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Reduction of Thermal Conductive Flux by Non-local Effects in the Presence of Turbulent Scattering

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Abstract

The heat flux in a plasma is determined by the degree of anisotropy in the particle distribution function, which is in turn driven by gradients in the ambient density and temperature. When the mean free path at the thermal speed is substantially smaller than the scale length associated with the temperature variation, the heat flux simply depends on the local value of the temperature gradient. However, when the temperature scale length and mean free path are comparable, heat conduction becomes substantially non-local in character: the magnitude of the heat flux now depends on the overall temperature profile and is generally smaller than the locally determined value. In the presence of angular scattering associated with turbulence, the mean free path (and its velocity dependence) can be significantly smaller than its collisional value; this makes the expression for the heat flux more local in character, but also results in a heat flux that is lower than that obtained through a purely collisional analysis. Therefore, whether or not turbulence is present, the heat flux is generally smaller than the value obtained from a local collisional analysis. We here present an analytic expression for the conductive heat flux in terms of a convolution of the local heat flux with a non-local kernel function that incorporates both Coulomb collisions and turbulent scattering. We comment on the need to include both non-local and turbulent scattering effects in the modeling of quasi-static active region loops and in the conductive cooling of post-flare loops.

Keywords

acceleration of particles; Sun: activity; Sun: flares; Sun: X-rays, gamma rays

1. Introduction

It has long been recognized in the plasma fusion community (see, e.g., Luciani et al. 1983, 1985; Albritton et al. 1986) that when the collisional mean free path is long compared to the size of the system under study, then thermal conduction becomes an inherently non-local phenomenon. The possibility that non-local heat conduction is appropriate in the context of very hot plasma created in solar flares has also been studied (Smith 1986; West et al. 2008). One approach (e.g., Brown et al. 1979; Smith & Lilliequist 1979) consists of simply limiting the heat flux to a fraction of the free-streaming value $F_c \sim (1/2)n m v_{te}^3$ (where m (g) is the electron mass, n (cm⁻³) is the density, and v_{te} (cm s⁻¹) is the thermal speed), with the

pertinent fraction possibly determined from consideration of the generation of turbulence by collective effects (Rudakov & Korabely 1966) in the ambient plasma. A more general approach (Karpen & DeVore 1987, their Equation (4)) involves taking a convolution,

$$q_H(z) = \int_{-\infty}^{\infty} Q(z - z') F_c(z') dz' \equiv Q(z) \star F_c(z), \quad (1)$$

where $F_c(z)$ is the local conductive heat flux (in general a function of both the local temperature and the local temperature gradient) and the kernel function $Q(z - z')$ represents the extent to which the heat flux at a given point z is determined by “sampling,” with varying weights, the temperature gradient at different distances $(z - z')$ from that point.

Such studies have heretofore assumed that the heat flux was driven by collisional processes only, and hence that the nonlocal kernel $Q(z - z')$ had a form that was a function of the collisional mean free path (Spitzer 1962). However, the idea that turbulence plays a key role in the conversion of magnetic free energy to the various manifestations of a solar flare has been around for some time (Miller et al. 1996, 1997), and there is a growing body of evidence, both theoretical (Gordovskyy et al. 2014; Fang et al. 2016) and observational (Kontar et al. 2017), in support of this contention. To the best of our knowledge, a straightforward expression for the non-local conductive heat flux under conditions of turbulent heat transport has not been published. The purpose of this work is to derive the form of the non-local kernel $Q(z - z')$ in an environment where both non-locality and turbulent deviations from collisional transport are considered.

Bian et al. (2016a) considered the reduction in thermal conductivity that results from turbulent angular scattering of *thermal* electrons. They obtained analytical expressions for the thermal heat flux via a Chapman–Enskog expansion of the electron kinetic equation where diffusion in pitch-angle space accounts for both collisional and turbulent processes. In addition, Bian et al. (2017) derived, through a Legendre polynomial expansion of the electron phase-space distribution function, an analytic form for the *particle* flux q ($\text{cm}^{-2} \text{s}^{-1}$) appropriate to a plasma with a mean free path that is comparable to the size of the region under consideration. They showed that the form of q was inherently non-local, involving a convolution of the local density gradient with a kernel that extends over a distance proportional to the (velocity-dependent) mean free path. Since the mean free path in general involved both collisions and turbulence, different expressions for the turbulent mean free path $\lambda_T(v)$ resulted in significantly different extents to which non-locality is important.

In this paper we combine the approaches in Bian et al. (2016a, 2017) to derive the form of the non-local *heat* flux q_H ($\text{erg cm}^{-2} \text{s}^{-1}$) in the presence of both collisional and turbulent scattering. We then use it, for a variety of parametric representations of $\lambda_T(v)$, to determine the heat flux for a given temperature profile. We apply the results to the temperature appropriate to static active region loops (Martens 2010), and we illustrate how to apply this methodology to the heating and cooling of flare loops.

In Section 2 we derive the form of the kernel $Q(z - z')$ for an environment in which collisional and turbulent scattering both play a role. In Section 3 we apply this kernel to deduce the form of the non-local heat flux for a typical temperature profile (Martens 2010) representative of hot, static active region loops, and we evaluate the extent to which non-local effects are significant, for various values and velocity dependencies of the turbulent mean free path $\lambda_T(v)$. In Section 4 we discuss the implications of non-local and turbulent effects on the cooling of post-flare loops, in particular its relevance to the long-standing conundrum that the observed cooling times of flare loops (Moore et al. 1980; Ryan et al. 2013) are much longer than would be expected from a local, collisional model (Spitzer 1962) of the conductive heat flux. In Section 5 we summarize the results and point toward future work that incorporates the revised form of the conductive heat flux derived herein.

