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Three-dimensional analytical poromechanical solutions for an arbitrarily inclined borehole subjected to fluid injection

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Hydraulic fracturing is the primary method of stimulation in unconventional reservoirs, playing a significant role in oil and gas production enhancement. A key issue for the analysis of hydraulic fracture initiation is to accurately determine the stress distributions in the vicinity of the borehole caused by the injection of pressurized fluids. This paper develops an exact, three-dimensional, poroelastic coupled analytical solution for such stress analysis of an arbitrarily inclined borehole subjected concurrently to a finite-length fluid discharge and *in situ* stresses, using Fourier expansion theorem and the Laplace–Fourier integral transform technique. The complicated boundary conditions, which involve the mixed boundary values at the borehole surface and the coupling between the total radial stress and injection-induced pore pressure over the sectioned borehole interval, as well as the fully three-dimensional far field *in situ* stresses, are addressed in a novel way and deliberately/elegantly decomposed into five fundamental, easier to handle modes. The rigorous and definitive nature of the proposed analytical methodology facilitates fundamental understanding of the mechanism underlying the stress responses of the borehole and porous medium. It can be and is used here as a benchmark for the numerical solutions obtained from the finite-element analysis commercial program (ABAQUS).

1. Introduction

The theory of poroelasticity, which takes into account full coupling between the quasi-static soil deformation and pore fluid diffusion processes, was established by

Biot [1,2] (now well known as Biot's theory). It plays important roles in a variety of disciplines including geomechanics, geotechnical engineering, biomechanics, petroleum engineering and environmental geosciences [3,4]. Particularly, in petroleum engineering, the subjects of poroelastic wellbore drilling instability [5–7], land subsidence [8,9] and hydraulic fracturing [10–13] demand such time-dependent solutions accounting for the coupled deformation–diffusion mechanisms and have been of special interest over the past few decades.

The hydraulic fracturing problem encountered is concerned with the stress perturbations in and around the wellbore in a three-dimensional stress field, subjected to a finite-length fluid source over its surface. Analytical investigations into such a fundamental boundary value problem are, however, rare. This is mainly due to the full three-dimensional nature of the problem (i.e. a finite segment of fluid injection combined with the inclined geometry of a borehole with respect to the *in situ* principal stresses) and to the coupling between the pumping-induced pore pressure and radial stress at the borehole surface, which makes the mathematical formulations involved formidably challenging. An example of the poroelastic stress analysis of a borehole under a symmetric fluid loading was given by Rajapakse [14]. The assumptions in [14], however, were that the porous medium is initially unstressed and that the total radial stress is simply zero over the fluid injection segment of the borehole wall. These two obvious drawbacks were recently rectified by Abousleiman & Chen [11], who proposed a more sophisticated analytical solution for the responses of an inclined borehole subjected to an *in situ* far-field state of stress and finite-length fluid discharge, by use of the Laplace and Fourier integral transforms technique. Nevertheless, it should be pointed out that, in Abousleiman and Chen's work [11], the solution corresponding to the stress boundary problem actually was obtained in a very approximate way by directly employing the results of Cui *et al.* [15] for an impermeable wellbore drilling (plane strain) problem. In this sense, this analytical solution is far from rigorous, and may cause considerable errors, as will be shown in this paper.

The focus of this paper is, therefore, to present an exact, rigorous, definitive analytical solution for the poromechanical responses of a borehole inclined with respect to the three-dimensional *in situ* stresses and subjected to a finite-length fluid discharge over its surface. The solution procedure proposed starts with solving directly the general Biot's poroelasticity governing equations by using the Fourier expansion theorem as well as the combined Laplace–Fourier integral transform technique. The hydraulic fracturing problem addressed is then decomposed into five fundamental modes that are easier to handle, based on the superposition principle. The mixed (in and outside the sectioned borehole interval) and coupled (over the injection segment) boundary value problems, for each of the individual modes, are found to be equivalent to solving a set of dual integral equations [16] with the transformed total stress at the borehole surface being the only unknown, which after meticulous treatment can be further reduced to a numerically solvable Fredholm integral equation of the second kind. Numerical analyses are finally carried out to examine the influences of consolidation and geometrical parameters on the calculated effective tangential stress and pore water pressure distributions. The analytical solution may be and has been used to validate the accuracy of the finite-element numerical results involving the poromechanics constitutive models and associated boundary value problems; it can also serve to back figure the minimum fluid flow rate or maximum packer distance required to initiate a fracture around the borehole and thus is of great practical value in petroleum engineering.

2. Problem statement and poroelasticity governing equations

Figure 1a shows schematically an inclined borehole drilled in an infinite, saturated porous medium. The borehole is assumed to be infinitely long and has a radius R , with a centroidal axis coinciding with the z -axis of the local coordinate system (x, y, z) . The *in situ* stress state in the formation, prior to borehole drilling, is represented by three compressive principal stresses $S_{x'}$, $S_{y'}$ and $S_{z'}$, which is coincident with the Cartesian coordinate system (x', y', z') . The *in situ* pore pressure is assumed to be p_0 . Other assumptions are that the borehole is deviated by an

Figure 1. Geometry of an inclined borehole: (a) a borehole inclined to principal stresses and (b) equivalent far-field stresses in a local coordinate system. (Online version in colour.)

inclination angle φ_y (from the z' axis) and an azimuth angle φ_z' , and subjected to an axisymmetric fluid injection at the volume rate Q_0 over a finite length $2b$ of its surface.

For the convenience of solution presentation, the local (well aligned with the borehole) coordinate system xyz is chosen to portray the solution domain (figure 1b). Using the cylindrical coordinate system (r, θ, z) , the governing equations for deformations of a homogeneous, isotropic saturated medium can be expressed as [1,2,17]

$$\nabla^2 u_r + \frac{1}{1-2\nu_u} \frac{\partial e_v}{\partial r} - \frac{1}{r} \left(\frac{2}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right) - \frac{2B(1+\nu_u)}{3(1-2\nu_u)} \frac{\partial \varepsilon_v}{\partial r} = 0, \quad (2.1)$$

$$\nabla^2 u_\theta + \frac{1}{1-2\nu_u} \frac{\partial e_v}{r \partial \theta} - \frac{1}{r} \left(\frac{u_\theta}{r} - \frac{2}{r} \frac{\partial u_r}{\partial \theta} \right) - \frac{2B(1+\nu_u)}{3(1-2\nu_u)} \frac{\partial \varepsilon_v}{r \partial \theta} = 0, \quad (2.2)$$

$$\nabla^2 u_z + \frac{1}{1-2\nu_u} \frac{\partial e_v}{\partial z} - \frac{2B(1+\nu_u)}{3(1-2\nu_u)} \frac{\partial \varepsilon_v}{\partial z} = 0 \quad (2.3)$$

$$\text{and} \quad \nabla^2 \varepsilon_v = \frac{1}{c} \frac{\partial \varepsilon_v}{\partial t}, \quad (2.4)$$

where $\nabla^2 = \partial^2/\partial r^2 + 1/r \partial/\partial r + 1/r^2 \partial^2/\partial \theta^2 + \partial^2/\partial z^2$; u_r , u_θ and u_z are the radial, circumferential and vertical displacements, respectively, of the soil matrix; $e_v = \partial u_r/\partial r + u_r/r + 1/r \partial u_\theta/\partial \theta + \partial u_z/\partial z$ is the matrix dilation and ε_v is the variation of fluid content per unit reference volume; ν_u is the undrained Poisson's ratio; B and c are, respectively, the Skempton pore pressure coefficient and the diffusion coefficient; and $c = 2G\kappa B^2(1-\nu)(1+\nu_u)^2/9(1-\nu_u)(\nu_u-\nu)$ with G denoting the shear modulus, ν the drained Poisson's ratio and κ the permeability coefficient. Note that κ can be expressed as $\kappa = k/\mu$, where k is the intrinsic permeability and μ is the fluid viscosity.

The constitutive equations can be written as [18]

$$\Delta p = -\frac{2GB(1+\nu_u)}{3(1-2\nu_u)} e_v - \frac{2GB^2(1-\nu)(1+\nu_u)}{9(\nu-\nu_u)(1-2\nu_u)} \varepsilon_v, \quad (2.5)$$

$$\Delta \sigma_{rr} = 2G \frac{\partial u_r}{\partial r} + \frac{2G\nu}{1-2\nu} e_v - \alpha \cdot \Delta p, \quad (2.6)$$

$$\Delta \sigma_{\theta\theta} = 2G \left(\frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{r \partial \theta} \right) + \frac{2G\nu}{1-2\nu} e_v - \alpha \cdot \Delta p, \quad (2.7)$$

$$\Delta \sigma_{zz} = 2G \frac{\partial u_z}{\partial z} + \frac{2G\nu}{1-2\nu} e_v - \alpha \cdot \Delta p, \quad (2.8)$$

$$\Delta\tau_{r\theta} = G \left(\frac{\partial u_r}{r\partial\theta} + \frac{\partial u_\theta}{\partial r} \right), \quad (2.9)$$

$$\Delta\tau_{rz} = G \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) \quad (2.10)$$

and

$$\Delta\tau_{z\theta} = G \left(\frac{\partial u_z}{r\partial\theta} + \frac{\partial u_\theta}{\partial z} \right), \quad (2.11)$$

where the parameter $\alpha = 3(v_u - \nu)/B(1 - 2\nu)(1 + \nu_u)$ is known as Biot's coefficient; σ_{rr} , $\sigma_{\theta\theta}$ and σ_{zz} are the total radial, tangential and vertical stresses, respectively; $\tau_{r\theta}$, τ_{rz} and $\tau_{z\theta}$ are the shear stresses; p is the pore pressure; and the symbol Δ is used to denote the changes in the stresses and pore pressure from their initial values. Note that Δp , in fact, represents the excess pore pressure.

