along the wire should be zero. This leads directly to the Pocklington integral equation (2.7).

In this paper, the analysis of the scattering problem of an incident plane wave by a single thin PEC wire in a free space, based on the first-order longwave approximation in the form of the linear Pocklington integral equation, has been applied to the corresponding scattering problems for finite and infinite arrays in the presence of the interface between two dielectric media.

Figure 1. The horizontal periodic array of six thin PEC wires over the interface separating two dielectric half-spaces. The perspective view of the array (a), and (b) the vertical cross section of the medium made up of two half-spaces with permittivity $\epsilon_1 = 81$ and $\epsilon_2 = 1$, correspondingly, and common permeability $\mu = 1$, with an array of six wires. The array with period d is illuminated by a plane normally incident electromagnetic wave. The array wires are directed along the X -axis, and parallel to the interface in the XY -plane separating the half-spaces.

(b) System of Pocklington integral equations for a finite array of thin wires

Let us assume that a horizontal finite array of N thin PEC wires, periodic with period d , is illuminated by a plane electromagnetic wave. Let all array wires be directed along the X -axis, and parallel to the interface in the XY -plane between two media with permittivity ϵ_1 and ϵ_2 , correspondingly, and common permeability $\mu = 1$. The array wires are located in the upper medium at height h (see figure 1 with a picture of six thin wires). Making use of the Sommerfeld exact solution for the horizontal point electric dipole located above the interface [17] and generalizing results of the previous section obtained for the single wire, one can derive a new system of Pocklington integral equations for N wires with total length $2L$ by adding additional terms of interaction between the wires. Thus, the system of N coupled integral equations for the unknown electric currents $I_n(x)$ ($n = 1, 2, \dots, N$) can be represented in the following form:

$$\begin{aligned}
 & \int_{-L}^L dx' I_n(x') \left(k_1^2 + \frac{d^2}{dx'^2} \right) \left(\frac{e^{ik_1 R_0}}{R_0} - \frac{e^{ik_1 R_1}}{R_1} + \int_0^{+\infty} J_0(\lambda r) e^{-2h\chi_1} F_x^{(1)}(\lambda) \frac{\lambda d\lambda}{\chi_1} \right) \\
 & + \sum_{m=1, m \neq n}^N \int_{-L}^L dx' I_m(x') \left(k_1^2 + \frac{d^2}{dx'^2} \right) \left(\frac{e^{ik_1 R_{nm}}}{R_{nm}} - \frac{e^{ik_1 \hat{R}_{nm}}}{\hat{R}_{nm}} + \int_0^{+\infty} J_0(\lambda r_{nm}) e^{-2h\chi_1} F_x^{(1)}(\lambda) \frac{\lambda d\lambda}{\chi_1} \right) \\
 & + \sum_{m=1}^N \int_{-L}^L dx' I_m(x') \frac{d^2}{dx'^2} \int_0^{+\infty} J_0(\lambda r_{nm}) e^{-2h\chi_1} F_z^{(1)}(\lambda) d\lambda \\
 & = ick_1 \sqrt{\epsilon_1} (\mathbf{E}_{in}(x, y_n, 0) + \mathbf{E}_{ref}(x, y_n, 0)) \cdot \mathbf{e}_x, \quad -L < x < L, \quad I_n(-L) = I_n(L) = 0,
 \end{aligned} \tag{2.12}$$

where

$$\begin{aligned}
 y_n &= d \left(n - \frac{N+1}{2} \right), \quad R_{nm} = \sqrt{(x - x')^2 + d^2(n - m)^2}, \\
 \hat{R}_{nm} &= \sqrt{(x - x')^2 + d^2(n - m)^2 + 4h^2}, \quad r_{nm} = R_{nm}.
 \end{aligned}$$

Using the method of moments (e.g. [15]), one can represent a discrete analogue of the system of integral equations (2.12) in the form of an inhomogeneous system of linear algebraic equations that is well known in antenna theory as the method of induced electro-motive forces. The collocation method by approximating the integral operator in (2.12) by means of the trapezoidal rule was used in the numerical analysis presented below. Thus, the unknown components of a

unified electric current $I_m^q = I_m(x_q)$ ($0 \leq q \leq P$, $1 \leq m \leq N$), made up of the electric currents of N wires which are computed at nodes $x_q = -L + 2Lq/P$, satisfy

$$\sum_{q=0, m=1}^{P, N} Z_{n, m}^{p, q}(k) I_m^q = U_n^p, \quad 1 \leq n, m \leq N, \quad 0 \leq p, q \leq P. \quad (2.13)$$

Here U_n^p is the vector of the induced electro-motive forces (voltages acting on the wire segments) whose components are determined by the values of the incident and the reflected from the interface plane waves at points x_q , where $Z_{n, m}^{q, p}(k)$ is the matrix of mutual impedances between the segments of the wires. Here $P + 1$ is the number of discretizations of the nodes of the wires obtained by means of the trapezoidal rule.

Solving the system of linear algebraic equations (2.13) numerically provides a discrete set of values for the currents $I_n(x)$, $1 \leq n \leq N$. The field, scattered from the horizontal finite array of N thin PEC wires in terms of the Hertz vector in the upper half-space for the observation point M with $z > 0$, can be written as follows:

$$\Pi_x(M) = \sum_{n=1}^N \frac{i}{ck_1 \sqrt{\epsilon_1}} \int_{-L}^L dx' I_n(x') \left(\frac{e^{ik_1 R_0}}{R_0} - \frac{e^{ik_1 R_1}}{R_1} + \int_0^{+\infty} J_0(\lambda r) e^{-(z+h)\chi_1} F_x^{(1)}(\lambda) \frac{\lambda d\lambda}{\chi_1} \right) \quad (2.14)$$

and

$$\Pi_z(M) = \sum_{n=1}^N \frac{-i}{ck_1 \sqrt{\epsilon_1}} \int_{-L}^L dx' I_n(x') \frac{d}{dx} \int_0^{+\infty} J_0(\lambda r) e^{-(z+h)\chi_1} F_z^{(1)}(\lambda) \frac{d\lambda}{\chi_1}, \quad (2.15)$$

where

$$R_0 = \sqrt{(x - x')^2 + (y - y_n)^2 + (z - h)^2}, \quad r = \sqrt{(x - x')^2 + (y - y_n)^2}, \\ R_1 = \sqrt{(x - x')^2 + (y - y_n)^2 + (z + h)^2}.$$

For the scattered field in the lower half-space ($z < 0$), one obtains that

$$\Pi_x(M) = \sum_{n=1}^N \frac{i}{ck_1 \sqrt{\epsilon_1}} \int_{-L}^L dx' I_n(x') \int_0^{+\infty} J_0(\lambda r) e^{z\chi_2 - h\chi_1} F_x^{(2)}(\lambda) \frac{\lambda d\lambda}{\chi_2} \quad (2.16)$$

and

$$\Pi_z(M) = \sum_{n=1}^N \frac{-i}{ck_1 \sqrt{\epsilon_1}} \int_{-L}^L dx' I_n(x') \frac{d}{dx} \int_0^{+\infty} J_0(\lambda r) e^{z\chi_2 - h\chi_1} F_z^{(2)}(\lambda) \frac{d\lambda}{\chi_2}. \quad (2.17)$$

In the next section, we shall demonstrate excitation of the localized modes for the array of six thin PEC wires over the interface by studying the magnetic component of the scattered field distributions

$$|\mathbf{H}_{sc}|(x, y, z) = \sqrt{|H_{sc,x}|^2 + |H_{sc,y}|^2 + |H_{sc,z}|^2},$$

within a horizontal rectangle located above the array. The Cartesian components of $\mathbf{H}_{sc}$ can be found from the following relation:

$$\mathbf{H}_{sc} = -i\sqrt{\epsilon_1} k_1 \nabla \times \boldsymbol{\Pi}, \quad \boldsymbol{\Pi} = (\Pi_x, 0, \Pi_z). \quad (2.18)$$

3. Numerical analysis for excitation of localized modes for an array of six thin wires over the interface

In this section, we discuss numerical results regarding the excitation of the localized guided modes described above for a 1D finite periodic array of six thin PEC wires shown in figure 1.

Figure 2. Resonance peaks of a scattered magnetic field evaluated at the fixed point $M = (0, 0, h + 0.01 \text{ m})$ and shown as a frequency dependence of the ratio $|H_{sc}(M, f)| / \max(|H_{sc}(M, f)|)$ (maximum is taken with respect to f) for modes 2 and 4, for $h = 0.005 \text{ m}$ and $h = 0.01 \text{ m}$, respectively. The resonance frequencies are $f = 64.653, 66.23, 66.96, 70.98 \text{ MHz}$. (Online version in colour.)

The wire radius is chosen as $a = 0.001 \text{ m}$, the total length $L = 0.1275 \text{ m}$, and the period is taken as $d = 0.01 \text{ m}$. The permittivities of the upper and lower half-spaces were chosen as $\epsilon_1 = 81$ and $\epsilon_2 = 1$. The value of 81 corresponds to distilled water and makes a high contrast between the two half-spaces. This makes the resonance frequencies $\sqrt{\epsilon_1}$ times less than in free space. The described model is suitable for MRI microwave tomography [16]. Slobozhanyuk *et al.* [16] suggested exactly these details for the formulation of the problem. The normally incident plane wave propagating downwards illuminates the array. Its electric component is directed along the axis X , and it has an amplitude of 1 V m^{-1} . As mentioned previously, owing to the symmetry of the scattering problem, from the data of the numerical analysis we observe only two types of mode excitations, namely modes 2 and 4. Definitions of modes 2 and 4 follow from the quantization conditions described in §4b (see formula (4.21)). The localized modes of other types can be excited by a skew incidence of the incident electromagnetic plane wave.

The first step in the analysis is to detect the frequencies of the localized modes 2 and 4 by performing a frequency $f = \omega/(2\pi)$ sweep for the magnitude $|\mathbf{H}_{sc}|(M, f)$ at fixed point $M = (0, 0, h + 0.01 \text{ m})$, located above the array. In general, it could be any point M situated in the near-field zone. The frequency dependencies of four narrow resonance peaks $f = 64.653 \text{ MHz}$, 66.23 MHz for $h = 0.01 \text{ m}$, and $f = 66.96 \text{ MHz}$, 70.98 MHz for $h = 0.005 \text{ m}$, normalized with respect to their maximum value, are shown in figure 2. The corresponding maximum values are: for mode 2 $|\mathbf{H}_{sc}| = 343.57 \text{ V m}^{-1}$, 998.57 V m^{-1} ($h = 0.005 \text{ m}$, 0.01 m), for mode 4 $|\mathbf{H}_{sc}| = 191.92 \text{ V m}^{-1}$, 24.98 V m^{-1} ($h = 0.005 \text{ m}$, 0.01 m).

