Quantum-like behavior without quantum physics II. A quantum-like model of neural network dynamics

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Abstract In earlier work, we laid out the foundation for explaining the quantum-like behavior of neural systems in the basic kinematic case of clusters of neuron-like units. Here we extend this approach to networks and begin developing a dynamical theory for them. Our approach provides a novel mathematical foundation for neural dynamics and computation which abstracts away from lower-level biophysical details in favor of information-processing features of neural activity. The theory makes predictions concerning such pathologies as schizophrenia, dementias, and epilepsy, for which some evidence has accrued. It also suggests a model of memory retrieval mechanisms. As further proof of principle, we analyze certain energy-like eigenstates of the 13 three-neuron motif classes according to our theory and argue that their quantum-like superpositional nature has a bearing on their observed structural integrity.

Keywords Neurons · Interneurons · Networks · Quasispin models · Memory

1 Introduction

Quantum-like aspects of certain human brain functions, such as cognition, the mental lexicon, memory, etc., have been noted since at least 1949 (von Foerster in [1]). Much recent work has been concerned with various quantum-like formulations and explications of such

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effects [2] but their provenance has remained something of a mystery. Considerations of the actual quantum mechanical operation of certain physical elements of brain anatomy seem unable to explain the manifold aspects of observed quantum effects, owing mainly to the rapidity of decoherence of superpositions in a hot brain [3].

In an ostensibly completely different line of research, workers on neural net paradigms have for decades found certain quantum theoretical models useful in reproducing certain brain behaviors as collective phenomena, without apparently being exercised by the quantum nature and provenance of their models. These models are Ising-like, pioneered in all but name in [4] and further developed in [5], where the name is explicitly invoked, and many others (see also [6, 7]). So we have at least two quantum-like paradigms of brain function, arrived at independently, whose provenances are mysterious to say the least.

It is the aim of this paper and its predecessor [8] to lay the foundations for a theory that might explain the emergence of these apparently non-classical behaviors. In [8], we argued that certain systems whose internal states may be deemed *confusable*, in the sense that similar states may not be distinguishable even in principle by external “observers” without disrupting the system, may indeed be expected under certain circumstances to exhibit quantum-like behaviors. We conjectured that brains, identified with networked clusters of neuron-like nodes, may exhibit such behavior. Our approach was via the logic such systems obey, which is a slight but significant deformation of ordinary Boolean logic called *orthologic* (**OL**), and we considered only the case of disconnected clusters of nodes. This was in order to provide a kinematical background for the more interesting case of connected nodes and their dynamics. We found, even in this disconnected case, that a version of that fundamental quantum phenomenon known as *superposition* obtained, even in the absence of any further assumptions, physical or non-physical.

In this paper we extend this approach to networks and begin developing a virtual dynamical theory for them. Such considerations provide prolegomena to a formal quantum-like computational treatment of such networks.

Our spaces will represent the ensembles of states of neural-like networks, namely the vectors of possible real values in the nodes of such a network. The actual values inside the nodes are considered to be *hidden* (until and unless some operation of measurement or observation is applied) as in a living brain, and the theoretical apparatus respects this degree of resolution. In this respect, our formalism has certain quantum theory-like attributes: ordinary quantum mechanics is surely not a theory of hidden variables, but a theory of hidden variables may appear to be quantum-like and the theory developed here is one of those. Its power resides precisely in its ability to hide otherwise intractable complexity.

Here we carry the program further by externalizing and multiplexing the spaces used in our earlier considerations. There we found that a single nodal neuronal unit cannot exhibit quantum-like behavior and we argue here that in fact actual neurons can be modeled more closely than is usually done by considering them to be two-node systems, one node representing the input “chamber” or zone, comprising the dendritic environment of the soma, and an output node or chamber, representing the trigger zone axonal initial segment, which may be considered to include the whole axon as well. The resulting object has the interesting quantum-like structure of a virtual fermion, with energy-like eigenstates corresponding to ON and OFF. Namely, OFF is the normalized state in which the output chamber is at zero, while ON is the normalized state in which the input chamber is at zero and the output chamber is “firable” (i.e., either firing or prepared to fire). This structure is thus similar to a qubit, but with the provisos mentioned in [8].

Thus we shall assume simplified versions of the usual structural properties of such networks: many dendritic inputs to a node; many axonal output terminals to other nodes;

communication via chemical synapses and a unidirectional flow of information from dendritic arbors through the soma to a trigger zone—*cum*—hillock—*cum*—axonal initial segment; etc. In addition, we shall postulate, for those networks of interest to us, the existence of a substrate of interneural tissue which promotes rapid connectivity of the nodes via electrical (gap) junctions, in addition to their principal connections via axon—dendrite synaptic action: cf. for instance [9]. This substrate has the effect in nature of inhibiting and synchronizing the firing of local groups of networked principal neurons, and we shall find this sort of phenomenon manifested in some of our conclusions. This is one of several ways our approach differs from the classical notion of a neural net (cf. Section 2).

The layout of the paper is as follows. Section 2 is a brief overview of our approach and its computational context. In Section 3, we briefly review the relevant results of [8] and in Section 4 we discuss the category of extensions of the spaces of states used there. In Section 5, we describe abstractly but in some detail the irreducible action of one network upon another in terms of transformations of their state spaces. These actions will provide the basic elements of the dynamics derived later. In Section 6, the two major combinatorial constructions upon networks are discussed and in Section 7 we introduce and treat at some length the crucial notion of the bicameral neuron. In Section 8, we start to address the issue of dynamics for a single isolated network, consequent upon the adoption of a bicameral neuron. We introduce the occupation number formalism and derive the time development operator, here dubbed the *pseudo-Hamiltonian*, or *p-Hamiltonian*. What emerges is an analogue of a quantum model of a many fermion-system known to physicists as being of the *quasispin* type [10]. (A general study of such a model will be significantly more difficult to conduct than in the physics context since the usual physical simplifying assumptions are entirely absent.) The essential attribute of such quantum-like systems, and one relevant to computational issues, is the phenomenon of quantum-like superposition which, we argue, is present in such systems. This is not superposition of physical quanta so the usual arguments concerning the rapidity of decoherence [3] in hot environments do not obtain. What emerges is a quantum-like theory of behavior in which the “quanta” are virtual, and do not represent actual particles but rather states of resource allocation, rendered in the formalism as states of “occupancy.”

As in the case of actual physics, the eigenstates of the pseudo-Hamiltonian are (relatively) stationary, or stable, and the oscillatory behavior of the complex phases of ordinary quantum mechanical stationary states is here replaced by an exponential scaling behavior. These states, which we believe to be the ones relevant to mental or cognitive processes in actual brains, are discussed in Section 8.2. We consider the two extreme eigenstates available to all such networks in isolation, namely the coma-like state of zero occupancy (all neurons OFF) and the hyperactive state of full occupancy (all neurons ON). The latter state seems relevant to the onset of seizures of epileptic type and various conjectures are put forward concerning this scenario. The last Section 9 is devoted to applications. In Sections 9.1 and 9.2, we discuss *context* and a simplified model of context formation and retrieval. We show that the retrieval of one of a contextualized set of certain states has the effect of retrieving all of them, and then returning them to the storage module. The returning states are possibly changed by this action while at the same time this process consolidates the context. It is in this section that outlines of an informal version of a quantum-like computation may be discerned among the various transformations of spaces of states. This approach will be formalized elsewhere. In Section 9.3, we consider the possible bearing of our considerations upon the provenance of such disorders as schizophrenia and dementias of Alzheimer’s type: they should be found to arise from disorganization of the interneural substrate of the brain regions involved. We also explain the origin and preponderance of von Domarus errors in

the logical processes of schizophrenic patients. In the final Section 9.4, as proof of principle, the theory is applied to an analysis of the spectral properties of the 13 three-neuron motif classes in which certain stable states are completely classified for the first nine entries in Sporns' list of these classes [7], and partially classified for the rest. It is argued that these superpositional states seem to have a bearing on the integrity of these networks as functional units. The superpositional nature of these states could not have emerged if only classical methods had been deployed.

There are two mathematical appendices briefly reviewing some of the algebra used, and collecting other results that would be distracting if left in the main text.

2 Preamble

Our approach in this article provides a novel mathematical foundation for neural computation. We describe neural networks in terms of the neuron's internal states, interconnections, and dynamical interactions. Our description abstracts away from lower-level biophysical details in favor of information processing features of neural activity. Like real neural networks, networks in our model have internal states and receive inputs that can carry information, process them, and yield outputs as a function of their inputs and internal states. Thus, neural networks process information-carrying variables in accordance with a rule. In separate work [11], the second author argued that physical computation is the processing of variables—that is, inputs and internal states that can carry information—in accordance with a rule. Therefore, our neural networks are computing systems, albeit of an unconventional sort.

The classical model of computation is digital; its inputs, internal states, and outputs are strings of digits [12]. Another historically influential model is analog computation; its inputs and outputs are real-valued functions, and its main operation is integration [13]. A more recent addition is quantum computation, which processes digital inputs and outputs but whose internal states can enter superpositions and be processed by quantum operators [14, 15]. Finally, standard models of neural computation focus primarily on the frequencies and timing of action potentials, which are neither digital nor analog variables in the strict sense [16]. Our approach derives action potential timing and frequencies from quantum-like internal states and processes. Thus, it lays the foundation for a quantum-like model of neural computation.

3 Review of previous work

In this section we briefly review the results of [8]. There we argued that the correct logic inherent to the systems of interest to us, namely those having confusable states, is *orthologic*, denoted **OL**. This logic differs from ordinary propositional logic in one respect only: namely, conjunction (AND) does not distribute over disjunction (OR). This single difference is enough to produce quantum-like behavior in certain Kripkean models of the logic if another condition is met: namely that the ortholattice of *propositions* or *regular sets*, in the model is in fact not Boolean. We showed in [8] that when realized in the space of states of a cluster of two or more nodes, each capable of holding a range of real values, the logic satisfies this latter condition if the vector with zeroes in all nodes is an interior point in the space of all vectors of the cluster. That is to say, within the space of vectors, there is a Euclidean open set containing the origin. (The origin itself is excluded as a state in these models.) As

a consequence, such a system, like Schrödinger's cat, may be in a superposition of pure states, without being in any component pure state, behavior we deem to be quantum-like. (Here pure means not itself a superposition, like being *dead* or being *alive*, or, more realistically, being ON or being OFF.) We showed that in the case of a single node, the state space can never satisfy the condition specifying the existence of quantum-like behavior: single nodes in isolation are necessarily not quantum-like. We call such non quantum-like behavior "classical."

Moreover, we showed that in clusters of two or more nodes of this type, at least one node is certain to behave classically, though we may not specify which it is. As the number of nodes increases we showed that the classical behavior of a specific node becomes less probable, since the effect of the increasing number of possible superpositions swamps the influence of a particular node.

4 Extending the state spaces

We recall from [8] that the state space models for the clusters of n nodes used there were rectangular sets of the form $W_n \subset \mathbb{R}^n$ constituting the product of the n closed intervals of $\mathbb{R}^n$ representing the ranges of the values in the nodes of the cluster, now network, with the origin omitted. We showed in case $n > 1$ that if the origin is an interior point, then the ortholattice of propositions in the associated Kripke model (based on the proximity relation given by the usual inner product restricted to subsets of the form W_n) is non-Boolean, and that this implies that the modeled system behaves in a quantum-like manner, the case of interest to us here. In fact, it is easy to see and easily proven that in these cases this ortholattice is isomorphic with the ortholattice of (necessarily closed) subspaces of the ambient space $\mathbb{R}^n$. This isomorphism is given by sending a Kripkean proposition in W_n , which is the intersection with W_n of a unique subspace of $\mathbb{R}^n$, to that subspace. The inverse map takes a subspace $V \subset \mathbb{R}^n$ to the set $V \cap W_n$. Note that in this isomorphism the bottom subspace $\{0\} \subset \mathbb{R}^n$ is mapped to the bottom Kripkean proposition $\emptyset \subset W_n$, since the origin has been omitted from W_n . Consequently, there is no loss of generality so far as our earlier logical considerations are concerned if we extend the state spaces of our nodal clusters or networks to the entire spaces of the form $\mathbb{R}^n$, considered as finite dimensional real Hilbert spaces in the usual way. The actual values inside the nodes at any moment, which are in any case presumed to be "hidden," are not relevant to the structural aspects we will exploit initially. In any particular instance of a network in which the nodal values are relevant, they can be restricted *ex post facto* if necessary.

We shall refer to these spaces as *extended* state spaces.

4.1 The naturality of the linear structure

A linear-like structure is inherent—arises naturally—in systems of the type we have been considering, and manifests as *disjunction* in any Kripkean model [8]. In the models of interest to us, namely those based on Hilbert spaces, disjunction amounts to superposition, or addition, of underlying vectors. However, such disjunctions depend upon choices of the states to be superposed (or propositions to be disjoined). In the case of interest to us here, namely that of neural-like networks, these choices are conditioned by the geometry of biological neuron connectivity. Thus we choose basic states corresponding to the nodes of the network as in Section 5 below, and this fixes the linear structure. The connectivity relation is one-way from one node to many nodes, and so *determines* a linear map of state spaces

without itself being linear in its dynamical parameters. Dually, the incoming connections to a node are many. These circumstances lead inevitably to the appearance of maps of vector spaces, though the linear structure is in a sense incidental. This perhaps explains retroactively the ubiquitous appearance of linear structures in such studies as [17, 18] and many others.

5 Network actions

We shall denote our networks by $N_A, N_B, \dots$ with the corresponding extended state spaces being denoted respectively by $\mathfrak{H}_A, \mathfrak{H}_B, \dots$. The underlying networks are postulated to interact with each other, and themselves, in certain ways which will be abstracted later in the form of maps induced upon their respective extended state spaces. For the network N_A , we shall denote by $\{n_i^A\}$ the set of nodes in a chosen ordering. We shall consider a cluster as a network with no internal connections. The extended state space $\mathfrak{H}_A \cong \mathbb{R}^n$ is equipped with a basis $\{e_i^A\}$ and elements of $\mathfrak{H}_A$ may be expressed as linear combinations of this basis (possibly with the ranges of coefficients suitably restricted). Thus a vector of the form λe_i^A is now interpreted as representing the presence of the value λ in the node n_i^A . It is the state vector in $\mathfrak{H}_A$ with λ in the i -th position and zeroes elsewhere. This interpretation of λe_i^A as the presence of the value λ in the node n_i^A , which is essentially forced upon us by the notation, gives a somewhat different interpretation to these spaces of states and their transformations than is the case in physics proper, where states can refer to physical attributes such as position, spin, etc.

The basic *effect* one putative network N_A can have upon another, N_B say, is for nodes of N_A to be connected via a single step axon-like link to certain nodes (or no nodes) of N_B , and upon their firing, should they fire, to determine certain values in these connected nodes. In the case of *neural-like* networks, which is the case we shall henceforth assume for all of our networks, a node n_i^A of N_A is generally connected to many nodes of N_B . Such an effect may be expressed by associating with each *state* e_i^A of N_A a superposition of states of the form $\sum_j \alpha_{ij} e_j^B$ where the coefficients reflect the number of connections $n_i^A \rightarrow n_j^B$ together with scalings of the values transferred. We note that the quantum-like imperative to superpose entails integrating nodes, in the case of quantum-like behavior. Such behavior is absent in the single node case but we shall assume that single nodes are also integrators.

In more detail, the firing of a particular node n_i^A in N_A into N_B entails the collapse of a superpositional state of N_A onto the “classical” state e_i^A and this may be interpreted as a *measurement* of N_A by N_B . The association

$$e_i^A \mapsto \sum_j \alpha_{ij} e_j^B \quad (5.1)$$

is then interpreted as representing the superposition of all such possible measurements. It represents a possible neural-like wiring of nodes of N_A to nodes of N_B along which firings may take place, together with information regarding the transmitted values, should they be transmitted. The unidirectional nature of the axonal connection of one neuron to others enables a mapping of each basis state of N_A (in its extended state space) to a (possibly null) superposition of basis states of the N_B extended state space. As noted, this *determines* a linear map from $\mathfrak{H}_A$ to the $\mathfrak{H}_B$ but is not itself linear in the relevant variables. Note that the oriented adjacency matrix of a network N_A realizes a map $\mathfrak{H}_A \rightarrow \mathfrak{H}_A$.

We shall now specify this relation in more detail. Namely, let $v_i^A(t)$ denote the value in n_i^A at time t . If this node is firing at t in the manner of a neuron, then the action potential has started traveling down the axon towards the axon terminals where it will induce neurotransmitters to be released into the synaptic clefts. The graded potential then induced by N_A upon the connected dendrites of the receiving nodes n_j^B —or “measured” by them—is some possibly very complicated nonlinear function, $p_{ij}^{AB}(v_i^A(t))$ of the value in n_i^A (the postsynaptic potential). Thus we shall think of a map $\mathfrak{H}_A \rightarrow \mathfrak{H}_B$ holding at time t , and find that the state e_i^A at time t is sent to the superpositional state

$$\sum_j \Lambda_{ij}^{AB}(t) \varphi_{ij}^{AB}(v_i^A(t)) e_j^B \quad (5.2)$$

of N_B , the relevant values to be received at the respective nodes at possibly different times later than t , since each axon terminal connection will have its own latency period or delay. Here $\Lambda_{ij}^{AB}(t)$ denotes the (*oriented*) *adjacency matrix* of the A to B connections at time t . Namely, $\Lambda_{ij}^{AB}(t)$ is the number of irreducible connections at time t from n_i^A to n_j^B , including zero if there are none. The other factors are defined as follows:

$$\varphi_{ij}^{AB}(v_i^A(t)) = \begin{cases} \frac{p_{ij}^{AB}(v_i^A(t))}{v_i^A(t)} & \text{if } v_i^A(t) \neq 0 \\ 0 & \text{if } v_i^A(t) = 0. \end{cases} \quad (5.3)$$

Here we note that the state of firing or not firing of the node n_i^A at t is encoded in the function p_{ij}^{AB} which would have to be zero if $v_i^A(t)$ were below the threshold in which case no postsynaptic potential is induced (by N_A 's nodes) in attached nodes. Thus, at time t we have the mapping

$$e_i^A \mapsto \sum_j \Lambda_{ij}^{AB}(t) \varphi_{ij}^{AB}(v_i^A(t)) e_j^B \quad (5.4)$$

with the concomitant value in n_i^A inducing the value

$$\Lambda_{ij}^{AB}(t) p_{ij}^{AB}(v_i^A(t)) \quad (5.5)$$

in each n_j^B at a later time, depending on i .

The measuring network N_B 's node n_j^B receives these values as follows: from n_1^A the value (5.5) with $i = 1$ at some time $t + \tau_1$; from n_2^A this value with $i = 2$ at some time $t + \tau_2$; and so on. That is to say, n_j^B receives $\Lambda_{1j}^{AB}(t) p_{1j}^{AB}(v_1^A(t))$ at time $t + \tau_1$; $\Lambda_{2j}^{AB}(t) p_{2j}^{AB}(v_2^A(t))$ at time $t + \tau_2$; and so on. (These values will be zero for the non-firing N_A nodes.) Dropping the double superscripts, with $\tau := \max\{\tau_1, \tau_2, \dots\}$ and superposing (or “time integrating”) the resulting states, we then have that at time $t + \tau$ the value transmitted by N_A to n_j^B is given by:

$$v_j^B(t + \tau) = \sum_i \Lambda_{ij}(t) p_{ij}(v_i^A(t)) \quad (5.6)$$

$$= \sum_i \Lambda_{ji}^T(t) p_{ij}(v_i^A(t)) \quad (5.7)$$

$$= (\Lambda^T(t) P(\mathbf{v}^A(t)))_{jj} \quad (5.8)$$

where $\Lambda(t)$ denotes the adjacency matrix ($\Lambda_{ij}(t)$) of A to B connections at time t , the superscript T denotes matrix transpose, and $P(\mathbf{v}^A(t)) := (p_{ij}(v_i^A(t)))$. (Here we assume

that the time scales are short enough so that fading out of the incoming signals does not occur before $t + \tau$.) Thus we may write

$$\mathbf{v}^B(t + \tau) = \text{diag}(\Lambda^T(t)P(\mathbf{v}^A(t))). \quad (5.9)$$

We note that despite the use of the matricial shorthand, this association is generally not linear in $\mathbf{v}^A(t)$. (We note that experiment has shown that the p_{ij} functions are generally smooth [5].)

This $\mathbf{v}^B(t + \tau)$ is a superpositional state of N_B which omits the additive, or superpositional, interference of other possible inputs into N_B from sources other than N_A during the period between t and $t + \tau$. This extraneous input is often simulated by adding a stochastic term, as in [5]. This is certainly consistent with a quantum-like underlying model and we shall return to it elsewhere.

In summary, we have interpreted the case of a network N_A being connected to a network N_B via irreducible (single) axonal connections in terms of a linear map $\mathfrak{H}_A \rightarrow \mathfrak{H}_B$ at time t . That is, both the wiring pattern and the values transferable at the time t are encoded in the expression (5.4) (while ignoring other sources of input to N_B). All of these may vary with t .

6 Network combinations

We shall consider two ways of combining networks. The first is to merely consider the network formed by juxtaposing, or putting together, networks without positing any connections between them. If this is done with networks N_A and N_B , say, then the extended state space of the combined system is easily seen to be

$$\mathfrak{H}_A \oplus \mathfrak{H}_B. \quad (6.1)$$

The second mode of combination of two networks N_A and N_B , say, is to consider the network obtained by *tightly* binding together each pair of constituent nodes, while leaving the respective connections in place. Here *tight* binding means that each bound pair of nodes, (n_i^A, n_j^B) , say, shares any value assigned to either member of the pair. In actual neuronal tissue, this simulates the function of a substrate of *interneurons*, which is independent of the principal axonal connections. Interneurons are cells of a vastly diversified group which includes basket cells and chandelier cells, which connect local principal neurons and induce very rapid inhibitory signalling between networks of them, having synapses of both chemical and electrical types (bidirectional gap junctions) [9]. Such interneuronal activity has been implicated in the formation of long-term memories [19]. The bound pairs (n_i^A, n_j^B) are the nodes of the combined network with the original connections being carried along. The basic state corresponding to such a pair is represented by a vector of some paired form, such as (e_i^A, e_j^B) , in which a nodal value is shared. Thus $(\lambda e_i^A, e_j^B) = (e_i^A, \lambda e_j^B) = \lambda(e_i^A, e_j^B)$ according to the convention forced upon us as in Section 5. Consequently, the extended state space of such a combined network is well represented by the tensor product

$$\mathfrak{H}_A \otimes \mathfrak{H}_B \quad (6.2)$$

in which we take $(e_i^A, e_j^B) := e_i^A \otimes e_j^B$. (See Appendix A for a definition of the tensor product). Thus, values are shared since $\lambda e_i^A \otimes \mu e_j^B = \mu e_i^A \otimes \lambda e_j^B$, and similarly for any number of tensorial tuples. That is to say, if the combined system is in this latter basic “classical” state, the nodal values may be distributed among the nodes connected in this manner and may vitiate the values already there: this is similar to the way in which interneural cells synchronize and orchestrate the activity of principal neurons. Moreover, should one of

these values momentarily become zero, then *all* of the interneurally connected nodes must also zero out at that moment. This is both synchronizing and inhibiting. We note the significant fact that superposition of such tensored states immediately entails the likelihood of quantum-like entanglement.

7 The bicameral neuron and the emergence of an Ising-like model

Actual neurons are extended in space, some of them having for instance very long axons, others having large somas, such as pyramidal cells, and this has consequences which are generally ignored in neural net models. In this work, we shall simulate such an extension by resolving our single node, or *unicameral*, neuron model into one having two “chambers.” The first is the input chamber corresponding to that part of the soma to which the dendrites are attached (depicted by the sub-oval on the left in Fig. 1), while the second is the output chamber, corresponding to the hillock (or trigger zone)–*cum*–axon system, or axon initial segment, along which the action potential is conducted upon its firing (depicted by the sub-oval on the right in Fig. 1). In this way, we can accommodate the values of the membrane potential at the two extreme ends of the neuron, which will generally be different upon its firing. Once the graded input potential achieves the threshold value, assuming it does, the action potential is triggered and occupies the output chamber, depicted by the right most oval in the diagram. The value, or amplitude, of this action potential will in general be different from, and independent of, the incoming somatic potential to the left of the trigger zone–hillock region subsequent to the firing. This arrangement includes the case of neurons that fire independently of the dendritic inputs, such as pacemaker cells.

The diagram in Fig. 1, and the model itself, is of course schematic and simplified. Real synapses may often impinge anywhere on a neuron. Since the basic neuron now comprises two nodes, its state space is a two-dimensional extended state space isomorphic to $\mathbb{R}^2 \cong \mathbb{R} \oplus \mathbb{R}$, where the first term accommodates the state space of the input node on the left of Fig. 1 and the second term accommodates the state space of the output node on the right, and will be interpreted as a two node system. As such, its states will be generally in a superposition, and we have something very like a qubit, though with the provisos discussed in [8].

Let us adopt the convention that lower case Romans $a, b, \dots$ will denote the extended state spaces of single nodes (i.e., the most trivial single node networks), namely the one-dimensional space isomorphic to $\mathbb{R}$. Since we wish to distinguish the two summands in the

Fig. 1 Schematic diagram for a bicameral neuron as implemented in the context of the present work

bicameral case, it will prove convenient to temporarily express the ambient space describing the setup in the diagram with the following convention:

$$\mathbb{R}e^{a^I} \oplus \mathbb{R}e^a. \quad (7.1)$$

Here we have explicitly written the generators of the two summands, namely, with a^I and a denoting respectively the state spaces of the input and output nodes, e^{a^I} and e^a denote respectively the basis elements of those spaces, in accordance with our earlier convention. We can now explain the dotted arrow. This system, represented by states of the form (λ, μ) , for real λ and μ , relative to the chosen basis $\{e^{a^I}, e^a\}$, admits a couple of operators, namely:

$$\alpha^T(\lambda, \mu) := (0, \lambda), \quad (7.2)$$

$$\alpha(\lambda, \mu) := (\mu, 0). \quad (7.3)$$

It is immediate that these operators are actually transposes of each other relative to the usual inner product, T denoting transpose. In addition, these operators satisfy the following *anticommutation relations*:

$$\alpha\alpha^T + \alpha^T\alpha = 1, \quad (7.4)$$

$$\alpha^2 = 0, \quad (7.5)$$

$$(\alpha^T)^2 = 0. \quad (7.6)$$

The dotted arrow in the diagram corresponds to the operator α^T restricted to the space $\mathbb{R}e^{a^I}$ with range $\mathbb{R}e^a$. It represents the internal transfer of the current value in the input system from that system to the output system and thus represents a *preparation* to fire the neuron: if λ has reached the threshold value then, at that moment, it becomes a value of the action potential and moves into the trigger zone or hillock, replacing any value μ that may be there (7.2). At the same time, there is still a graded potential registering upon the input node, representing the soma and dendritic structure to the left of the trigger zone-hillock in the orientation depicted in Fig. 1. That is to say, the action potential state $(0, \lambda)$ must at this moment be superposed with the current state of the input node which is of the form $(\nu, 0)$ for some ν . We note that the operator α^T is not an observable of the system, but rather an internal unobservable operation that in a sense stands for the complex ionic flows, etc., that physically transfer the potential across the cell body.

The anticommutation relations listed above are formally the same as the relations satisfied by the *creation* operator, corresponding to our α^T , and the *annihilation* operator, corresponding to our α , for a fermionic (spin 1/2) quantum of one type, although acting in a real space (which may be complexified). The analogy may be misleading, since there are no actual physical fermions involved, but it reveals the underlying quantum-like behavior inherent in the logic of “hidden” variables. We shall adopt the deeply ingrained convention of choosing the un-transposed operator as the corresponding annihilation operator for these *virtual* fermions. Although there are no actual physical particles implied by our picture we note that the fermionic *number operator* $N := \alpha^T\alpha$ has the two eigenvalues 0 (with eigenspace the set of vectors $(\lambda, 0)$ having the output node at zero: the OFF state when normalized), and 1 (with eigenspace the set of vectors of the form $(0, \mu)$ having the input node at zero: the ON, or firable, or excited state when normalized) so that quantum mechanically an array of many such *b-neurons* may be expected to exhibit large scale behavior similar to that of an Ising-like model with two-state qubit-like systems at the lattice points, or more generally, that of a *quasispin* model. (This was noted in the highly influential [5] where the

Ising-like nature of the models adopted there by fiat are explicitly indicated: thus, quantum-like behavior is built into them from the beginning, though their quantum nature is there unremarked. See also [7].)

We consider the fermionic attributes of a single b-neuron in more detail. (For readers conversant with the uniqueness theorem for the CAR representation in finite dimensions the following argument is redundant.) Let us consider the subspace of the algebra of operators on $\mathbb{R}^2$, namely $\text{End } \mathbb{R}^2$, generated by the identity operator I and $\mathfrak{a}^T$, which may be written

$$\mathcal{A} := \mathbb{R}I \oplus \mathbb{R}\mathfrak{a}^T \subset \text{End } \mathbb{R}^2. \quad (7.7)$$

Here we have again explicitly written in the two generators I and $\mathfrak{a}^T$. It is immediately seen that $\mathcal{A}$ is in fact a (commutative) *subalgebra* of $\text{End } \mathbb{R}^2$ in view of (7.6): this will be significant later (cf. Appendix A.2). As a space $\mathcal{A}$ is of course isomorphic to the state space of a b-neuron as above. It may be noted that $(\mathfrak{a}^T)^2 = 0$ is the only property of this operator that is relevant to the structure of the algebra $\mathcal{A}$. Algebraically inclined readers will notice that the canonical such algebra is represented by the *ring of dual numbers* $\mathbb{R}[x]/(x^2) \cong \mathbb{R} \oplus \mathbb{R}\varepsilon$, where $\varepsilon := x \bmod (x^2)$ has square zero, which is obviously isomorphic with $\mathcal{A}$. This last property is the defining *leitmotif* of exterior algebra and it is implicit in the anticommutation relations which stand behind the usual Fermi–Dirac representation for physical fermions, though we do not have those here. (cf. Appendix A.2). From the above it is clear that the state space of a b-neuron is a commutative algebra isomorphic with the exterior algebra $E(\mathbb{R}e)$, where e denotes the state of the output node, subject to the relation $e^2 = e \wedge e = 0$. The last property expresses the impossibility of a simultaneous double firing of the b-neuron, in the second quantized, or occupation number formulation described below (Section 8.1).

For a single b-neuron, let us suppose, as in ordinary quantum mechanics, that the (kinetic) energy observable, or (kinetic) Hamiltonian, is proportional to the number operator. Then the two energy eigenspaces are exactly the “classical” state spaces of the individual nodes, and this is consistent with the quantum mechanical view that these eigenstates are exactly the ones with definite energy values, which the system collapses into when the energy is measured or observed (possibly by another neuron). Note that N is a projection. Moreover, the minimal energy states of a b-neuron are the ones in which the output node is at *zero*. Consequently, under the usual physical rubric of minimizing energy, b-neurons would tend towards a state of having released their action potentials, or having *fired*. However the overall behavior of a network of many b-neurons will be very different, as in the case of a physical many-quanta system. In Section 8.1, we will derive the analogue of the Hamiltonian for such a network and in Section 9.4 classify the spectra in the simplest non-trivial cases of the 13 three-neuron network *motif classes*.

Let us note at this point that our considerations have produced a hybrid object, namely one whose external behavior is apparently binary—there are exactly two observable energy eigenstates—but whose internal behavior is not.

8 Network dynamics

In elementary non-relativistic quantum mechanics the action of time upon states is taken to be an operator, $U(t)$, say, necessarily unitary since normalized states are assumed to develop into normalized states. A state ξ develops in time $t \in \mathbb{R}$ to

$$\xi(t) = U(t)\xi. \quad (8.1)$$

Under mild constraints, a theorem of Stone and von Neumann applies to yield the representation

$$U(t) = \exp(-itH), \quad (8.2)$$

where H is an Hermitian operator, and this leads immediately to the Schrödinger equation

$$\frac{\partial}{\partial t} \xi(t) = -iH\xi(t). \quad (8.3)$$

Here H is identified with the (time independent) Hamiltonian of the system in units in which $\hbar = 1$. (If H is time dependent, (8.2) is replaced by one of the form

$$U(t) = \mathcal{T} \exp \left(-i \int_0^t H(t') dt' \right) \quad (8.4)$$

where $\mathcal{T}$ denotes the time ordering operator.)

In Section 8.1 we consider this Schrödinger-like dynamics and derive an operator implementing the time evolution of the states of such a network, which is the analogue of the Hamiltonian in physics proper. Our dynamics differs fundamentally from that of ordinary quantum mechanics in at least the following respects:

- the state spaces are real finite dimensional spaces;
- inner products between vectors in these spaces are not conserved since the vectors represent actual values in the network nodes, and these can change in time. The *state* represented by a vector is the ray it determines, which, as in ordinary quantum mechanics, may be associated uniquely with its normalized element. Since probabilities of transitions $\mathbf{w} \rightarrow \mathbf{v}$ are given by inner products of the associated states or normalized vectors, as in

$$\text{prob}(\mathbf{w} \rightarrow \mathbf{v}) = \left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} \right)^2, \quad (8.5)$$

the tautologous statement that the transition $\mathbf{v} \rightarrow \mathbf{v}$ is certain still obtains at all times. It is clear from this that it is not necessarily the case that $\text{prob}(\mathbf{w}(t) \rightarrow \mathbf{v}(t)) = \text{prob}(\mathbf{w} \rightarrow \mathbf{v})$. In fact, the time evolution operator we shall derive (Section 8.1) is not orthogonal;

- the time development operator and states are highly time dependent;
- the simplifications inherent in most of the physical analogues of our system due to spatial symmetries, known force laws and their symmetries, etc., are not available here.

Finally, we note that the connection between the quantities of interest in a quantum theory of many degrees of freedom, namely certain amplitudes, and the quantities of interest in the associated statistical theory, namely partition functions etc., is given by the replacement of the time variable (t , say) in the quantum theory by the corresponding imaginary variable (it). This leads to expressions of the type we obtain here. This statistical connection will be developed elsewhere.

8.1 Derivation of the pseudo-Hamiltonian for a network

As noted, a b-neuron as specified in Section 7 is equivalent to a (real) fermionic system with two pure states we have characterized as ON, corresponding to it having some non-zero value in its output node so that it is in a state of firing, and 0 in its input node, and OFF, in which there is a non-zero value in its input node and 0 in its output node: the state of not firing. We consider a single unnamed such network of b-neurons, $n_i, i = 1, \dots, N$, with adjacency matrix Λ , possibly time dependent. As a collection of b-neurons its state space is the tensor product of the individual b-neuron states spaces, which are respectively

isomorphic to the corresponding exterior algebras $E(\mathbb{R}e_i)$ so that we have for the whole network $\bigotimes_{i=1}^N E(\mathbb{R}e_i) \cong E(\bigoplus_{i=1}^N \mathbb{R}e_i) \cong E(\mathbb{R}^N)$ by (9.35). That is, the Fermi–Dirac space of the extended state space $\mathfrak{H} := \bigoplus_{i=1}^N \mathbb{R}e_i$. Here we note that, since $E(\mathbb{R}e_i)$ is itself isomorphic to the extended state space of the single b-neuron considered as a trivial network, in view of (9.35) the Fermi–Dirac space will itself be realizable as the extended state space of a *network* in the presence of an interneural substrate. Namely, the network obtained by tightly binding the single b-neuron networks as described in Section 6, since in the presence of the interneural substrate such formations are possible. Below we shall interpret this space also as the space of *firing patterns*: this equivalence of interpretations will later have computational significance.

We denote by $a_i, a_i^\dagger$, the usual annihilation and creation operators on this space. We have adopted the usual notations for these operators for cosmetic reasons: the dagger here denotes transpose since we are working over the real numbers. Thus

$$a_i^\dagger(e_{j_1} \wedge \dots \wedge e_{j_m}) = e_i \wedge e_{j_1} \wedge \dots \wedge e_{j_m} \quad (8.6)$$

from which it follows that

$$\begin{aligned} a_i(e_{j_1} \wedge \dots \wedge e_{j_m}) &= \delta_{ij_1}(e_{j_2} \wedge \dots \wedge e_{j_m}) - \delta_{ij_2}(e_{j_1} \wedge e_{j_3} \wedge \dots \wedge e_{j_m}) + \dots \\ &\quad \dots + (-1)^{m+1} \delta_{ij_m}(e_{j_1} \wedge \dots \wedge e_{j_{m-1}}). \end{aligned} \quad (8.7)$$

The last equation may be described as follows: a_i has the effect on an element $e_{j_1} \wedge \dots \wedge e_{j_m}$ of giving 0 if e_i does not appear. If it does appear, shuffle it to the front keeping track of the sign changes, and then remove it. These operators obey the canonical anticommutation relations:

$$\{a_i, a_j^\dagger\} = a_i a_j^\dagger + a_j^\dagger a_i = \delta_{ij} I, \quad (8.8)$$

$$\{a_i, a_j\} = 0, \quad (8.9)$$

$$\{a_i^\dagger, a_j^\dagger\} = 0, \quad \text{from which it follows that}$$

$$a_i^2 = (a_i^\dagger)^2 = 0. \quad (8.10)$$

These circumstances lead to a useful paradigm known as the *occupation number* formalism [10]. The idea is to take account of the right-hand side of the identity (9.35) to develop a compact vector-like notation in which each i -entry may take values characterizing the ON and OFF states in $E(\mathbb{R}e_i)$. For each number operator, $a_i^\dagger a_i$ has the eigenvalues 0 and 1. The eigenvalue 0 has as eigenvector the state $|\mathbf{0}\rangle := 1 \in \mathbb{R}$ this $\mathbb{R}$ being the first summand in $E(\mathfrak{H})$, which is regarded as the state of zero occupancy, containing no “particles,” also called the *vacuum* state. (It is an unfortunate accident of history that the state containing no particles should be represented by the scalar we denote by 1.) In our context, $|\mathbf{0}\rangle$ is the state in which no b-neurons are firing (though there may be “hidden” values in the input nodes). The other eigenstate is the i^{th} ON state, in which there is a (necessarily) single i^{th} occupancy, corresponding to the state of firing of the i^{th} b-neuron.

Let us write a general state of occupancy of the network system in the form

$$|v_1, v_2, \dots, v_N\rangle \quad (8.11)$$

where

$$v_i = \begin{cases} \mathbf{1}_i & \text{if b-neuron } n_i \text{ is ON; or occupied;} \\ & \text{or firing;} \\ \mathbf{0}_i & \text{if b-neuron } n_i \text{ is OFF; or unoccupied;} \\ & \text{or not firing.} \end{cases} \quad (8.12)$$

If all the b-neurons are OFF in (8.11), we have

$$|v_1, v_2, \dots, v_N\rangle = |\mathbf{0}\rangle. \quad (8.13)$$

We shall, further, adopt the convention that any $\mathbf{0}_i$ will simply be dropped from the list $\{v_1, \dots, v_N\}$ in (8.11), since their inclusion just amounts to multiplying an element of $E(\mathfrak{H})$ by instances of the algebra unit $1 \in \mathbb{R}$. Then we note that for instance,

$$|\mathbf{1}_{j_1}, \mathbf{1}_{j_2}, \dots, \mathbf{1}_{j_m}\rangle = a_{j_1}^\dagger a_{j_2}^\dagger \dots a_{j_m}^\dagger |\mathbf{0}\rangle \quad (8.14)$$

$$= e_{j_1} \wedge e_{j_2} \wedge \dots \wedge e_{j_m}. \quad (8.15)$$

All of these notations will be prove helpful in what follows. (We shall generally use lower case Greek characters to denote generic elements of $E(\mathfrak{H})$ in order to distinguish them from elements in the original extended state space $\mathfrak{H}$. *States* of the now multipartite system, comprising N b-neurons, will, as usual, be associated with rays in $E(\mathfrak{H})$, classified by their normalized elements). Note the alternating nature of the $\wedge$ -product precludes more than one simultaneous firing of any b-neuron.

From the last two equations, it is apparent that the basic states depicted there may also be regarded as *firing patterns* since they denote the “classical” states of network occupancies, namely the set or pattern of b-neurons that are ON and therefore firing at an implicit time. We shall extend this terminology to all the states in $E(\mathfrak{H})$. Thus, superpositions of “classical” firing patterns will also be considered to be firing patterns.

Before moving on to a consideration of the dynamics of such a system we shall digress briefly to discuss an aspect of such state spaces which is highly relevant to their computational capacity. Thus consider a system, such as the one just described, as a classical system: that is, its classical states are not subject to superposition. Then the number of (classical) subsystems, or (classical) firing patterns, is 2^N . In order to store such patterns, perhaps for future use in computations, we would require a store whose size increases exponentially with N . In view of the fundamental identity (9.35), however, we have

$$E(\mathfrak{H}) \cong E\left(\bigoplus_{i=1}^N \mathbb{R}e_i\right) \cong \bigotimes_{i=1}^N E(\mathbb{R}e_i) \quad (8.16)$$

so that the number of b-neurons required to house these firing patterns in the presence of quantum-like superposition (and, in this case, also concomitant entanglement) only increases linearly in N . This benefit presumably underlies some of the efficacy of Ising-like model simulations *à la* [5, 6].

Basic to the construction of a p-Hamiltonian are the analogues of those expressions which, in the case of the physics of many-fermion systems, are called the (*one particle*) *exchange integrals*, which express the interaction between the fermions, such as electrons, which are exchanged between the nodes or sites of a lattice. In our case, they are not of course actual particles, and, moreover, the sites do not constitute a lattice. If b-neuron n_i is axonally connected by single links to b-neuron n_j and fires at time $t = 0$, say, then the value in n_j 's input node changes, after a short delay, t , by the addition of $\theta_{ij}(0)$, the post

synaptic potential, where

$$\theta_{ij}(s) := \Lambda_{ij}(s)p_{ij}(v_i(s)) \quad (8.17)$$

as in (5.5) with N_A and N_B here being the same network. So, if $v_j(t)$ denotes the value in the n_j input node when this happens, it is updated to $v_{ij}(t) := v_j(t) + \theta_{ij}(0)$. If this latter value is less than the n_j threshold value, denoted Θ_j , then the firing of n_j is not initiated and the state of occupancy of n_j , which may be firing, is not changed at this moment. (We are assuming for the moment that there are no inputs extraneous to the network at hand.) If n_j happens to be firing when the input node value reaches threshold, we shall assume that this input value has no impact on the n_j output node during the duration of n_j 's firing, since actual neurons do not initiate another firing while already firing, a circumstance modeled precisely by (8.10). If the n_j threshold is reached at time t , and if n_j is not firing at that moment so that the network is in a state of the form $|v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_N\rangle$, then this state moves to the state

$$a_j^\dagger |v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_N\rangle = |v_1, \dots, v_{j-1}, \mathbf{1}_j, v_{j+1}, \dots, v_N\rangle. \quad (8.18)$$

That is to say the firing of n_j is initiated. We shall assume that the delay between the initiation of firing and firing itself is negligible. