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Constructive realizable multiresolution wavelet-like systems based on multi-windows spline-type spaces

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Abstract

In this paper, we present a constructive realizable procedure for obtaining a multiresolution wavelet-like system in the framework of stability for spline-type spaces, considering only the generators that prominently intersect the spectrum of the signal to be approximated and stable under deformations through frequency shifts. The realizable system constructed in this way, can analyse jointly three different features of a signal namely time, scale and frequency. We present numerical tests and a detailed complexity analysis to compare the performances of our method against standard constructions.

I Introduction

Wavelet analysis has shown since its early discovery prominent application in signal analysis [1]. The possibility to employ a multiscale analysis became a milestone for compression and learning of a signal. On the other hand Gabor systems are used to analyse the joint time-frequency features of a function [2]. Both can be reformulated as particular cases of shift invariant systems (SI) [3], [4]. We prefer to use a more general approach in here, defining SI spaces as a particular case of spline-type spaces (ST). SI spaces are defined over $\mathbb{R}^d$ [5], [6], [7], [8], [9], while ST spaces are their natural generalization over locally compact groups. This paper has only a short introductory exposition since we are mainly concern with two numerical aspects: establishing the stability of the proposed algorithms and overcoming the problems of standard orthogonal expansion in multiresolution analysis[10]. The structure of this paper is as follows: in the Section II the mathematical preliminaries and the connection between different functions systems is established, in Section III we present the spline-type reformulation of multiresolution-like analysis including a detailed ST multiresolution algorithm and its complexity analysis (III-B) while in the Section IV we present the numerical experiments. The last section is dedicated to conclusions and the outline of future research.

II Mathematical preliminaries

A topological group is called locally compact (LC) if every element has a compact neighbourhood. For an Abelian LC groups we use the acronym LCA. The important feature of a LC group is the possibility to define a convolution structure over them. To do this, we

define the left shift operator as the operator acting on a function or distribution f defined over the LC group G as

$$L_y f(x) := f(y^{-1}x), \quad y, x \in G.$$

Since a LC group possesses a unique Haar measure, we can define for the space $\mathcal{K}(G)$ of compactly supported distributions or functions the convolution

$$f * g := \int_G f(x) L_x g(y) dx$$

which is commutative only for LCA groups. The left shift is continuous with respect to L^p norms hence, thanks to the convergence theorems, it is possible to extend the convolution to much wider classes of function or distributions such as $L^1(G)$ and $L^2(G)$.

The morphisms from G into the torus $\mathbb{T}$ are called characters of the group. The set of characters of a LC group G is a LC group $\hat{G}$ called the dual group of G .

Due to the duality principle, which states that the topological dual of $\hat{G}$ is G and vice versa, we can represent the characters as $x \mapsto \langle x, \hat{x} \rangle$ for each $x \in G$, and the element of G as $\hat{x} \mapsto \langle x, \hat{x} \rangle$ for $\hat{x} \in \hat{G}$. The Fourier transform of a function in $L^1(G)$ is defined as

$$\hat{f}(\hat{x}) := \int_G f(x) \overline{\langle x, \hat{x} \rangle} dx \quad \hat{x} \in \hat{G},$$

while the inverse Fourier transform of a function in $L^1(\hat{G})$ is $\hat{f}(x) := \int_{\hat{G}} \hat{f}(\hat{x}) \langle x, \hat{x} \rangle dx$ for $x \in G$. Selecting $G = \mathbb{R}^d$, the characters have the form $\langle x, \omega \rangle := e^{2\pi i x \cdot \omega}$, so the usual Fourier transform is obtained.

An important object to analyse in LC theory is the orthogonal group of a subgroup. Given a LCA group G and H a subgroup, the orthogonal of H is the subgroup of $\hat{G}$ whose elements are the functions which are equal to the 1-function (the zero function over the torus) on H

$$H^\perp := \{ \hat{x} \in \hat{G} \mid \forall x \in H, \langle x, \hat{x} \rangle = 1 \}.$$

The Haar measures on G , on the dual group $\hat{G}$, on a normal subgroup H , on the quotient G/H and on the perpendicular group $H^\perp$ are said to be compatible since the Poisson summation formula and the Weil formula holds. We can finally introduce ST spaces.