In figures 3–6, the 2D pictures of the scattered magnetic component magnitudes $|\mathbf{H}_{sc}|(M)$ and $|H_{sc,z}|(M)$ are demonstrated. These are plotted within a rectangle in the AB field plane at $z = h + 0.01 \text{ m}$ (figure 1b) for the two types of the mode excitations, 2 and 4. The figures were computed for the corresponding resonance frequencies at $h = 0.005 \text{ m}$, $h = 0.01 \text{ m}$, shown in figure 2. The six thin wires are shown in figures 3a–6a as thin white horizontal strips. In figures 3b–6b, $|H_{sc,z}|(x, y, z)$ mode 2 is defined by a single nodal line, whereas mode 4 is defined by three nodal lines. In figures 3 and 5, mode 2 for the resonance frequencies $f = 70.98 \text{ MHz}$, $f = 66.23 \text{ MHz}$, for $h = 0.005 \text{ m}$, 0.01 m , is represented as $|\mathbf{H}_{sc}|$ (a) and $|H_{sc,z}|$ (b). In figures 4 and 6, mode 4 for the resonance frequencies $f = 66.96 \text{ MHz}$, $f = 64.653 \text{ MHz}$, for $h = 0.005 \text{ m}$, 0.01 m , is shown in a similar way. It is worth remarking that $f = k_1 c / (2\pi \sqrt{\epsilon_1})$. In all figures, the magnitudes of $|\mathbf{H}_{sc}|$ and

Figure 3. Magnetic field distributions $|H_{sc}|$ (a) and $|H_{sc,z}|$ (b), for mode 2 resonance frequency $f = 70.98$ MHz, computed for the rectangle $(x, y, h + 0.01 \text{ m})$ with $-0.15 < x < 0.15$, $-0.05 < y < 0.05$, $h = 0.005 \text{ m}$. (Online version in colour.)

Figure 4. Magnetic field distributions $|H_{sc}|$ (a) and $|H_{sc,z}|$ (b), for mode 4 resonance frequency $f = 66.96$ MHz, computed for the rectangle $(x, y, h + 0.01 \text{ m})$ with $-0.15 < x < 0.15$, $-0.05 < y < 0.05$, $h = 0.005 \text{ m}$. (Online version in colour.)

$|H_{sc,z}|$ are measured in V m^{-1} units. It is surprising that for all the indicated resonance frequencies one should clearly see a significant localization of the magnetic component of the scattered field in the domain covering the array. It can be demonstrated that this localization takes place for any horizontal or vertical cross sections made inside the near-field zone with respect to the array.

In figure 7, the modes' resonance frequency dependence on the height within the range $h \in [0.005 \text{ m}, 0.1 \text{ m}]$ is shown. This was computed on the basis of the Pocklington integral equation formalism and numerical modelling using the CST Studio Suite. The data obtained by the Pocklington integral equation for mode 2 are shown by the solid curve and for mode 4 by the dashed curve. The data obtained with the CST Studio Suite for mode 2 are represented by stars and for mode 4 by circles. It is peculiar for the modes' behaviour, demonstrated in the paper, that their resonance frequencies coalesce at a height of approximately $h = 0.015 \text{ m}$. This effect takes place in close proximity to the interface separating the two media. It can be clearly seen that there is quite good agreement between both types of data. It is worth remarking that the numerical computations for the array of six wires were performed in Matlab. Analysis for the array of the thin PEC wires that are much greater in length than the wavelength ($L \gg \lambda$) will require more computer resources and time for numerical calculations.

Figure 5. Magnetic field distributions $|H_{sc}|$ (a) and $|H_{sc,z}|$ (b), for mode 2 resonance frequency $f = 66.23$ MHz, computed for the rectangle $(x, y, h + 0.01 \text{ m})$ with $-0.15 < x < 0.15$, $-0.05 < y < 0.05$, $h = 0.01 \text{ m}$. (Online version in colour.)

Figure 6. Magnetic field distributions $|H_{sc}|$ (a) and $|H_{sc,z}|$ (b), for mode 4 resonance frequency $f = 64.653$ MHz, computed for the rectangle $(x, y, h + 0.01 \text{ m})$ with $-0.15 < x < 0.15$, $-0.05 < y < 0.05$, $h = 0.01 \text{ m}$. (Online version in colour.)

Figure 7. Resonance frequency dependence on the height within the range $h \in [0.005 \text{ m}, 0.1 \text{ m}]$ computed on the basis of the Pocklington integral equation formalism and numerical modelling using the CST Studio Suite for modes 2 and 4. The data obtained by the Pocklington integral equation for mode 2 are shown by a solid curve and for mode 4 by a dashed curve. The data from CST Studio Suite for mode 2 are represented by stars and for mode 4 by circles. (Online version in colour.)

