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Direct 2-D reconstructions of conductivity and permittivity from EIT data on a human chest

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Abstract

A novel direct D-bar reconstruction algorithm is presented for reconstructing a complex conductivity distribution from 2-D EIT data. The method is applied to simulated data and archival human chest data. Permittivity reconstructions with the aforementioned method and conductivity reconstructions with the D-bar method based on [22] depicting ventilation and perfusion in the human chest are presented. This constitutes the first fully nonlinear D-bar reconstructions of human chest data and the first D-bar permittivity reconstructions of experimental data. The results of the human chest data reconstructions are compared on a circular domain versus a chest-shaped domain.

Keywords

electrical impedance tomography; reconstruction algorithm; D-bar method; scattering transform; ventilation; perfusion

I. INTRODUCTION

ELECTRICAL impedance tomography (EIT) is a noninvasive, nonionizing imaging technique with significant promise for visualizing functional changes associated with ventilation and perfusion at the bedside in the ICU [8], [18]. EIT images depict the conductivity distribution of the medium, or more generally, the conductivity and permittivity, which is modeled as a complex coefficient $\gamma = \sigma + i\omega\epsilon$ in the generalized Laplace equation

$$\nabla \cdot (\gamma(x, y) \nabla u(x, y)) = 0, \quad (x, y) \in \Omega. \quad (1)$$

The conductivity σ is assumed to be bounded away from zero $0 < c \leq \sigma(x, y)$, and the permittivity is assumed to be non-negative $\epsilon(x, y) \geq 0$. The variable ω represents the angular frequency of the applied current, and $u(x, y)$ represents the electric field. In this work the domain Ω represents the region enclosed in the plane of electrodes placed around the circumference of a subject's chest. Low frequency, low amplitude current is applied on the electrodes, and the knowledge of the current density on the boundary is modeled as a Neumann boundary condition

$$\gamma(x, y) \frac{\partial u}{\partial \nu}(x, y) = j(x, y), \quad (x, y) \in \partial\Omega,$$

where ν is the unit outward normal to the boundary, and the resulting voltage is measured on the electrodes. We will denote the unit tangent vector by $\tau = \nu^\perp$. The Dirichlet boundary condition

$$u(x, y) = f(x, y), \quad (x, y) \in \partial\Omega,$$

corresponds to knowledge of the (measured) voltage on the boundary. The reader is referred to [6], [25] for discussions of electrode models in EIT.

The data for the reconstructions is the voltage-to-current density, or Dirichlet-to-Neumann (DN), map denoted by Λ_γ and defined by

$$\Lambda_\gamma: u|_{\partial\Omega} \rightarrow \gamma \frac{\partial u}{\partial \nu},$$

and the current density-to-voltage mapping (ND), will be denoted by R_γ

$$R_\gamma: \gamma \frac{\partial u}{\partial \nu} \rightarrow u|_{\partial\Omega}.$$

The construction of matrix approximations to R_γ and Λ_γ is described in Section II, which also includes the description of the (archival) data collection.

D-bar methods are a class of direct (noniterative) reconstruction methods that make use of complex geometrical optics (CGO) solutions to partial differential equations known as D-bar, or $\bar{\partial}$, methods because they are of the form $\bar{\partial}u = f$ where f may depend on u and the $\bar{\partial}$ operator is defined by $\bar{\partial} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$. For an introduction to D-bar methods in EIT, see [25]. A common feature of D-bar methods is the need to compute an intermediate function $\mathbf{t}(k)$ known as the scattering transform as part of the reconstruction algorithm. The scattering transform is a nonlinear Fourier Transform that plays a key role in the method and arises as a coefficient in the D-bar equation for the CGO solutions. These steps can be expressed in general terms by

$$\Lambda_\gamma \rightarrow \mathbf{t}(k) \rightarrow \text{CGO solution} \rightarrow \gamma$$

In this work a new direct nonlinear reconstruction algorithm using the D-bar method is presented for the computation of permittivity on a chest-shaped domain, and the fully nonlinear D-bar method for reconstructing conductivity based on the global uniqueness proof by Nachman [28] is further developed for use on human subject data on a chest-shaped domain. The new method for reconstructing permittivity is based on the uniqueness proof by Francini [12], but equations relating the Dirichlet-to-Neumann to the scattering transform and the exponentially growing solutions are not present in that work. Such equations are derived in [15] with a new expression for the scattering transform derived here, which differs from what is used in [14]–[16]. The algorithm in [14]–[16] requires the computation of two sets of intermediate CGO solutions, both requiring the solution to an integral equation on the boundary of the domain, and both involving the DN map. Hence the inherent ill-posedness of the problem arises at two steps in the computation. In contrast, the method presented here requires the solution of only one set of CGO solutions, and hence only one set of sensitive integral equations to be solved. (We refer to sets of CGO solutions because the method is based on a 2×2 elliptic system and so the CGO solutions are 2×2 matrices of exponentially growing functions.) The difference in the scattering transforms is explained in Section IV. One significant difference is that the scattering transform derived here makes use of the tangential derivative map and the DN map, while that of [14]–[16] does not use the tangential derivative map.

To illustrate that the algorithm is sufficiently robust for experimental data including uncertainty in the boundary shape and electrode positions, 2-D reconstructions of conductivity and permittivity changes due to ventilation and perfusion in the chest of a healthy adult male subject sitting upright are presented. The real part of the admittivity images using the permittivity method of reconstruction were inferior to those computed by the D-bar method for reconstructing the conductivity only. This is the motivation for including both methods. However, the complex admittivity method is effective in reconstructing both conductivity and permittivity in simulated data as demonstrated in Section V. The real part of the perfusion data was previously used with the direct reconstruction algorithms in [3], [9], [20] on a circular approximation to the domain. In [20], a first-order approximation to the scattering transform, denoted in the literature by $\mathbf{t}^{\text{exp}}$, was employed to obtain D-bar reconstructions of conductivity changes due to perfusion. In [9] a second-order approximation to the scattering transform was used, and in [3] Calderón's method was applied to these data sets. Here we compute fully nonlinear approximations to the scattering transform to obtain difference images of the conductivity on a chest-shaped domain. These are the first fully nonlinear reconstructions of human data using the D-bar method with the full scattering transform. Reconstructions of simulated and tank data using the $\mathbf{t}^{\text{exp}}$ approximation on noncircular domains were computed and studied in [26], [27]. The reconstructions of the permittivity are the first direct D-bar reconstructions of permittivity from experimental data and illustrate the feasibility and potential of this method.