Referring to figure 1b, the boundary conditions for the borehole problem can be imposed as follows. At the far field, $r \rightarrow \infty$,

$$\sigma_{xx} = -S_x, \sigma_{yy} = -S_y, \sigma_{zz} = -S_z, \quad (2.12a)$$

$$\tau_{xy} = -S_{xy}, \tau_{yz} = -S_{yz}, \tau_{xz} = -S_{xz} \quad (2.12b)$$

and

$$p = p_0. \quad (2.12c)$$

At the wellbore surface, $r = R$,

$$\sigma_{rr} = \begin{cases} -S_r H(-t) - p H(t) & 0 \leq |z| \leq b, \\ -S_r H(-t) & b < |z| < \infty, \end{cases} \quad (2.13a)$$

$$\tau_{r\theta} = -S_{r\theta} H(-t), \tau_{rz} = -S_{rz} H(-t) \quad (2.13b)$$

and

$$q_r = \begin{cases} \frac{Q_0}{4\pi Rb} H(t) & 0 \leq |z| \leq b, \\ 0 & b < |z| < \infty, \end{cases} \quad (2.13c)$$

where σ_{xx} , σ_{yy} , σ_{zz} and τ_{xy} , τ_{yz} , τ_{xz} are the total normal stresses and shear stresses in the directions of three coordinate axes x , y and z ; S_x , S_y , S_z , S_{xy} , S_{yz} and S_{xz} are the six stress components in the far field; S_r , $S_{r\theta}$ and S_{rz} are the corresponding far-field values in the cylindrical coordinate system, i.e.

$$S_r = \frac{S_x + S_y}{2} + \frac{S_x - S_y}{2} \cos 2\theta + S_{xy} \sin 2\theta, \quad (2.14a)$$

$$S_{r\theta} = S_{xy} \cos 2\theta - \frac{S_x - S_y}{2} \sin 2\theta \quad (2.14b)$$

and

$$S_{rz} = S_{xz} \cos \theta + S_{yz} \sin \theta, \quad (2.14c)$$

where q_r is the radial fluid flow rate and H is the Heaviside step function. It should be emphasized that the boundary condition for radial stress σ_{rr} presented herein, i.e. equation (2.13a), has taken into full account its coupling relationship with the pore water pressure p generated at the same borehole surface. This differs substantially from the formulation in [14] where a simpler but incorrect one was adopted, i.e. $\sigma_{rr} = 0$ for the entire borehole surface $|z| < \infty$, so the solution to be sought is rigorous in this sense.

The hydraulic fracturing problem now mathematically reduces to solving one set of differential equations (2.1)–(2.4) under the given boundary conditions (2.12a)–(2.12c) and (2.13a)–(2.13c). This can be equivalently treated by removing at time $t=0$ the stresses and pore pressure that were acting on the borehole boundary (i.e. $\Delta\tau_{r\theta} = S_{r\theta}$ and $\Delta\tau_{rz} = S_{rz}$ for $|z| < \infty$, and $\Delta\sigma_{rr} + \Delta p = S_r - p_0$ for $0 \leq |z| \leq b$; $\Delta\sigma_{rr} = S_r$ for $b < |z| < \infty$) yet simultaneously assigning a fluid discharge of $q_r = Q_0/(4\pi Rb)$ over the segment $0 \leq |z| \leq b$, and then superpose the solution results obtained with the initial stress state.

3. General solutions

It is noteworthy that the present study abandons the conventional heuristic decomposition technique [6,11] in the development of solution formulations for the borehole hydraulic fracturing problem, which actually is no longer applicable for the current finite-length fluid injection analysis. Instead, the solution procedure proposed will start with directly solving the general three-dimensional governing equations (2.1)–(2.4), in strict accordance with the complete boundary conditions at both the wellbore surface and the far field.

Application of Fourier series expansion with respect to the circumferential coordinate θ , the displacement, strain and stress components as well as the pore pressure and flow rate can be expressed as [19,20]

$$u_r(r, \theta, z, t) = \sum_{m=0}^{\infty} u_{r,m1}(r, z, t) \cos m\theta - u_{r,m2}(r, z, t) \sin m\theta, \quad (3.1a)$$

$$u_{\theta}(r, \theta, z, t) = \sum_{m=0}^{\infty} u_{\theta,m1}(r, z, t) \sin m\theta + u_{\theta,m2}(r, z, t) \cos m\theta, \quad (3.1b)$$

$$u_z(r, \theta, z, t) = \sum_{m=0}^{\infty} u_{z,m1}(r, z, t) \cos m\theta - u_{z,m2}(r, z, t) \sin m\theta, \quad (3.1c)$$

$$e_v(r, \theta, z, t) = \sum_{m=0}^{\infty} e_{v,m1}(r, z, t) \cos m\theta - e_{v,m2}(r, z, t) \sin m\theta, \quad (3.1d)$$

$$\varepsilon_v(r, \theta, z, t) = \sum_{m=0}^{\infty} \varepsilon_{v,m1}(r, z, t) \cos m\theta - \varepsilon_{v,m2}(r, z, t) \sin m\theta, \quad (3.1e)$$

$$\Delta p(r, \theta, z, t) = \sum_{m=0}^{\infty} \Delta p_{m1}(r, z, t) \cos m\theta - \Delta p_{m2}(r, z, t) \sin m\theta, \quad (3.1f)$$

$$\Delta \sigma_{rr}(r, \theta, z, t) = \sum_{m=0}^{\infty} \Delta \sigma_{rr,m1}(r, z, t) \cos m\theta - \Delta \sigma_{rr,m2}(r, z, t) \sin m\theta, \quad (3.1g)$$

$$\Delta \sigma_{\theta\theta}(r, \theta, z, t) = \sum_{m=0}^{\infty} \Delta \sigma_{\theta\theta,m1}(r, z, t) \cos m\theta - \Delta \sigma_{\theta\theta,m2}(r, z, t) \sin m\theta, \quad (3.1h)$$

$$\Delta \sigma_{zz}(r, \theta, z, t) = \sum_{m=0}^{\infty} \Delta \sigma_{zz,m1}(r, z, t) \cos m\theta - \Delta \sigma_{zz,m2}(r, z, t) \sin m\theta, \quad (3.1i)$$

$$\Delta \tau_{r\theta}(r, \theta, z, t) = \sum_{m=0}^{\infty} \Delta \tau_{r\theta,m1}(r, z, t) \sin m\theta + \Delta \tau_{r\theta,m2}(r, z, t) \cos m\theta, \quad (3.1j)$$

$$\Delta \tau_{rz}(r, \theta, z, t) = \sum_{m=0}^{\infty} \Delta \tau_{rz,m1}(r, z, t) \cos m\theta - \Delta \tau_{rz,m2}(r, z, t) \sin m\theta, \quad (3.1k)$$

$$\Delta \tau_{z\theta}(r, \theta, z, t) = \sum_{m=0}^{\infty} \Delta \tau_{z\theta,m1}(r, z, t) \sin m\theta + \Delta \tau_{z\theta,m2}(r, z, t) \cos m\theta \quad (3.1l)$$

$$\text{and} \quad q_r(r, \theta, z, t) = \sum_{m=0}^{\infty} q_{r,m1}(r, z, t) \cos m\theta - q_{r,m2}(r, z, t) \sin m\theta, \quad (3.1m)$$

where $e_{v,mk} = \partial u_{r,mk} / \partial r + u_{r,mk} / r + m/r \partial u_{\theta,mk} / \partial \theta + \partial u_{z,mk} / \partial z$ ($k=1, 2$).

Substituting equations (3.1a)–(3.1l) into equations (2.1)–(2.11) and introducing the non-dimensional constants and variables: $\bar{r} = r/R$, $\bar{z} = z/R$, $\bar{t} = ct/R^2$, $\bar{u}_r = u_r/R$, $\bar{u}_{r,mk} = u_{r,mk}/R$, $\bar{u}_{\theta} = u_{\theta}/R$, $\bar{u}_{\theta,mk} = u_{\theta,mk}/R$, $\bar{u}_z = u_z/R$, $\bar{u}_{z,mk} = u_{z,mk}/R$, $\Delta \bar{\sigma}_{rr} = \Delta \sigma_{rr}/2G$, $\Delta \bar{\sigma}_{rr,mk} = \Delta \sigma_{rr,mk}/2G$, $\Delta \bar{\sigma}_{\theta\theta} =$

$\Delta\sigma_{\theta\theta}/2G$, $\Delta\bar{\sigma}_{\theta\theta,mk} = \Delta\sigma_{\theta\theta,mk}/2G$, $\Delta\bar{\sigma}_{zz} = \Delta\sigma_{zz}/2G$, $\Delta\bar{\sigma}_{zz,mk} = \Delta\sigma_{zz,mk}/2G$, $\Delta\bar{\tau}_{r\theta} = \Delta\tau_{r\theta}/2G$, $\Delta\bar{\tau}_{r\theta,mk} = \Delta\tau_{r\theta,mk}/2G$, $\Delta\bar{\tau}_{rz} = \Delta\tau_{rz}/2G$, $\Delta\bar{\tau}_{rz,mk} = \Delta\tau_{rz,mk}/2G$, $\Delta\bar{\tau}_{z\theta} = \Delta\tau_{z\theta}/2G$, $\Delta\bar{\tau}_{z\theta,mk} = \Delta\tau_{z\theta,mk}/2G$, $\Delta\bar{p} = \Delta p/2G$, $\Delta\bar{p}_{mk} = \Delta p_{mk}/2G$, $\bar{q}_r = q_r R/2G\kappa$, $\bar{q}_{r,mk} = q_{r,mk} R/2G\kappa$, one obtains the following equations for $m = 1, 2, 3, \dots$ and $k = 1, 2$:

$$\left(\nabla_m^2 - \frac{1}{\bar{r}^2}\right)\bar{u}_{r,mk} - \frac{2m}{\bar{r}^2}\bar{u}_{\theta,mk} + \frac{1}{1-2\nu_u}\frac{\partial e_{v,mk}}{\partial \bar{r}} - \frac{2B(1+\nu_u)}{3(1-2\nu_u)}\frac{\partial \varepsilon_{v,mk}}{\partial \bar{r}} = 0, \quad (3.2)$$

$$\left(\nabla_m^2 - \frac{1}{\bar{r}^2}\right)\bar{u}_{\theta,mk} - \frac{2m}{\bar{r}^2}\bar{u}_{r,mk} - \frac{m}{\bar{r}}\frac{1}{1-2\nu_u}e_{v,mk} + \frac{m}{\bar{r}}\frac{2B(1+\nu_u)}{3(1-2\nu_u)}\varepsilon_{v,mk} = 0, \quad (3.3)$$

$$\nabla_m^2\bar{u}_{z,mk} + \frac{1}{1-2\nu_u}\frac{\partial e_{v,mk}}{\partial \bar{z}} - \frac{2B(1+\nu_u)}{3(1-2\nu_u)}\frac{\partial \varepsilon_{v,mk}}{\partial \bar{z}} = 0, \quad (3.4)$$

$$\nabla_m^2\varepsilon_{v,mk} = \frac{\partial e_{v,mk}}{\partial \bar{r}}, \quad (3.5)$$