4. Infinite periodic array of thin wires above the interface between two dielectric media

(a) Pocklington integral equation approach

Let us assume that the horizontal infinite array of thin PEC wires is illuminated by a plane electromagnetic wave. The array is periodic with period d . All wires with the total length $2L$ are directed along the X -axis, and are parallel to the interface in the XY -plane between two media with permittivity ϵ_1 and ϵ_2 , correspondingly, and common permeability $\mu = 1$. The infinite array is located in the upper medium at a distance h from the interface. Applying the Sommerfeld exact solution for a horizontal point electric dipole locating over the interface [17], generalizing the results of §2 for a single thin PEC wire, and using the quasi-periodicity Floquet's condition with quasi-momentum β ($0 < \beta < \pi/d$), written for the components of the Hertz vector in the form

$$\mathbf{\Pi}(x, y + d, z) = e^{i\beta d} \mathbf{\Pi}(x, y, z),$$

one could obtain a Pocklington integral equation for the electric current of the central wire of the infinite array by adding extra terms of interaction between the wires. Thus, the equation could be written in the following form:

$$\begin{aligned} & \int_{-L}^L dx' I(x') \left(k_1^2 + \frac{d^2}{dx^2} \right) \frac{e^{ik_1 R_0}}{R_0} + \sum_{m=-\infty, m \neq 0}^{+\infty} \frac{1}{2\pi} e^{im\beta d} \int_{-L}^L dx' I(x') \left(k_1^2 + \frac{d^2}{dx^2} \right) \int_0^{2\pi} d\varphi \frac{e^{ik_1 R_m}}{R_m} \\ & + \sum_{m=-\infty}^{+\infty} e^{im\beta d} \int_{-L}^L dx' I(x') \left(k_1^2 + \frac{d^2}{dx^2} \right) \left(-\frac{e^{ik_1 \hat{R}_m}}{\hat{R}_m} + \int_0^{+\infty} J_0(\lambda r_m) e^{-2h\chi_1} F_x^{(1)}(\lambda) \frac{\lambda d\lambda}{\chi_1} \right) \\ & + \sum_{m=-\infty}^{+\infty} e^{im\beta d} \int_{-L}^L dx' I(x') \frac{d^2}{dx^2} \int_0^{+\infty} J_0(\lambda r_m) e^{-2h\chi_1} F_z^{(1)}(\lambda) d\lambda \\ & = i\kappa_1 \sqrt{\epsilon_1} (\mathbf{E}_{\text{in}}(x, 0, 0) + \mathbf{E}_{\text{ref}}(x, 0, 0)) \cdot \mathbf{e}_x, \quad -L < x < L, \quad I(-L) = I(L) = 0, \end{aligned} \quad (4.1)$$

where

$$\begin{aligned} R_0 &= \sqrt{(x - x')^2 + a^2}, \quad R_m = \sqrt{(x - x')^2 + |\mathbf{r}' + m \mathbf{d} \mathbf{e}_y|^2}, \quad \mathbf{r}' = (0, a \cos \varphi, a \sin \varphi), \\ \hat{R}_m &= \sqrt{(x - x')^2 + (dm)^2 + 4h^2}, \quad r_m = \sqrt{(x - x')^2 + (dm)^2}. \end{aligned}$$

It is worth remarking that these series in (4.1) include the so-called lattice sums and they are extremely slowly convergent. Thus, this form of integral equation is not suitable for numerical analysis. Moreover, the corresponding kernel of the integral operator is complex.

Using the collocation method with the trapezoidal rule for the excitation of the infinite array of thin wires similar to (2.13), one can obtain an inhomogeneous system of linear algebraic equations (the method of induced electro-motive forces) representing a discrete analogue of the integral equation (4.1). Thus, the components of the vector of the electric current of the central wire satisfy

$$\sum_{q=0}^P Z_{p,q}(k_1, \beta) I_q = U_p, \quad 0 \leq p \leq P, \quad (4.2)$$

where $P + 1$ is the number of nodes of the central wire discretization.

The system of linear algebraic equations (4.2) is to be solved numerically. If the current $I(x)$ is known, the field scattered from the horizontal infinite array of thin PEC wires in terms of the Hertz vector in the upper half-space for the observation point $M = (x, y, z)$, $z > 0$, could be represented

as follows:

$$\Pi_x(M) = \sum_{m=-\infty}^{+\infty} e^{im\beta d} \frac{i}{ck_1\sqrt{\epsilon_1}} \int_{-L}^L dx' I(x') \left(\frac{e^{ik_1 R_m}}{R_m} - \frac{e^{ik_1 \hat{R}_m}}{\hat{R}_m} + \int_0^{+\infty} J_0(\lambda r_m) e^{-(z+h)\chi_1} F_x^{(1)}(\lambda) \frac{\lambda d\lambda}{\chi_1} \right) \quad (4.3)$$

and

$$\Pi_z(M) = \sum_{m=-\infty}^{+\infty} e^{im\beta d} \frac{-i}{ck_1\sqrt{\epsilon_1}} \int_{-L}^L dx' I(x') \frac{d}{dx} \int_0^{+\infty} J_0(\lambda r_m) e^{-(z+h)\chi_1} F_z^{(1)}(\lambda) \frac{d\lambda}{\chi_1}, \quad (4.4)$$

where

$$R_m = \sqrt{(x-x')^2 + (y-md)^2 + (z-h)^2}, \quad r_m = \sqrt{(x-x')^2 + (y-md)^2}, \\ \hat{R}_m = \sqrt{(x-x')^2 + (y-md)^2 + (z+h)^2}.$$

For the scattered field in the lower half-space ($z < 0$), one could derive

$$\Pi_x(M) = \sum_{m=-\infty}^{+\infty} e^{im\beta d} \frac{i}{ck_1\sqrt{\epsilon_1}} \int_{-L}^L dx' I(x') \int_0^{+\infty} J_0(\lambda r_m) e^{z\chi_2 - h\chi_1} F_x^{(2)}(\lambda) \frac{\lambda d\lambda}{\chi_2} \quad (4.5)$$

and

$$\Pi_z(M) = \sum_{m=-\infty}^{+\infty} e^{im\beta d} \frac{-i}{ck_1\sqrt{\epsilon_1}} \int_{-L}^L dx' I(x') \frac{d}{dx} \int_0^{+\infty} J_0(\lambda r_m) e^{z\chi_2 - h\chi_1} F_z^{(2)}(\lambda) \frac{d\lambda}{\chi_2}. \quad (4.6)$$