Definition 1. Given G a LC group, H a subgroup of G , and $\Phi = \{ \phi_i; G \rightarrow \mathbb{C} \}_{i=1}^R$ a finite set of functions or distributions in a translation invariant Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$, the collection of left shift $(\Phi, H) := \{ L_a \phi_i; a \in H, i = 1, \dots, R \}$ is called **spline-type system of generating**

set Φ and subgroup H , while its closed span in $\mathcal{B}$ is called **spline-type space** generated by Φ and H , which will be indicated as $\mathcal{S}(\Phi, H)$.

We decided to use a much more general definition of ST space over LC group than [11], [12], [13], making use of arbitrary subgroup H , since constructive approximations based on Theorem 1 do not require a discrete subgroup.

Infinite linear combinations of the shifts can be represented by the synthesis operator. Given the functions $\mathbf{c} = \{c_i: H \mapsto \mathbb{C}\}_{i=1}^R$ we define it as:

$$U_{\Phi, H}\mathbf{c} := \sum_{i=1}^R \int_H c_i(a) L_a \phi_i d_H a = \sum_{i=1}^R c_i *_H \phi_i. \quad (1)$$

In signal analysis, another important operator is the analysis operator. It measures the correlation between a given signal and the shifts of the atoms over the subgroup through the L^2 product $\langle \cdot, \cdot \rangle_2$:

$$V_{\Phi, H}f := (c_i(a))_{\substack{i \in I \\ a \in H}} = \left(\langle f, L_a \phi_i \rangle_2 \right)_{\substack{i=1, \dots, R \\ a \in H}}.$$

Usual $L^p(H)$ spaces for sequence of functions are defined through the sup

$$\|\mathbf{c}\|_{(L^p(H))^I} := \max_{i=1, \dots, R} \left(\int_H |c_i(a)|^p d\xi \right)^{\frac{1}{p}} < \infty.$$

Injectivity of the synthesis operator means uniqueness of expression in the spline type space. The following theorem holds.

Theorem 1. *Let G an LCA group, H a cocompact subgroup and a finite generating set $\Phi = \{\phi_i\}_{i=1}^R$ of compactly supported distributions.*

There is equivalence between

- (i) $\mathcal{A}f(x) = (f_1(a), \dots, f_R(a)) \in (L^\infty(H))^R$ such that $U_{\Phi, H}f = 0$
- (ii) $\mathcal{A}\xi \in H^\perp$ such that the functions on $H^\perp$, $\{\hat{\phi}_j(\cdot \xi)\}_{j=1, \dots, R}$ are linearly dependent
- (iii) $\exists \delta > 0$ such that $\forall 1 \leq p < \infty, \forall f \in (L^p(H))^R$

$$\delta \|f\|_{(L^p(H))^R} \leq \|U_{\Phi, H}f\|_{L^p(G)} \quad (2)$$

In other words the Theorem 1 characterise whether the ST system is a Riesz Projection basis. Interesting feature of the previous theorem is that it incorporates the construction of a biorthogonal ST system, e.g. a set of function $\tilde{\Phi} = \{\tilde{\phi}_i\}_{i=1, \dots, R}$ such that

$$\delta_{x,x'}\delta_{i,j} = \langle L_x\phi_i, L_{x'}\tilde{\phi}_j \rangle_2. \quad (3)$$

For such pairs the reproduction formula

$$f = U_{\Phi,H}V_{\tilde{\Phi},H}f = U_{\tilde{\Phi},H}V_{\Phi,H}f \quad (4)$$

holds for each element f in the ST space. This construction is based on the inversion of the Gramian

$$(\mathcal{G})_{i,j} = \hat{\phi}_i \hat{\phi}_j.$$

We aim to control the Frobenius norm of the difference matrix hence we need to find the proper neighbourhood $\mathcal{U}$ of $\hat{0} \in \hat{G}^r$ where to chose $\hat{y}$.

