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Low-temperature anomaly in disordered superconductors near B_{c2} as a vortex-glass property

Benjamin Sacépé^{1,*}, Johanna Seidemann¹, Frédéric Gay¹, Kevin Davenport², Andrey Rogachev², Maoz Ovadia³, Karen Michaeli³, and Mikhail V. Feigel'man^{4,5}

¹Univ. Grenoble Alpes, CNRS, Grenoble INP, Institut Néel, 38000 Grenoble, France

²Department of Physics and Astronomy, University of Utah, Salt Lake City, Utah 84112, USA

³Department of Condensed Matter Physics, The Weizmann Institute of Science, Rehovot 76100, Israel

⁴L. D. Landau Institute for Theoretical Physics, Chernogolovka, 142432, Moscow region, Russia

⁵Skolkovo Institute of Science and Technology, Moscow 143026, Russia

Abstract

Strongly disordered superconductors in a magnetic field display many characteristic properties of type-II superconductivity—except at low temperatures, where an anomalous linear temperature dependence of the resistive critical field B_{c2} is routinely observed. This behavior violates the conventional theory of superconductivity, and its origin has posed a long-standing puzzle. Here we report systematic measurements of the critical magnetic field and current on amorphous indium oxide films with various levels of disorder. Surprisingly, our measurements show that the B_{c2} anomaly is accompanied by mean-field-like scaling of the critical current. Based on a comprehensive theoretical study we argue that these observations are a consequence of the vortex-glass ground state and its thermal fluctuations. Our theory further predicts that the linear-temperature anomaly occurs more generally in both films and disordered bulk superconductors, with a slope that depends on the normal-state sheet resistance, which we confirm experimentally.

The magnetic-field tuned transition of disordered superconductors continues to surprise as well as to pose intriguing and challenging puzzles. A wealth of experimental results obtained over decades of study still defies current theoretical understanding. The anomalous temperature dependence of the resistive critical field $B_{c2}(T)$ near the $T = 0$ quantum critical point (QCP) between superconductor and normal metal is a well known example. Within the

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* benjamin.sacepe@neel.cnrs.fr.

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conventional Bardeen-Cooper-Schrieffer theory, $B_{c2}(T)$ is expected to saturate at low temperatures^{1,2}. In contrast, a strong upturn of B_{c2} with a *linear- T* dependence as $T \rightarrow 0$ has been observed in numerous disordered superconductors. These systems range from alloys and oxides, both in thin films^{3–9} and bulk¹⁰, to boron-doped diamond¹¹ as well as gallium monolayers¹².

Substantial theoretical efforts^{13–17} have been unable to fully resolve the origin of this anomalous behavior. The main challenge lies in the complexity of these systems and the subtle interplay between strong fluctuations, disorder and vortex physics. The prevailing explanation for the low- T anomaly of $B_{c2}(T)$ is based on mesoscopic fluctuations^{13,14}, which result in a spatially inhomogeneous superconducting order parameter. A recent alternative interpretation invokes a quantum Griffiths singularity to account for the upturn in $B_{c2}(T)$ observed in ultra-thin gallium films¹². While these theoretical approaches are generally plausible, they predict an exponential increase of $B_{c2}(T)$ at very low T , which manifestly does not capture the specific *linear* dependence measured in disordered superconductors^{3–12}.

To gain new insights on the underlying physical mechanism it is desirable to not only study $B_{c2}(T)$, but to also extract information on additional characteristic quantities such as the superfluid stiffness. We therefore conducted systematic measurements of both B_{c2} and the critical current j_c in films of amorphous indium oxide (a:InO), a prototypical disordered superconductor. In the absence of magnetic field or when vortices are strongly pinned by disorder (i.e., form a vortex glass) the superfluid stiffness can be directly related to the critical current. Since this is expected to apply to all materials that exhibit the low-temperature B_{c2} anomaly^{3–12}, measurements of $j_c(B, T \approx 0)$ provide access to the critical behavior of the superfluid density $\rho_s(B)$ near the QCP $B_{c2}(0)$.

The key experimental finding of this work is that the linear T -dependence of $B_{c2}(T)$ at low temperatures is accompanied by a *power-law* dependence of the critical current on B , i.e., $j_c(B) \sim |B - B_{c2}|^\nu$ with $\nu \simeq 1.6$. As explained below, this is consistent with the mean-field (MF) value $\nu = 3/2$ (but not with the mesoscopic fluctuation scenario^{13,14} which predicts an exponential dependence¹⁴). Our unexpected finding has direct implication for the critical behavior of $\rho_s(B)$, and demands a revised theory of disordered superconductors in the presence of magnetic field. We therefore complement our experimental work with a comprehensive theoretical study, which identifies the key to understanding the low- T anomaly in the vortex glass. When vortices are strongly pinned by impurities, their presence only weakly affects the $T = 0$ limit of superfluid stiffness and critical current. As a result, both exhibit MF-like dependence on magnetic field. In contrast, the temperature variation of the superfluid stiffness is strongly affected by thermal fluctuations of the vortex glass. This gives rise to the observed linear T -dependence of $B_{c2}(T)$ near the QCP. Moreover, we predict a strong dependence of the slope $dB_{c2}(T)/dT|_{T \rightarrow 0}$ on the sheet resistance, which we confirm experimentally.