In this section, for the infinite array we are going to study a homogeneous problem to describe the dispersion curves of the localized guided modes similar to [3,4,12,14]. We assume that there is no incident plane wave, nor any other sources of external incident fields. Thus, the right-hand side of equation (4.1) is identical to zero. The localization of the guided modes is guaranteed in our case if the dispersion curves are located below the light line (e.g. [3,4,12,14]), namely if $0 < k_1 < \beta < \pi/d$. It will be shown that after some transformations the initial Pocklington integral equation (4.1) for electric current turns into the new form with a real kernel with fast convergent series instead. This new form of the integral equation is much more suitable for performing numerical analysis.

Consider firstly the second term in the first line of equation (4.1). Applying the following well-known integral representation [20]:

$$\frac{e^{ik\sqrt{x^2+d^2}}}{\sqrt{x^2+d^2}} = \frac{i}{2} \int_{-\infty}^{+\infty} e^{iwx} H_0^{(1)}(vd) dw, \quad (4.7)$$

where $H_0^{(1)}(v)$ is the Hankel function of the first kind, $v = \sqrt{k^2 - w^2}$ if $|w| < k$, and $v = i\sqrt{w^2 - k^2}$ if $|w| > k$, one could obtain

$$\sum_{m=-\infty, m \neq 0}^{+\infty} a e^{im\beta d} \int_{-L}^L dx' I(x') \int_0^{2\pi} d\varphi \frac{e^{ik_1 R_m}}{R_m} \\ = \frac{i}{2} \sum_{m=-\infty, m \neq 0}^{+\infty} e^{-im\beta d} \int_{-L}^L dx' I(x') \int_{-\infty}^{+\infty} e^{iwx(x-x')} H_0^{(1)}(v|m|d) J_0(va) dv.$$

While deriving the last relation, the following relation was used:

$$H_0^{(1)}(v|m\mathbf{de}_y - \mathbf{r}'|) = \sum_{s=-\infty}^{+\infty} J_s(va) H_s^{(1)}(v|m|d) e^{is(\arg(m\mathbf{de}_y) - \varphi)},$$

as $d > a$. Taking into account the following representations:

$$\begin{aligned} \left(k^2 + \frac{d^2}{dx^2}\right) \frac{\sin kR_0}{R_0} &= \int_0^k \cos w(x-x')v^2 J_0(va) dw, \\ \sum_{m=1}^{+\infty} J_0(vmd) \cos m\beta d &= -\frac{1}{2}, \quad k < \beta < \frac{\pi}{d}; \quad J_0(iva) = I_0(va), \\ H_0^{(1)}(iva) &= \frac{-2i}{\pi} K_0(va), \quad |w| > k, \end{aligned}$$

the two terms in the first line of equation (4.1) can be written as follows:

$$\left(k_1^2 + \frac{d^2}{dx^2}\right) \int_{-L}^L I(x') \left[\frac{\cos k_1 R_0}{R_0} + K_1(x-x') + K_2(x-x') \right] dx', \quad (4.8)$$

where

$$K_1(x) = -2 \int_0^{k_1} \cos(wx) J_0(va) S_1(\beta d, vd) dw, \quad v = \sqrt{k_1^2 - w^2} \quad (4.9)$$

and

$$K_2(x) = \frac{4}{\pi} \int_{k_1}^{+\infty} \cos(wx) I_0(va) S_2(\beta d, vd) dw, \quad v = \sqrt{w^2 - k_1^2}. \quad (4.10)$$

Here the lattice sums that were mentioned above are represented as follows:

$$S_1(\beta d, vd) = \sum_{m=1}^{\infty} \cos(m\beta d) Y_0(mvd), \quad 0 < w < k, \quad v = \sqrt{k^2 - w^2} \quad (4.11)$$

and

$$S_2(\beta d, vd) = \sum_{m=1}^{\infty} \cos(m\beta d) K_0(mvd), \quad k < w < +\infty, \quad v = \sqrt{w^2 - k^2}. \quad (4.12)$$

Their convergence could be made faster on the basis of the following identities [20]:

$$\begin{aligned} S_1(\beta d, vd) &= -\frac{1}{\pi} \left(\gamma + \log \frac{vd}{4\pi} \right) - \frac{1}{\sqrt{(\beta d)^2 - (vd)^2}} - \sum_{n=1}^{+\infty} \left[\frac{1}{\sqrt{(2\pi n + \beta d)^2 - (vd)^2}} - \frac{1}{2\pi n} \right] \\ &\quad - \sum_{n=1}^{+\infty} \left[\frac{\sigma_n}{\sqrt{(2\pi n - \beta d)^2 - (vd)^2}} - \frac{1}{2\pi n} \right] \end{aligned} \quad (4.13)$$

and

$$\begin{aligned} \sigma_n &= 0, \quad \text{if } (2\pi n - \beta d)^2 < (vd)^2, \\ \sigma_n &= 1, \quad \text{if } (2\pi n - \beta d)^2 > (vd)^2, \\ S_2(\beta d, vd) &= \frac{1}{2} \left(\gamma + \log \frac{vd}{4\pi} \right) + \frac{\pi}{2\sqrt{(\beta d)^2 + (vd)^2}} + \frac{\pi}{2} \sum_{n=1}^{+\infty} \left[\frac{1}{\sqrt{(2\pi n + \beta d)^2 + (vd)^2}} - \frac{1}{2\pi n} \right] \\ &\quad + \frac{\pi}{2} \sum_{n=1}^{+\infty} \left[\frac{1}{\sqrt{(2\pi n - \beta d)^2 + (vd)^2}} - \frac{1}{2\pi n} \right]. \end{aligned} \quad (4.14)$$

Here $\gamma = 0.5772 \dots$ is the Euler constant.