$$\Delta\bar{p}_{mk} = -\frac{B(1+\nu_u)}{3(1-2\nu_u)}e_{v,mk} - \frac{B^2(1-\nu)(1+\nu_u)}{9(\nu-\nu_u)(1-2\nu_u)}\varepsilon_{v,mk}, \quad (3.6)$$

$$\Delta\bar{\sigma}_{rr,mk} = \frac{\partial \bar{u}_{r,mk}}{\partial \bar{r}} + \frac{\nu}{1-2\nu}e_{v,mk} - \alpha \cdot \Delta\bar{p}_{mk}, \quad (3.7)$$

$$\Delta\bar{\sigma}_{\theta\theta,mk} = \left(\frac{\bar{u}_{r,mk}}{\bar{r}} + \frac{m\bar{u}_{\theta,mk}}{\bar{r}}\right) + \frac{\nu}{1-2\nu}e_{v,mk} - \alpha \cdot \Delta\bar{p}_{mk}, \quad (3.8)$$

$$\Delta\bar{\sigma}_{zz,mk} = \frac{\partial \bar{u}_{z,mk}}{\partial \bar{z}} + \frac{\nu}{1-2\nu}e_{v,mk} - \alpha \cdot \Delta\bar{p}_{mk}, \quad (3.9)$$

$$\Delta\bar{\tau}_{r\theta,mk} = \frac{1}{2}\left(-\frac{m\bar{u}_{r,mk}}{\bar{r}} + \frac{\partial \bar{u}_{\theta,mk}}{\partial \bar{r}} - \frac{\bar{u}_{\theta,mk}}{\partial \bar{r}}\right), \quad (3.10)$$

$$\Delta\bar{\tau}_{rz,mk} = \frac{1}{2}\left(\frac{\partial \bar{u}_{r,mk}}{\partial \bar{z}} + \frac{\partial \bar{u}_{z,mk}}{\partial \bar{r}}\right) \quad (3.11)$$

and
$$\Delta\bar{\tau}_{z\theta,mk} = \frac{1}{2}\left(-\frac{m\bar{u}_{z,mk}}{\bar{r}} + \frac{\partial \bar{u}_{\theta,mk}}{\partial \bar{z}}\right), \quad (3.12)$$

where $\nabla_m^2 = \partial^2/\partial \bar{r}^2 + 1/\bar{r}\partial/\partial \bar{r} - m^2/\bar{r}^2 + \partial^2/\partial \bar{z}^2$.

The primary governing equations (3.2)–(3.5) with $\bar{u}_{r,mk}$, $\bar{u}_{\theta,mk}$, $\bar{u}_{z,mk}$, $e_{v,mk}$ and $\varepsilon_{v,mk}$ being the basic unknowns can be solved by means of the Laplace and Fourier transform techniques [11,21]. By application of Laplace and Fourier integral transforms with respect to $\bar{t}$ and $\bar{z}$, respectively, equation (3.5) becomes

$$\left[\frac{d^2}{d\bar{r}^2} + \frac{1}{\bar{r}}\frac{d}{d\bar{r}} - \frac{m^2}{\bar{r}^2} - (\xi^2 + s)\right]\hat{\varepsilon}_{v,mk}(r, \xi, s) = 0, \quad (3.13)$$

where the tilde ($\sim$) stands for the Laplace transform and the circumflex ($\hat{}$) for the Fourier transform; s and ξ are the corresponding transformation parameters.

The solution of equation (3.13) can be readily found to take the form

$$\hat{\varepsilon}_{v,mk}(r, \xi, s) = A_{mk}(\xi, s)K_m(\eta\bar{r}), \quad (3.14)$$

where $\eta = \sqrt{\xi^2 + s}$; A_{mk} is an arbitrary function of ξ and s ; and K_m denotes the modified Bessel function of the second kind of order m .

Performing $\partial/\partial r$ equation (3.2) + $1/r$ equation (3.2) + m/r equation (3.3) + $\partial/\partial z$ equation (3.4), and then applying the combined Laplace–Fourier transform to the result gives

$$\left(\frac{d^2}{d\bar{r}^2} + \frac{1}{\bar{r}}\frac{d}{d\bar{r}} - \frac{m^2}{\bar{r}^2} - \xi^2\right)\hat{\varepsilon}_{v,mk}(r, \xi, s) = \frac{B(1+\nu_u)}{3(1-\nu_u)}\left(\frac{d^2}{d\bar{r}^2} + \frac{1}{\bar{r}}\frac{d}{d\bar{r}} - \frac{m^2}{\bar{r}^2} - \xi^2\right)\hat{\varepsilon}_{v,mk}(r, \xi, s), \quad (3.15)$$

which yields the following solution:

$$\hat{e}_{v,mk}(r, \xi, s) = C_{mk}(\xi, s) |\xi| K_m(\rho) + \frac{B(1 + v_u)}{3(1 - v_u)} A_{mk}(\xi, s) K_m(\eta \bar{r}), \quad (3.16)$$

where $\rho = |\xi| \bar{r}$ and C_{mk} is another arbitrary function of ξ and s .

On combining equations (3.14) and (3.16) with equation (3.6), the generated excess pore pressure, in the transformed domain, therefore, can be expressed as

$$\Delta \hat{p}_{mk}(r, \xi, s) = -\frac{B^2(1 - v)(1 + v_u)^2}{9(1 - v_u)(v - v_u)} A_{mk}(\xi, s) K_m(\eta \bar{r}) - \frac{B(1 + v_u)}{3(1 - 2v_u)} C_{mk}(\xi, s) |\xi| K_m(\rho), \quad (3.17)$$

and hence

$$\begin{aligned} \hat{q}_{r,mk}(r, \xi, s) &= -\frac{d\hat{p}_{mk}(r, \xi, s)}{d\bar{r}} = -\frac{d[\Delta \hat{p}_{mk}(r, \xi, s)]}{d\bar{r}} \\ &= -\frac{B^2(1 - v)(1 + v_u)^2}{18(1 - v_u)(v - v_u)} \eta A_{mk}(\xi, s) [K_{m+1}(\eta \bar{r}) + K_{m-1}(\eta \bar{r})] \\ &\quad - \frac{B(1 + v_u)}{3(1 - 2v_u)} \xi^2 C_{mk}(\xi, s) [K_{m+1}(\rho) + K_{m-1}(\rho)]. \end{aligned} \quad (3.18)$$

Adding equation (3.2) to equation (3.3), followed by application of the Laplace–Fourier transform, gives

$$\begin{aligned} &\left(\frac{d^2}{d\bar{r}^2} + \frac{1}{\bar{r}} \frac{d}{d\bar{r}} - \frac{(m+1)^2}{\bar{r}^2} - \xi^2 \right) [\hat{u}_{r,mk}(r, \xi, s) + \hat{u}_{\theta,mk}(r, \xi, s)] \\ &+ \frac{1}{1 - 2v_u} \left(\frac{d}{d\bar{r}} - \frac{m}{\bar{r}} \right) \hat{e}_{vm,k}(r, \xi, s) - \frac{2B(1 + v_u)}{3(1 - 2v_u)} \left(\frac{d}{d\bar{r}} - \frac{m}{\bar{r}} \right) \hat{e}_{vm,k}(r, \xi, s) = 0. \end{aligned} \quad (3.19)$$

Similarly, subtraction of equation (3.3) from equation (3.2) and using the Laplace–Fourier transform leads to

$$\begin{aligned} &\left(\frac{d^2}{d\bar{r}^2} + \frac{1}{\bar{r}} \frac{d}{d\bar{r}} - \frac{(m-1)^2}{\bar{r}^2} - \xi^2 \right) [\hat{u}_{r,mk}(r, \xi, s) - \hat{u}_{\theta,mk}(r, \xi, s)] \\ &+ \frac{1}{1 - 2v_u} \left(\frac{d}{d\bar{r}} + \frac{m}{\bar{r}} \right) \hat{e}_{vm,k}(r, \xi, s) - \frac{2B(1 + v_u)}{3(1 - 2v_u)} \left(\frac{d}{d\bar{r}} + \frac{m}{\bar{r}} \right) \hat{e}_{vm,k}(r, \xi, s) = 0, \end{aligned} \quad (3.20)$$

and the transformed version of equation (3.4) can be written as

$$\left(\frac{d^2}{d\bar{r}^2} + \frac{1}{\bar{r}} \frac{d}{d\bar{r}} - \frac{m^2}{\bar{r}^2} - \xi^2 \right) \hat{u}_{z,mk}(r, \xi, s) - \frac{i\xi}{1 - 2v_u} \hat{e}_{vm,k}(r, \xi, s) + \frac{2B(1 + v_u)}{3(1 - 2v_u)} i\xi \hat{e}_{vm,k}(r, \xi, s) = 0. \quad (3.21)$$

Substituting equations (3.14) and (3.16) into equations (3.19)–(3.21) and after tedious derivation and manipulations, the resulting solutions for the three displacement components (in the transformed domain) can be obtained in a neat and compact form as follows:

$$\begin{aligned} \hat{u}_{r,mk}(\bar{r}, \xi, s) &= -\frac{B(1 + v_u)}{6(1 - v_u)} \frac{\eta}{s} [K_{m+1}(\eta \bar{r}) + K_{m-1}(\eta \bar{r})] A_{mk}(\xi, s) - \frac{|\xi| r}{4(1 - 2v_u)} [K_{m+2}(\rho) \\ &\quad + K_m(\rho)] C_{mk}(\xi, s) + \frac{1}{2} K_{m+1}(\rho) |\xi| E_{mk}(\xi, s) + \frac{1}{2} K_{m-1}(\rho) |\xi| F_{mk}(\xi, s), \end{aligned} \quad (3.22)$$

$$\begin{aligned} \hat{u}_{\theta,mk}(\bar{r}, \xi, s) &= -\frac{B(1 + v_u)}{6(1 - v_u)} \frac{\eta}{s} [K_{m+1}(\eta \bar{r}) - K_{m-1}(\eta \bar{r})] A_{mk}(\xi, s) - \frac{|\xi| r}{4(1 - 2v_u)} [K_{m+2}(\rho) \\ &\quad - K_m(\rho)] C_{mk}(\xi, s) + \frac{1}{2} K_{m+1}(\rho) |\xi| E_{mk}(\xi, s) - \frac{1}{2} K_{m-1}(\rho) |\xi| F_{mk}(\xi, s) \end{aligned} \quad (3.23)$$

$$\text{and } \hat{\hat{u}}_{z,mk}(\bar{r}, \xi, s) = -\frac{B(1+v_u)}{3(1-v_u)} \frac{i\xi}{s} K_m(\eta\bar{r}) A_{mk}(\xi, s) - \left[\frac{i\xi r}{4(1-2v_u)} K_{m+1}(\rho) \right. \\ \left. + \frac{|\xi|}{2i\xi} \frac{3-4v_u}{1-2v_u} K_m(\rho) \right] C_{mk}(\xi, s) + \frac{i\xi}{2} K_m(\rho) E_{mk}(\xi, s) + \frac{i\xi}{2} K_m(\rho) F_{mk}(\xi, s), \quad (3.24)$$

where E_{mk} and F_{mk} are two additional as yet undetermined functions of ξ and s .