Low- T anomaly near $B_{c2}(0)$

In this study we focus on a series of a:InO samples, which exhibit critical temperatures $T_c = 3\text{K} - 3.5\text{K}$ and normal state sheet resistances $R = 2\text{ k}\Omega - 1.2\text{ k}\Omega$ (see Table S1 in SI). Those samples are far from the disorder-tuned superconductor-insulator transition and behave in many ways as standard dirty superconductors⁹. Moreover, to demonstrate the universality of the low- T anomaly, we extended our measurements to amorphous molybdenum-germanium films¹⁸ (MoGe) (for sample characterization see SI)

Figure 1a displays the magneto-resistance isotherms of sample J033 measured down to 0.03 K. We define the critical magnetic field B_{c2} through the resistive transition, i.e., the onset of superconducting phase coherence. This critical field in general differ from the onset of pairing. To determine B_{c2} , we used three different criteria: 1, 10 and 50% of the normal state resistance at high field as indicated by open dots on magneto-resistance isotherms. The resulting B_{c2} versus T curves are shown in Fig. 1b together with a fit (solid-line) of the high-temperature data with the theory for dirty superconductors^{1,2}. We see that $B_{c2}(T)$ deviates below $\lesssim 1\text{K}$ from the fit and increases linearly with decreasing T down to our base temperature of 0.03 K (see SI). This deviation, which is the focus of this work, is independent of the criterion used to determine B_{c2} . Notice that the linear dependence of $B_{c2}(T)$ persists down to our lowest T approaching the quantum phase transition. This is a crucial point in our analysis below.

The critical current density

We turn to the study of the B -evolution of the critical current density j_c at our lowest temperature and focus on three samples J033, ITb1 and J038. We systematically measured the differential resistance dV/dI versus current-bias I at fixed B 's. As shown in Fig. 2a, on increasing I , a sudden, non-hysteretic jump occurs in the dV/dI curve, indicating the critical current. The resulting j_c of the samples is plotted versus B in Fig. 2b. Interestingly, the continuous suppression of j_c with increasing B tails off prior to vanishing at $B_c^{jc} = 12.8\text{ T}$, 12.1 T and 11.9 T for samples J033, ITb1 and J038, respectively. Such a resilience of j_c to the applied B when approaching $B_{c2}(0)$ is reminiscent of the anomalous upturn of the $B_{c2}(T)$ line at low T . We also notice that the critical field B_c^{jc} at which j_c vanishes slightly differ from $B_{c2}(0)$ obtained in Fig. 1 ($B_{c2}(0) = 13.3\text{ T}$, 12.4 T and 12.6 T determined with the 50 % criterion for samples J033, ITb1 and J038 respectively). This stems from the finite-resistance criterion used to determine $B_{c2}(T)$, which does not coincide with the termination of superconducting current at B_c^{jc} .

The key result of this work is shown in Fig. 3 where j_c is plotted in logarithmic scale as a function of $|B_c^{jc} - B|$. By adjusting the value of B_c^{jc} to 12.8 , 12.1 and 11.9 T for samples J033, ITb1 and J038 respectively, one obtains clear straight lines that unveil a scaling relation of the form:

$$j_c(B) = j_c(0) \left| 1 - \frac{B}{B_c^{j_c}} \right|^{\nu}, \quad (1)$$

where the fitted values for the exponent are $\nu_{J033} = 1.62 \pm 0.02$, $\nu_{ITb1} = 1.67 \pm 0.02$ and $\nu_{J038} = 1.65 \pm 0.02$. A similar critical exponent has been obtained for the MoGe sample (see SI). The prefactor $j_c(0)$ falls in the range $(2.5 - 4) \cdot 10^3 \text{ A/cm}^2$ for all three samples. The inset of Fig. 3 shows the sensitivity of the straight line to $B_c^{j_c}$, where a small variation of 0.05T yields a significant deviation from linearity. It is noteworthy that the values of the exponent are within 10% of the MF value 3/2 of the classical temperature-dependent critical current. MF theory predicts $j_c^{GL} \propto \rho_s / \xi_{GL} \propto (T_c - T)^{3/2}$, with the Ginzburg-Landau superconducting coherence length $\xi_{GL} \propto (T_c - T)^{-1/2}$ and $\rho_s \propto (T_c - T)$. This striking similarity suggests that both the scaling of $B_{c2}(T)$ and $j_c(B)$ low temperature may be captured by MF theory of the bulk material.

Interpretation of experimental results within MF theory

The MF critical exponent of $j_c(B)$ at $T = 0$ near $B_{c2}(0)$ can be extracted from the Ginzburg-Landau free energy:

$$F = \alpha |\Delta(\mathbf{r})|^2 + \beta |\Delta(\mathbf{r})|^4 + \gamma \left| \left(-i\nabla - \frac{2e}{\hbar c} \mathbf{A}(\mathbf{r}) \right) \Delta(\mathbf{r}) \right|^2. \quad (2)$$

Within MF theory, the coefficient α strongly depends on temperature and magnetic field¹⁹:

$$\alpha = \nu \left[\ln \frac{T}{T_{c0}} + \psi \left(\frac{1}{2} + \frac{eDB}{2\pi cT} \right) - \psi \left(\frac{1}{2} \right) \right], \quad (3)$$

while β and $\gamma \propto \nu D$ depend only weakly on either. Here, D and ν are the electron diffusion coefficient and density-of-states respectively, and $\psi(x)$ is the digamma function. B in Eq. (3) is the magnetic field penetrating the superconductor. While for type-II superconductors such as our a:InO films this magnetic field can be non-uniform at low B , close to B_{c2} the spatial fluctuations of B are negligible, and the average magnetic field is equal to the externally applied one. Consequently, Eq. (2) captures the two effects of magnetic field on superconductors: The suppression of the transition point due to pair-breaking (through the parameter α), and the diamagnetic response captured by the vector potential $\mathbf{A}$. Note that the expression for α in Eq. (3) is typical for superconductors in the presence of a pair-breaking mechanism²⁰. The MF treatment is performed under the assumption that vortices are strongly pinned. As a result, the presence of magnetic field induced vortices can be neglected, and j_c is proportional to the depairing current. A detailed explanation of the origin and the consequences of strong pinning is given in the next section.