Using the integral representation (4.7), the well-known plane-wave decomposition for the Hankel function of zero order [20] and the Poisson summation formula, for the first series in the second line of equation (4.1) we obtain an integral representation with exponential convergence

that takes only real values. If it is denoted by $K_3(x - x')$, one can write that

$$K_3(x - x') = - \sum_{m=-\infty}^{+\infty} \frac{e^{im\beta d + ik_1 \hat{R}_m}}{\hat{R}_m} = - \frac{2}{d} \int_0^{+\infty} \cos w(x - x') \sum_{m=-\infty}^{+\infty} \frac{e^{-2hv_m}}{v_m} dw, \quad (4.15)$$

where $v_m = \sqrt{w^2 + (2\pi m/d + \beta)^2 - k_1^2}$. This value of the square root is real owing to the fact that in our case $\beta > k_1$.

The last slowly convergent series in equation (4.1) involving Bessel functions $J_0(\lambda r_m)$ with the help of the plane-wave decomposition for the Hankel function of zero order and the Poisson summation formula is converted into a real and finite sum

$$\begin{aligned} G(x - x', \lambda, \beta) &= \sum_{m=-\infty}^{+\infty} J_0(\lambda r_m) e^{im\beta d} = \text{Re} \sum_{m=-\infty}^{+\infty} H_0^{(1)}(\lambda r_m) e^{im\beta d} \\ &= \frac{2}{k_1 d} \sum_{\lambda_m < \lambda} \frac{\cos v_m k_1 |x - x'|}{v_m}, \\ G(x - x', \lambda, \beta) &= 0, \quad \text{for } \lambda < \lambda_0, \quad \lambda_0 = \frac{\beta}{k_1} > 1, \end{aligned} \quad (4.16)$$

where

$$v_m = \sqrt{\lambda^2 - \lambda_m^2}, \quad \lambda_m = \left| \frac{2\pi m}{k_1 d} + \frac{\beta}{k_1} \right|.$$

Thus, we obtain a new form of the Pocklington integral equation for the electric current of the central wire of the infinite array with the real kernel that involves fast convergent lattice sums

$$\begin{aligned} &\left(k_1^2 + \frac{d^2}{dx^2} \right) \int_{-L}^L dx' I(x') \left[\frac{\cos k_1 R_0}{R_0} + K_1(x - x') + K_2(x - x') + K_3(x - x') \right. \\ &\quad \left. + \int_0^{+\infty} G(x - x', \lambda, \beta) e^{-2h\chi_1} F_x^{(1)}(\lambda) \frac{\lambda d\lambda}{\chi_1} \right] \\ &\quad + \int_{-L}^L dx' I(x') \frac{d^2}{dx^2} \int_0^{+\infty} G(x - x', \lambda, \beta) e^{-2h\chi_1} F_z^{(1)}(\lambda) d\lambda = 0, \quad -L < x < L, \\ &I(-L) = I(L) = 0. \end{aligned} \quad (4.17)$$

This homogeneous integral equation is the main result of this section. This form of the Pocklington integral equation is very efficient for numerical analysis. It is worth remarking that in the case when the array is excited by an incident plane wave ($\beta < k_1$) all the transformations described above will have to be modified in a different way. This new analysis could be a subject of future publications.

(b) Anomalous dispersion for the infinite array of thin wires

In this section, we discuss the numerical analysis of the dispersion curves computed for the localized guided modes described above for a 1D infinite periodic array of thin PEC wires and how these results are connected with excitations of the localized modes for the corresponding finite array. It is well known that the localized guided waves of the linear arrays of thin PEC wires exist if $k_1 < \beta$, that is, below the light line. To compute the dispersion curves for the localized guided modes for a 1D infinite periodic array, one should study the corresponding homogeneous system of linear algebraic equations (4.2). Similar to [3,12,14], the solvability condition leads to the dispersion equation $\det Z(k_1, \beta) = 0$, $k_1 < \beta < \pi/d$. This describes the dispersion dependencies $k_1 = k_1^{(\nu)}(\beta)$ and $\nu = 1, 2, \dots$. In this paper, we deal with the lowest half-wave resonance dispersion $k_1 = k_1^{(1)}(\beta)$ (below we shall omit the superscript). For the finite array of N wires, the quantization

equation for the frequencies of localized modes, derived from the homogeneous system (2.13), is given by

$$\det Z_{nm}^{pq}(k_1) = 0. \quad (4.18)$$

This provides N solutions of complex resonances $k_{1,n}$, $n = 1, 2, \dots, N$, as a result of splitting of the lowest half-wave single-wire resonance.