Finally, the corresponding stress components can be derived straightforwardly by combining equations (2.5)–(2.11) with equations (3.1a)–(3.1l), which yield the following:

$$\Delta \hat{\hat{\sigma}}_{rr,mk}(\bar{r}, \xi, s) = \frac{B(1+v_u)}{3(1-v_u)} \frac{1}{\bar{r}^2 s} [\eta\bar{r} K_{m-1}(\eta\bar{r}) + (\xi^2 \bar{r}^2 + m^2 + m) K_m(\eta\bar{r})] A_{mk}(\xi, s) \\ + \left\{ \frac{v_u}{1-2v_u} |\xi| K_m(\rho) + \frac{1}{1-2v_u} \frac{1}{2\bar{r}} [\xi^2 \bar{r}^2 + (m+1)^2] K_{m+1}(\rho) \right\} C_{mk}(\xi, s) \\ - \frac{|\xi|}{2\bar{r}} [\rho K_m(\rho) + (m+1) K_{m+1}(\rho)] E_{mk}(\xi, s) \\ - \frac{|\xi|}{2\bar{r}} [\rho K_m(\rho) - (m-1) K_{m-1}(\rho)] F_{mk}(\xi, s), \quad (3.25)$$

$$\Delta \hat{\hat{\sigma}}_{\theta\theta,mk}(\bar{r}, \xi, s) = -\frac{B(1+v_u)}{3(1-v_u)} \frac{1}{\bar{r}^2 s} [\eta\bar{r} K_{m-1}(\eta\bar{r}) + (\bar{r}^2 s + m^2 + m) K_m(\eta\bar{r})] A_{mk}(\xi, s) \\ - \frac{1}{2} \left\{ |\xi| K_m(\rho) + \frac{1}{1-2v_u} \frac{(m+1)^2}{\bar{r}} K_{m+1}(\rho) \right\} C_{mk}(\xi, s) \\ + \frac{|\xi|}{2\bar{r}} (m+1) K_{m+1}(\rho) E_{mk}(\xi, s) - \frac{|\xi|}{2\bar{r}} (m-1) K_{m-1}(\rho) F_{mk}(\xi, s), \quad (3.26)$$

$$\Delta \hat{\hat{\sigma}}_{zz,mk}(\bar{r}, \xi, s) = -\frac{B(1+v_u)}{3(1-v_u)} \eta^2 K_m(\eta\bar{r}) A_{mk}(\xi, s) \\ + \left\{ \left[\frac{1}{2} |\xi| \frac{3-4v_u}{1-2v_u} + |\xi| \frac{v_u}{1-2v_u} \right] K_m(\rho) - \frac{\xi^2}{2} \frac{1}{1-2v_u} \bar{r} K_{m+1}(\rho) \right\} C_{mk}(\xi, s) \\ + \frac{\xi^2}{2} K_m(\rho) E_{mk}(\xi, s) + \frac{\xi^2}{2} K_m(\rho) F_{mk}(\xi, s), \quad (3.27)$$

$$\Delta \hat{\hat{\tau}}_{r\theta,mk}(\bar{r}, \xi, s) = \frac{B(1+v_u)}{6(1-v_u)} \frac{m}{\bar{r}^2 s} [2\eta\bar{r} K_{m-1}(\eta\bar{r}) + 2(m+1) K_m(\eta\bar{r})] A_{mk}(\xi, s) \\ + \frac{1}{4} \frac{1}{1-2v_u} \left\{ |\xi| (2m+1) K_m(\rho) + \frac{2(m+1)^2}{\bar{r}} K_{m+1}(\rho) \right\} C_{mk}(\xi, s) \\ - \left[\frac{\xi^2}{4} K_m(\rho) + \frac{|\xi|}{2\bar{r}} (m+1) K_{m+1}(\rho) \right] E_{mk}(\xi, s) \\ + \left[\frac{\xi^2}{4} K_m(\rho) - \frac{|\xi|}{2\bar{r}} (m-1) K_{m-1}(\rho) \right] F_{mk}(\xi, s), \quad (3.28)$$

$$\Delta \hat{\hat{\tau}}_{rz,mk}(\bar{r}, \xi, s) = \frac{B(1+v_u)}{(1-v_u)} \frac{i\xi\eta}{s} \left[K_{m-1}(\eta\bar{r}) + \frac{m}{\eta\bar{r}} K_m(\eta\bar{r}) \right] A_{mk}(\xi, s) + \frac{1}{4} \frac{i\xi}{\rho} \\ \times \left\{ \left[\frac{3-4v_u}{1-2v_u} m + \frac{2}{1-2v_u} \rho^2 \right] K_m(\rho) + 2\rho \left[\frac{m}{1-2v_u} - 1 \right] K_{m+1}(\rho) \right\} C_{mk}(\xi, s) \\ - \frac{i\xi|\xi|}{4} \left[2K_{m+1}(\rho) - \frac{m}{\xi} K_m(\rho) \right] E_{mk}(\xi, s) - \frac{i\xi|\xi|}{8} [K_{m+1}(\rho) + 3K_{m-1}(\rho)] F_{mk}(\xi, s) \quad (3.29)$$

$$\begin{aligned}
 \text{and } \hat{\bar{\tau}}_{z\theta, mk}(\bar{r}, \xi, s) = & \frac{B(1 + \nu_u)}{3(1 - \nu_u)} \frac{i\xi m}{\bar{r}s} K_m(\eta\bar{r}) A_{mk}(\xi, s) \\
 & + \left\{ \frac{1}{4} \frac{i\xi}{1 - 2\nu_u} (2m + 1) K_{m+1}(\rho) - \frac{i|\xi|}{4\xi} \frac{m}{\bar{r}} \frac{3 - 4\nu_u}{1 - 2\nu_u} K_m(\rho) \right\} C_{mk}(\xi, s) \\
 & - \left[\frac{i\xi}{2} \frac{m}{\bar{r}} K_m(\rho) + \frac{i\xi}{2} |\xi| K_{m+1}(\rho) \right] E_{mk}(\xi, s) \\
 & - \left[\frac{i\xi}{2} \frac{m}{\bar{r}} K_m(\rho) - \frac{i\xi}{2} |\xi| K_{m-1}(\rho) \right] F_{mk}(\xi, s).
 \end{aligned} \tag{3.30}$$

At this point, all the Fourier components of the displacements and stresses for the solid matrix as well as of the pore pressure and radial fluid flow rate have been formally obtained in the Fourier–Laplace transform domain. The four unknown functions $A_{mk}(\xi, s)$, $C_{mk}(\xi, s)$, $E_{mk}(\xi, s)$ and $F_{mk}(\xi, s)$ need to and can be determined from the appropriate boundary conditions adopted at the wellbore surface.

4. Boundary conditions and decomposition

As mentioned earlier, the stress and hydraulic boundary conditions (2.12a)–(2.13c) for the present problem may be equivalently expressed as follows, to determine the changes of stresses and pore pressure due to the borehole drilling and hydraulic fracturing:

$$\begin{cases} \Delta \bar{\sigma}_{rr} + \Delta \bar{p} = \bar{S}_r - \bar{p}_0 = \frac{\bar{S}_x + \bar{S}_y}{2} + \frac{\bar{S}_x - \bar{S}_y}{2} \cos 2\theta + \bar{S}_{xy} \sin 2\theta - \bar{p}_0 & 0 \leq |\bar{z}| \leq \bar{b}, \\ \Delta \bar{\sigma}_{rr} = \bar{S}_r = \frac{\bar{S}_x + \bar{S}_y}{2} + \frac{\bar{S}_x - \bar{S}_y}{2} \cos 2\theta + \bar{S}_{xy} \sin 2\theta & \bar{b} < |\bar{z}| < \infty, \end{cases} \tag{4.1a}$$

$$\Delta \bar{\tau}_{r\theta} = \bar{S}_{r\theta} = \bar{S}_{xy} \cos 2\theta - \frac{\bar{S}_x - \bar{S}_y}{2} \sin 2\theta, \tag{4.1b}$$

$$\Delta \bar{\tau}_{rz} = \bar{S}_{rz} = \bar{S}_{xz} \cos \theta + \bar{S}_{yz} \sin \theta \tag{4.1c}$$

$$\text{and } \bar{q}_r = \begin{cases} \frac{\bar{Q}_0}{4\pi\bar{b}} & 0 \leq |\bar{z}| \leq \bar{b}, \\ 0 & \bar{b} < |\bar{z}| < \infty, \end{cases} \tag{4.1d}$$

where $\bar{S}_x = S_x/2G$, $\bar{S}_y = S_y/2G$, $\bar{S}_{xy} = S_{xy}/2G$, $\bar{S}_{xz} = S_{xz}/2G$, $\bar{S}_{yz} = S_{yz}/2G$, $\bar{S}_r = S_r/2G$, $\bar{S}_{r\theta} = S_{r\theta}/2G$, $\bar{S}_{rz} = S_{rz}/2G$, $\bar{p}_0 = \frac{p_0}{2G}$, $\bar{Q}_0 = Q_0/2GR\kappa$ and $\bar{b} = b/R$. Note that the induced stresses and pore pressure calculated with the above borehole surface conditions shall be superposed with the *in situ* stress state to yield the final solutions. The boundary conditions (4.1)–(4.2d) nevertheless possess an obvious advantage over the original ones of equations (2.12a)–(2.13c) in that in this equivalent case only the conditions at the borehole surface need to be handled, which indeed may greatly simplify the derivations of the desired analytical solution.

To facilitate the solution procedure, the boundary conditions (4.1a)–(4.1d) will be decomposed into five fundamental, easier to handle modes to determine separately the Fourier components of the field variables, as follows.