The $T = 0$ limit of Eq. (2) yields the magnetic-field dependence of the order parameter and the coherence length near the QCP, $| \langle B \rangle | \sim | B - B_{c2} |^{1/2}$ and $\xi_{GL}(B) \approx \xi_0 | 1 - B/B_{c2} |^{-1/2}$ (where $\xi_0 \approx 5$ nm in our a:InO samples⁹.) The latter, together with ρ_s determines the MF value of the critical de-pairing current: $j_c^{GL} \propto \rho_s / \xi_{GL}$. To find the superfluid stiffness, we match the superconducting current extracted from the free energy, $\mathbf{j} = -c \nabla F / A$, with the London equation, $\mathbf{j} = -4\rho_s e^2 A / \hbar^2 c$. We find that

$$\rho_s(B) = \frac{12}{\pi} \rho_{s0} \left(1 - \frac{B}{B_{c2}(0)} \right), \quad (4)$$

and the relation between the superfluid stiffness and the critical current yields

$$j_c^{GL}(B) = \frac{8e}{3\sqrt{3}\pi\hbar\rho_s(B)\xi_{GL}(B)} = j_c^{GL}(0) \left(1 - \frac{B}{B_{c2}} \right)^{3/2}. \quad (5)$$

The corresponding critical exponent $\nu = 3/2$ is in excellent agreement with our experimental findings. The prefactor $j_c^{GL}(0)$ can be estimated using the experimental data of Ref. 21, where the superfluid stiffness of 20 nm thick a:InO films was measured. From their experimental results at low magnetic field, we estimate the critical current to be $j_c^{GL}(0) \sim 10^4$ A/cm² — larger than our experimentally observed values only by a factor of ~ 4 . This is a non-trivial observation which is in contrast to a weakly pinned vortex state where the critical current is set by the de-pinning current $j_c^{\text{depin}} \ll j_c^{GL}$. It also provides an important hint to the origin of the anomalous critical magnetic field and, as we show below, follows naturally from our theory.

To complete the comparison between the experimental results and MF theory, we extract the low-temperature critical field from the condition $\alpha(B, T) = 0$. Within MF theory, the resistive critical magnetic field coincides with the onset of pairing, and $B_{c2}(0) - B_{c2}(T) \sim T^2$. This is inconsistent with the experimentally observed linear dependence $B_{c2}(0) - B_{c2}(T) \sim T$ (see Fig. 2). Power-laws with exponent smaller than two are known to arise in strongly correlated superconductors^{22–26}. In the context of high T_c cuprates, the deviation from MF result has been attributed²⁴ to an extended region of strong fluctuations around $B_{c2}(T)$. However, in conventional superconductors, such as the a:InO films studied here, the Ginzburg-Landau theory is expected to describe the onset of pairing at all temperatures and magnetic fields²⁴. Indeed, MF theory captures the scaling of the critical temperature with magnetic field at low B . This indicates that to understand the low temperature behavior of $B_{c2}(T)$ other beyond-MF effects should be considered. In particular, thermal fluctuations of the vortex-glass, which are essential in the finite-temperature transition to the normal state. As we show below, these fluctuations are the key ingredient to understanding the linear- T dependence of B_{c2} ; however, they do not change the scaling behavior of $j_c(B, 0)$.

Vortex-glass fluctuations

In low-dimensional superconductors the pairing instability is known to differ from the onset of phase coherence. A prominent example is the Berezinskii-Kosterlitz-Thouless (BKT) transition in thin films^{27,28}. Similar decoupling occurs in moderately-disordered type-II superconductors near B_{c2} , where the magnetic field gives rise to the formation of a weakly pinned vortex lattice^{29,30}. There, the superconducting state becomes resistive when the force applied on the vortex lattice by the current exceeds the pinning forces. The corresponding de-pinning current j_c^{depin} is significantly lower than the pair-breaking critical current extracted from the Ginzburg-Landau theory, j_c^{GL} , and it is not expected to obey simple scaling behavior^{30,31} close to B_{c2} .

In contrast, in highly disordered superconductors such as our a:InO films, we expect a strongly pinned vortex-glass to form^{30,32}. This is caused by large spatial fluctuations of the order parameter^{33,34}, which are predicted by the theory of "fractal" superconductors³⁵. According to this theory,³⁵ in the absence of a magnetic field, the superconducting condensation energy E_g fluctuates strongly in space, $\delta E_g \sim E_g$, over distances comparable to the coherence length ξ . Correspondingly, core energies of vortices, which are induced by an applied magnetic field, exhibit similar fluctuations, and hence become strongly pinned. In fact, vortex pinning in such systems resembles the one found in models of columnar defects³⁶. Upon applying a current, vortices de-pin only when the superconducting order parameter is sufficiently reduced, i.e., within MF theory j_c^{depin} scales like the Ginzburg-Landau de-pairing current. Thus, in our systems $j_c^{\text{depin}} = \Upsilon j_c^{GL}$ where $\Upsilon < 1$ (for example, in the model of Ref.36, $\Upsilon \approx 1/3$), and the I-V curves are expected to follow those studied theoretically in Ref. 37. Consequently, Eq. (5) still applies even in the presence of vortices.

The above analysis implies that the MF values of the critical field and current in highly disordered superconductors are modified primarily by long-wavelength fluctuations of the vortex glass³². As we show in detail in the Methods, this results in renormalization of the superfluid stiffness, which at low temperatures takes the form

$$\delta\rho_s(T, B) = \rho_s(T, B) - \rho_s(0, B) = -C \frac{\hbar\sigma_n}{e^2} \frac{T^2}{3\pi\rho_s(B)a_0}. \quad (6)$$

Here, σ_n is the normal state conductance, a_0 is inter-vortex spacing, and C is a material specific coefficient of order unity. The last equality holds in the low-temperature limit $T \ll \pi\rho_s(B)a_0$; in the opposite limit one recovers the classical result $\delta\rho_s(T, B) \propto T$.