2. Expressions for Non-local Heat Flux in the Presence of Turbulence

There have been several investigations into whether it is reasonable to apply a theory originally developed in the context of inertial fusion to the solar flare domain (see, e.g., Karpen & DeVore 1987). Here, we will show that the physics of nonlocality is very general, being inherently encapsulated in the streaming term appearing in the Fokker–Planck equation describing the transport of particles (here electrons). We consider the Lorentz plasma model (see, e.g., Kruskal & Bernstein 1964), where the effects of diffusion in energy space and of the neutralizing electric field are both neglected. In a stationary state the equation governing the evolution of the electron phase-space distribution function $f(z, v, \mu)$ (electrons $\text{cm}^{-3} (\text{cm s}^{-1})^{-1}$) in a Lorentz plasma is

$$\mu v \frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu} \left(\nu(v)(1 - \mu^2) \frac{\partial f}{\partial \mu} \right), \quad (2)$$

where $\nu(v) = v/\lambda(v)$ is the scattering frequency, which depends on the physical process(es) responsible for the scattering. In the presence of both collisional and turbulent scattering, $\lambda(v)$ satisfies (Bian et al. 2016a)

$$\frac{1}{\lambda(v)} = \frac{1}{\lambda_C} + \frac{1}{\lambda_T} = \frac{1}{\lambda_{ei}(v/v_{te})^4} + \frac{1}{\lambda_{Te}(v/v_{te})^\alpha}, \quad (3)$$

so that

$$\lambda(v) = \lambda_{ei} \frac{(v/v_{te})^4}{1 + R(v/v_{te})^{4-\alpha}}, \quad (4)$$

where $R = \lambda_{ei}/\lambda_{Te}$ is the ratio of collisional to turbulent mean free paths at $v = v_{te}$ and we have expressly included the v^4 dependence of the collisional mean free path λ_C .

In Equation (3) we have also introduced the parameter α , which reflects the velocity dependence of the turbulent mean free path λ_T . We now discuss the range of values of α that will be considered herein. First, as discussed by Bian et al. (2016a), a possible source of turbulent pitch-angle scattering lies in magnetostatic fluctuations $\delta \mathbf{B}_\perp$ perpendicular to the guiding field of the loop $B_0 \mathbf{z}$. We consider the wavenumber spectrum

$$W(k_\parallel) = C(q) (\delta B_\perp)^2 \lambda_B [1 + (k_\parallel \lambda_B)^2]^{-q}, \quad (5)$$

where λ_B is the correlation length of the turbulence and $C(q)$ is a normalization constant such that $\int dk_\parallel W(k_\parallel) = (\delta B_\perp)^2$. The turbulent mean free path corresponding to such a turbulence model can be shown (see Bian et al. 2016a) to take the form

$$\lambda_T(v) = \lambda_{Te} \left(\frac{v}{v_{te}} \right)^{2-2q}, \quad (6)$$

where (for $q < 1$)

$$\lambda_{Te} = \frac{3}{4\pi C(q) (1-q) (2-q)} \lambda_B \left(\frac{B_0}{\delta B_\perp} \right)^2 \left(\frac{v_{te}}{\Omega_{ce} \lambda_B} \right)^{2-2q} \quad (7)$$

is the turbulent mean free path at the thermal speed v_{te} . Hence, the parameter α in Equation (4) is related to the power-law index q of the magnetic turbulence spectrum by

$$\alpha = 2 - 2q. \quad (8)$$

Contrary to collisional scattering, scattering by magnetic fluctuations is generally not isotropic, except for the case $q = 1/2$ (see Equation (77) of Bian et al. 2016a), corresponding to $\alpha = 1$. For a Kolmogorov spectrum with $2q = 5/3$, the dependence of the turbulent mean free path on the electron speed is very weak, i.e., $\alpha = 1/3$. The quantity λ_{Te} (Equation (7)) diverges as $q \rightarrow 1$ ($\alpha \rightarrow 0$), which suggests that in such cases turbulence plays a negligible role in suppressing transport below the collisional value. Indeed, in practice the divergence in λ_{Te} for $q = 1$ is avoided by considering the effect of collisions (Bian et al. 2016a).

The integrated spectrum of W over k is convergent only for $q > 1/2$, which implies that the physically admissible range of α is $\alpha < 1$. Notwithstanding this restriction, however, we have herein (see below) considered values of α over the much wider range $[0, 6]$. We have done this for two reasons: (1) other turbulence models could have a different value of α , and (2) to parametrically explore the effect of the value of α on the results. The natural reference value in the latter case is $\alpha = 4$, corresponding to the velocity dependence of the collisional mean free path λ_C . Since electrons with velocities substantially greater than the thermal speed are responsible for the bulk of the heat flux, the variations of λ_C and λ_T with velocity are therefore of considerable importance. For example, for small values of λ , the turbulent

mean free path increases much more slowly with velocity than does the collisional mean free path, so that even for modest values of R the conductive flux restriction is quite large. On the other hand, for $\alpha = 4$, the ratio of collisional to turbulent mean free paths is independent of velocity, so the heat flux limitation is more modest. At the other extreme, for values of $\alpha > 4$ (we have below used the example $\alpha = 6$), the turbulent mean free path increases so quickly with velocity that turbulence has a negligible limiting effect on the heat flux, even when the turbulent mean free path at the thermal velocity may be dominant.