(a) Mode 1 (zeroth order of the Fourier expansion of boundary conditions, $m = 0, k = 1$)

$$\begin{cases} \Delta \bar{\sigma}_{rr} + \Delta \bar{p} = \frac{\bar{S}_x + \bar{S}_y}{2} - \bar{p}_0 & 0 \leq |\bar{z}| \leq \bar{b}, \\ \Delta \bar{\sigma}_{rr} = \frac{\bar{S}_x + \bar{S}_y}{2} & \bar{b} < |\bar{z}| < \infty, \end{cases} \tag{4.2a}$$

$$\Delta \bar{\tau}_{r\theta} = 0, \quad (4.2b)$$

$$\Delta \bar{\tau}_{rz} = 0 \quad (4.2c)$$

and

$$\bar{q}_r = \begin{cases} \frac{\bar{Q}_0}{4\pi \bar{b}} & 0 \leq |\bar{z}| \leq \bar{b}, \\ 0 & \bar{b} < |\bar{z}| < \infty. \end{cases} \quad (4.2d)$$

(b) Mode 2 (first order of the Fourier expansion of boundary conditions, $m = 1, k = 1$)

$$\begin{cases} \Delta \bar{\sigma}_{rr} + \Delta \bar{p} = 0 & 0 \leq |\bar{z}| \leq \bar{b}, \\ \Delta \bar{\sigma}_{rr} = 0 & \bar{b} < |\bar{z}| < \infty, \end{cases} \quad (4.3a)$$

$$\Delta \bar{\tau}_{r\theta} = 0, \quad (4.3b)$$

$$\Delta \bar{\tau}_{rz} = \bar{S}_{xz} \cos \theta \quad (4.3c)$$

and

$$\bar{q}_r = 0. \quad (4.3d)$$

(c) Mode 3 (first order of the Fourier expansion of boundary conditions, $m = 1, k = 2$)

$$\begin{cases} \Delta \bar{\sigma}_{rr} + \Delta \bar{p} = 0 & 0 \leq |\bar{z}| \leq \bar{b}, \\ \Delta \bar{\sigma}_{rr} = 0 & \bar{b} < |\bar{z}| < \infty, \end{cases} \quad (4.4a)$$

$$\Delta \bar{\tau}_{r\theta} = 0, \quad (4.4b)$$

$$\Delta \bar{\tau}_{rz} = \bar{S}_{yz} \sin \theta \quad (4.4c)$$

and

$$\bar{q}_r = 0. \quad (4.4d)$$

(d) Mode 4 (second order of the Fourier expansion of boundary conditions, $m = 2, k = 1$)

$$\begin{cases} \Delta \bar{\sigma}_{rr} + \Delta \bar{p} = \frac{\bar{S}_x - \bar{S}_y}{2} \cos 2\theta & 0 \leq |\bar{z}| \leq \bar{b}, \\ \Delta \bar{\sigma}_{rr} = \frac{\bar{S}_x - \bar{S}_y}{2} \cos 2\theta & \bar{b} < |\bar{z}| < \infty, \end{cases} \quad (4.5a)$$

$$\Delta \bar{\tau}_{r\theta} = -\frac{\bar{S}_x - \bar{S}_y}{2} \sin 2\theta, \quad (4.5b)$$

$$\Delta \bar{\tau}_{rz} = 0 \quad (4.5c)$$

and

$$\bar{q}_r = 0 \quad (4.5d)$$

(e) Mode 5 (second order of the Fourier expansion of boundary conditions, $m = 2, k = 2$)

$$\begin{cases} \Delta \bar{\sigma}_{rr} + \Delta \bar{p} = \bar{S}_{xy} \sin 2\theta & 0 \leq |\bar{z}| \leq \bar{b}, \\ \Delta \bar{\sigma}_{rr} = \bar{S}_{xy} \sin 2\theta & \bar{b} < |\bar{z}| < \infty, \end{cases} \quad (4.6a)$$

$$\Delta \bar{\tau}_{r\theta} = \bar{S}_{xy} \cos 2\theta, \quad (4.6b)$$

$$\Delta \bar{\tau}_{rz} = 0 \quad (4.6c)$$

and

$$\bar{q}_r = 0. \quad (4.6d)$$

Evidently, it can be observed that, after the addition of equations (4.2a)–(4.5d), the original boundary conditions (4.1a)–(4.1d) are identically recovered.

5. Solutions for individual modes

(a) Mode 1 ($m = 0, k = 1$)

For this loading mode, it is obvious that the resulting solution shall only include the zeroth Fourier expansion terms with $m = 0$ and $k = 1$. The boundary conditions (4.2a)–(4.2d), therefore, can be rewritten as (in the transformed form)

$$\begin{cases} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\Delta \hat{\hat{\sigma}}_{rr,01}(1, \xi, s) + \Delta \hat{\hat{p}}_{01}(1, \xi, s)] e^{-i\xi \bar{z}} d\xi = \frac{\bar{S}_x + \bar{S}_y}{2} - \frac{\bar{p}_0}{s} & 0 \leq |\bar{z}| \leq \bar{b}, \\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Delta \hat{\hat{\sigma}}_{rr,01}(1, \xi, s) e^{-i\xi \bar{z}} d\xi = \frac{\bar{S}_x + \bar{S}_y}{2} & \bar{b} < |\bar{z}| < \infty, \end{cases} \quad (5.1a)$$

$$\Delta \hat{\hat{v}}_{r\theta,01}(1, \xi, s) = 0, \quad (5.1b)$$

$$\Delta \hat{\hat{t}}_{rz,01}(1, \xi, s) = 0 \quad (5.1c)$$

$$\text{and} \quad \hat{\hat{q}}_{r,01}(1, \xi, s) = \frac{\bar{Q}_0}{2\pi \bar{b}} \frac{\sin(\xi \bar{b})}{\sqrt{2\pi} \xi s} = \Omega(\xi, s), \quad (5.1d)$$

which may be eventually reduced to a set of Bessel function dual integral equations as follows (see appendix A):

$$\int_0^{\infty} \xi^{1/2} [1 + f_{1,01}(\xi, s)] \Delta \hat{\hat{\sigma}}_{rr,01}^*(1, \xi, s) J_{-1/2}(\xi \bar{z}) d\xi = \sqrt{\frac{2}{\pi \bar{z}}} g(\bar{z}, s) \quad 0 \leq |\bar{z}| \leq \bar{b} \quad (5.2a)$$

and

$$\int_0^{\infty} \xi^{1/2} \Delta \hat{\hat{\sigma}}_{rr,01}^*(1, \xi, s) J_{-1/2}(\xi \bar{z}) d\xi = 0 \quad \bar{b} < |\bar{z}| < \infty, \quad (5.2b)$$

where $\Delta \hat{\hat{\sigma}}_{rr,01}^*(1, \xi, s) = \Delta \hat{\hat{\sigma}}_{rr,01}(1, \xi, s) - (\hat{\hat{S}}_x + \hat{\hat{S}}_y)/2$, with $\hat{\hat{S}}_x = \bar{S}_x \sqrt{2\pi} \delta(\xi)/s$ and $\hat{\hat{S}}_y = \bar{S}_y \sqrt{2\pi} \delta(\xi)$ ($\delta(\xi)/s$ denoting the Dirac delta function), is an auxiliary variable specifically introduced to arrive at the above Noble [16]-type dual integral equations in their desired and numerically solvable form,

$$g(\bar{z}, s) = -\frac{\sqrt{2\pi}}{2s} \frac{\bar{S}_x + \bar{S}_y}{2} f_{1,01}(0, s) - \int_0^{\infty} f_{2,01}(\xi, s) \Omega(\xi, s) d\xi - \frac{\sqrt{2\pi} \bar{p}_0}{s}, \quad (5.3)$$

and $f_{1,01}(\xi, s)$ and $f_{2,01}(\xi, s)$ are two explicit functions of ξ and s as defined by equations (A 3) and (A 4) in appendix A. Note that the integral of $f_{2,01}(\xi, s)$ for $g(\bar{z}, s)$ is valid and converges.

As demonstrated in Noble [16], the solution of dual integral equations (5.2a) and (5.2b) may be sought by using the so-called multiplying-factor method. By employing the following integral representation:

$$\Delta \hat{\hat{\sigma}}_{rr,01}^*(1, \xi, s) = \frac{2}{\pi} \int_0^{\bar{b}} \theta_{01}(x, s) \cos(x\xi) dx, \quad (5.4)$$

equations (5.2a) and (5.2b) are found identical to solving the following Fredholm integral equation of the second kind for $\theta_{01}(x, s)$:

$$\theta_{01}(x, s) + \frac{2}{\pi} \int_0^{\bar{b}} M_{01}(x, y, s) \theta_{01}(y, s) dy = g(x, s), \quad (5.5)$$

where

$$M_{01}(x, y, s) = \int_0^{\infty} f_{1,01}(\xi, s) \cos(x\xi) \cos(y\xi) d\xi \quad (5.6)$$

is known as the kernel of the Fredholm integral equation and the infinite integral involved can be found existing and convergent.

Equations (5.4)–(5.6) numerically determine $\Delta\hat{\hat{\sigma}}_{rr,01}^*(1, \xi, s)$ and subsequently

$$\Delta\hat{\hat{\sigma}}_{rr,01}(1, \xi, s) = \Delta\hat{\hat{\sigma}}_{rr,01}^*(1, \xi, s) + \frac{\bar{S}_x + \bar{S}_y}{2s} \sqrt{2\pi} \delta(\xi) \quad (5.7)$$

in the Laplace–Fourier transformed space.

It must be remarked that, at first sight, the transformed boundary condition equation (5.1a) with $\Delta\hat{\hat{\sigma}}_{rr,01}(1, \xi, s)$ being the unknown appears to fall into the generalized category of dual integral equations considered by Noble [16], i.e. having a non-zero term of $(\bar{S}_x + \bar{S}_y)/2$ on the right side of the second equation. Unfortunately, with the inclusion of such a non-zero term (in fact constant with respect to $\bar{z}$), the integral equations (5.1a) turn out to no longer be directly solvable for $\Delta\hat{\hat{\sigma}}_{rr,01}(1, \xi, s)$ by means of the standard multiplying-factor method as originally developed by Noble [16]. This is mainly because the contribution from $(\bar{S}_x + \bar{S}_y)/2$ to the non-homogeneous term in the Fredholm integral equation will result in a divergent infinite integral, which makes the formulations essentially invalid.

