Complex dynamics of a neuron model with discontinuous magnetic induction and exposed to external radiation

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Abstract

The last two decades have seen many literatures on the mathematical and computational analysis of neuronal activities resulting in many mathematical models to describe neuron. Many of those models have described the membrane potential of a neuron in terms of the leakage current and the synaptic inputs. Only recently researchers have proposed a new neuron model based on the electromagnetic induction theorem, which considers inner magnetic fluctuation and external electromagnetic radiation as a significant missing part that can participate in neural activity. While the flux coupling of the membrane is considered equivalent to a memductance function of a memristor, standard memductance model of $\alpha + 3\beta\phi^2$ has been used in the literatures, but in this paper we propose a new memductance function based on discontinuous flux coupling. Various dynamical properties of the neuron model with discontinuous flux coupling are studied and interestingly the proposed model shows hyperchaotic behavior which was not identified in the literatures. Furthermore, we consider a ring network of the proposed model and investigate whether the chimera state can emerge. The chimera state relates to the state with simultaneously coherence and incoherence in oscillatory networks and has received much attention in recent years.

Keywords HR model · Discontinuous flux · Bifurcation · Hyperchaos · Chimera state

Introduction

Human Brain is the most complex dynamical system composed of multidimensional neuronal activities. Neurons are quiescent but can be excited to fire spikes when

they are stimulated. Even though research on neurons are many fold, electrical activities characterizations of neurons have attracted many neurophysiologists. To capture the membrane potential, DC pulses are injected to the neuron using an electrode. The results show different mode transitions from quiescent state to spiking, bursting and chaotic states (Gu et al. 2014). Dynamical behaviors of a single neuron or neuronal network are crucial to understand the neuronal behavior in the brain and also to understand the causes of serious diseases in nervous system. To investigate the neuron characteristics, various mathematical models have been proposed by researchers (Izhikevich 2004). Equivalent nonlinear circuits have also been proposed to produce complex sampled time series that are consistent with biological experimental data, by setting appropriate parameters and external forcing. Using nonlinear oscillators and its properties many neuron models have been developed to describe the dynamical behavior in electrical activities of neurons (Wang et al. 2017). Ma and Jun (2015, 2017) discussed the influence of functional

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connection and anatomic connection of neuron on model setting and effect of pattern formation and synchronization in neuronal network using mean field theory. In 1984 Hindmarsh and Rose proposed a model motivated by the experiment conducted on pond snail *Lymnaea*, which generated a burst after being depolarized by a short current pulse. Though they proposed a third order dimensionless model, the phenomenon of bursting was modelled in two-dimensional space. The model was efficient in analyzing the mode transition, phase synchronization and spatial pattern formation in neuronal networks (Lv and Ma 2016; Ma et al. 2017).

Even though there have been many dimensionless models for neuronal activities, Moujahid et al. (2011) introduced a more controllable bifurcating model, so that dynamical behaviors could be extensively investigated. Neural firing patterns showing period-doubling bifurcation to chaotic bursting, chaotic bursting to chaotic spiking and an inverse period-doubling bifurcation of spiking patterns were investigated with the bifurcation analysis carried out in the hypothalamus (Braun et al. 1997, 2011; Coombes and Osbaldestin 2000) and on an experimental neural pacemaker (Gu et al. 2003; Lu et al. 2008; Yang et al. 2009). The behavior of the membrane potential manifests different oscillations for various applied current values. Bursting and Inter Spike Interval (ISI) are the two important characters defining the behavior of the neurons. Bursting is an intricate behavior in which bursts are detached by intervals of quiescence, and the ISI defines the frequency of spiking pattern. Most of the existing literatures show that the HR model is effective in characterizing the neuronal activities (bursting and spiking) and hence can better serve as a benchmark neuron model (Dtchetgnia Djeundam et al. 2013).

In Belykh et al. (2000), González-Miranda (2003, 2007), Innocenti et al. (2007), Terman (1991, 1992) and Wang (1993) dynamical behaviors like Quiescence, Spiking, Bursting, Irregular spiking and Irregular bursting were analyzed with respect to the bifurcation parameters. The bifurcation analysis explained the transitions between stable bursting solutions and continuous spiking regimes with a period halving mechanism, to the transitions between quiescent asymptotic behaviors and bursting regimes (Storace et al. 2008).