We can formulate the following stability theorem for a set of generating functions shifted in frequency domain:

Theorem 2. *Given a LCA group G and a generating set $\Phi = \{\phi_1, \dots, \phi_r\}$ of compactly supported distributions defined over G , then for every $\epsilon > 0$ there exist a vector of shift $\hat{y} \in \hat{G}$ such that the Frobenius norm of the matrix*

$$\mathcal{D} := L_{\hat{y}}\mathcal{G} - \mathcal{G} \quad (5)$$

can be bounded as

$$\|\mathcal{D}\|_F < 2 \epsilon r \max_{j=1, \dots, r} \|\phi_j\|_1. \quad (6)$$

III Spline-type reformulation of multiresolution-like analysis

A Modulations and dilations

The ST analysis measure the correlation of a signal with different shifts of a fixed set of atoms (a dictionary). Together with time-localization, signal and image analysis is concerned with scale and frequency localization. Two important operators are defined: the modulation and dilation operator.

In LC theory modulation is character multiplication since, by defining for $\hat{x} \in \hat{G}$

$$M_{\hat{x}}f := \langle \cdot, \hat{x} \rangle f,$$

we obtain that $\widehat{M_{\hat{x}}f} = \hat{L}_{\hat{x}}\hat{f}$, where we have denote with $\hat{L}_{\cdot}$ the left shift over the LC group $\hat{G}$. The usual frequency interpretation for the group $G = \mathbb{R}^d$ becomes straightforward once the particular form or the related characters are considered. To define the contraction (or dilation) operator is much more arduous work, since its construction is not as straightforward as for modulation. To mimic the case of the real group, we would need to give the LC group an external product with scalar e.g. a vectorial structure, such that standard dilation

$$D_a f := f(a \cdot)$$

can be employed.

In [4] we have shown that the Gabor systems can be exactly reformulated in a biorthogonal construction problem; this construction becomes a powerful and easy to use tool in signal analysis. This is due to the favourable case for which it is possible to periodize both time and frequency domain. The same is not possible for wavelet systems since the collection of generating functions is infinite, not periodizable, and defined over different lattices. Wavelets can indeed be defined as a collection of function $\{\phi_i\}_{i \in I}$ which are characterized by being the contraction of a unique mother function ϕ and their shifts over a related lattice H_i such that $H_i \subset H_{i+1}$.

The wavelet system becomes

$$\{L_x(D^{(i)}\psi) : i \in I, x \in H_i\},$$

as a Gabor systems can be defined as the collection of shifts over the same lattice of different modulations

$$\{L_x(M_{\omega}\phi) : x \in H, \omega \in K\},$$

being K a discrete subset of $\hat{G}$. For any arbitrary set of functions $X = \{x_n\}$, not only for the particular case of spline type systems, we can define the so-called system operator.

$$Sf := \sum_n \langle f, x_n \rangle_2 x_n.$$

For a union of ST spaces $\bigcup_i (\phi_i, H_i)$

$$Sf := \sum_i U_{\phi_i, H_i} V_{\phi_i, H_i}^* f,$$

With this reformulation Gabor systems can be rewritten as ST spaces being ϕ_i different modulations and H_i a constant sequence, while for Wavelets one uses contractions and an ascending monotone sequence of lattices. The Gabor system operator commutes with both shifts and modulations; this is not the case for wavelets for which dilation do not commutes. If the system operator is invertible, it is possible to build the functions $S^{-1}\phi_i$, called the canonical duals, which behaves as the biorthogonal in the reproduction formula (4).

For Gabor systems built on a uniform rational sampling in both space and modulation the inversion procedure is made computationally feasible through the Walnut representation which partition the system operator into a block circulant matrix. Thanks to its multiwindow feature, the spline type reformulation gives an analogous partitioning, hence the construction of biorthogonal systems (3) do not suffer from uniform frequency partitioning while maintaining the same algorithmic complexity.

To solve the wavelet construction in a sampling theory, generalized shift invariant systems (GSI) were introduced [9], [14], allowing the construction of ST space as infinite unions of ST spaces, each of them generated by a unique function (in the wavelet case a different dilation) and a different subgroup. This reformulation leads to good result for theoretical purpose [9] but the complexity of the related algorithm raises [14]. The nimbleness of Theorem 1 is lost, so the complexity of the algorithm raise up to usual operator inversion.

Here we use the wavelet scheme in its typical multiresolution reformulation. In its early formulation [10], the mathematical justification of multiresolution analysis (MRA) relies in it an ascending chain of ST spaces in $L^2(\mathbb{R}^d)$

$$V_{-\infty} \subset \dots \subset V_{-1} \subset V_0 \subset V_1 \subset \dots \subset V_{\infty} = L^2(\mathbb{R}^d)$$

each of them generated by a scaling function $D_{a^j}\phi$ over the lattice $a^j\mathbb{Z}^d$.

At each level j is then considered W_j the orthogonal complement of V_j in V_{j+1} . Therefore, it is possible to consider any element of $L^2(\mathbb{R}^d)$ as expansion

$$L^2(\mathbb{R}^d) = V_{j_0} \oplus W_{j_0} \oplus W_{j_0+1} \oplus \dots$$

for an arbitrary fixed index j_0 .

At each scale only one generating function is considered and the orthogonality of the W_j spaces is used to express subsequent projections in the ST spaces $\mathcal{S}(D_{a_i}\phi, H_i)$. Our interest is focussed in biorthogonal expansions (4) which can express different features of a signal. In this particular case

- Time = through the ST structure
- Scale = through the multiresolution iteration
- Frequency = through the multiwindow nature.

Using multiwindow spline type spaces, we can improve the accuracy with a small drawback in complexity. Fixed a scale, we chose to modify the window obtaining different atoms, having the same scale. The main operator that does not change the scale of a signal, e.g. it's localization in time is the modulation operator.

B The ST multiresolution algorithm

During the first step of a MRA the low frequency content of the signal is analysed. For this purpose in Step 2 of Algorithm 1 we use few modulations $\Phi := \{M_{\hat{x}_i} \phi\}$ of a Gaussian function ϕ , being $\hat{x}_i$ close to 0. In the function BIORTH of Step 3 we compute the related biorthogonal system and compute the low resolution approximation in Step 4.

To handle the three different behaviours that we stated above, we have to modify the multiresolution algorithm building at each scale a stable generating set for which the corresponding ST well represent the given signal, with the less possible effort. As shown in [4], it is possible to approximate a signal in ST spaces considering only those generators that better intersect its spectrum: the close translation invariant subspaces of $L^1(G)$ coincide with the ideals of the convolution algebra over $L^1(G)$, which is isomorphic to the Fourier algebra $\mathcal{AL}^1(G)$, whose ideals are characterized by the support of its elements. Hence we cover the whole spectrum through a modified $\bar{M}$. modulation operator which will be exploited in Section IV. After the whole set of AVAILABLE generating atoms are built we rank the outcome of the multiwindow-analysis operator as in Steps 9 and 8. We consider the best $\#at$ atoms which ensure stability as in Algorithm 2, which is the pseudocode to check condition (ii) in Theorem 1, compute the biorthogonal system and the approximation. A row echelon reduction is used to track the windows which give instability. It is important to notice that we have to check the linear independence of the Fourier transforms of the atoms for only one representative of each element of $\hat{G}/H^\perp$. Finally the biorthogonal system and the approximation are computed. The procedure is iterated on the reminder.

Algorithm 1 Multiresolution ST approximation

Precondition: $\left\{ \begin{array}{l} \text{Signal } XX, \\ \text{Scaling function } \phi, \\ \text{Windows } \Gamma = \{D^{(i)}\psi : i \in I\}, \\ \text{subgroups } H := \{H_i\}, \\ \text{Fixed number of atoms } \#at \end{array} \right.$