Thermal fluctuation corrections to the critical current

To substantiate the correction to the superfluid stiffness given in Eq. (6), we conducted additional measurements of $j_c(T)$ in the vicinity of $B_{c2}(0)$. The differential resistance dV/dI as a function of current measured on sample ITb1 at various temperatures and fixed $B = 11.25$ T is shown in Figure 4a. A clear jump in the resistance, similar to that in Fig. 1a,

develops at ultra-low temperatures and indicates the position of j_c . At higher temperatures, j_c decreases and a non-zero resistance is found already before the jump. This resistance rises above the noise level for $T > 0.05$ K, and exhibits a clear exponential increase with the current, which is highlighted by the black dashed-line in this semi-log plot. Such resistance curves are expected to be observed when the vortex glass is strongly pinned: The resistance at low current is a typical signature of vortex creep, where the Lorentz force induced by the current reduces the barrier. Above j_c , the current-voltage characteristics show an excess current^{38,39} (see SI), which implies that thermal creep persists there, in agreement with recent strong pinning theory^{37,40}. At temperatures above 0.07 K strong thermal fluctuations cause the sharp jump to be replaced by a smooth crossover³⁹. Moreover, the resistance jump that is seen here only at very low T points to a collective de-pinning of the vortex-glass. Notice that Joule overheating is in play here but mainly above the critical current (see detailed analysis in SI).

As we showed before, the (zero-temperature) B -dependence of the critical current scales like j_c^{GL} near $B_{c2}(0)$, indicating that $j_c^{\text{depin}} \propto j_c^{GL}$. Correspondingly, we expect the T -dependence of the critical current to be determined by the thermal corrections to the superfluid stiffness

$$\delta j_c^{GL}(T, B) \propto \frac{\delta \rho_s(T, B)}{\xi_{GL}} \propto \frac{T^2}{\sqrt{B_{c2}(0) - B}} \quad (7)$$

To test this predicted scaling of j_c , we measured additional resistance curves at different magnetic fields (see SI). The critical current near $B_{c2}(0)$ extracted from the jump in the resistance is plotted as a function of temperature in Fig. 4b. The dashed lines are fits of the $j_c(T, B)$ data to Eq. (7), which were performed by setting $B_{c2}(0) = 12.1$ T (deduced from Fig. 3), by adjusting the $T = 0$ value $j_c(0, B)$ for each B and finding one global pre-factor. The fit reproduces remarkably well the T -dependence of the data for the $B = 10.5, 1.75$ and 11 T, confirming the T^2 correction to the critical current as well as its B -dependence. Deviations from the fit occur for the highest T data points as well as in the immediate vicinity of $B_{c2}(0)$. This is not surprising since our theoretical derivation of the correction to the superfluid stiffness given in Eq. (6) is valid so long as the fluctuations are small $\delta \rho_s(B, T) \ll \rho_s(B)$. The thermal fluctuations, however, become strong at lower T as the magnetic field is increased.

The excellent agreement between theory and measurement has important implications: Together with the observation of the MF scaling of the critical current shown in Fig. 3, it confirms that the vortex depinning current, which causes the jump in the resistance, is indeed proportional to the de-pairing critical current j_c^{GL} . Moreover, this result validates our prediction for renormalization of the superfluid stiffness by thermal fluctuations of the vortex glass, and suggests that this should affect other observables such as magnetotransport near the QCP. In the following we show these fluctuations can account for the linear upturn of $B_{c2}(T)$ at low temperature.

Theory for the low-temperature anomaly

The boundary between superconductor and normal state follows from $\rho_s(T, B) \approx 0$ in bulk systems, or from the BKT formula in films. As we show in the Methods, combining these conditions with the renormalized ρ_s given by Eq. (6) yields the scaling of the transition temperature with magnetic field as a function of thickness

$$1 - \frac{B_{c2}(T)}{B_{c2}(0)} = \left[1 + \sqrt{1 + C_1^2 \frac{d^2}{\epsilon a_0^2 \chi^2}} \right] \frac{\pi \chi T}{24 \rho_{s0} d}, \quad (8)$$

where $C_1^2 = 4C\hbar\sigma_n a_0/3\pi e^2$ and ϵ are numerical coefficients of order unity, d is the film thickness and $\chi^{-1} \approx 2$. We thus obtain a linear temperature dependence of $B_{c2}(T)$ at low T , which well describes the experimental data shown in Fig. 1b. In addition, this expression agrees with numerous experiments in films^{3–9,12} and bulk^{10,11} disordered superconductors. Eqs. (6) and (8) are the main theoretical results of this work.

Finally, we provide a quantitative comparison between the theoretical prediction and experimental data from eight samples of various thicknesses and resistances. To eliminate the non-universal dependence of T_c and $B_{c2}(0)$ on disorder, we plot in Figure 5 $b(t) = B_{c2}(T)/(-T_c dB_{c2}/dT|_{T=T_c})$ versus the reduced temperature $t = T/T_c$, as well as the theoretical MF curve (solid line). We see that, for $t \gtrsim 0.2 - 0.3$, all high-temperature data collapse on the theoretical curve. At smaller t , the low-temperature anomaly of B_{c2} develops as a linear deviation from the MF curve. Our theory explains the anomalous slope in this regime and, as we show below, captures the dependence on the sample parameters.