In this paper we propose a new neuron model based on Hindmarsh–Rose model, by using a discontinuous memductance function as the magnetic flux coupling and with an external electromagnetic excitation. Then the dynamical properties of the model such as Lyapunov spectrum and bifurcation diagrams are analyzed. Recently, an interesting phenomenon named chimera state has been studied extensively in coupled oscillators (Abrams and Strogatz 2004; Dudkowski et al. 2016; Kapitaniak et al. 2014;

Parastesh et al. 2018). Chimera states emerge when there is coexistence of coherence and incoherence in a network of identical coupled oscillators (Dudkowski et al. 2014). It has been proved that this state can be related to many real phenomena such as uni-hemispheric sleep in animals or some irregularities in the brain like Epilepsy seizure and Parkinson (Majhi et al. 2016, 2017). Therefore, there are many studies investigating chimera states in neuronal networks (Hizanidis et al. 2014; Mishra et al. 2017; Omelchenko et al. 2015; Schmidt et al. 2017). Here, we also consider a network of new model and examine the emergence of chimera state.

The paper is organized as follows. “**Modified Hindmarsh–Rose neuron model (MHRN)**” section introduces the new modified HR neuron model. Dynamical analysis of the model is described in “**Dynamic analysis of MNRN**” section. In “**Dynamics of MHRN network**” section, dynamics of the model network and especially chimera state is investigated and finally the conclusions are presented in “**Conclusion**” section.

Modified Hindmarsh–Rose neuron model (MHRN)

A new neuron model which describes the effect of electromagnetic induction on neuronal activities was presented in Lv et al. (2016), which is derived by modifying the well-known Hindmarsh–Rose. The dimensionless mathematical model is given by

$$\begin{aligned}\dot{x} &= y - ax^3 + bx^2 - z + I_{ext} - kx\rho(\phi) \\ \dot{y} &= c - dx^2 - y \\ \dot{z} &= r(s(x + 1.6) - z) \\ \dot{\phi} &= k_1x - k_2\phi\end{aligned}\quad (1)$$

where the variables x, y, z and ϕ describe the membrane potential, slow current for recovery variable, adapting current, and magnetic flux across the membrane, respectively. I_{ext} denotes the external forcing current and $\rho(\phi)$ denotes the magnetic flux coupling given by $\rho(\phi) = \alpha + 3\beta\phi^2$.

In this paper we modify the model (1) by introducing an external electromagnetic excitation $\phi_{ext} = E \cos(2\pi ft)$ where E is the intensity of the external excitation and f is the frequency of excitation. We also assume that the magnetic flux coupling to be discontinuous, and hence propose using a discontinuous memductance function to understand the complete magnetic flux effects on the neurons. Hence the memory conductance $\rho(\phi)$ is assumed to be a monotonically increasing piecewise linear function given by

$$\rho(\phi) = \frac{dq(\phi)}{d\phi} = \beta + 0.5(\alpha - \beta)(\text{sgn}(\phi + 1) - \text{sgn}(\phi - 1)) \quad (2)$$

where $\alpha, \beta, \delta > 0$.

Hence the new dimensionless mode with the external electromagnetic radiation and discontinuous flux coupling is derived as,

$$\begin{aligned} \dot{x} &= y - ax^3 + bx^2 - z + I_{ext} - kx\rho(\phi) \\ \dot{y} &= c - dx^2 - y \\ \dot{z} &= r(s(x + 1.6) - z) \\ \dot{\phi} &= k_1x - k_2\phi + \phi_{ext} \end{aligned} \quad (3)$$

where ϕ_{ext} is the external radiation, $\rho(\phi)$ is the magnetic flux coupling, $a, b, c, d, r, s, k, k_1, k_2$ are the system parameters. Figure 1 shows the 2D phase portraits of the system for the parameters given in (4) and initial conditions $[0.5, -2, 4, 0.1]$ for the conditions $E = 0$ hereinafter mentioned as case-A and Fig. 2 shows the 2D phase portraits for $E = 1$ herein after mentioned as case-B. The other system parameters are defined as,

$$\begin{aligned} a &= 1; \quad b = 3; \quad k = 1; \quad c = 1; \quad d = 5; \quad r = 0.006; \\ s &= 4; \quad k_1 = 0.1; k_2 = 0.5; \alpha = 0.1; \quad \beta = 0.06; \\ f &= 0.01; \quad I_{ext} = 3.3. \end{aligned} \quad (4)$$

Dynamic analysis of MNRN

The dynamical properties of the MHRN system such as equilibrium points, eigenvalues, Lyapunov spectrum and bifurcation plots are derived and discussed in this section.