- 1: **function** ST_MULTIREOLUTION($XX, \phi, \Psi, H, \#at$)
- 2: $\Phi \leftarrow \{M_{\hat{x}_i} \hat{\phi} : \hat{x}_i \in \hat{G}\}$
- 3: $\tilde{\Phi} \leftarrow \text{BIORTH}(\Phi, H_0)$
- 4: $XXa \leftarrow U_{\Phi, H_0} V_{\tilde{\Phi}, H_0}^T XX \quad \triangleright \text{Formula (4)}$
- 5: **RR** $\leftarrow XX - XXa$

```

6:   for  $i \in I$  do
7:      $\Psi \leftarrow \left\{ \overline{M}_{\hat{x}_i} D^{(i)} \psi : \hat{x}_i \in \hat{G} \right\}$ 
8:      $A \leftarrow V_{\phi_i, H_i}(RR)$ 
9:      $P_I \leftarrow \text{SORT}(A)$ 
10:     $\text{AVAILABLE} \leftarrow \Phi(P_I)$ 
11:     $\Psi^{(i)} \leftarrow \emptyset$ 
12:    while  $\text{card}(\Psi^{(i)}) < \#at$  do
13:       $\text{GOOD} \leftarrow \text{AVAILABLE}(1, \dots, \#at)$ 
14:       $(\Psi^{(i)}, \text{BAD}) \leftarrow \text{ST\_STABLE}(\text{GOOD}, H_i)$ 
15:       $\text{AVAILABLE} \leftarrow \text{AVAILABLE} \setminus \text{BAD}$ 
16:    end while
17:     $\widetilde{\Psi}^{(i)} \leftarrow \text{BIORTH}(\Psi^{(i)}, H_i)$ 
18:     $XXa \leftarrow XXa + U_{\widetilde{\Psi}^{(i)}, H_i}^V \Psi^{(i), H_i} RR \triangleright \text{Formula (4)}$ 
19:     $RR \leftarrow RR - XXa$ 
20:  end for
21:  return  $XXa$ 
22: end function

```

1) *Complexity analysis*: The input signal XX in Algorithm 1 is a sampled function, e.g. an element of $\mathbb{C}^L$ for some $L \in \mathbb{N}$. We consider as LCA group G the periodic group of integers modulo L , $\bar{L} := \mathbb{Z}/L\mathbb{Z}$. We are not using the notation $\mathbb{Z}_L$ since it also indicates the group of L -adic numbers, another LC group that can be studied with the same theory here developed.

The subgroups of $\bar{L}$ are $H_a = \{0, a, \dots, L-a : L \equiv 0 \pmod{a}\}$. The dual of $\bar{L}$ is $\bar{L}$ itself, and the perpendicular subgroup of H_a is the subgroup $\widehat{H}_{\frac{L}{a}}$, pointing out through the *hat*

superscript that the related object belong to the dual structure.

For our multiresolution analysis, we consider a sequence of lattices

$$H := \left\{ H_{a_i} : a_i < a_i + 1 \right\}_{i=1, \dots, r}$$

So we do not impose the constrain that $H_i \subset H_{i+1}$, but we just assume that levels with higher index have higher resolution. Since we want to have an $\mathcal{O}(1)$ redundancy at each level, the

number of available atoms will be an $\mathcal{O}\left(\frac{L}{a_i}\right)$ at level H_{a_i} . In this way we showed [4] that the complexity of the biorthogonality construction take an $\mathcal{O}(\mathcal{L})$ in time.

Algorithm 2 Stability Analysis

Precondition: $\begin{cases} \text{Windows } \Gamma = \{\gamma_i : i \in I\}, \\ \text{subgroup } H \end{cases}$

```

1: function ST_STABLE( $\Gamma, H$ )
2:    $H^\perp \leftarrow \text{PERPENDICULAR}(H)$ 
3:   for  $\hat{x} \in \hat{G}/H^\perp$  do
4:      $C_{i,j} \leftarrow (\hat{\gamma}_i(j\hat{x})) \ j \in H^\perp$ 
5:      $A \leftarrow \text{ROW\_ENCHELOR\_FORM}(C)$ 
6:     for  $j = 1, \dots, \text{card}(\Gamma)$  do
7:       if  $a_{i,j} = 0$  then
8:          $\text{INDEPENDENTS} \leftarrow \text{INDEPENDENTS} \cup \{\gamma_j\}$ 
9:       else
10:         $\text{DISCARDED} \leftarrow \text{DISCARDED} \cup \{\gamma_j\}$ 
11:      end if
12:    end for
13:  end for
14:  return ( $\text{INDEPENDENTS}, \text{DISCARDED}$ )
15: end function

```

We finish our complexity analysis considering the stability check. Since

$$\text{card}\left((H)^\perp\right) = \text{card}\left(H \frac{L}{a_i}\right) = a_i,$$

the matrix computed in Step 4 is a $a_i \times \#at$ matrix. Its row echelon form reduction, computed through Gauss-Jordan elimination, has a time cost of $\mathcal{O}\left(\min(\#at, a_i)^2 \max(\#at, a_i)\right)$ in time.

Considering the number of atoms as a constant, Algorithm 2 has $\mathcal{O}(a_i)$ time complexity.