The description of the archival human chest data and the construction of the DN map are described in Section II. The computational method for the reconstruction of conductivity is given in Section III, and the method for the computation of the permittivity is presented in

Section IV. The reconstructions from the human chest data are found and discussed in Section V. Movies of the reconstructions are found online at [36].

II. DESCRIPTION OF THE DATA COLLECTION

We consider two sets of 100 frames of archival data collected at 18 frames/s on the chest of an adult male sitting upright, using the ACT3 system at RPI [10]. One set was collected during breathholding to image perfusion, and the other set consisted of a deep slow inhalation followed by slow exhalation to image ventilation. In both cases, the trigonometric current patterns

$$T_l^m = C \begin{cases} \cos(m\theta_l), & m = 1, \dots, \frac{L}{2} \\ \sin\left(\left(m - \frac{L}{2}\right)\theta_l\right), & m = \frac{L}{2} + 1, \dots, L - 1 \end{cases} \quad (2)$$

with $C = 0.85$ mA were applied on 32 electrodes of size 2.9 cm high and 2.4 cm wide placed around the circumference of the chest, which measured 90 cm. The real part of the breathholding data was also used in the studies [9], [20] to reconstruct the conductivity using first and second-order approximations to the scattering transform, respectively.

We model the current density $j(x, y)$ by the gap model in the computation of the DN map so that

$$j(x, y) = \begin{cases} \frac{T_l^m}{A_l}, & (x, y) \in e_l \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

where A_l is the area of the l th electrode, e_l , which were of uniform size. For construction of the DN map, the current patterns and the resulting voltages must be normalized. Let t_l^m denote the $(l, m)^{th}$ entry of the matrix of applied currents with each column normalized with respect to the ℓ^2 vector norm. Let $\mathbf{T}$ denote the L by $L - 1$ matrix of current patterns where $\mathbf{T}(l, m) = t_l^m$. Let v_l^m denote the entries of the m^{th} voltage matrix normalized so that

$\sum_{l=1}^L v_l^m = 0$ for each m , and $v_l^m = \frac{V_l^m}{\|\mathbf{T}(:, m)\|_2}$ where V_l^m is the measured voltage on electrode l corresponding to current pattern m .

Finite dimensional approximations to the DN and ND maps will be denoted by Λ_γ^M and R_γ^M , respectively, where the M stands for matrix. They are size $L - 1 \times L - 1$. The ND map R_γ maps functions from the Sobolev spaces $\tilde{H}^{-1/2}(\partial\Omega)$ to $\tilde{H}^{1/2}(\partial\Omega)$ where the tilde is to indicate that the functions must have mean value zero. For a boundary voltage arising from a current pattern $j^n(x, y)$, the $(m, n)^{th}$ entry of R_γ^M is given by

$$R_{\gamma}^M(m, n) = \int_{\partial\Omega} u^n(x, y) \overline{j^m(x, y)} d\tau.$$

For a general domain with boundary parameterized by $r(\theta)$, the arclength differential is given by

$$d\tau = s(\theta)d\theta = \sqrt{r^2(\theta) + (r'(\theta))^2} d\theta.$$

So

$$R_{\gamma}^M(m, n) \approx \sum_{l=1}^L \frac{1}{A_l} t_l^m v_l^n ds_l$$

where ds_l is the arclength differential evaluated at the angle θ_l corresponding to the center of the l^{th} electrode. Note that the current patterns are real-valued, and so the complex conjugate is omitted. Since the voltages sum to zero, we can then compute the matrix approximation to Λ_{γ} from $\Lambda_{\gamma}^M = (R_{\gamma}^M)^{-1}$.

In the computational work, the domain is scaled to have a maximal radius of one. Let R denote the maximal radius of the domain. Since the human chest data is archival and the actual domain geometry and electrode positions are not available, we model the domain by both a circle of radius $90/(2\pi)$ and a representative chestshaped domain of the same perimeter, which was 90 cm. Both domains are depicted in Figure 1. We assumed the electrodes were equally spaced around the chest perimeter. Let $\Lambda_{\gamma, R}$ denote the Dirichlet-to-Neumann map for a domain with conductivity γ and of maximal radius R . Then $\Lambda_{\gamma, 1}$ and $\Lambda_{\gamma, R}$ are related by $\Lambda_{\gamma, 1} = R\Lambda_{\gamma, R}$. In the following, we use the notation Λ_{γ} for $\Lambda_{\gamma, 1}$ and a subscript of R for the unscaled domain.

The reconstructions depict changes in conductivity and permittivity from a reference data set. We will denote the DN map for the reference data set by Λ_{ref} . For the perfusion data, the reference data set was chosen to be a frame collected at mid-systole, that is, when the heart was halfway through a contraction. For the ventilation data set, the reference frame was the first frame, which was at the end of a full expiration.

III. FULLY NONLINEAR RECONSTRUCTION OF THE CONDUCTIVITY

Fully nonlinear reconstructions of the conductivity were computed with the 2-D D-bar method based on the 1996 uniqueness proof of A. Nachman [28], which was developed as a practical algorithm in [19]–[22], [27], [31]. A nonlinear regularization method for the algorithm was established in [22]. The reader is referred to these papers and to the text [25] for a thorough description of the method. Here we provide a brief outline and the methods used to compute on the noncircular domain.

The D-bar method of this section assumes the conductivity is twice differentiable, and the starting point is the change of variables $q(x, y) = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}$ and $\tilde{u} = \gamma^{1/2}u$ in (1). This results in

$$-\Delta\tilde{u} + q(x, y)\tilde{u} = 0 \quad (x, y) \in \Omega. \quad (4)$$

The assumption that γ is constant in a neighborhood of the boundary implies that $q = 0$ in that neighborhood, and allows one to extend (4) to the whole plane, and construct solutions with a certain type of asymptotic behavior, known as complex geometrical optics, or CGO, solutions. The CGO solutions depend on a complex parameter $k = k_1 + ik_2$, and have the property that they are asymptotic to e^{ikz} in a certain sense for large $|z|$ or large $|k|$, where $z = x + iy$ is the spatial variable in 2-D identified with the corresponding point in the complex plane. We will denote the exponentially growing solutions by $\psi(z, k)$. Then $\psi(z, k)$ satisfies

$$-\Delta\psi(z, k) + q(z)\psi(z, k) = 0, \quad z \in \mathbb{R}^2$$

with $e^{-ikz}\psi(z, k) - 1 \in W^{1,p}(\mathbb{R}^2)$, $p > 2$ where $W^{1,p}(\mathbb{R}^2)$ is the Sobolev space of functions with one weak derivative in $L^p(\mathbb{R}^2)$ [1].

It was shown in [28] that $\psi|_{\Omega}$ satisfies a Fredholm integral equation of the second kind, and the computation of $\psi|_{\Omega}$ is the first step of the reconstruction algorithm. A first-order approximation to $\psi|_{\Omega}$ is e^{ikz} , and taking this approximation to $\psi|_{\Omega}$ results in a first-order approximation to the scattering transform known as t^{exp} . This approximation, introduced in [31] has been successfully used on a variety of experimental data, including the perfusion data set studied here in [20]. To understand the second-order approximation, we must look at the formula for $\psi|_{\Omega}$. The Faddeev Green's function [11] is a special Green's function for the Laplacian defined by $-G_k(z) = \delta(z)$, where

$$G_k(z) = e^{ikz}g_k(z) = e^{ikz} \int_{\mathbb{R}} \frac{e^{iz \cdot \xi}}{2\xi(\xi + 2k)} d\xi.$$

The integral equation for $\psi|_{\Omega}$ is given for each $k \in \mathbb{C}$ and $z \in \Omega$ by [28]

$$\psi(z, k) = e^{ikz} - \int_{\partial\Omega} G_k(z - \zeta(s)) \delta\Lambda\psi(\zeta(s), k) d\zeta(s). \quad (5)$$

The Faddeev Green's function differs from the standard Green's function for the Laplacian $G_0(z)$ by a harmonic function $H_k(z)$. That is, $G_k(z) = \frac{1}{2\pi} \log|z| + H_k(z)$. (See [25], [30] for properties of the Faddeev's Green's function.) Replacing G_k by G_0 in (5) results in second-order approximations to $\psi|_{\Omega}$ and the scattering transform, which we will denote by $\psi^0|_{\Omega}$ and $t^0(k)$, respectively. The second-order approximations were used with the perfusion data sets in [9] on a circular domain.

The formula relating the scattering transform $\mathbf{t}(k)$ to the measured data is [28]

$$\mathbf{t}(k) = \int_{\partial\Omega} e^{\overline{ikz(s)}} (\delta\Lambda) \psi(z(s), k) dz(s). \quad (6)$$

In practice, the numerically computed scattering transform blows up in the presence of noise and is therefore computed on a restricted domain in the k plane. In [22], this was shown to comprise a nonlinear regularization scheme for the D-bar method. Here we compute the scattering transform on a disk of radius R , and truncate the values to zero outside that disk.

The reconstruction of γ comes from solving the D-bar equation for the function $\mu(z, k)$ defined by

$$\mu(z, k) \equiv e^{-ikz} \psi(z, k), \quad (7)$$

where $k = k_1 + ik_2 \in \mathbb{C}$ and $z = x + iy \in \mathbb{C}$. Let

$$e_{-z}(k) = e^{-i(kz + \bar{k}\bar{z})} \quad (8)$$

The D-bar equation satisfied by μ is

$$\frac{\partial \mu(z, k)}{\partial \bar{k}} = \frac{\mathbf{t}(k)}{4\pi k} e_{-z}(k) \overline{\mu(z, k)}. \quad (9)$$

Using the Generalized Cauchy Integral Formula, this can be written as an integral equation, and we define μ to be the solution of the equation in integral form:

$$\mu(z, k') = 1 + \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{\mathbf{t}(k)}{(k' - k)k} e_{-z}(k) \overline{\mu(z, k)} dk_1 dk_2.$$

Finally, an approximation to the conductivity difference can be recovered from $\mu(z, k)$ using the formula

$$\gamma(z) = (\mu(z, 0))^2, \quad z \in \Omega. \quad (10)$$

A. Computational considerations

The computation of the DN map was described in Section II. Throughout this work, we will replace $\Lambda_\gamma - \Lambda_1$ by $\Lambda_\gamma - \Lambda_{ref}$ as was done in [3], [9], [20]. Let $\delta\Lambda \equiv R(\Lambda_\gamma - \Lambda_{ref})$, and let $\delta\Lambda_\gamma^M$ denote the finite dimensional approximation $R(\Lambda_\gamma^M - \Lambda_{ref}^M)$. Here we describe briefly the

computations on the chest-shaped domain, and the computations for the circular domain follow as a special case.

Let $z(\theta) = r(\theta)e^{i\theta}$ represent a point on the boundary of the scaled domain expand $\psi(z, k)$ and e^{ikz} in the trigonometric basis

$$\psi(z(\theta), k) \approx \sum_{j=1}^{L-1} b_j(k) r^j(\theta), \quad e^{ikz(\theta)} \approx \sum_{j=1}^{L-1} c_j(k) r^j(\theta), \quad (11)$$

where the coefficients are given by

$$b_j(k) = \int_{\partial\Omega} \psi(z, k) r^j(z) d\tau \approx \sum_{l=1}^L \psi(z_l, k) r_l^j ds_l$$

$$c_j(k) = \int_{\partial\Omega} e^{ikz} r^j(z) d\tau \approx \sum_{l=1}^L e^{iz_l k} r_l^j ds_l.$$

Let $\hat{r}^j(\theta)$ denote the action of $\delta\Lambda$ on the j^{th} basis function, evaluated at angle θ . Let r_l^j denote the j^{th} basis function evaluated at angle $\theta_l = 2\pi l/L$. Evaluate equation (5) at the center of the l^{th} electrode $z_l = r(\phi_l)e^{i\phi_l}$ and approximate the integral over the boundary by a sum of integrals over the electrodes to obtain

$$\sum_{m=1}^{L-1} b_m(k) r_l^m = \sum_{m=1}^{L-1} c_m(k) r_l^m - \sum_{j=1}^{L-1} b_j(k) \sum_{\ell'=1}^L \int_{e_{\ell'}} s(\theta) G_k(z_{\ell} - \zeta(\theta)) f^j(\theta) d\theta. \quad (12)$$

The convolution of G_k with $\hat{r}^j(\theta)$ was computed using a fine discretization over electrode ℓ' when $\ell = \ell'$ and the value at the center when $\ell' = \ell$. This results in

$$\int_{e_{\ell'}} s(\theta) G_k(z_l - \zeta(\theta)) f^j(\theta) d\theta \approx \begin{cases} ds_{\ell'} G_k(z_{\ell} - \zeta_{\ell'}) f^j(\theta_l), & \ell' \neq \ell \\ \frac{1}{(N_e - 1)} \sum_{n=1}^{N_e} ds_{\ell_n} G_k(z_{\ell} - \zeta_{\ell_n}) f^j(\theta_l) & \ell' = \ell. \end{cases} \quad (13)$$

Let the matrix $\tilde{\mathbf{G}}_k$ be defined such that $\tilde{\mathbf{G}}_k(\ell, \ell')$ is the right-hand side of (13).

The function G_k was computed as in [2] using

$$G_k(z) = e^{ikz} g_1(kz) = \frac{1}{4\pi} \Re(EI(ikz))$$

where $E\mathcal{I}(z)$ is the exponential integral function. Let $\mathbf{b}_k$ be the vector of unknown coefficients of $\psi(\zeta, k)$ and $\mathbf{c}_k$ be the vector of coefficients in the expansion of e^{ikz} from (11). Then equation (12) can be expressed in matrix form as

$$\mathbf{T}\mathbf{b}_k = \mathbf{T}\mathbf{c}_k - \tilde{\mathbf{G}}_k \mathbf{T} \delta \Lambda_\gamma^M \mathbf{b}_k.$$

Since the basis $\mathcal{I}$ is orthonormal, $\mathbf{T}^T \mathbf{T} = \mathbf{I}$ and so defining $\mathbf{A} \equiv \mathbf{T}^T \tilde{\mathbf{G}}_k \mathbf{T} \delta \Lambda_\gamma^M$, we obtain the following linear system, which was solved using GMRES [29]

$$(\mathbf{I} + \mathbf{A})\mathbf{b}_k = \mathbf{c}_k.$$

For the computation of the scattering transform, expand $e^{ik\bar{z}}$ on the scaled domain with $z(\theta) = \frac{r(\theta)}{R} e^{i\theta}$ as in equation (11) with coefficients $d_j(k)$. Then from (6)

$$t(k) \approx R \sum_{j=1}^{L-1} \sum_{m=1}^{L-1} b_m d_j \sum_{l=1}^L \int_{e_l} r^j(\theta) \delta \Lambda_\gamma^M t^m(\theta) d\tau \approx R \sum_{j=1}^{L-1} \sum_{m=1}^{L-1} b_m(k) d_j(k) \sum_{l=1}^L r_l^j f_l^m ds_l$$

The D-bar equation in integral form

$$\mu(z, k') = 1 + \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{\mathbf{t}(k)}{\bar{k}(k' - k)} e^{-k(z)} \overline{\mu(z, k)} dk \quad (14)$$

was solved for $\mu(z, k)$ at each point z in the scaled domain using the method in [23] in the region of interest on a k mesh with $|k| \leq R$, an empirically chosen truncation radius. The reconstruction of $\chi(z)$ was then obtained from (10).

IV. RECONSTRUCTION OF THE COMPLEX ADMITTIVITY

A direct reconstruction algorithm for computing complex admittivities based on the uniqueness proof in [12] was presented in [15]. Here we present a different algorithm, in which only one set of CGO solutions is computed and a different formula is derived for the scattering transform matrix.

The theory in [12] begins with a change of variables. Let

$$Q_{12}(z) = -\frac{1}{2} \partial_z \log \gamma(z) \quad (15)$$

$$Q_{21}(z) = -\frac{1}{2} \bar{\partial}_z \log \gamma(z),$$

and define a vector $[v, w]^T = \gamma^{1/2} [\partial u, \bar{\partial} u]^T$, in terms of the solution u to (1). Then $\vec{v} \equiv (v, w)^T$ satisfies $D\vec{v} - Q_\gamma \vec{v} = 0$, where

$$D = \begin{pmatrix} \bar{\partial} & 0 \\ z & \partial_z \end{pmatrix},$$

and Q_γ is defined to be the matrix with diagonal elements 0 and off-diagonal entries defined by (15).

Introducing the complex variable $k \in \mathbb{C}$, Francini shows [12] there exists a unique matrix $\Psi(z, k)$ defined by

$$\Psi(z, k) = M(z, k) \begin{pmatrix} e^{izk} & 0 \\ 0 & e^{-i\bar{z}k} \end{pmatrix}$$

that is a solution to $(D - Q_\gamma)\Psi = 0$, where the matrix $M(z, k)$ is asymptotic to the identity matrix I in the sense that $M(\cdot, k) - I \in L^p(\mathbb{R}^2)$, for some $p > 2$. The scattering transform matrix $S(k)$ with entries $S_{ij}(k)$, $i, j = 1, 2$ is defined by matrix with zeros on the diagonal and off-diagonal entries

$$S_{12}(k) = \frac{i}{\pi} \int_{\mathbb{R}^2} Q_{12}(z) e_z(-\bar{k}) M_{22}(z, k) dz$$

$$S_{21}(k) = -\frac{i}{\pi} \int_{\mathbb{R}^2} Q_{21}(z) e_z(k) M_{11}(z, k) dz$$

where $e_A(k)$ is defined as in (8).

The formula for the scattering transform requires knowledge of two special CGO solutions u_1 and u_2 to (1) on the boundary of Ω . These solutions were derived in [15], [35] and have the property that $u_1 \sim e^{ikz}/(ik)$ and $u_2 \sim e^{-ik\bar{z}}/(-ik)$. They relate to the DN data through the boundary integral equations for $z \in \Omega$

$$u_1(z, k) = \frac{e^{ikz}}{ik} - \int_{\partial\Omega} G_k(z - \zeta) \delta \Lambda u_1(\zeta, k) d\tau,$$

$$u_2(z, k) = \frac{e^{-ik\bar{z}}}{-ik} - \int_{\partial\Omega} G_k(-\bar{z} + \bar{\zeta}) \delta \Lambda u_2(\zeta, k) d\tau,$$

where $G_k(z)$ is the Faddeev Green's function defined in the previous section. These solutions are related to the CGO solutions ψ_{ij} , $i, j = 1, 2$ defined by Francini in [12] by [15]

$$\gamma^{1/2} \begin{bmatrix} \partial u_1 \\ \bar{\partial} u_1 \end{bmatrix} = \begin{bmatrix} \psi_{11} \\ \psi_{21} \end{bmatrix}, \quad \gamma^{1/2} \begin{bmatrix} \partial u_2 \\ \bar{\partial} u_2 \end{bmatrix} = \begin{bmatrix} \psi_{12} \\ \psi_{22} \end{bmatrix}.$$

The definition of the scattering transform is not practical for the inverse problem since Q_γ is unknown. Using integration by parts, the scattering transform can then be written directly in terms of u_1 and u_2 [17]. The formula requires the tangential derivative map which we denote by τ_γ , defined by

$$\tau_\gamma u = \gamma \nabla u \cdot \nu^\perp \quad (16)$$

with $\nu^\perp = (-\nu_y, \nu_x)$ and $\nu = \nu_x + i\nu_y$. We then have:

$$\begin{aligned} S_{12}(k) &= \frac{i}{2\pi} \int_{\partial\Omega} e^{-i\bar{k}z} \Psi_{12}(z, k) \nu \, d\tau \quad (17) \\ &= \frac{i}{2\pi} \int_{\partial\Omega} e_z(-\bar{k}) e^{-ikz} \gamma^{1/2} \bar{\partial}_z u_2(z, k) \nu \, d\tau \\ &= \frac{i}{2\pi} \int_{\partial\Omega} e_z(-\bar{k}) e^{-ikz} \frac{1}{\sqrt{\gamma}} \frac{\gamma}{2} (\nabla u_2 \cdot \nu - i \nabla u_2 \cdot \nu^\perp) \end{aligned}$$

$$= \frac{i}{4\pi} \int_{\partial\Omega} e_z(-\bar{k}) e^{-ikz} \gamma^{-1/2} (\Lambda_\gamma + i\tau_\gamma) u_2 \, d\tau \quad (18)$$

Similarly,

$$S_{21}(k) = \frac{-i}{2\pi} \int_{\partial\Omega} e^{i\bar{k}\bar{z}} \Psi_{21}(z, k) \bar{\nu} \, d\tau \quad (19)$$

$$= \frac{-i}{4\pi} \int_{\partial\Omega} e^{i\bar{k}\bar{z}} \gamma^{-1/2} (\Lambda_\gamma - i\tau_\gamma) u_1 \, d\tau \quad (20)$$

The scattering transform equations here differ from those in [14]–[16]. In [14]–[16] formulas (17) and (19) were used for the computation of $S_{12}(k)$ and $S_{21}(k)$, respectively, after solving BIE's for Ψ_{12} and Ψ_{21} , while here we use formulas (18) and (20).

The scattering transform arises as a coefficient in the D-bar equation for $M(z, k)$

$$\bar{\partial}_k M(z, k) = M(z, \bar{k}) \begin{pmatrix} e_z(\bar{k}) & 0 \\ 0 & e_z(-k) \end{pmatrix} S_\gamma(k). \quad (21)$$

Finally, γ is reconstructed from the CGO solutions $M(z, k)$ from the formulas derived in [14], [15]. Namely,

$$\gamma(z) = \exp\left(-\frac{2}{\pi} \int_{\mathbb{C}} \frac{1}{z-\zeta} \frac{\partial_z M_-(z, 0)}{M_+(z, 0)} d\zeta\right), \quad (22)$$

where

$$M_+(z, k) = M_{11}(z, k) + e_z(-k)M_{12}(z, k)$$

$$M_-(z, k) = M_{22}(z, k) + e_z(k)M_{21}(z, k).$$

A. Computational considerations

The computation of the complex admittivity utilizes the “exp” approximation in which the CGO solutions u_1 and u_2 on Ω are replaced by their asymptotic behavior. Let $\Gamma_\gamma \equiv \Lambda_\gamma + i\tau_\gamma$ and $\bar{\Gamma}_\gamma \equiv \Lambda_\gamma - i\tau_\gamma$. Define

$$\tilde{S}_{12}(k) = \frac{1}{4\pi k} \int_{\partial\Omega} \frac{e_z(-\bar{k})e^{-ikz}}{\gamma^{1/2}} \Gamma_\gamma e^{-ikz\bar{z}} dz(s) \quad (23)$$

$$\tilde{S}_{21}(k) = \frac{-1}{4\pi k} \int_{\partial\Omega} \frac{e^{ik\bar{z}}}{\gamma^{1/2}} \bar{\Gamma}_\gamma e^{ikz} dz(s) \quad (24)$$

This scattering transform also blows up in the presence of noise, and so we also compute its values in a disk of radius R . Expand the exponentials in the basis of applied current patterns. For example,

$$e_{z(\theta)}(-\bar{k})e^{-ikz(\theta)} = \sum_{m=1}^{L-1} a_m(k) t^m(\theta). \quad (25)$$

Then on an arbitrary domain shape scaled by R , and assuming $\gamma = 1$ on the boundary, equation (23) becomes

$$\begin{aligned}\tilde{S}_{12}(k) &\approx \\ &\frac{R}{4\pi k} \int_0^{2\pi} s(\theta) \sum_{m=1}^{L-1} a_m(k) t^m(\theta) \Gamma_\gamma \sum_{n=1}^{L-1} d_n(-\bar{k}) t^n(\theta) d\theta \\ &\approx \frac{R}{4\pi k} \sum_{m=1}^{L-1} \sum_{n=1}^{L-1} a_m(k) d_n(-\bar{k}) \int_0^{2\pi} s(\theta) t^m(\theta) \Gamma_\gamma t^n(\theta) d\theta.\end{aligned}$$

The map τ_γ is approximated by the matrix P_γ with entries

$$P_\gamma(m, n) = \left(\frac{\partial t^m(\theta)}{\partial \tau}, t^n(\theta) \right).$$

In the case of a circular domain, $\tau = \theta$. In the general case $\tau = s(\theta)d\theta$. Taking the finite-dimensional approximations to Λ_γ and τ_γ , and defining $\Gamma_\gamma^M = \Lambda_\gamma^M + iP_\gamma$ and $\bar{\Gamma}_\gamma^M = \Lambda_\gamma^M - iP_\gamma$, we have

$$\tilde{S}_{12}(k) \approx \frac{\Delta\theta R}{4\pi k} \sum_{m=1}^{L-1} \sum_{n=1}^{L-1} a_m(k) d_n(-\bar{k}) \Gamma_\gamma^M(m, n)$$

$$\tilde{S}_{21}(k) \approx \frac{-\Delta\theta R}{4\pi k} \sum_{m=1}^{L-1} \sum_{n=1}^{L-1} d_m(k) c_n(k) \bar{\Gamma}_\gamma^M(m, n)$$

Under the assumption that for each frame, the values of γ on the boundary of Ω are equal to the value of γ (which is assumed to be constant on the boundary) for the reference frame, define

$$(\mathbf{S}_{dif}^{\exp})_{12} \equiv (\mathbf{S}_\gamma^{\exp})_{12} - (\mathbf{S}_{\gamma_{ref}}^{\exp})_{12} \quad (26)$$

$$(\mathbf{S}_{dif}^{\exp})_{21} \equiv (\mathbf{S}_\gamma^{\exp})_{21} - (\mathbf{S}_{\gamma_{ref}}^{\exp})_{21}. \quad (27)$$

In light of (16) and the assumption that γ remains constant on Ω , $(\tau_\gamma - \tau_{\gamma_{ref}})e^{ikz} = 0$. Now, using the same type of calculation as above, equations (26) and (27) become

$$\tilde{S}_{12}(k) \approx \frac{\Delta\theta R}{4\pi k} \sum_{m=1}^{L-1} \sum_{n=1}^{L-1} a_m(k) d_n(-\bar{k}) \delta \Lambda^M(m, n) \quad (28)$$

$$\tilde{S}_{21}(k) \approx -\frac{\Delta\theta R}{4\pi k} \sum_{m=1}^{L-1} \sum_{n=1}^{L-1} d_m(k) c_n(k) \delta\Lambda^M(m, n) \quad (29)$$

where the coefficients $b_n(k)$, $c_m(\bar{k})$ and $d_n(k)$ are obtained analogously to $a_m(\bar{k})$ in (25) for $|k| < R$.

The solution of the system of D-bar equations (21) was computed as in [15] using a fast method based on the method by Vainikko [33] for integral equations with weakly singular kernels. FFT's are used to compute the convolutions and GMRES was used to solve the resulting linear systems.

The admittivity was then computed from (22), using centered finite differences to compute the first-order derivatives in the numerator and an FFT to compute the convolution.

B. Reconstructions from simulated data

Since this is a new method of reconstructing permittivity, we also include reconstructions from simulated data, depicting (a) heart with admittivity $1 + 0.6i$, lungs with admittivity $0.75 + 0.4i$, and spine with admittivity $0.2 + 0.08i$ in a 2-D circular domain of admittivity $0.55 + 0.25i$, and (b) a numerical phantom with high conductivity and low permittivity fluid in one lung. In this case the heart has admittivity $0.88 + 0.65i$, the lungs have admittivity $0.5 + 0.22i$ except for the simulated fluid which has admittivity $1 + 0.08i$, and the background has admittivity $0.75 + 0.33i$.

The reconstructions for case (a) are found in Figure 2 and for case (b) in Figure 3. The truncation radius in the reconstructions was $R = 6.0$. The reconstructions identify the shape, position, and locations of the phantom organs quite well, although the spine is barely visible in the permittivity reconstruction in case (a). The values of the heart and lungs are overestimated for both conductivity and permittivity, while that of the spine is underestimated in both conductivity and permittivity. Error plots in the sup norm are found in the third column of Figures 2 and 3. As to be expected, the largest errors are found near the organ boundaries. This is due to the necessary regularization of the method by truncation of the scattering transform, which has a smoothing effect [22].

V. RECONSTRUCTIONS OF HUMAN CHEST DATA

The human chest data was collected as described in section 2. For the ventilation data set, the reference data set used to construct Λ_{ref} was chosen to be the first frame, which was at end-expiration. For the perfusion data set, the reference data set was a frame selected at mid-systole. since the actual domain shape for these archival data sets is not available, the domain was approximated by (a) a circle with the same perimeter as the subject's chest in the plane of the electrodes and (b) a chest-shaped domain of the same perimeter representative of the domain shape, as shown in Figure 1. Note that no forward simulations are necessary for computing the difference images of human chest data.

Reconstructions of conductivity, computed by the method in Section 3, and reconstructions of permittivity, computed by the method of Section 4, on the circular and chest-shaped domains are found in the figures in this section. Figure 4 contains the 16th, 18th, 20th, 25th, 31st, 33rd, 37th, and 38th image in the perfusion sequence of 100 frames reconstructed on the circular domain with the 14th frame as reference. Figure 5 depicts the same 8 frames with the conductivity and permittivity computed on the chest-shaped domain. Videos of the full 100 frames on the circular and noncircular domains are found at [36]. The real part of these data sets were also used in [3], [9], [20] on a circular approximation to the domain, and the reconstructions here of the real part on the circular domain are very similar to the results obtained in those works, demonstrating the stability of the fully nonlinear computation of the scattering transform, but also the possibility that the information provided by the full nonlinear solution is not being used due to the truncation, which is related to the level of precision and the accuracy of the data. For both conductivity and permittivity the blood flow changes in the heart and lungs are clearly visible, and the temporal resolution is excellent. The reconstructions on the noncircular domain seem to provide less distortion of the heart and lungs, although ground truth is not available for this archival healthy human subject data. The lungs are less elliptical than they appear in the circular domain, and the heart also appears to have less spreading.

Reconstructions of conductivity, computed by the method in Section 3, and reconstructions of permittivity, computed by the method of Section 4, for the ventilation data sets are found in Figure 6 on the noncircular domain. The reconstructions are remarkably similar to each other, showing good agreement between the two methods, and similar properties to the perfusion reconstructions. Airflow changes are clearly visible and coincide with the subject's breathing. The reconstructions on the circular domain are similar in nature, and so are omitted for space considerations. Videos of ventilation and perfusion for the full 100 frames can be found at [36].

VI. CONCLUSION

We have presented a direct D-bar method for computing reconstructions of both conductivity and permittivity from human chest data on a 2-D cross-sectional domain. The conductivity was computed using the fully nonlinear D-bar method based on [22], [28], [31], and the permittivity was computed using a new D-bar method based on the elliptic system introduced in the global uniqueness proof of Francini [12]. The difference images of ventilation and perfusion in a healthy human subject do not exhibit boundary artefacts and clearly show changes due to blood flow between the heart and lungs and gas exchange. Further research on animal subjects with induced lung injury is needed to better assess the clinical usefulness of the algorithm, but this first application of the D-bar method with the fully nonlinear scattering transform and new algorithm for obtaining the permittivity shows that the method is sufficiently robust to handle human data with incomplete knowledge of domain shape and electrode position.

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References

- [1]. Adams RA, Sobolev Spaces, Academic Press (1975).
- [2]. Astala K, Mueller JL, Päiväranta L, Perämäki A, and Siltanen S Direct electrical impedance tomography for nonsmooth conductivities, *Inv. Prob.Imag*, 5 (2011), pp. 531–550.
- [3]. Bikowski J and Mueller JL. 2D EIT reconstructions using Calderón's method. *Inv. Prob.Imag*, 2(1):43–61, 2008.
- [4]. Brown RM and Uhlmann G, Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions, *Comm. Partial Differential Equations*, 22: 1009–1027, 1997.
- [5]. Calderón AP On an inverse boundary value problem In *Seminar on Numerical Analysis and its Applications to Continuum Physics*, Soc. Brasileira de Matematica: 65–73, 1980.
- [6]. Cheney M, Isaacson D and Newell JC, Electrical Impedance Tomography, *SIAM Review*, 41:85–101, 1999.
- [7]. Cheney M, Isaacson D, Newell JC, Goble J, and Simske S, Noser: and algorithm for solving the inverse conductivity problem, *Internat. J. Imaging Systems and Tech*, 2:66–75, 1990.
- [8]. Costa ELV, Gonzalez Lima R and Amato MBP. Electrical impedance tomography. *Curr. Op. Crit. Care*, 15:18–24, 2009.
- [9]. DeAngelo M and Mueller JL. D-bar reconstructions of human chest and tank data using an improved approximation to the scattering transform, *Physiol. Meas* 31: 221–232, 2010. [PubMed: 20057005]
- [10]. Edic PM, Saulnier GJ, Newell JC, and Isaacson D. A realtime electrical impedance tomograph. *IEEE Trans. Biomed. Eng*, 42(9):849–859, 1995. [PubMed: 7558059]
- [11]. Faddeev LD. Increasing solutions of the Schrödinger equation. *Sov.-Phys. Dokl*, 10:1033–1035, 1966.
- [12]. Francini E, Recovering a complex coefficient in a planar domain from the Dirichlet-to-Neumann map. *Inverse Problems* 6:107–119, 2000.
- [13]. Fuks LF, Cheney M, Isaacson D, Gisser DG, and Newell JC. Detection and imaging of electric conductivity and permittivity at low frequency. *IEEE Trans. Biomed. Engr* 38:1106–1110, 1991.
- [14]. Hamilton SJ A direct D-bar reconstruction algorithm for complex admittivities in $W^{2,\infty}(\Omega)$ for the 2-D EIT problem. PhD thesis, Colorado State University, 2012.
- [15]. Hamilton SJ, Herrera CNL, Mueller JL and Von Herrmann A. A direct D-bar reconstruction algorithm for recovering a complex conductivity in 2-D. *Inv. Prob*, 28 (2012) 095005.
- [16]. Hamilton SJ and Mueller JL. Direct EIT reconstructions of complex conductivities on a chest-shaped domain in 2-D. *IEEE Transactions on Medical Imaging*, 32(4):757–769, 2013. [PubMed: 23314771]
- [17]. Herrera CNL A D-bar method for estimating admittivity in 2-D for electrical impedance tomography PhD thesis, University of São Paulo, Brazil, 2012.
- [18]. Holder DS Electrical impedance tomography first edition, Institute of Physics Publishing, Bristol and Philadelphia, 2005.
- [19]. Isaacson D, Mueller JL, Newell JC, and Siltanen S. Reconstructions of chest phantoms by the d-bar method for electrical impedance tomography. *IEEE Trans Med Im*, 23:821–828, 2004.
- [20]. Isaacson D, Mueller JL, Newell JC, and Siltanen S. Imaging cardiac activity by the D-bar method for electrical impedance tomography. *Physiol. Meas*, 27:S43–S50, 2006. [PubMed: 16636419]
- [21]. Knudsen K, Lassas M, Mueller JL and Siltanen S, D-bar method for electrical impedance tomography with discontinuous conductivities, *SIAM J. Appl. Math*, 67:893–913, 2007.
- [22]. Knudsen K, Lassas M, Mueller JL and Siltanen S, Regularized D-bar method for the inverse conductivity problem, *Inverse Problems and Imaging*, 3:599–624, 2009.
- [23]. Knudsen K, Mueller JL, and Siltanen S, Numerical solution method for the dbar-equation in the plane. *J. Comp. Phys*, 198:500–517, 2004.

- [24]. Mueller JL and Siltanen S. Direct reconstructions of conductivities from boundary measurements. *SIAM J. Sci. Comput.*, 24(4):1232–1266 (electronic), 2003.
- [25]. Mueller JL and Siltanen S, Linear and nonlinear inverse problems with practical applications, SIAM, 2012.
- [26]. K Murphy E, Mueller JL, and Newell JC. Reconstructions of conductive and insulating targets using the D-bar method on an elliptical domain. *Physiol. Meas.*, 28(7):S101–114, 2007. [PubMed: 17664628]
- [27]. K Murphy E and Mueller JL. Effect of domain shape modeling and measurement errors on the 2-D D-bar method for EIT. *IEEE Trans. Med. Im.*, 28(10):1576–1584, 2009.
- [28]. Nachman AI. Global uniqueness for a two-dimensional inverse boundary value problem. *Ann. of Math. (2)*, 143(1):71–96, 1996.
- [29]. Saad Y and Schultz GMRES MH: A Generalized Minimal Residual Algorithm for Solving Nonsymmetric Linear Systems *SIAM J. Sci. and Stat. Computing.*, 7(3):856–869, 1986.
- [30]. Siltanen S 1999, Electrical impedance tomography and Faddeev’s Green functions, *Ann. Acad. Sci. Fenn. Mathematica Dissertationes* 121.
- [31]. Siltanen S, Mueller JL, and Isaacson D. An implementation of the reconstruction algorithm of A. Nachman for the 2D inverse conductivity problem. *Inverse Problems*, 16(3):681–699, 2000.
- [32]. C Trigo F, R Gonzalez-Lima, and M.B.P. Amato. Electrical impedance tomography using the extended kalman filter. *IEEE Trans. on Biomed. Eng.*, 51(1):72–81, 2004.
- [33]. Vainikko G. Fast Solvers of the Lippmann-Schwinger Equation Helsinki University of Technology, Institute of Mathematics Research Reports A,387, 1997
- [34]. Victorino JA, Borges JB, Okamoto VN, Matos GFJ, Tucci MR, Caraméz MPR, Tanaka H, Santos DCB, Barbas CSV, Carvalho CRR, Amato MBP. Imbalances in regional lung ventilation: a validation study on electrical impedance tomography. *Am. J. Respir. Crit. Care Med.*, 169:791–800, 2004. [PubMed: 14693669]
- [35]. Von Herrmann A. Properties of the reconstruction algorithm and associated scattering transform for admittivity $\gamma \in W^1 \cdot P(\Omega)$ in the plane. PhD thesis, Colorado State University, 2009.
- [36]. www.math.colostate.edu/~mueller/eit_lab_movies.html.

Fig. 1.

The circular and chest-shaped domains used in the reconstructions of the human chest. $r_{\max}^{\text{circle}} = 0.1432 \text{ m}$. $r_{\max}^{\text{chest}} = 0.1702 \text{ m}$.

Fig. 2. Tests with simulated data. First column: Numerical phantom including lungs, heart, and spine. Second column: Reconstruction. Third column: Error = |Phantom – Reconstruction|. Top row: Conductivity. Bottom row: Permittivity.

Fig. 3.

Tests with simulated data. First column: Numerical phantom including high conductivity, low permittivity fluid in one lung. Second column: Reconstruction Third column: Error = |Phantom – Reconstruction|. Top row: Conductivity. Bottom row: Permittivity.

Fig. 4.

Selection of 8 reconstructed frames of the conductivity and permittivity distribution in a healthy human subject during breathholding (images displayed with the heart at the top). The images depict changes in blood flow, with red representing high conductivity/permittivity and blue low conductivity/permittivity. Moving from left to right, the first four frames depict diastole, the 5th and 6th depict systole, and in the last two frames, diastole is resuming.

Fig. 5.

Selection of 8 reconstructed frames of the conductivity and permittivity distribution in a healthy human subject during breathholding (images displayed with the heart at the top). The images depict changes in blood flow, with red representing high conductivity/permittivity and blue low conductivity/permittivity. Moving from left to right, the first four frames depict diastole, the 5th and 6th depict systole, and in the last two frames, diastole is resuming.

Fig. 6.

Selection of 5 reconstructed frames of the conductivity and permittivity distributions in a healthy human subject during a slow ventilation maneuver (images displayed with the heart at the top.) Top row: Red is high resistivity and blue low resistivity. Bottom row: Red is high permittivity and blue low permittivity.