Once $\Delta\hat{\hat{\sigma}}_{rr,01}(1, \xi, s)$ is known from equation (5.7), the desired functions $A_{01}(\xi, s)$, $C_{01}(\xi, s)$, $E_{01}(\xi, s)$ and $F_{01}(\xi, s)$ can be obtained because they are all explicitly expressible in terms of the basic unknown $\Delta\hat{\hat{\sigma}}_{rr,01}(1, \xi, s)$ (see equation (A 1) in appendix A). Substitution of these functions into equations (3.17), (3.18) and (3.22)–(3.30) completes the transformed solutions (pertaining to the first loading mode) for the excess pore pressure, radial flow rate and the displacement and stress components. The corresponding physical quantities, therefore, may be evaluated through the inversion of the Laplace and Fourier transforms.

(b) Mode 2 ($m = 1, k = 1$)

This mode is associated with the first order of Fourier expansion of $m = 1$ and $k = 1$. The solution can be derived in a similar manner to that outlined in the previous section. However, for this loading case, it is interesting to find that the dual integral equations governing the transformed radial stress at the borehole surface $\Delta\hat{\hat{\sigma}}_{rr,11}(1, \xi, s)$ turn out to be

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [1 + f_{1,11}(\xi, s)] \Delta\hat{\hat{\sigma}}_{rr,11}(1, \xi, s) e^{-i\xi\bar{z}} d\xi = 0 \quad 0 \leq |\bar{z}| \leq \bar{b} \quad (5.8a)$$

and

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Delta\hat{\hat{\sigma}}_{rr,11}(1, \xi, s) e^{-i\xi\bar{z}} d\xi = 0 \quad \bar{b} < |\bar{z}| < \infty, \quad (5.8b)$$

where $f_{1,11}(\xi, s)$ is a known function of ξ and s , and can be expressed in a form similar to equation (A 3) as obtained for $f_{0,11}(\xi, s)$ associated with the mode 1 solution. Equations (5.8a) and (5.8b) yield a trivial solution of $\Delta\hat{\hat{\sigma}}_{rr,11}(1, \xi, s) = 0$, and so does the pore pressure $\Delta\hat{\hat{p}}_{11}(1, \xi, s)$ in the transformed domain. Here, the subscript ‘11’ represents the corresponding components of $m = k = 1$ in equations (3.1a)–(3.1m). Since $\Delta\hat{\hat{\sigma}}_{rr,11}(1, \xi, s) \equiv 0$, the formulation of the solution for this mode will be greatly simplified and the stress components and pore pressure can be explicitly obtained from the Laplace and Fourier inversions as follows:

$$\Delta\bar{\sigma}_{rr,11}(\bar{r}, \bar{z}, \bar{t}) = \Delta\bar{\sigma}_{\theta\theta,11}(\bar{r}, \bar{z}, \bar{t}) = \Delta\bar{\sigma}_{zz,11}(\bar{r}, \bar{z}, \bar{t}) = \Delta\bar{\tau}_{r\theta,11}(\bar{r}, \bar{z}, \bar{t}) = 0, \quad (5.9a)$$

$$\Delta\bar{\tau}_{rz,11}(\bar{r}, \bar{z}, \bar{t}) = -\bar{S}_{xz} \left(1 - \frac{1}{\bar{r}^2}\right), \quad \Delta\bar{\tau}_{z\theta,11}(\bar{r}, \bar{z}, \bar{t}) = \bar{S}_{xz} \left(1 + \frac{1}{\bar{r}^2}\right) \quad (5.9b)$$

$$\text{and} \quad \Delta\bar{p}_{11}(\bar{r}, \bar{z}, \bar{t}) = 0, \quad (5.9c)$$

which are identical to the heuristic solutions proposed in Cui *et al.* [6] for the elastic anti-plane shear problem, but herein these same results have been derived in a mathematically more straightforward and rigorous way.

(c) Mode 3 ($m = 1, k = 2$)

In like manner, the solution corresponding to this loading mode is shown below by making the replacement $\bar{S}_{xz} \rightarrow -\bar{S}_{yz}$. This gives

$$\Delta \bar{\sigma}_{rr,12}(\bar{r}, \bar{z}, \bar{t}) = \Delta \bar{\sigma}_{\theta\theta,12}(\bar{r}, \bar{z}, \bar{t}) = \Delta \bar{\sigma}_{zz,12}(\bar{r}, \bar{z}, \bar{t}) = \Delta \bar{\tau}_{r\theta,12}(\bar{r}, \bar{z}, \bar{t}) = 0, \quad (5.10a)$$

$$\Delta \bar{\tau}_{rz,12}(\bar{r}, \bar{z}, \bar{t}) = \bar{S}_{yz} \left(1 - \frac{1}{\bar{r}^2}\right), \quad \Delta \bar{\tau}_{z\theta,12}(\bar{r}, \bar{z}, \bar{t}) = -\bar{S}_{xz} \left(1 + \frac{1}{\bar{r}^2}\right) \quad (5.10b)$$

and
$$\Delta \bar{p}_{12}(\bar{r}, \bar{z}, \bar{t}) = 0. \quad (5.10c)$$

(d) Mode 4 ($m = 2, k = 1$)

Only the second-order components of the Fourier expansions need to be considered in this case. And, again, the solution procedure used above for mode 1 is equally applicable to the present mode. One, therefore, arrives at the following system of dual integral equations with $\Delta \hat{\hat{\sigma}}_{rr,21}^*(1, \xi, s)$ being unknown:

$$\begin{aligned} & \int_{-\infty}^{\infty} \xi^{1/2} [1 + f_{1,21}(\xi, s)] \Delta \hat{\hat{\sigma}}_{rr,21}^*(1, \xi, s) J_{-1/2}(\xi \bar{z}) d\xi \\ &= \frac{1}{\sqrt{\bar{z}}} \frac{\bar{S}_x - \bar{S}_y}{2s} \{f_{2,21}(0, s) - f_{1,21}(0, s)\} \quad 0 \leq |\bar{z}| \leq \bar{b} \end{aligned} \quad (5.11a)$$

and

$$\int_{\infty}^{\infty} \xi^{1/2} \Delta \hat{\hat{\sigma}}_{rr,21}^*(1, \xi, s) J_{-1/2}(\xi \bar{z}) d\xi = 0 \quad \bar{b} < |\bar{z}| < \infty, \quad (5.11b)$$

where $\Delta \hat{\hat{\sigma}}_{rr,21}^*(1, \xi, s) = \Delta \hat{\hat{\sigma}}_{rr,21}(1, \xi, s) - \frac{\hat{S}_x - \hat{S}_y}{2}, f_{1,21}(\xi, s)$ and $f_{2,21}(\xi, s)$ are defined in equations (B 1) and (B 2) in appendix B.

Once $\Delta \hat{\hat{\sigma}}_{rr,21}^*(1, \xi, s)$ is solved from equations (5.11a) and (5.11b) through their reduction to a Fredholm integral equation of the second kind, one has

$$\Delta \hat{\hat{\sigma}}_{rr,21}(1, \xi, s) = \Delta \hat{\hat{\sigma}}_{rr,21}^*(1, \xi, s) + \frac{\bar{S}_x - \bar{S}_y}{2s} \sqrt{2\pi} \delta(\xi). \quad (5.12)$$

The coefficient functions $A_{21}(\xi, s)$, $C_{21}(\xi, s)$, $E_{21}(\xi, s)$ and $F_{21}(\xi, s)$ thus may be evaluated by simple algebraic manipulations on equations (4.5b) and (4.5c) (see appendix B for their expressions). And thereafter the field variables for this mode, i.e. $\bar{u}_{r,21}(\bar{r}, \bar{z}, \bar{t})$, $\Delta \bar{\sigma}_{rr,21}(\bar{r}, \bar{z}, \bar{t})$, $\Delta \bar{\sigma}_{\theta\theta,21}(\bar{r}, \bar{z}, \bar{t})$, etc., can be retrieved through the standard numerical Laplace–Fourier inversion methods.

(e) Mode 5 ($m = 2, k = 2$)

Comparison of boundary conditions (4.5a)–(4.5d) and (4.6a)–(4.6d) indicates that the above analysis for mode 4 can be directly employed for the solution of mode 5, with the only exception being that $(\bar{S}_x - \bar{S}_y)/2$ should be replaced by $-\bar{S}_{xy}$ in the appropriate places. Detailed formulations are not presented herein due to the page limit.

(f) Superposition

Finally, the solution of the inclined borehole problem subjected to fluid injection and *in situ* stresses can be obtained by superposition. For example, the pore pressure and most important

Figure 2. Comparisons of (a) pore pressure and (b) effective tangential stress distributions with radial distance, $b/R = 1$. (Online version in colour.)

effective tangential stress can be formally shown as follows:

$$\bar{p}(\bar{r}, \bar{z}, \bar{t}) = \Delta \bar{p}_{01}(\bar{r}, \bar{z}, \bar{t}) + \Delta \bar{p}_{21}(\bar{r}, \bar{z}, \bar{t}) \cos 2\theta - \Delta \bar{p}_{22}(\bar{r}, \bar{z}, \bar{t}) \sin 2\theta + \bar{p}_0, \quad (5.13)$$

$$\begin{aligned} \bar{\sigma}_{\theta\theta}(\bar{r}, \bar{z}, \bar{t}) = & \Delta \bar{\sigma}_{\theta\theta,01}(\bar{r}, \bar{z}, \bar{t}) + \Delta \bar{\sigma}_{\theta\theta,21}(\bar{r}, \bar{z}, \bar{t}) \cos 2\theta - \Delta \bar{\sigma}_{\theta\theta,22}(\bar{r}, \bar{z}, \bar{t}) \sin 2\theta \\ & - \frac{\bar{S}_x + \bar{S}_y}{2} + \frac{\bar{S}_x - \bar{S}_y}{2} \cos 2\theta + \bar{S}_{xy} \sin 2\theta \end{aligned} \quad (5.14)$$

$$\text{and} \quad \bar{\sigma}'_{\theta\theta}(\bar{r}, \bar{z}, \bar{t}) = \bar{\sigma}_{\theta\theta}(\bar{r}, \bar{z}, \bar{t}) + \alpha \cdot \bar{p}(\bar{r}, \bar{z}, \bar{t}), \quad (5.15)$$

where the prime denotes effective stress.

Figure 3. Mutual verifications between analytical and ABAQUS numerical results: (a) pore pressure and (b) effective tangential stress, $b/R = 1$. (Online version in colour.)