Equilibrium points

To derive the equilibrium points of the system, we consider that there is no external electromagnetic radiation (case-A). The equilibrium points of the system can be solved by the relations:

$$\begin{aligned} y - ax^3 + bx^2 - z + I_{ext} - kx\rho(\phi) &= 0 \\ y &= -c + dx^2 \\ z &= s(x + 1.6) \\ x &= \frac{k_2}{k_1}\phi \end{aligned} \quad (5)$$

Fig. 1 2D phase portraits of the MHRN for $E = 0$

Fig. 2 2D phase portraits of the MHRN for $E = 1$

Simplifying (5) and using the parameters as in (4) with I_{ext} taken as control parameter we get

$$\begin{aligned}
 x &= \frac{\left\{ \sqrt[3]{\frac{I_{ext}}{2}} + \sqrt{\left[\left(\frac{I_{ext}}{2} - 1.6296\right)^2 + 0.7843}\right]} - 1.6296 \right\} - 83}{\left\{ 90 \times \sqrt[3]{\frac{I_{ext}}{2}} + \sqrt{\left[\left(\frac{I_{ext}}{2} - 1.6296\right)^2 + 0.7843}\right]} - 1.6296 \right\} - 0.666} \\
 y &= -1 + 5 \left\{ \frac{\left\{ \sqrt[3]{\frac{I_{ext}}{2}} + \sqrt{\left[\left(\frac{I_{ext}}{2} - 1.6296\right)^2 + 0.7843}\right]} - 1.6296 \right\} - 83}{\left\{ 90 \times \sqrt[3]{\frac{I_{ext}}{2}} + \sqrt{\left[\left(\frac{I_{ext}}{2} - 1.6296\right)^2 + 0.7843}\right]} - 1.6296 \right\} - 0.666} \right\}^2 \\
 z &= 4 \left\{ \frac{\left\{ \sqrt[3]{\frac{I_{ext}}{2}} + \sqrt{\left[\left(\frac{I_{ext}}{2} - 1.6296\right)^2 + 0.7843}\right]} - 1.6296 \right\} - 83}{\left\{ 90 \times \sqrt[3]{\frac{I_{ext}}{2}} + \sqrt{\left[\left(\frac{I_{ext}}{2} - 1.6296\right)^2 + 0.7843}\right]} - 1.6296 \right\} - 0.666} \right\} + 1.6 \\
 \phi &= 2 \left\{ \frac{\left\{ \sqrt[3]{\frac{I_{ext}}{2}} + \sqrt{\left[\left(\frac{I_{ext}}{2} - 1.6296\right)^2 + 0.7843}\right]} - 1.6296 \right\} - 83}{\left\{ 90 \times \sqrt[3]{\frac{I_{ext}}{2}} + \sqrt{\left[\left(\frac{I_{ext}}{2} - 1.6296\right)^2 + 0.7843}\right]} - 1.6296 \right\} - 0.666} \right\}
 \end{aligned} \tag{6}$$

For the fixed parameters as in (4) and $I_{ext} = 3.5$ the system shows two set of equilibrium points depending on the value of the third state variable ϕ as given in (7):

$$\begin{aligned} x &= -0.5883, y = 0.5192, z = 4.0467, \\ \phi &= -1.1767 \quad \text{for } |\phi| < 1 \\ x &= -0.5799, y = 0.5044, z = 4.0804, \\ \phi &= -1.1598 \quad \text{for } |\phi| > 1 \end{aligned} \quad (7)$$

The characteristic equation of the system is derived as:

$$\begin{aligned} \lambda^4 + 6.0941\lambda^3 + 1.6439\lambda^2 - 0.5599\lambda + 0.0084 & \quad \text{for } |\phi| < 1 \\ \lambda^4 + 6.1344\lambda^3 + 1.6200\lambda^2 - 0.5821\lambda + 0.0082 & \quad \text{for } |\phi| > 1 \end{aligned} \quad (8)$$

Figure 3 shows the stability of the MHRN system for the change in the parameter I_{ext} .

Bifurcation analysis

To understand the complete dynamical behavior of the system, it is important to investigate the impact of the

Fig. 3 Stability of equilibrium for various values of I_{ext} for two conditions, **a** $|\phi| < 1$; **b** $|\phi| > 1$; the remaining two real part of Eigen values are negative in the region of interest and hence have not been displayed in this figure

Fig. 4 a Bifurcation diagram of the MHRN system with parameter I_{ext} considering no external electromagnetic radiation. The bifurcation plot shows two y-axis (left and right) to show the bifurcation plots clearly as the MHRN shows two areas of bifurcation with one in positive y-axis (shown in right axis) and the other region in negative y-axis (shown in left axis); **b** the corresponding maximum two LEs

Fig. 5 **a** Bifurcation diagram of the MHRN system with parameter I_{ext} considering an external electromagnetic radiation $\phi_{ext} = E \cos(2\pi ft)$. The bifurcation plot shows two y-axis (left and right) to show the bifurcation plots clearly as the MHRN shows two areas of bifurcation with one in positive y-axis (shown in left axis) and the other region in negative y-axis (shown in right axis); **b** the corresponding maximum two LEs

Fig. 6 Spatiotemporal patterns (left panel) and time snapshots (right panel) of the network Eq. (9). **a** $d = 0.01$, **b** $d = 0.25$, **c** $d = 0.9$. The initial conditions are chosen randomly in the interval $x, y, z, \emptyset \in [-1.1]$

parameters on the MHRN system. We consider the control parameter as I_{ext} while the other parameters are taken as in (4). Firstly, we discuss the bifurcation of the system for case-A wherein we consider that there is no external electromagnetic excitation. The bifurcation of the system is shown in Fig. 4a. The system shows two regions of chaotic behavior and the same is confirmed by the existence of a positive Lyapunov exponent (LE) calculated using the Wolf algorithm (Wolf et al. 1985) for a finite time of 20,000 s shown in Fig. 4b. In the second case (case-B) we consider that the neuron is exposed to an external electromagnetic radiation $\phi_{ext} = E \cos(2\pi ft)$. Figure 5a shows the bifurcation of the system for case-B and similar to case-A, we could see two regions of chaotic oscillations. To examine further, the LEs are derived and presented in Fig. 5b which shows that the system shows two positive LEs for $3.613 \leq I_{ext} \leq 3.73$. Furthermore, the chaotic regions are wider when analyzed in the presence of external excitation. This clearly shows that the complex dynamics of the neuron is altered when exposed to an external radiation.

Dynamics of MHRN network

For further analyzing the proposed model dynamics, we construct a ring network of Eq. (3). The equation of the network is given as:

$$\begin{aligned}\dot{x}_i &= y_i - x_i^3 + 3x_i^2 - z_i + 3 - x_i\rho(\phi_i) + \frac{d}{2p} \sum_{j=i-p}^{i+p} (x_j - x_i) \\ \dot{y}_i &= 1 - 5x_i^2 - y_i \\ \dot{z}_i &= 0.006(4(x_i + 1.6) - z_i) \\ \phi_i &= 0.1x_i - 0.5\phi_i + 0.1\cos(0.02\pi t) \\ &\text{for } i = 1 \text{ to } N\end{aligned}\quad (9)$$

where d is the coupling strength and the neurons are coupled non-locally to their $2P$ nearest neighbors.

To investigate the network dynamics, we choose $N = 100$ and $P = 25$ and change the coupling strength. The numerical simulations are done by using the fourth order Runge–Kutta method with time step 0.05. As can be seen in Fig. 6, the network has a common dynamic change from asynchronous states with very small couplings to synchronous state for $d > 0.7$. In this transition, for special range of coupling strength, chimera states emerge as it is shown in Fig. 6b for $d = 0.25$.

Conclusion

A HR neuron model with a discontinuous flux coupling is derived and investigated. Dynamical analysis of the MHRN using tools like the equilibrium points, stability of equilibrium, Eigen values, bifurcation and Lyapunov exponents are presented. It is interesting to note that the proposed model shows hyperchaotic behavior for selected values of the control parameter and these complex oscillations were not investigated earlier in the literatures. We have also considered a network of proposed model. Investigating the network dynamics, synchronous, asynchronous and chimera states were observed. Capability of this network to represent chimera state, confirms that the new model can be used in neuronal networks and it can provide useful information about brain activities. In the future work the model can be analyzed by changing magnetic flux relations to exponential flux coupling.

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