To compare the plot in Fig. 5 with our theory, we extract $b(t)$ from Eq. (8) and find:

$$-\frac{db}{dt} = \begin{cases} KR_{\square}/R_Q & d \ll a_0 \\ \tilde{g}_0 \sqrt{\frac{1}{\sigma_n R_Q a_0}} + \tilde{K} \frac{R_{\square}}{R_Q} & d \gg a_0 \end{cases}, \quad (9)$$

where $R_Q = h/4e^2$ is the quantum of resistance for electron pairs (for details see the Methods). Neglecting higher order corrections to ρ_s beyond Eq. (8), and setting $\rho_{s0}/T_{c0} = 1.76\pi\hbar\sigma_n/4e^2$ yields $K = 0.4\pi \approx 1$, $\tilde{K} \approx 0.1$, and $\tilde{g}_0 \approx 0.05\sqrt{C}$. The slopes $db(t)/dt|_{t=0}$ of four samples: 30 nm, 50 nm, and two 60 nm thick films are shown in the inset of Figure 5. We clearly see that the slope of the linear- T anomaly increases with sheet resistance as predicted by Eq. (9), demonstrating the consistency of our theoretical description. Furthermore, the linear dependence of $db(t)/dt|_{t=0}$ on the sheet resistance does not extrapolate to zero in the bulk limit ($d \rightarrow \infty$), confirming again our theoretical result Eq. (9). However, fitting the data to Eq. (9) gives $-db/dt = 0.4R_{\square}/R_Q + 0.44$, i.e., $\tilde{g}_0 \approx 0.45$ and $\tilde{K} \approx 0.4$. These values exceed the ones found via our simplified approximation by a factor of about 4-10. Partially, it might be due to the overestimation of the ratio ρ_{s0}/T_{c0} , see the Methods. Obtaining the

correct numerical coefficients presumably requires including corrections neglected above, which is beyond the scope of this work. A fully quantitative analysis should be based upon extension of the theory of classical gauge glasses^{41–43} to the quantum limit.

In conclusion, we have conducted a systematic study of the critical current near $B_{c2}(T=0)$ in disordered a:InO and MoGe films, and uncovered a power-law dependence of j_c on magnetic field. We have shown theoretically that the behavior of *both* $B_{c2}(T)$ and $j_c(B)$ can be attributed to properties of the vortex-glass, which is characteristic state of disordered films in the presence of magnetic field. Although this mechanism is quite generally applicable for any disordered superconductor, both 2D and 3D, the magnitude of the effect is appreciable for superconductors with low superfluid density, $\rho_s \xi_0 \ll T_{c0}$, where phase fluctuations are strong⁴⁴. Our analysis provides sharp predictions for $B_{c2}(T)$ and $j_c(B)$ which allows a clear distinction between the physical mechanisms in play. It would be very interesting to look for similar scaling in other superconducting films where a similar B_{c2} anomaly has been observed^{3–8,10–12}. Moreover, the theory developed here relates the anomaly to the absence of any vortex lattice order. Decay of vortex-correlations over a very short length may have already been experimentally observed in amorphous W-based thin films⁴⁵, however the connection to the quantum critical behavior needs to be further investigated.

Methods

Sample fabrication and measurement setup

Disordered a:InO films were prepared by e-gun evaporation of 99.99 % In_2O_3 pellets on a Si/SiO₂ substrate in a high-vacuum chamber with a controlled O₂ partial pressure. Films were patterned into Hall bar geometry (100 μm wide for all samples except ITb1 which is 20 μm wide) by optical lithography, enabling four-terminal transport measurements using standard low-frequency lock-in amplifier and DC techniques. Measurements were performed in dilution refrigerators equipped with superconducting solenoid. Multi-stage filters, including feed-through π -filters at room temperature, highly dissipative shielded stainless-steel twisted pairs down to the mixing chamber, copper-powder filters at the mixing chamber stage, and 47 nF capacitors to ground on the sample holder, were installed on each dc lines of the fridge. This careful filtering of low and high frequency noises was crucial to measure well defined critical currents as low as ~ 50 nA (see Fig. 4), which would have been otherwise disguised by spurious noise. Furthermore, a calibrated RuO₂ thermometer was installed directly on the sample holder to precisely monitor the sample temperature. This accurate thermometry eliminates small temperature gradients below 0.1 K.

Derivation of the renormalized superfluid stiffness

To estimate the renormalization of the superfluid stiffness, ρ_s , we focus exclusively on phase-fluctuations of the order parameter, i.e., $\langle \Phi(\mathbf{r}) \rangle = \langle \int_0^{2\pi} e^{i\Phi(\mathbf{r})} d\theta \rangle$. Inserting this into the free energy (Eq. 2 in the main text) yields $F[\Phi] = \rho_s \left[\nabla \Phi(\mathbf{r}) - \frac{2e}{\hbar c} A(\mathbf{r}) \right]^2 / 2$. It is convenient to further separate $\Phi(\mathbf{r})$ into smooth phase fluctuations (the superfluid mode) $\varphi(\mathbf{r})$ with $\nabla \times \nabla \varphi = 0$ and fluctuations of the vortex-glass $\psi(\mathbf{r})$, with $\Phi(\mathbf{r}) = \varphi(\mathbf{r}) + \psi(\mathbf{r})$. We determine the renormalized superfluid stiffness via the static current-current correlation function

$\langle J_{\text{sc}}^i(\mathbf{r}, \omega_n = 0) \rangle = \int d\mathbf{r} \langle J_{\text{sc}}^i(\mathbf{r}, 0) J_{\text{sc}}^j(0, 0) \rangle A^j(0, 0)$, where as before $\mathbf{J}_{\text{sc}}(\mathbf{r}) = -c \nabla \times \mathbf{A}(\mathbf{r})$. Since ρ_s is determined by the long-wavelength properties, it is sufficient to focus on length-scales larger than the (typical) inter-vortex spacing a_0 . Moreover, within such a coarse-grained description, the coupling between superfluid and vortex fluctuations is local.