6. Numerical results and discussions

The inversions of Laplace and Fourier transforms for the pore pressure, stress components, etc. involve an infinite integral with respect to the Fourier transform parameter ξ ,

$$f_{mk}(\bar{r}, \bar{z}, \bar{t}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}_{mk}(\bar{r}, \xi, \bar{t}) e^{-i\xi \bar{z}} d\xi, \quad (6.1)$$

and a Bromwich integral to the Laplace transform parameter s ,

$$f_{mk}(\bar{r}, \bar{z}, \bar{t}) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \tilde{f}_{mk}(\bar{r}, \bar{z}, s) e^{s\bar{t}} ds, \quad (6.2)$$

Figure 4. (a) One-eighth finite-element mesh (three dimensional) for borehole drilling/fluid injection analysis and (b) detailed mesh around the flux zone. (Online version in colour.)

where $f_{mk}(\bar{r}, \bar{z}, \bar{t})$ represents the mk th Fourier expansion term of a generic function $f(\bar{r}, \bar{z}, \bar{t})$, and γ is greater than the real part of all singularities of $\tilde{f}_{mk}(\bar{r}, \bar{z}, s)$ [21]. In the computation procedure, the efficient Stehfest's formula [22] will be adopted in inverting the Laplace transform, while the Fourier transform inversion ($\xi \rightarrow \bar{z}$) can be furnished by first truncating the infinite integrals at certain sufficiently large values and then being numerically evaluated using a five-point Gaussian quadrature formula with a small step size. However, it should be pointed out that for the latter Fourier inversion, due to the presence of Dirac delta function $\delta(\xi)$ in the transformed total radial stresses at the borehole surface (e.g. $\Delta\hat{\sigma}_{rr,01}(1, \xi, s)$ and $\Delta\hat{\sigma}_{rr,21}(1, \xi, s)$ in equations (5.7) and (5.12), respectively) and its subsequent propagation via the four coefficient functions A_{mk} , C_{mk} , E_{mk} and F_{mk} , meticulous attention must be given in handling the infinite integrals with respect to ξ . Here, to ensure the accuracy and convergence of the numerical integrations, the integrand terms including the $\delta(\xi)$ factor will be extracted and directly evaluated on leverage of the following important feature of the Dirac delta function:

$$\int_{-\infty}^{\infty} \varphi(\xi) \delta(\xi) d\xi = \varphi(\xi)|_{\xi=0} = \varphi(0). \quad (6.3)$$

Note that the expressions for pore pressure and stress components in the transformed domain indeed involve the modified Bessel function of the second kind, $K_n(\xi)$, which approaches infinity at $\xi = 0$. Such a singularity needs to and can be eliminated by taking a sufficiently small value of ξ in calculating the right-hand side of equation (6.3), and it is found that a selection of $\xi = 10^{-8}$ usually will give satisfactory results.

In this section, comparisons will be first made for the distributions of effective tangential stress $\sigma'_{\theta\theta}$ and pore pressure p around an inclined borehole, between the approximate solution reported by Abousleiman & Chen [11] (assuming zero total radial stress at the entire borehole surface for the stress boundary sub-problem) and the current one developed in a truly rigorous manner. The poromechanical properties of Ruhr sandstone considered in the numerical analyses as well as the borehole geometric parameters and *in situ* stress/pore pressure conditions are as follows: $G = 13$ GPa, $\nu = 0.12$, $\nu_u = 0.3$, $B = 0.849$, $\alpha = 0.645$, $\kappa = 0.0173 \text{ m}^4 (\text{MN} \cdot \text{d})^{-1}$, $k = 0.2 \text{ mD}$, $c = 424 \text{ m}^2 \text{ d}^{-1}$; $R = 0.1 \text{ m}$, $\varphi_{z'} = 30^\circ$, $\varphi_y = 60$, $Q_0 = 1.41 \text{ min}^{-1}$, $b/R = 1$; and $S_{x'} = 20 \text{ MPa}$, $S_{y'} = 18 \text{ MPa}$, $S_{z'} = 25 \text{ MPa}$, $p_0 = 9.8 \text{ MPa}$. It can be seen from figure 2 that, although introducing no appreciable errors in the calculated pore pressure, Abousleiman and Chen's approximate solution profoundly underestimates the effective tangential stress near the borehole. This indicates that [11] may lead to considerable overestimation in terms of the required injection rate for triggering the fracture, and in turn highlights the imperative demand of a complete rigorous analytical

Figure 5. Isochrones of (a) pore pressure and (b) effective tangential stress variations with radial distance, $b/R = 1$. (Online version in colour.)

solution, as formulated/developed in the present paper, for the hydraulic fracturing initiation analysis.

Figure 3 further provides the comparisons of the variations of pore pressure and effective tangential stress versus the radial distance between the analytical solutions and ABAQUS finite-element numerical results, to check not only the validity of the proposed analytical approach but also the accuracy of the numerical computations associated with the Fredholm integral equations of the second kind. The parameters used for this verification purpose are the same as those considered for figure 2, with the borehole azimuth angle $\varphi_z = 30^\circ$ and inclination angle $\varphi_y = 60^\circ$. Note that, in the ABAQUS simulations, a full three-dimensional model needs to be constructed because the stress and strain fields generated due to the inclined borehole drilling and finite-length fluid discharge cannot be modelled using two-dimensional or axisymmetric mesh

Figure 6. Impacts of fluid discharge length on (a) pore pressure and (b) effective tangential stress variations with radial distance, $t = 0.01$ day. (Online version in colour.)

domains. As shown in figure 4 (for clearer illustration only one-eighth of the finite-element model has been presented therein), the whole calculation domain adopted is $R_e = 300R$ and $2D = 600R$, where R_e and $2D$ denote the external bounds of the mesh in the radial and vertical directions, respectively. The finite-element model includes a total number of 178 560 C3D8P (eight-node brick, trilinear displacement and pore pressure) elements with a refined mesh near the injection segment of the wellbore surface. A sensitivity analysis shows that such a choice of the calculation domain and mesh size is large and fine enough to obtain satisfactory numerical results.

Three steps are required in the ABAQUS simulations to model the borehole drilling/hydraulic fracturing problem, i.e. the initial geostatic step, the subsequent step of instant drilling and lastly the fluid injection step. The excellent agreement between the analytical and numerical results ($\theta = -4.8^\circ$ and $t = 0.1$ day), as shown in figure 3, clearly justifies the reliability and accuracy of

the analytical approach. It is remarkable that, in the ABAQUS modelling, the applications of the coupled fluid injection boundary conditions at the borehole surface have to be accomplished through the programming of two user sub-routines URDFIL and DLOAD. This is because the applied surface (radial) stress in ABAQUS by default should be the total stress σ_{rr} , which unfortunately is unknown and itself needs to be determined for the implicit problem of borehole injection. With URDFIL, the pore pressure results over the flux zone can be extracted and stored at the end of each increment during the fluid injection step, while DLOAD assigns the pore pressure thus obtained as the radial total stress (i.e. $p = \sigma_{rr}$) exerting on the injection segment of the borehole surface, from the beginning of the immediately following analysis increment.

Figure 5 shows the variations of the maximum effective tangential stress $\sigma'_{\theta\theta}$, which occurs at the $\theta = -4.8^\circ$ direction for the current *in situ* stresses and borehole inclination involved (i.e. the maximum compressive principal stress direction in the plane normal to the borehole axis), and the pertinent pore pressure with the normalized radial distance r/R , each for three different stages of the fluid injection $t = 0.0001, 0.01, 1$ day but the same location $z = 0$. As is expected, both the effective tangential stress and pore pressure increase as time goes on, yet decrease against the radial distance towards their respective *in situ* values $\sigma'_{\theta\theta} = (\sigma_x + \sigma_y)/2 - \sqrt{S_{xy}^2 + (\sigma_x - \sigma_y)^2/4} - \alpha p_0 = -12.14$ MPa (compression negative) and $p_0 = 9.8$ MPa. Also noted in this figure is the same slope dp/dr featured at the borehole wall for the three presented pore pressure curves, a direct consequence of the constant fluid discharge boundary condition $q_r(R, \theta, 0, t) = Q_0/4\pi Rb$. Overall, these observed trends are similar to those demonstrated in [11], although the calculated magnitudes may differ particularly when $\sigma'_{\theta\theta}$ is concerned.

The impacts of the injection length b on the calculated maximum effective tangential stress and the pore pressure (at $\theta = -4.8^\circ$) are presented in figure 6, for $t = 0.01$ day and again at the middle of the fluid discharge $z = 0$. Since the fluid flow rate per unit area is inversely proportional to b for the constant volume rate of $Q_0 = 1.4 \text{ l min}^{-1}$ as given in the current numerical example, it is not surprising to observe that both $\sigma'_{\theta\theta}$ and p decrease rapidly with the discharge length increasing from $b/R = 1$ to 5, especially in the vicinity of the borehole. On the leverage of the relationship between the fluid discharge length and the induced maximum effective tangential stress at the wellbore surface, as indicated in figure 6, one may easily determine the minimum fluid flow rate and desired packer distance (sealing the pressurized section from the rest of borehole) that are required to initiate the hydraulic fracturing.

7. Conclusion

Development of an exact, fully analytical, coupled solution for the poromechanical responses of an inclined borehole, subjected to far-field three-dimensional *in situ* stresses in addition to a fluid discharge over a finite length of its surface, remains a complex challenge in theoretical geomechanics. Such a general mixed, coupled boundary value problem has been successfully tackled in this paper by using the Fourier expansion theorem as well as the combined Laplace–Fourier integral transform technique for solving directly the governing equations of poroelasticity, and by elegantly separating the quite complicated boundary conditions into five fundamental modes that are easier to deal with. Each of these individual modes is essentially equivalent to a set of dual integral equations and, after meticulous treatment, can be solved through the multiplying-factor method and standard numerical procedure. The complete solution for the addressed borehole problem subjected to fluid injection and *in situ* stresses therefore is finally obtainable via the superposition scheme.

Excellent agreement between the analytical solutions and ABAQUS finite-element numerical results is obtained. Comparisons with an earlier reported approximate solution indicates that neglecting the coupling between the total radial stress and injection-induced pore pressure at the borehole surface in relation to the stress boundary problem (*in situ* stress release) may profoundly underestimate the effective tangential stress in the vicinity of the borehole, and hence a considerable overestimation of the required injection rate for triggering the hydraulic fracturing.

Numerical results also show that both the induced effective tangential stress and pore pressure increase as time progresses and/or the injection length decreases, and that the maximum effective tangential stress occurs along the direction of the maximum compressive principal stress in the plane normal to the borehole axis. The rigorous analytical solution proposed in the present paper not only may serve as a benchmark to validate the reliability of finite-element numerical results, but also can be used for hydraulic fracture initiation analysis, especially for predicting the fluid injection rate and the packer distance desired to trigger the hydraulic fracturing.