We now return to the Fokker-Planck Equation (2). Following Bian et al. (2017), and noting (see, e.g., Gombosi et al. 1993) that the Legendre polynomials $P_n(\mu)$ are eigenfunctions of the Lorentz scattering operator, we decompose the full distribution into a complete set of orthogonal Legendre polynomials:

$$f(z, v, \mu) = \sum_n f_n(z, v) P_n(\mu). \quad (9)$$

Now, the particle¹ and heat fluxes are respectively given by

$$q_P(z) = 2\pi \int_0^\infty dv v^2 \int_{-1}^1 d\mu \mu v f(z, v, \mu) \quad (10)$$

and

$$q_H(z) = 2\pi \int_0^\infty dv v^2 \int_{-1}^1 d\mu \mu v \left(\frac{1}{2} m_e v^2 \right) f(z, v, \mu). \quad (11)$$

Since both these fluxes involve the projection of the full distribution $f(z, v, \mu)$ on $P_1(\mu) = \mu$, then the orthogonality of Legendre polynomials implies that the particle and heat fluxes involve only the component $f_1(z, v)$ in the expansion (9). Determination of the functional relationship between these fluxes and the surrounding density and temperature gradients therefore reduces to finding an expression for $f_1(z, v)$ in terms of gradients in the isotropic core distribution $f_0(z, v)$. Substituting Equation (9) into Equation (2), applying both Legendre's differential equation

$$\frac{d}{d\mu} \left((1 - \mu^2) \frac{dP_n(\mu)}{d\mu} \right) + n(n+1)P_n(\mu) = 0 \quad (12)$$

and Bonnet's recursion relation

¹We note that often the term "particle flux" is used to designate a quantity that is differential in velocity (i.e., particles $\text{cm}^{-2} \text{s}^{-1}$ per unit velocity (cm s^{-1}) range). Here, the term refers to the hydrodynamical (velocity-integrated) quantity.

$$(2n + 1) \mu P_n(\mu) = (n + 1) P_{n+1}(\mu) + n P_{n-1}(\mu), \quad (13)$$

and equating coefficients of $P_n(\mu)$. gives (Equation (18) of Bian et al. 2017)

$$f_n = D \left[\frac{f_{n-1}}{(n+1)(2n-1)} + \frac{f_{n+1}}{n(2n+3)} \right], \quad (14)$$

where

$$D \equiv -\lambda(v) \frac{\partial}{\partial z} \quad (15)$$

is the spatial derivative operator. Equation (14) can be solved by recurrence to any order to find (Equation (25) of Bian et al. 2017)

$$\left[1 - \frac{1}{1.5.3.3} D^2 - \frac{1}{2.5.7.4.5.3.3} D^4 - \dots \right] f_1 = \frac{1}{2} D f_0, \quad (16)$$

which is the sought-after relation between the first-order anisotropic term $f_1(z, v)$ and the gradient of the isotropic Maxwellian component $f_0(z, v)$. It should be noted that this relation involves derivatives of progressively higher order as the number of terms in the P_n decomposition is increased. However, due to the rapid decrease in the coefficients of successive powers of D on the left side, we may set D^4 and higher terms equal to zero, resulting in the expression

$$\left[1 - \frac{\lambda^2(v)}{45} \frac{\partial^2}{\partial z^2} \right] f_1 = -\frac{1}{2} \lambda(v) \frac{\partial f_0}{\partial z}, \quad (17)$$

with a solution (Equation (32) of Bian et al. 2017)

$$f_1(z, v) = -\frac{\sqrt{45}}{4} \int dz' \exp\left(-\frac{\sqrt{45}}{\lambda(v)} |z - z'| \right) v \frac{\partial f_0(z', v)}{\partial z'}. \quad (18)$$

Equation (18) is the central result that drives the non-local aspect of heat conduction: the first-order anisotropic component $f_1(z, v)$ of the particle distribution function responsible for the flux (both particle and heat) is determined² by a weighted average of the spatial gradient of the isotropic component $f_0(z, v)$ over a range of values z' that is proportional to

²The degree of anisotropy of the particle distribution is also determined by the form of the wave/particle interactions that lead to turbulent reduction of the local heat flux, in addition to the gradients in macroscopic quantities. Such considerations are beyond the scope of the present work.

the mean free path $\lambda(v)$. It should first be noted that the factor $\sqrt{45}$ in the exponential term implies that the value of $f_1(z, v)$ (and hence the heat flux $q_H(z)$; Equation (11)) involves the effective sampling of the gradient $f_0(z, v)/z$ only over distances that extend a small fraction of a mean free path on either side of the point in question; therefore, a local relation $f_1(z, v) \sim -f_0(z, v)/z$ is a rather good approximation even when the mean free path is significantly larger than the extent L of the region under consideration. However, it is also important to note that the mean free path is dependent on velocity and, as we shall see below, because particles with relatively high velocities carry the majority of the conductive flux, there is a very significant degree of non-locality in the expression for $f_1(z, v)$, even when the extent L of the region is comparable to the mean free path evaluated at the thermal speed.

Equation (18) can now be used in Equations (10) and (11) to compute the particle and heat fluxes. We here focus on the heat flux, and from Equations (4), (11) and (18), we obtain (cf. Equation (45) of Bian et al. 2016b)

$$q_H(z) = -2\pi \left(\frac{\sqrt{45}}{4} \right) \int_{-1}^1 \mu^2 d\mu \int_0^\infty dv v^2 \left(\frac{m_e v^2}{2} \right) \int dz' \times \exp \left(- \frac{\sqrt{45} (1 + R(v/v_{te})^4 - \alpha) |z - z'|}{\lambda_{ei}(v/v_{te})^4} \right) v \frac{\partial f_0(z', v)}{\partial z'}. \quad (19)$$

Next, using the Maxwellian result for a uniform pressure environment

$$f_0(v, z) = \frac{P}{2k_B T(z)} \left(\frac{m_e}{2\pi k_B T(z)} \right)^{3/2} e^{-m_e v^2 / 2k_B T(z)}, \quad (20)$$

and the chain rule $f_0(z', v)/z' = (f_0(z', v)/T)(dT/dz)$, Equation (19) can be written as³

$$q_H(z) = -\frac{1}{2} \sqrt{\frac{10k_B}{\pi m_e}} P \int_0^\infty dx x^5 \left(x^2 - \frac{5}{2} \right) e^{-x^2} \int_{-\infty}^\infty \exp \left(- \frac{\sqrt{45} |z - z'| (1 + R x^4 - \alpha)}{\lambda_{ei} x^4} \right) \frac{1}{\sqrt{T(z')}} \frac{dT}{dz'} dz', \quad (21)$$

where $x = v/v_{te}$, $v_{te} = \sqrt{2k_B T / m_e}$ being the thermal velocity.

³For a uniform density environment, one obtains a very similar result, with $(x^2 - \frac{3}{2})$ replacing $(x^2 - \frac{5}{2})$. This does not qualitatively change the results to be discussed herein.

By setting $R = 0$ and letting the term $|z - z'|$ approach zero, while maintaining the value of the resulting delta-function-like integral of the non-local kernel over z' , we obtain the result for a local, collision-dominated, heat flux, namely

$$\begin{aligned} q_H(z) &= -\frac{1}{2} \sqrt{\frac{10k_B}{\pi m_e}} \left(\frac{2\lambda_{ei}}{\sqrt{45}} \right) \\ &\quad \times P \left[\int_0^\infty x^9 \left(x^2 - \frac{5}{2} \right) e^{-x^2} dx \right] \frac{1}{\sqrt{T}} \frac{dT}{dz} \\ &= -15 \sqrt{\frac{10k_B}{\pi m_e}} \left(\frac{2\lambda_{ei}}{\sqrt{45}} \right) P \frac{1}{\sqrt{T}} \frac{dT}{dz} \equiv -\kappa_o T^{5/2} \frac{dT}{dz}. \end{aligned} \quad (22)$$

We can thus make the identification, valid at all points z' ,

$$\frac{1}{2} \sqrt{\frac{10k_B}{\pi m_e}} P \frac{1}{\sqrt{T(z')}} \frac{dT}{dz'} \equiv \frac{\sqrt{45}}{60} \frac{\kappa_o}{\lambda_{ei}} T^{5/2}(z') \frac{dT(z')}{dz'}, \quad (23)$$

and substituting this in Equation (21) gives

$$\begin{aligned} q_H(z; R, \alpha, \lambda_{ei}) &= -\frac{\sqrt{45}}{60} \frac{k_0}{\lambda_{ei}} \int_0^\infty dx x^5 \left(x^2 - \frac{5}{2} \right) e^{-x^2} \\ &\quad \times \int_{-\infty}^\infty \exp \left(-\frac{\sqrt{45} |z - z'| (1 + R x^{4-\alpha})}{\lambda_{ei} x^4} \right) T^{5/2} \frac{dT}{dz'} dz'. \end{aligned} \quad (24)$$

This is the central result of the present work. It can be formally written as the convolution product

$$q_H \left(\frac{z}{\lambda_{ei}}; R, \alpha \right) = Q \left(\frac{z}{\lambda_{ei}}; R, \alpha \right) \star F_c(z), \quad (25)$$

where the local collision-dominated (Spitzer 1962) heat flux

$$F_c(z) \equiv -\kappa_o T^{5/2} \frac{dT}{dz} \quad (26)$$

and the kernel

$$Q\left(\frac{z}{\lambda_{ei}}; R, \alpha\right) = \frac{\sqrt{45}}{60 \lambda_{ei}} \int_0^\infty x^5 \left(x^2 - \frac{5}{2}\right) \times \exp\left[-\left(x^2 + \frac{\sqrt{45}(1 + Rx^4 - \alpha)}{x^4} \left|\frac{z}{\lambda_{ei}}\right|\right)\right] dx. \quad (27)$$

The quantity $Q(z/\lambda_{ei}; R, \alpha)$ represents deviations both from locality and from a wholly collisional model for thermal conduction (see, for comparison, Equation (4) of Karpen & DeVore 1987). For numerical convenience, the substitution $y = x^2$ gives

$$Q\left(\frac{z}{\lambda_{ei}}; R, \alpha\right) = \frac{\sqrt{45}}{120 \lambda_{ei}} \int_0^\infty y^2 \left(y - \frac{5}{2}\right) \times \exp\left[-\left(y + \frac{\sqrt{45}(1 + Ry^2 - \alpha/2)}{y^2} \left|\frac{z}{\lambda_{ei}}\right|\right)\right] dy. \quad (28)$$

As a check on the numerical evaluation of this integral, we note that for all values of R and α ,

$$\begin{aligned} Q(0; R, \alpha) &= \frac{\sqrt{45}}{120 \lambda_{ei}} \int_0^\infty y^2 \left(y - \frac{5}{2}\right) e^{-y} dy \\ &= \frac{\sqrt{45}}{120 \lambda_{ei}} = \frac{0.056}{\lambda_{ei}}, \end{aligned} \quad (29)$$

and that for $R=0$, $\int_{-\infty}^\infty Q(z/\lambda_{ei}; 0, \alpha) dz = 1$. It should also be noted that for $R=0$ the derivative for $z > 0$ is

$$\begin{aligned} Q'(z/\lambda_{ei}; R, \alpha) &= -\frac{45}{120 \lambda_{ei}} \int_0^\infty \left(y - \frac{5}{2}\right) e^{-y} \\ &\times \exp\left(-\frac{\sqrt{45} z}{\lambda_{ei} y^2}\right) dy. \end{aligned} \quad (30)$$