We note that a charge encircling a vortex acquires phases from both the external and the vortex field, $\oint [\nabla \psi(\mathbf{r}) - 2e/\hbar c \mathbf{A}(\mathbf{r})] \cdot d\mathbf{l}$. If the vortices were uniformly spaced (a vortex lattice), the two contributions to the phase would cancel at a length scale a_0 . In the coarse grained description and in the symmetric gauge for the vector potential, this amounts to $\nabla \psi(\mathbf{r}) = \mathbf{A}(\mathbf{r})$. In a vortex-glass this cancellation is not exact. Still, near B_{c2} and at the length scales of $a_0 = \sqrt{\Phi_0/B} \approx \xi_0 \sqrt{2\pi}$, the density of vortices is nearly uniform. We introduce the field $\mathbf{R}(\mathbf{r}) = (R_x(\mathbf{r}), R_y(\mathbf{r}))$ that describes the deviation of the vortex positions from uniformity. $\mathbf{R}$ thus encodes the particular realization of the vortex-glass, and, in the absence of forces, its spatial average vanishes, $V^{-1} \int d\mathbf{r} \mathbf{R}(\mathbf{r}) = 0$. It is thus appropriate to expand

$$\begin{aligned} \nabla \psi(\mathbf{r} - \mathbf{R}(\mathbf{r})) - \frac{2e}{\hbar c} \mathbf{A}(\mathbf{r}) &\approx -\frac{2e}{\hbar c} (\mathbf{R}(\mathbf{r}) \cdot \nabla) \mathbf{A} \\ &= \frac{\mathbf{u}(\mathbf{r}) \times \hat{z}}{2a_0}, \end{aligned} \quad (10)$$

where $\mathbf{u}(\mathbf{r}) = 2\pi \mathbf{R}(\mathbf{r})/a_0$ is the dimensionless displacement field. It follows that the leading coupling terms in the free energy between φ and $\mathbf{u}$ are

$$\delta F = \rho_s a_0^{-1} \hat{z} \cdot [\nabla \varphi(\mathbf{r}) \times \mathbf{u}(\mathbf{r})] - C \rho_s (\nabla \varphi(\mathbf{r}))^2 \mathbf{u}^2(\mathbf{r}), \quad (11)$$

where C is a material specific coefficient of order unity. Note that here we assume isotropy in the $x - y$ plane.

The first term in Eq. (11) corresponds to the lowest order expansion with respect to gradients in the Ginzburg-Landau free energy (Eq. (2) in the main text). It gives rise to a temperature and magnetic-field independent correction to the superfluid stiffness that depends on the realization of the vortex-glass. This reduction enters the measurable quantity ρ_{s0} which we treat as a phenomenological parameter. As we show below, higher-order gradients (like the second term in Eq. (11)) become important at non-zero temperature. The leading contribution of such terms to ρ_s is given by

$$\delta \rho_s^{x,y} = -\frac{C \rho_s}{a_0^3} T \sum_n \int d\mathbf{r} \langle \mathbf{u}(\mathbf{r}, \omega_n) \cdot \mathbf{u}(0, -\omega_n) \rangle. \quad (12)$$

It remains to evaluate the $\mathbf{u}-\mathbf{u}$ correlation function. In the strong pinning regime, a restoring force acts to keep the vortex structure near its local energy minimum. Consequently, it is

sufficient to reduce the equation of motion to its local form for $\mathbf{u}(\mathbf{r}, t)$, which describes individual vortices³⁰, and in addition neglect spatial gradients of the $\mathbf{u}(\mathbf{r})$ field

$$\eta \partial_t \mathbf{u}(\mathbf{r}, t) + \kappa (\mathbf{u}(\mathbf{r}, t) - \mathbf{u}_0(\mathbf{r})) = \mathbf{f}(t) \equiv \frac{\hbar a_0}{2e} \mathbf{j} \times \hat{\mathbf{z}}, \quad (13)$$

where $\mathbf{u}_0(\mathbf{r})$ is the static displacement at zero current. The r.h.s. of Eq.(13) is the Lorentz force acting on a segment of a vortex of length a_0 in presence of a supercurrent $\mathbf{j}$. We emphasize the absence of gradients of $\mathbf{u}$ in the equation of motion—this manifests the locality of the vortex dynamics. Effectively, the vortex fluctuations in the glass state at low temperature resemble a (damped) optical phonon mode. The parameter $\eta = (\hbar/2e)^2 \sigma_n / 2\pi = \frac{\pi \hbar^2}{2e^2} \sigma_n$ reflects the presence of normal electrons in the vortex cores, whose resistance σ_n^{-1} gives friction to the vortex motion (note that we defined friction coefficient η w.r.t. dynamics of dimensionless coordinate $\mathbf{u} = 2\pi \mathbf{R}/a_0$). For strongly disordered materials $\sigma_n a_0 \sim e^2/\hbar$ and thus $\eta \sim \hbar/a_0$. This overdamped character of the vortex motion leads us to neglect the inertial term $\propto \partial_t^2 \mathbf{u}$ with respect to friction. Likewise, we neglect the contribution to the Lorentz force due to vortex velocity since it does not affect the relevant current-current correlation function. The parameter κ can be determined similar to the penetration length in pinned vortex systems, also known as the Campbell length^{46–48} (for a recent review see Ref.49). According to Eq.(13), the shift of $\mathbf{u}$ due to current $\mathbf{j}$ is equal to $\mathbf{u}(\mathbf{r}) - \mathbf{u}_0(\mathbf{r}) = \hbar a_0 (\mathbf{j} \times \mathbf{n}) / 2e\kappa$. The corresponding shift of the vortex magnetic flux can be expressed through $\delta A = -\hbar \Phi_0 \mathbf{j} / 4e\kappa a_0$. Using the London relation, we get $\kappa = \pi \rho_s$. From Eq. (13) we obtain the Matsubara Green's function of $\mathbf{u}$

$$G(\mathbf{r}, \omega_n \neq 0) = a_0^2 \delta(\mathbf{r}) [\eta |\omega_n| + \kappa]^{-1}. \quad (14)$$

We thus arrive at the reduction of the superfluid stiffness due to thermal fluctuations of the vortex-glass. Combining Eqs. (12) and (14) yields the following correction to the superfluid stiffness $\delta \rho_s(T, B) = \rho_s(T, B) - \rho_s(0, B)$:

$$\begin{aligned} \delta \rho_s(T, B) = & - \left[T \sum_n [\eta |\omega_n| + \kappa]^{-1} \right. \\ & \left. - \int \frac{d\omega}{2\pi} [\eta |\omega| + \kappa]^{-1} \right] = -C \frac{\hbar \sigma_n}{e^2} \frac{T^2}{3\pi \rho_s(B) a_0}. \end{aligned} \quad (15)$$

The last equality holds in the low-temperature limit $T/\hbar \ll \kappa/\eta$, in the opposite limit one recovers the classical result $\langle \mathbf{u}^2 \rangle = T/\kappa$. The smallness of the low temperature result $\delta \rho_s(T, B) \propto T^2$ is compensated by the small $\rho_s(B) \propto 1 - B/B_{c2}(0)$ in the denominator, see Eq. (4) in the main text. This new result is unique for disordered superconductors, in which the

fluctuations at $T > 0$ are controlled by the dissipation in the gapless vortex cores, and it provides the key to understanding the low- T linear upturn of B_{c2} .

Derivation of the critical field at low temperature

The superconducting transition in the bulk limit can be estimated from the condition $\rho_s(T, B) \approx 0$, or equivalently when $\delta\rho_s(T, B) = \epsilon\rho_s(0, B)$, with ϵ being a number of the order unity (similar to the Lindemann criterium for melting of solids). Under this condition and using Eqs. (4) and (6) from the main text, we find

$$1 - \frac{B_{c2}(T)}{B_{c2}(0)} = C_1 \frac{\pi T}{24\rho_{s0}a_0\epsilon^{1/2}}, \quad (16)$$

where $C_1^2 = 4C\hbar\sigma_n a_0/3\pi e^2$ is of order unity for our a:InO films. In films, the transition temperature is set instead by the BKT condition:

$$\rho_s(B, T_{\text{BKT}}) = \frac{\chi}{d} T_{\text{BKT}}(B) \quad (17)$$

where d is the film thickness and χ^{-1} was numerically found to be between 1.5 to 2.2 (this holds for *any value of* d 50–53). Moreover, kinetic inductance measurements of thin a:InO films have observed a universal jump in the two-dimensional superfluid density per square even in the presence of a magnetic field, indicating that the BKT transition persists near the quantum critical point²¹. We note that Eq. (17) in the limit $d \rightarrow \infty$ reproduces the condition for the bulk transition [$\rho_s(T, B) \approx 0$], and thus this equation describes the scaling of $B_c(T)$ for any d . Thus, combining the BKT condition with the renormalized ρ_s given by Eq. (6) yields Eq. (8) of the main text.

We emphasize that the linear- T dependence of $B_{c2}(T)$ does not depend on the sample dimensionality and, in particular, remains valid for bulk superconductors. For films, $|\delta\rho_s(B, T)/\rho_s(B)|$ grows with the film thickness d , however it remains much below unity so long as $d \ll a_0$ (As is the case for the experiments reported in Ref.12). In this thin limit our theory predicts that the slope $-dB_{c2}(T)/dT$ grows linearly with $(\rho_{s0}d)^{-1}$. For thicker films, however, $|\delta\rho_s(B, T)/\rho_s(B)|$ increases and higher order corrections to $\rho_s(B)$ can become important. Still, $dB_{c2}(T)/dT|_{T \rightarrow 0}$ maintains the structure $(\rho_{s0}a_0)^{-1}(g_0 + g_1 a_0/d)$ with numerical coefficients $g_{0,1}$ that (may) deviate from those given in Eq. (8). To conclude, while our analysis provides an exact expression for the slope only in the thin-film limit, it predicts a distinct d -dependence that holds for any thickness.

Derivation of the normalized slope $db(t)/dt|_{t \rightarrow 0}$

Here we present the transformation between Eq.(8) and Eq.(9) of the main text. Inserting Eq. (8) into $b(t) = B_{c2}(T)/(-T_c dB_{c2}/dT|_{T=T_c})$ yields

$$\frac{b(0) - b(t)}{t} = \left[1 + \sqrt{1 + \frac{C_1^2 d^2}{\epsilon a_0^2}} \right] \frac{\pi \chi}{24 \rho_{s0} d} \frac{B_{c2}(0)}{|dB_{c2}/dT|_{T=T_c}}. \quad (18)$$

It remains only to determine B_{c2} in two limits, $T \rightarrow 0$ and $T \rightarrow T_c$, starting with the latter. As discussed in the main text, the transition in the films is of the BKT type with the condition $\rho_s(T, B_c) = \chi T/d$. Near T_c the superfluid stiffness of a disordered superconductor is given by⁵⁴

$$\rho_s(T, B) = \frac{\pi \sigma_n \hbar}{8 T e^2} |\Delta(T, B)|^2. \quad (19)$$

The gap $\Delta(T, B)$ near T_c can be found by minimizing the free energy in Eq.(2) of the main text, without the gradient term: $|\Delta|^2 = \alpha/2\beta$. Here α is given by Eq.(3) of the main text, which we expand to first order in B , and $\beta = \nu \psi^{(2)}(1/2)/32 \pi^2 T^2$ [$\psi^{(2)}(x) = d^2 \psi(x)/dx^2$ is the second derivative of the digamma function]. Thus, in this limit the superfluid stiffness as a function of temperature and magnetic field is

$$\rho_s(T, B) = \frac{2\pi^2 T \sigma_n \hbar}{e^2 \psi^{(2)}(1/2)} \left[\ln \frac{T}{T_{c0}} + \psi' \left(\frac{1}{2} \right) \frac{e B D}{2\pi c T} \right]. \quad (20)$$

Inserting this expression into the BKT condition and taking a derivative with respect to the temperature yields

$$\left| \frac{dB_{c2}(T)}{dT} \right|_{T=T_c} = \frac{2\pi \hbar c}{e D \psi'(1/2)}. \quad (21)$$

The critical magnetic field at zero temperature, in contrast, is determined by the mean field condition $\alpha = 0$

$$B_{c2}(T=0) = \frac{2\pi \hbar c}{e D} T_{c0} e^{\psi(1/2)}. \quad (22)$$

Using the expressions in Eqs. (21) and (22), we can write Eq. (18) as

$$\frac{b(0) - b(t)}{t} = \left[1 + \sqrt{1 + \frac{C_1^2 d^2}{\epsilon a_0^2}} \right] \frac{\pi \chi T_{c0}}{24 \rho_{s0} d} \psi' \left(\frac{1}{2} \right) e^{\psi(1/2)}. \quad (23)$$

The final step in estimating $b(t)$ is to find the ratio of ρ_{s0} to the mean-field transition temperature T_{c0} . For moderately disordered superconductors, in the absence of any pair-breaking mechanism, semiclassical theory yields^{1,54}

$$\rho_{s0} = \pi \Delta \hbar \sigma_n / 4e^2 = 1.76\pi \hbar \sigma_n T_{c0} / 4e^2. \quad (24)$$

We emphasize that the ratio ρ_{s0}/T_{c0} is expected to be reduced for strongly disordered superconductors^{21,35,55} with respect to the semiclassical formula in Eq. (24).

For ultra-thin films with $d \ll a_0$ we find, using Eqs. (23) and (24),

$$\frac{b(0) - b(t)}{t} = 0.8 \frac{e^2}{\hbar \sigma_n d} = 0.4\pi \frac{R_{\square}}{R_Q} \equiv K \frac{R_{\square}}{R_Q}. \quad (25)$$

Here $R_Q = h/4e^2$ is the quantum of resistance for electron pairs and we used $\chi = 2/\pi$, as appropriate for the two-dimensional limit. Since this result relies on Eq. (24) that overestimates the ratio ρ_{s0}/T_{c0} , we expect to experimentally observe larger values for K . For thick films with $d \gg a_0$, Eqs. (23) and (24) imply that $|db/dt|$ grows linearly with $R_{\square}/R_Q$ i.e.,

$$\frac{b(0) - b(t)}{t} = \tilde{g}_0 \sqrt{\frac{e^2}{\hbar \sigma_n a_0}} + \tilde{K} \frac{R_{\square}}{R_Q}, \quad (26)$$

where $\tilde{g}_0 \approx 0.05\sqrt{C}$, and $\tilde{K} \approx 0.1$. Note that $|db/dt|$ does not extrapolate to zero as $R_{\square} = 1/\sigma_n d \rightarrow 0$. Similar to the result in the thin-film limit, the true numerical values of both $\tilde{g}_0$ and $\tilde{K}$ are expected to be larger due to the lower value of ρ_{s0}/T_{c0} . Moreover, as explained in the main text, the coefficients $\tilde{g}_0$ and $\tilde{K}$ can deviate from the values given by straightforward expansion of Eq. (23) for large d/a_0 . Such a deviation would reflect higher-order corrections to $\rho_s(T, B)$, which may become important in thick films.

Supplementary Material

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FIG. 1. Low- T anomaly of the upper critical field $B_{c2}(T)$.

a, $R_{\square}$ versus B measured at fixed temperatures for sample J033. Open circles indicate the determination of $B_{c2}(T)$ using three different criteria, namely 50, 10 and 1% of the high field normal state resistance. **b**, Extracted $B_{c2}(T)$ values from the measurement shown in **a**, plotted versus T . The solid line is a high temperature fit using the theory for dirty superconductors^{1,2}.

FIG. 2. Critical current density j_c near $T = 0$ and $B_{c2}(0)$.

a, dV/dI versus I of sample J033 measured at $T = 0.03$ K and at magnetic fields of 10.4, 10.8, 11.2, 11.6, 12, 12.5 and 13 T (green to red curves respectively). **b**, j_c versus B of both samples J033, ITb1 and J038. Each values of $j_c(B)$ were extracted from dV/dI versus I measurements as shown in **a** at the resistance value of $dV/dI = 10 \Omega/\square$.

FIG. 3. Scaling of the critical current density with magnetic field.

j_c versus $|B_c^{j_c} - B|$. The $B_c^{j_c}$ values are adjusted to obtain straight lines that are emphasized by black solid-lines. Inset: the dark grey and light grey curves are the data of sample J033 plotted with $B_c^{j_c} \pm \delta$, where $\delta = 0.05$ T.

FIG. 4. Vortex de-pinning and thermal creep.

The differential resistance dV/dI versus I of sample IT1b measured at $B = 11.25$ T at different temperatures is plotted in panel **a**. The black dashed line indicates the exponential increase of the differential resistance with the current, which is caused by vortex creep. The critical current density j_c as a function of T and B is shown in panel **b**. Each value of $j_c(B)$ was extracted from the differential resistance curves, similar to the one shown in (a), by finding the threshold to the high resistance state. Dashed lines are fits using Eq. (7) with $B_{c2}(0) = 12$ T and adjusting $j_c(T=0, B)$ for each curve using a single prefactor for all B 's.

FIG. 5. Disorder dependence of the low- T anomaly.

$b = B_{c2}(T)/(-T_c \cdot dB_{c2}/dT|_{T=T_c})$ versus reduced temperature $t = T/T_c$ for 8 samples of different thicknesses. The solid gray line is a fit using the MF theory for dirty superconductors emphasizing the deviations at low T . Inset: Slope $-db/dt$ at zero temperature versus $R_{\square}/R_Q$. Error bars indicate the standard deviation of the linear fit of $b(t)$ at low temperatures.