Data accessibility. The work does not have any experimental data. The Fredholm integral equations of the second kind for the individual modes were numerically solved using the Wolfram Mathematica software. The finite-element numerical simulations were conducted by using the commercial program ABAQUS, through the implementation of two user subroutines URDFIL and DLOAD that have been specifically developed to fulfil the application of the implicit fluid injection boundary conditions at the borehole surface. All computational results are reproducible.

Competing interests. I have no competing interests.

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Appendix A. Formulation of dual integral equations for mode 1

Substituting equations (3.18), (3.28) and (3.29) into equations (5.1*b*)–(5.1*d*) and making use of equation (3.25), one can readily solve the four unknown functions in terms of $\Delta\hat{\sigma}_{rr,01}(1, \xi, s)$, giving

$$\begin{Bmatrix} A_{01}(\xi, s) \\ C_{01}(\xi, s) \\ E_{01}(\xi, s) \\ F_{01}(\xi, s) \end{Bmatrix} = \begin{bmatrix} \beta_{11,01}(\xi, s) & \beta_{12,01}(\xi, s) \\ \beta_{21,01}(\xi, s) & \beta_{22,01}(\xi, s) \\ \beta_{31,01}(\xi, s) & \beta_{32,01}(\xi, s) \\ \beta_{41,01}(\xi, s) & \beta_{42,01}(\xi, s) \end{bmatrix} \cdot \begin{Bmatrix} \Delta\hat{\sigma}_{rr,01}(1, \xi, s) \\ \Omega(\xi, s) \end{Bmatrix}, \quad (\text{A } 1)$$

where $\beta_{ij,01}(\xi, s) = \alpha_{(3j-2)i,01}^*(\xi, s)/|\alpha_{01}(\xi, s)|$ ($i = 1, 2, 3, 4$; $j = 1, 2$), $|\alpha_{01}(\xi, s)|$ denotes the determinant of the matrix

$$\alpha_{01}(\xi, s) = \begin{bmatrix} \alpha_{11,01}(\xi, s) & \alpha_{12,01}(\xi, s) & \alpha_{13,01}(\xi, s) & \alpha_{14,01}(\xi, s) \\ \alpha_{21,01}(\xi, s) & \alpha_{22,01}(\xi, s) & \alpha_{23,01}(\xi, s) & \alpha_{24,01}(\xi, s) \\ \alpha_{31,01}(\xi, s) & \alpha_{32,01}(\xi, s) & \alpha_{33,01}(\xi, s) & \alpha_{34,01}(\xi, s) \\ \alpha_{41,01}(\xi, s) & \alpha_{42,01}(\xi, s) & \alpha_{43,01}(\xi, s) & \alpha_{44,01}(\xi, s) \end{bmatrix},$$

while $\alpha_{(3j-2)i,01}^*$ represents the algebraic complement of the element $\alpha_{(3j-2)i,01}(\xi, s)$, and

$$\begin{aligned} \alpha_{11,01}(\xi, s) &= \frac{B(1 + \nu_u)}{3(1 - \nu_u)} \frac{1}{s} [\eta K_1(\eta) + \xi^2 K_0(\eta)], \\ \alpha_{12,01}(\xi, s) &= \frac{\nu_u}{1 - 2\nu_u} |\xi| K_0(|\xi|) + \frac{1}{1 - 2\nu_u} \frac{1}{2} (\xi^2 + 1) K_1(|\xi|), \\ \alpha_{13,01}(\xi, s) &= \alpha_{14,01}(\xi, s) = -\frac{|\xi|}{2} [|\xi| K_0(|\xi|) + K_1(|\xi|)], \end{aligned}$$

$$\begin{aligned}
\alpha_{21,01}(\xi, s) &= 0, \quad \alpha_{22,01}(\xi, s) = \frac{1}{4} \frac{1}{1 - 2\nu_u} [|\xi| K_0(|\xi|) + 2K_1(|\xi|)], \\
\alpha_{23,01}(\xi, s) &= -\alpha_{24,01}(\xi, s) = -\left[\frac{\xi^2}{4} K_0(|\xi|) + \frac{|\xi|}{2} K_1(|\xi|) \right] \\
\alpha_{31,01}(\xi, s) &= \frac{B(1 + \nu_u)}{(1 - \nu_u)} \frac{i\xi\eta}{s} K_1(\eta), \quad \alpha_{32,01}(\xi, s) = \frac{1}{2} i\xi |\xi| \frac{1}{1 - 2\nu_u}, \\
\alpha_{33,01}(\xi, s) &= \alpha_{34,01}(\xi, s) = -\frac{i\xi |\xi|}{2} K_1(|\xi|), \\
\alpha_{41,01}(\xi, s) &= -\frac{B^2(1 - \nu)(1 + \nu_u)^2}{9(1 - \nu_u)(\nu - \nu_u)} K_1(\eta), \quad \alpha_{42,01}(\xi, s) = -\frac{2B(1 + \nu_u)}{3(1 - 2\nu_u)} \xi^2 K_1(|\xi|) \\
\text{and} \quad \alpha_{43,01}(\xi, s) &= \alpha_{44,01}(\xi, s) = 0.
\end{aligned}$$

The pore pressure change $\Delta \hat{\bar{p}}_{0,1}(1, \xi, s)$ now can be written out by incorporating equation (A 1) into equation (3.17)

$$\Delta \hat{\bar{p}}_{01}(1, \xi, s) = f_{1,01}(\xi, s) \Delta \hat{\bar{\sigma}}_{rr,01}^*(1, \xi, s) + f_{2,01}(\xi, s) \Omega(\xi, s), \quad (\text{A } 2)$$

where

$$f_{1,01}(\xi, s) = -\frac{B^2(1 - \nu)(1 + \nu_u)^2}{9(1 - \nu_u)(\nu - \nu_u)} K_0(\eta) \beta_{11,01}(\xi, s) - \frac{B(1 + \nu_u)}{3(1 - 2\nu_u)} |\xi| K_0(|\xi|) \beta_{21,01}(\xi, s) \quad (\text{A } 3)$$

and

$$f_{2,01}(\xi, s) = -\frac{B^2(1 - \nu)(1 + \nu_u)^2}{9(1 - \nu_u)(\nu - \nu_u)} K_0(\eta) \beta_{12,01}(\xi, s) - \frac{B(1 + \nu_u)}{3(1 - 2\nu_u)} |\xi| K_0(|\xi|) \beta_{22,01}(\xi, s). \quad (\text{A } 4)$$

So the boundary condition (5.1a) becomes

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [1 + f_{1,01}(\xi, s)] \Delta \hat{\bar{\sigma}}_{rr,01}^*(1, \xi, s) e^{-i\xi \bar{z}} d\xi \\
&= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ f_{1,01}(\xi, s) \frac{\hat{\bar{S}}_x + \hat{\bar{S}}_y}{2} + f_{2,01}(\xi, s) \Omega(\xi, s) \right\} e^{-i\xi \bar{z}} d\xi - \frac{\bar{p}_0}{s} \quad 0 \leq |\bar{z}| \leq \bar{b}, \quad (\text{A } 5a)
\end{aligned}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Delta \hat{\bar{\sigma}}_{rr,01}^*(1, \xi, s) e^{-i\xi \bar{z}} d\xi = 0 \quad \bar{b} < |\bar{z}| < \infty, \quad (\text{A } 5b)$$

where $\Delta \hat{\bar{\sigma}}_{rr,01}^*(1, \xi, s) = \Delta \hat{\bar{\sigma}}_{rr,01}(1, \xi, s) - (\hat{\bar{S}}_x + \hat{\bar{S}}_y)/2$.

Noting that $f_{1,01}(\xi, s)$, $f_{2,01}(\xi, s)$, $\Delta \hat{\bar{\sigma}}_{rr,01}^*(1, \xi, s)$ and $\Omega(\xi, s)$ are all even functions of ξ and on account of the relationship $J_{-\frac{1}{2}}(\xi \bar{z}) \equiv \sqrt{2/\pi} \xi \bar{z} \cos(\xi \bar{z})$, the above two equations can be further reduced to a pair of dual integral equations originally considered by Noble [16], as shown in equations (5.2a) and (5.2b) of the main text.

Appendix B. Expressions of $f_{1,21}(\xi, s)$ and $f_{2,21}(\xi, s)$

The two functions $f_{1,21}(\xi, s)$ and $f_{2,21}(\xi, s)$ in equation (5.11a) take the following form:

$$f_{1,21}(\xi, s) = -\frac{B^2(1 - \nu)(1 + \nu_u)^2}{9(1 - \nu_u)(\nu - \nu_u)} K_2(\eta) \beta_{11,21}(\xi, s) - \frac{B(1 + \nu_u)}{3(1 - 2\nu_u)} |\xi| K_2(|\xi|) \beta_{21,21}(\xi, s) \quad (\text{B } 1)$$

and

$$f_{2,21}(\xi, s) = -\frac{B^2(1 - \nu)(1 + \nu_u)^2}{9(1 - \nu_u)(\nu - \nu_u)} K_2(\eta) \beta_{12,21}(\xi, s) - \frac{B(1 + \nu_u)}{3(1 - 2\nu_u)} |\xi| K_2(|\xi|) \beta_{22,21}(\xi, s), \quad (\text{B } 2)$$

where $\beta_{11,21}(\xi, s)$, $\beta_{12,21}(\xi, s)$, $\beta_{21,21}(\xi, s)$, $\beta_{22,21}(\xi, s)$ are the elements of a 4 by 2 matrix relating the four unknown functions $A_{21}(\xi, s)$, $C_{21}(\xi, s)$, $E_{21}(\xi, s)$ and $F_{21}(\xi, s)$ to the specified borehole surface

boundary conditions (mode 4)

$$\begin{Bmatrix} A_{21}(\xi, s) \\ C_{21}(\xi, s) \\ E_{21}(\xi, s) \\ F_{21}(\xi, s) \end{Bmatrix} = \begin{bmatrix} \beta_{11,21}(\xi, s) & \beta_{12,21}(\xi, s) \\ \beta_{21,21}(\xi, s) & \beta_{22,21}(\xi, s) \\ \beta_{31,21}(\xi, s) & \beta_{32,21}(\xi, s) \\ \beta_{41,21}(\xi, s) & \beta_{42,21}(\xi, s) \end{bmatrix} \cdot \begin{Bmatrix} \Delta \hat{\sigma}_{rr,21}(1, \xi, s) \\ -\frac{\hat{s}_x - \hat{s}_y}{2} \end{Bmatrix}. \quad (\text{B3})$$

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