The right derivative at $Z=0$ is thus

$$\begin{aligned} Q'(0; R, \alpha) &= -\frac{45}{120 \lambda_{ei}} \int_0^\infty \left(y - \frac{5}{2}\right) e^{-y} dy \\ &= +\frac{135}{240 \lambda_{ei}}, \end{aligned} \quad (31)$$

which is positive. However, for sufficiently large positive z , the factor $1/y^2$ in the exponential term in z makes this term negligible except at large values of y ; the integral over such values of y is positive and hence the derivative $Q(z/\lambda_{ei}; R, \alpha)$ is negative. We therefore see that there must be a local maximum in Q at a finite value of z . Similar considerations for $z < 0$ imply that $Q(z/\lambda_{ei}; R, \alpha)$ has a local minimum at $z = 0$ (see Figure 1).

Figure 1 shows the form of $Q(z/\lambda_{ei}; R, \alpha)$ for various values of the collisional-to-turbulent mean free path ratio R and its velocity dependence α . The local minimum in $Q(z/\lambda_{ei}; R, \alpha)$, as well as its value at $z = 0$, are both verified. The area under the curve is unity for $R = 0$. For $R > 0$, the width of the kernel function becomes smaller and its area becomes correspondingly less than unity, reflecting the role of turbulence in decreasing the local (and hence overall) heat flux. Indeed, as R increases the reduced mean free path generally makes the heat flux become more and more local in character. The exception is for large α (>4), for which the turbulent mean free path increases faster with velocity than does the collisional mean free path (Equation (3)), so that the role of turbulence in determining the effective mean free path (Equation (4)) becomes negligible and the heat flux reverts to its collision-dominated form.

3. Application

The convolution (25) can be evaluated by calculating the Fourier transforms $\hat{Q}(\lambda_{ei}; R, \alpha)$ of the kernel $Q(z/\lambda_{ei}; R, \alpha)$, and the heat flux $\hat{F}_c(\lambda_{ei}k)$ corresponding to a prescribed temperature profile $T(z)$, respectively, and then evaluating the product

$$\hat{q}_H(\lambda_{ei}k; R, \alpha) = \hat{Q}(\lambda_{ei}k; R, \alpha) \cdot \hat{F}_c(\lambda_{ei}k). \quad (32)$$

Carrying out an inverse Fourier transform on the quantity $\hat{q}_H(k; R, \alpha, \lambda_{ei})$ then gives the heat flux profile q_H . Alternatively, the convolution can be evaluated by directly calculating the integral (25). We illustrate both approaches below.

3.1. Constant Collisional Heat Flux

We first illustrate with the temperature profile

$$T(z) = T_M \left[1 - \left\{ 1 - \left(\frac{T_0}{T_M} \right)^{7/2} \right\} \frac{|z|}{L} \right]^{2/7}; \quad 0 \leq |z| \leq L \quad (33)$$

that corresponds, in a local collisional heat flux approximation, to a heat flux

$$\begin{aligned} F_c(z) &= -\kappa_o T^{5/2} \frac{dT}{dz} = \frac{2\kappa_o}{7L} (T_M^{7/2} - T_0^{7/2}) \operatorname{sgn}(z) \\ &\equiv F_0 \operatorname{sgn}(z), \end{aligned} \quad (34)$$

representing a uniform (divergenceless) heat flux that emanates symmetrically from $z = 0$ and extends over a temperature range from $T = T_M$ at $z = 0$ to $T = T_0$ at $z = \pm L$. Such a temperature profile has a Fourier transform

$$\begin{aligned}\hat{F}_c(\lambda_{ei}k) &= F_0 \int_{-L}^L \text{sgn}(z) e^{-ikz} dz \\ &= (2F_0 / i\lambda_{ei}k)(1 - e^{-ikL}),\end{aligned}\quad (35)$$

so that

$$\hat{q}_H(\lambda_{ei}k; R, \alpha) = \frac{2F_0}{i\lambda_{ei}k}(1 - e^{-ikL}) \hat{Q}(\lambda_{ei}k; R, \alpha). \quad (36)$$

Upon performing the inverse Fourier transform using standard tables of transforms, we obtain

$$q_H(z / \lambda_{ei}; R, \alpha) = F_0 \left[2 \int_{-L}^z Q(z / \lambda_{ei}; R, \alpha) dz - 1 \right]. \quad (37)$$

In the extreme collisional limit $\lambda_{ei} \rightarrow 0$, $Q(z / \lambda_{ei}; R, \alpha)$ approaches a delta-function-like bi-exponential form and we recover the original collisional heat flux profile

$$q_H(z / \lambda_{ei}; R, \alpha) = \begin{cases} -F_0; & -L < z < 0 \\ F_0; & 0 < z < L. \end{cases} \quad (38)$$

However, if $Q(z / \lambda_{ei}; R, \alpha)$ has an extended form, the expression for $q_H(z / \lambda_{ei})$ is more complicated, involving a sampling of positive and negative heat flux components from both sides of the loop, weighted by the area under the $Q(z / \lambda_{ei}; R, \alpha)$ kernel that lies in the pertinent half of the loop; the flux thus varies smoothly from zero at the top of the loop to $\pm F_0$ at distances $|z| \gg \lambda_{ei}$.