There is another method to derive an approximate quantization equation for an N thin wire array that is usually used for the Fabry–Perot open resonators theory, namely

$$R^2 e^{2iNd\beta} = 1 = e^{2\pi in}, \quad n \in \mathbb{Z}, \quad (4.19)$$

where R is the reflection coefficient of the guided mode propagating back and forward that reflects from the array ends. The physical sense of this relation is quite clear. While the guided wave propagates the whole loop between the two ends of the array, its phase acquires an increment of $2iNd\beta$. Taking into account that the two reflections form the ends, the requirement for the corresponding wave function to be single-valued leads to the quantization condition (4.19). The reflection coefficient is not known—it depends on the frequency. It is natural to obtain R from solving a separate problem of scattering of an incident localized guided mode that is approaching the end of the semi-infinite array of thin wires. A similar quantization condition was studied in a 2D problem of scattering of water waves by a long array of evenly spaced, rigid, vertical circular cylinders that was analysed under the usual assumptions of linear acoustic theory [4].

Taking into account that $0 \leq \beta \leq \pi/d$, one can write

$$Nd\beta - i \log R = \pi n, \quad k_1 = k_1(\beta), \quad n = 1, 2, \dots, N. \quad (4.20)$$

We note that here the dispersion dependence has to be taken into consideration. The frequencies obtained from (4.20) must agree approximately with the values evaluated from (4.18). The presence of the term $-i \log R$ in (4.20) makes solutions of the quantization condition $k_{1,n}$ acquire a small imaginary part that describes the Q -factor of the eigenfrequency of the open resonator of the finite array of thin PEC wires. This small imaginary part is due to radiation of the localized mode of the finite array.

For the localized modes of the finite arrays with a very high Q -factor, the reflection coefficient is almost unity. This fact was discussed in detail in [4]. If the radiation is neglected, then for the array of six thin wires one could obtain a quite simple quantization formula,

$$6d\beta = \pi n, \quad k_1 = k_1(\beta), \quad n = 1, \dots, 6. \quad (4.21)$$

Thus, the frequencies of the localized modes for the array of six thin wires could be derived as follows:

$$k_{1,n} = k_1(\beta_n), \quad \beta_n = \frac{\pi n}{6d}, \quad 0 < \beta_n \leq \frac{\pi}{d}. \quad (4.22)$$

In the symmetrical case of the normally incident plane wave, the states with $n = 2, 4$ correspond to the excitations of the second and the fourth modes that were discussed in §§2 and 3. Thus, for $n = 2, 4$ we obtain $\beta_{2,4} = 104.72\text{m}^{-1}$, 209.44m^{-1} . Using the dispersion curve for $h = 0.005\text{m}$, one could evaluate the frequencies $f_{2,4} = 71.24\text{MHz}$, 67.06MHz , correspondingly, that are very close to the values $f_{2,4} = 70.98\text{MHz}$, 66.96MHz , computed numerically in the previous section (figures 3 and 4). For the dispersion curve with $h = 0.01\text{m}$ we obtain $f_{2,4} = 66.29\text{MHz}$, 64.68MHz , correspondingly, versus the values $f_{2,4} = 66.23\text{MHz}$, 64.653MHz , computed numerically (figures 5 and 6). When $h = 0.1\text{m}$, we again observe quite good coincidence between the values of resonance frequencies, computed numerically, and those evaluated by means of the quantization condition (4.21), with a difference smaller than 0.3%. This position is very close to a free space medium. The coincidence is not exact as it does not take into account the radiation of the excited modes through the finite array edges into the outer space as the reflection coefficient ($\log R$) has been neglected in the quantization condition (4.20).

For the symmetrical case with the normally incident plane wave for the six thin wires, taking into account the periodic boundary conditions (the electric currents of the first and the

Figure 8. Dispersion curves for the infinite array of thin PEC wires for different values of $h = 0.005, 0.01, 0.05, 0.1$ m, including the light line $k_1 = \beta$ (red), computed by the Pocklington integral equation for the parameters $a = 0.001$ m, $L = 0.1275$ m, $d = 0.01$ m. (Online version in colour.)

sixth wires must be the same), a different approximate quantization condition for $k_{1,n}$ may be obtained. It leads to the following relation: $e^{i6d\beta} = 1$. It is similar to the Born–Carman periodicity condition that is frequently used in the quantum electronics of solids and the quantum physics of semiconductors [21]. Solutions to these quantization conditions for guided modes 2 and 4 look exactly the same as (4.21) with $n = 1, 2$.

It is well known from scattering theory that, when studying scattering from a finite array of thin wires, if the value of k_1 for the incident plane wave coincides with the real part of one of $k_{1,n}$, then we observe a significant increase in the scattering pattern in the far field for any observation angles and also for the total scattered power. When the period d of linear finite arrays of thin PEC wires becomes smaller, the wires start interacting more strongly. All these complex resonances for the finite array of N wires as $N \rightarrow +\infty$ turn into real k -zones or pass-bands, as really happens in the quantum electronics of crystal solids. The smaller the period, the wider the pass-bands that one can observe.

In figure 8, the dispersion curves for the lowest half-wave resonances for the infinite 1D periodic array of thin PEC wires are shown; these were computed on the basis of the Pocklington integral equation for the following values of the problem parameters: $L = 0.1275$ m, $a = 0.001$ m and $d = 0.01$ m. It is worth remarking that both dispersion curves computed for the values of $h = 0.01$ m and $h = 0.005$ m demonstrate anomalous dispersion for the corresponding parts of the curves with negative group velocity,

$$v_{gr} = \frac{d\omega}{d\beta} = 2\pi \frac{df}{d\beta} < 0, \quad f = \frac{k_1(\beta)c}{2\pi\sqrt{\epsilon_1}}.$$

This fact is in agreement with the resonance frequencies behaviour shown in figure 7.