Finally, since the number of AVAILABLE atoms is a $\mathcal{O}\left(\frac{L}{a_i}\right)$, building a stable ST space takes a $\mathcal{O}(L)$ at each level. This proves

Proposition 3. *Let Γ be a set of windows having different scale features suitable for the analysis of signal in $\mathbb{C}^L$.*

The multiresolution ST procedure in Algorithm 1 has time Complexity

$$\mathcal{O}(\text{card}(\Gamma) \cdot L)$$

IV Numerical experiments

A function that is often used in wavelet scheme is the following radial function

$$\varphi(x) = (1 - \|x\|^2)e^{-\frac{\|x\|^2}{2}} \quad x \in \mathbb{R} \quad (7)$$

In Figure 1 two shifts of this window over the periodized domain $\bar{L}$ are shown. The high frequency region of $\hat{L}$ is located in the middle of it, so we understand how modulation operator does not handle well the multi-frequency analysis in a wavelet scheme. Mother wavelet is traditionally chosen [1] such that

$$\int |\hat{x}|^{-1} |\hat{\varphi}(\hat{x})|^2 d\hat{x} < \infty$$

this imply that $\int \hat{\varphi}(\hat{0}) = \int \varphi(x) dx = 0$, so we cannot have good frequency localization. This typical behaviour is shown in Figure 2.

The ST scheme has the great flexibility to work with modulation as different generating atoms, so we can build a generating set that does not rely on the modulation operator. We build a modified modulation operator

$$\bar{M}_{\hat{x}} = \left(\hat{L}_{\hat{x}} \hat{f}\left(0, \dots, \frac{L}{2}\right) + \hat{L}_{\hat{x}} \hat{f}\left(\frac{L}{2} + 1, \dots, L\right) \right)^V$$

which shifts the positive and negative parts of the Fourier transform in opposite direction. It is easy to check that this modified modulation build a new window sharing the same decay of the original one, so same scale parameter. In this way we can consider different frequency features of a signal at the same resolution level. An exemplification can be seen in Figure 3. As stated in III-B we complete the scheme with some low frequency shift of a Gaussian, covering the low region of the frequency domain.

As an example, consider the periodized domain $\overline{1024}$. The low frequency region is analysed by $\mathcal{S}(\Phi, H_{128})$, being Φ a set of 7 modulated Gaussians

$$\Phi := \left\{ M_{\hat{x}} e^{-\frac{x^2}{2}} : \hat{x} = 0, \pm 8, \pm 16, \pm 24 \right\}. \text{ We then considered for each } a = 64, 32, 16, 8 \text{ the ST}$$

space generated over H_a by a mother wavelet φ_a of the type (7) having support of radius $3a$ and by the modified modulations $\bar{M}_{\hat{x}} \varphi_a, x = \frac{2}{3} \frac{L}{a} k, k = 0, \dots, \left\lfloor \frac{3}{4} a \right\rfloor - 1$, with $\lfloor \cdot \rfloor$ the usual round toward negative infinity operation; in this way the spectrum of the ST space covers the whole Fourier domain and the modified modulations show good overlap.

We have performed 3 separate runs of the code:

- Generating only the $\bar{M}_0$ modified modulation to reproduce standard wavelet system
- Generating all the modulations and selecting the best correlated one
- Generating all the modulations and selecting the 4 best windows

The outcome of this test is displayed in Figure 4. At each modification of the original multiresolution algorithm, the approximation improves.

V Conclusions

We presented in this paper a numerical procedure to construct wavelet-like systems based on multiwindow spline-type spaces under the stability argumentation. The next step is to reproduce standard multiresolution wavelet constructions, without the use of generalized spline type spaces, and considering instead only the recurrency structure that comes from the refinement equation upon which these ascending sequences of shift invariant spaces are built.

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Figure 1.
Wavelet (7) in time and periodized time domain

Figure 2.
Fourier transform of Wavelet (7), a modulation and the Fourier transform of a contraction

Figure 3.
Fourier transform of Wavelet (7), a modulation and the Fourier transform on another scale

Figure 4. Comparison of different runs of algorithm 1: 1 window to reproduce standard multilevel approach, 1 window having maximum correlation, 4 best correlated windows