3.2. Active Region Loop Temperature Profile

(Martens 2010, cf. his Equation (38)) has shown that for very general heating functions the temperature profile in a static coronal loop is well represented by

$$T(z) = T_M \left(1 - \frac{z^2}{L^2} \right)^{1/3}; \quad -L \leq z \leq L. \quad (39)$$

For such a temperature profile

$$F_c(z) = -\kappa_o T^{5/2} \frac{dT}{dz} = \frac{2\kappa_o T_M^{7/2}}{3L^2} z \left(1 - \frac{z^2}{L^2}\right)^{1/6}. \quad (40)$$

We have evaluated the integral (25) directly for different values of the Knudsen number $\text{Kn} = \lambda_{ei}/L$. This is best done by first rescaling the kernel $Q(z/\lambda_{ei}; R, \alpha)$ to $Q(z/L; R, \alpha) = Q(\text{Kn} \times z/\lambda_{ei}; R, \alpha)$ and then directly evaluating the discretized convolution sum

$$q_H(z_k) = \sum_i F_c(z_k - i) Q_i. \quad (41)$$

Here, the index i labels a set of spatial grid points throughout the loop and the summation extends over the width of the kernel, here truncated at $z = \pm 5\lambda_{ei} = \pm 5(\text{Kn} \times L)$. We normalize the value of q_H to the quantity

$$F_0 = \frac{2}{7} \kappa_o \frac{T_M^{7/2}}{L}, \quad (42)$$

representing the approximate value of $k_0 T^{5/2} (dT/dz) = (2/7) k_0 (dT^{7/2}/dz)$ for a loop of maximum temperature T_M and half-length L . In cases where the kernel extended beyond $z = \pm L$ we used a value $F_c = 0$ for the heat flux at points $|z| > L$ in the evaluation of the convolution (41).

Results for various values of the Knudsen number $\text{Kn} = \lambda_{ei}/L$, are shown in Figures 2 through 5. For $R = 0$ (Figure 2) the results reflect purely collisional conduction, but with a non-local kernel. Since the non-local kernel is a function of z/λ_{ei} (Equation (27) and Figure 1), as the Knudsen number $\text{Kn} = \lambda_{ei}/L$ is increased, the width of the kernel relative to the loop half-length L increases. As a result, the heat flux now becomes a kernel-weighted average over a substantial fraction of the loop, including locations where the heat flux is less than the local value, and even locations in the other half of the loop where the heat flux has the opposite sign. Consequently, the actual heat flux is, for modest values of the Knudsen number Kn , in most cases only a fraction of the (Spitzer 1962) value calculated using local values of the temperature and its gradient (the exception to this, of course, is near the ends of the loop, where the local heat flux vanishes, but the non-local heat flux is finite).

Even for values as low as $R = 0.3$, turbulent limitation of the heat flux can be a significant factor (Figure 3). Since the term involving R in the kernel (27) prefixes a velocity-dependent term $x^{4-\alpha}$, where α is the exponent representing the velocity dependence of the turbulent mean free path (Equation (4)), larger values of α cause the term involving R to become more weighted to smaller values of the dimensionless velocity $x = v/v_{te}$, which have a much smaller influence on the size of the kernel $Q(z/\lambda_{ei}; R, \alpha)$ (Equation (27)). As a result, the non-local heat flux values approach the values for the purely collisional ($R = 0$) case (cf. Figure 2). For smaller values of α , however, the term in R becomes more important and

hence the heat flux is reduced considerably. Physically, this is because for small values of α the turbulent mean free path λ_T increases only slowly with velocity (Equation (3)), and so is much smaller than the collisional mean free path at the high velocities that are responsible for most of the heat transport. For values of R greater than unity and values of $\alpha \lesssim 4$, the width of the kernel function (27) becomes sufficiently small (see also Figures 4 and 5) that non-local effects cease to be important. However, the reduced value of λ at high velocities acts to reduce the value of the heat flux by a large factor (Figures 4 and 5).

4. Cooling of Post-flare Loops

It has long been known from analysis of soft X-ray time profiles during the decay phase of solar flares (e.g., Moore et al. 1980; Ryan et al. 2013) that post-flare loops cool on a timescale much longer than expected on the basis of collision-dominated thermal conduction, and this is often cited (e.g., Emslie et al. 2012) as evidence for some form of post-impulsive-phase heating. On the other hand, such an increase in the cooling time could instead be due to turbulent limitation of the conductive flux. Using a static loop model, Bian et al. (2016a) found that, allowing for the possible presence of turbulence, the cooling time can be expressed as the transit time for an electron with the thermal speed divided by the collisional Knudsen number, multiplied by a factor $(1 + R)$, where R is the usual turbulence-related heat flux limitation factor:

$$\tau_c \simeq \left(\frac{L}{v_{te}} \right) \left(\frac{L}{\lambda_{ei}} \right) (1 + R). \quad (43)$$

Hence, in the presence of turbulence the cooling time τ_c is increased by a factor $(1 + R)$, which significantly reduces the requirement (Moore et al. 1980; Emslie et al. 2012) for post-impulsive phase heating of the coronal plasma.

As noted above, the results of Bian et al. (2016a) were based on a static loop model. However, it is well known that mass motions play a significant role in the evolution of flare loops. Klimchuk et al. (2008) and Cargill et al. (2012) have developed the “zero-dimensional” Enthalpy-Based Thermal Evolution of Loops (EBTEL) formalism to describe the evolution of loop-averaged temperature and densities. The essential element in this model is enthalpy balance: heating to high coronal temperatures produces large conductive fluxes toward the chromosphere. This in turn drives evaporative flows of material upward into the corona, increasing coronal radiative losses and stabilizing the temperature profile. On the other hand, during the cooling phase, the reduction in conductive energy input relative to radiative cooling causes a temperature decrease. This leads to draining of coronal material, reducing the coronal density and thus radiative losses, and again stabilizing the temperature profile.

Bian et al. (2018) have recently used the EBTEL model to explore the cooling of post-flare loops in the presence of a heat conductive term that includes both collisional and turbulent scattering. They compared their model results with observed peak temperatures and cooling times (Ryan et al. 2013) and hence showed the following:

1. Consistent with earlier studies (e.g., Moore et al. 1980), even with the inclusion of mass motions, collisional conduction produces cooling times (using, as a reference, the cooling from $T \simeq 1.5 \times 10^7$ to 10^7 K) much shorter than observed.
2. If an impulsive heating term was assumed, then cooling times consistent with those observed could only be achieved with an unreasonably high peak temperature $\simeq 10^8$ K.
3. Spreading the heating over a finite time (including substantial heating during the decay phase) did help bring model results into alignment with observations; however, the relatively rapid cooling associated with collision-dominated conduction required an unreasonably large amount of energy input $\simeq 3000 \text{ erg cm}^{-3}$ to sustain the temperature for the 200 s or so observed (Ryan et al. 2013). This corresponds to more than the typical magnetic energy available.

These results led Bian et al. (2018) to conclude that the EBTEL model could be brought into consistency with observed peak temperatures and cooling times, with a reasonable amount of energy input required, only if the conductive cooling was suppressed by invoking a relatively short (velocity-averaged) turbulent mean free path λ_T . Specifically, they showed that the turbulent mean free path λ_T must be of the order of 1.4×10^7 cm. Using $\lambda_{ei} \simeq 2 \times 10^4 T^2/n$, with $T = 10^7$ K and $n = 5 \times 10^{10} \text{ cm}^{-3}$ gives $\lambda_{ei} \simeq 4 \times 10^7$ cm, so that the inferred value for λ_T corresponds to $R = \lambda_{ei}/\lambda_T \simeq 3$.

For $R = 3$ the behavior of the heat flux depends critically on the value of the exponent α that governs the dependence of the turbulent mean free path with velocity.

1. For the small values $\alpha \ll 1$ discussed in Section 2, the very narrow form of the non-local kernel $Q(z/\lambda_{ei}, R, \alpha)$ (Figure 1) means that non-local effects are generally not important in determining the conductive flux; however the lower values of the effective mean free path in the presence of turbulent angular scattering mean that the heat flux is significantly reduced from its collisional value.
2. For $\alpha = 4$ (so that the collisional and turbulent mean free paths have the same velocity dependence), the *local* heat flux is straightforwardly reduced by a factor $\sim (1 + R) = 4$. Furthermore, non-local effects have an effect that is similar to the collision-dominated ($R = 0$) case. In general, this weighted averaging of the local heat flux results in a reduction in the heat flux throughout most of the loop, except at the ends of the loop (Figure 5), where the heat flux is no longer its local value of zero, but approximately 20% of the peak value.
3. For values of $\alpha > 4$ (so that the turbulence mean free path λ_T increases more steeply with velocity than the collisional mean free path λ_C), the very large turbulent mean free paths appropriate to the high velocities that carry the bulk of the heat flux mean that the non-local kernel is driven toward its collisional value (see Equation (27)). In such a case the flux reduction factor resulting from non-local effects is very similar to the collisional case, straightforwardly scaling with the Knudsen number $\text{Kn} = \lambda_{ei}/L$.

We see that for $R \approx 3$, the heat flux is, for any value of α , significantly changed from its collision-dominated value. For low values of α , this is due primarily to the reduction in the local mean free path λ (Equation (4)), while for high values of α this change is primarily due to non-local effects.

5. Discussion and Conclusions

On physical grounds, it is natural to expect (e.g., Albritton et al. 1986; Karpen & DeVore 1987) that in the absence of turbulence non-local effects on the magnitude of the heat flux extend over a distance of the order of the collisional mean free path λ_{ei} calculated at the thermal velocity. A quantitative analysis (Section 2) essentially confirms this: although the effective sampling distance involved in the expression for the particle flux turns out to be significantly smaller than the mean free path ($\sim \lambda(v) / \sqrt{45}$), high-velocity particles dominate the expression for the heat flux, and since, for collisions, $\lambda(v) \propto v^4$, non-local effects are important even for values of the Knudsen number λ_{ei}/L that are somewhat smaller than unity (Figure 2).

(Bian et al. 2016a) have emphasized that in conditions appropriate to the solar corona after flare energy release, turbulent angular scattering can be as important as, if not more important than, Coulomb collisions in determining the appropriate value of the mean free path λ (Equation (4)) and hence in the magnitude of the thermal conductive flux (see Equation (18)). We have therefore derived a general expression for the conductive heat flux in the presence of both non-local effects and turbulent scattering; this expression (Equation (25)) takes the form of a convolution of the local (Spitzer 1962) heat flux with a kernel function $Q(z/\lambda_{ei}, R, \alpha)$ (Equation (27)) that reflects the roles of both collisional and turbulent scattering. The shape of this non-local kernel depends on the relative magnitudes R of turbulent and collisional scattering at the thermal velocity, and on the velocity dependence of the turbulent mean free path (α , Equation (4)).

In general, for large values of the turbulence limiting factor R (Equation (4)) and small values of its velocity dependence α (e.g., $\alpha \approx 1/3$ for a Kolmogorov spectrum of magnetic fluctuations), the mean free path (Equation (4)) becomes so small that conduction is effectively local in character. However, for such high values of R the heat flux is much less than its collisional (Spitzer 1962) value (Figures 3 through 5). On the other hand, for low values of the turbulent reduction factor R and/or high values of the velocity exponent α , collisional conduction dominates at the velocities that carry the bulk of the heat flux. In such cases non-local effects can be substantial, which again results in a reduction of the heat conduction relative to its Spitzer (1962) value (Figure 2). Only for collision-dominated conduction ($R \ll 1$) with a very low value of the Knudsen number $\text{Kn} = \lambda_{ei}/L$ is a local, collisional, expression (Spitzer 1962) for the heat flux valid; in all other cases the actual heat flux is substantially less than this reference value over most of the volume.

For active region loops, the changed form of the relationship between heat flux and temperature gradient will lead to revised forms of the scaling laws relating loop temperature, pressure, and length (e.g., Rosner et al. 1978; Martens 2010). For heated post-flare loops, the reduction in the heat flux caused by turbulent and/or non-local effects will have the

combined effects of (1) increasing the coronal temperature and (2) reducing the heat flux into the chromosphere, thus suppressing chromospheric evaporation. As a result, the thermal and hydrodynamic behaviors in the loop (Bian et al. 2018) will be significantly altered. Quantitative evaluation of both these effects requires numerical modeling (Allred et al. 2015) that is beyond the scope of the present paper, but which will also be the subject of future works.

References

- Albritton JR, Williams EA, Bernstein IB, & Swartz KP 1986, *PhRvL*, 57, 1887
- Allred JC, Kowalski AF, & Carlsson M 2015, *ApJ*, 809, 104
- Bian N, Emslie AG, Horne D, & Kontar EP 2018, *ApJ*, 852, 127
- Bian NH, Emslie AG, & Kontar EP 2017, *ApJ*, 835, 262
- Bian NH, Kontar EP, & Emslie AG 2016a, *ApJ*, 824, 78
- Bian NH, Watters JM, Kontar EP, & Emslie AG 2016b, *ApJ*, 833, 76
- Brown JC, Spicer DS, & Melrose DB 1979, *ApJ*, 228, 592
- Cargill PJ, Bradshaw SJ, & Klimchuk JA 2012, *ApJ*, 752, 161
- Emslie AG, Dennis BR, Shih AY, et al. 2012, *ApJ*, 759, 71
- Fang X, Yuan D, Xia C, Van Doorselaere T, & Keppens R 2016, *ApJ*, 833, 36
- Gombosi TI, Jokipii JR, Kota J, Lorencz K, & Williams LL 1993, *ApJ*, 403, 377
- Gordovskyy M, Browning PK, Kontar EP, & Bian NH 2014, *A&A*, 561, A72
- Karpen JT, & DeVore CR 1987, *ApJ*, 320, 904
- Klimchuk JA, Patsourakos S, & Cargill PJ 2008, *ApJ*, 682, 1351
- Kontar EP, Perez JE, Harra LK, et al. 2017, *PhRvL*, 118, 155101
- Kruskal MD, & Bernstein IB 1964, *PhFl*, 7, 407
- Luciani JF, Mora P, & Pellat R 1985, *PhFl*, 28, 835
- Luciani JF, Mora P, & Virmont J 1983, *PhRvL*, 51, 1664
- Martens PCH 2010, *ApJ*, 714, 1290
- Miller JA, Cargill PJ, Emslie AG, et al. 1997, *JGR*, 102, 14631
- Miller JA, Larosa TN, & Moore RL 1996, *ApJ*, 461, 445
- Moore R, McKenzie DL, Svestka Z, et al. 1980, in *Skylab Solar Workshop II*, ed. Sturrock PA (Boulder, CO: Colorado Associated Univ. Press), 341
- Rosner R, Tucker WH, & Vaiana GS 1978, *ApJ*, 220, 643
- Rudakov LI, & Korabev LV 1966, *JETP*, 23, 145
- Ryan DF, Chamberlin PC, Milligan RO, & Gallagher PT 2013, *ApJ*, 778, 68
- Smith DF 1986, *ApJ*, 302, 836
- Smith DF, & Lilliequist CG 1979, *ApJ*, 232, 582
- Spitzer L 1962, *Physics of Fully Ionized Gases* (New York: Interscience)
- West MJ, Bradshaw SJ, & Cargill PJ 2008, *SoPh*, 252, 89

Figure 1.

Shape of the convolution kernel $Q(z, R, \alpha, \lambda_{ei})$ (Equation (27)), for different values of the mean free path ratio R and velocity exponent α (Equation (4)).

Figure 2. Non-local modification to the heat flux profile for an active region loop, for $R = 0$ and different values of the Knudsen number $\text{Kn} = \lambda_{ei}/L$.

Figure 3.

Non-local modification to the heat flux profile for an active region loop, for $R = 0.3$ and various values of the exponent α and the Knudsen number $\text{Kn} = \lambda_{ei}' L$.

Figure 4.

Non-local modification to heat flux profile for an active region loop, for $R = 1$ and various values of the exponent α and the Knudsen number $\text{Kn} = \lambda_{ei}/L$.

Figure 5.

Non-local modification to heat flux profile for an active region loop, for $R = 3$ and various values of the exponent α and the Knudsen number $\text{Kn} = \lambda_{ei}/L$.