In figure 9a, an expanded part of figure 8 near to the light line $k_1 = \beta$ is presented. The dispersion curves terminate on the left-hand side at some points of the light line with finite values of k_1 and not at the point $k_1 = \beta = 0$. In general, computing the dispersion curves close to the light line is an extremely difficult task as all lattice sums described in the previous section become singular. In figure 9b, we compare the data of two dispersion curves for the infinite array of thin PEC wires computed on the basis of the Pocklington integral equation with 200 and 400 nodes for the parameters $a = 0.001$ m, $L = 0.1275$ m, $d = 0.01$ m, $h = 0.01$ m. In this case, we may observe a very good agreement between the curves. This means that the Pocklington integral equation algorithm is quite stable and robust.

In figure 10, the dispersion curves for the infinite array of thin PEC wires for two values of $h = 0.01$ m and $h = 0.05$ m for problem parameters $a = 0.001$ m, $L = 0.1275$ m, $d = 0.01$ m are presented to demonstrate a comparison between the results obtained on the basis of the Pocklington integral equation approach and the CST Studio Suite numerical simulations. It can be clearly seen that a very good agreement between both types of data occurs again.

Figure 9. (a) Expanded part of figure 8 near to the light line. (b) Comparison of two dispersion curves for the infinite array of thin PEC wires for the values of $h = 0.01$ m computed by the Pocklington integral equation with 200 (solid line) and 400 nodes (circles) for the parameters $a = 0.001$ m, $L = 0.1275$ m, $d = 0.01$ m. (Online version in colour.)

Figure 10. Dispersion curves for the infinite array of thin PEC wires for the two values of $h = 0.01$ m and $h = 0.1$ m for the parameters $a = 0.001$ m, $L = 0.1275$ m, $d = 0.01$ m to compare the results of the Pocklington integral equation approach with the CST Studio Suite simulations. (Online version in colour.)

The effect of the anomalous dispersion is due to the proximity of the array location with respect to the interface between the two dielectric media. It is worth remarking that this always takes place when the array is embedded into the space with $\epsilon_1 > \epsilon_2$, even for a smaller contrast. This can be shown by analysing the results of numerous computations on the basis of the method of the Pocklington integral equation. In the opposite case, we always observe normal dispersion behaviour as well as in the case when the array is embedded into a homogeneous space bounded with a PEC screen.

It is worth noting that, owing to the absence of dissipation losses in the structure and the fact that we study the regime below the light line, the propagating modes for the infinite array of thin wires do not experience any radiation losses. This means that, despite the finite length of each wire, our dispersion curves are real, and the propagating guided wave modes experience no decay while travelling along the array.

5. Conclusion

In this paper, guided electromagnetic waves propagating along 1D arrays of thin metallic parallel wires, finite and infinite, were studied. The cylindrical wires are assumed to be perfectly conducting, their length is of the order of the wavelength, and the radius of the wires is much smaller than the wavelength (longwave approximation). The array is embedded in the upper dielectric half-space with ϵ_1 and μ close to the interface separating the upper dielectric from the

lower medium with ϵ_2 and μ . In the first part of this work, we studied the behaviour of the localized modes for a finite array of parallel thin wires located above the interface and excited by a normally incident plane electromagnetic wave. It can be clearly seen that the finite array of thin wires being considered in the paper represents an open resonator. In the numerical analysis, we studied an example in which the incident wave propagates from the top to the bottom, and $\epsilon_1 = 81$, $\epsilon_2 = 1$ and $\mu = 1$. Owing to the symmetry only two main modes are excited. We applied analysis based on the system of the Pocklington integral equations to compute the resonance frequencies and plotted for the excited localized modes their 2D portraits, parallel to the XY-plane, above the array and compared our results with numerical simulations of the CST Studio Suite. It has been demonstrated that the order of the modes' resonances changes if the height of the array becomes sufficiently small. It is worth remarking that the analysis is also capable of treating the oblique incidence of the plane electromagnetic wave as well as thin wires of real metals with high conductivity. These studies will be the subject of our future publications.

The second part of the work described the analysis of the guided electromagnetic wave propagation along the infinite 1D array of parallel thin metallic wires located near the interface between two different dielectric media. The quasi-periodic wave field is constructed as a superposition of vector wave fields irradiated by linear electric currents. Then applying the Pocklington integral equation formalism leads to an infinite system of real linear algebraic equations. Vanishing of the determinant of the associated infinite matrix provides the condition for the guided waves to exist, and these are determined numerically after truncation of the infinite system. We presented the dispersion curves for the lowest case of the half-wave resonance of thin wires for different heights of the array above the interface but below the light line (the cut-off frequency). What is happening above the light line, especially the effects of bound states in continuum, or trapped modes, represents separate research work, and these will be considered in forthcoming publications. We note that anomalous dispersion occurs if the array comes very close to the interface. Additional numerical analysis for weaker contrast $\epsilon_1 = 4$, $\epsilon_2 = 1$ shows that the localization of the modes for a finite array takes place, and the effect of the anomalous dispersion takes place for the infinite array as well.

Data accessibility. This article does not contain any additional data.

Authors' contributions. V.V.Z. contributed to the derivation of the analytical model and performed all the numerical calculations using the Matlab model. S.Yu.K. performed the numerical simulations on the basis of the CST Studio Suite. Both authors contributed to writing the text of the paper.

Competing interests. We declare we have no competing of interests.

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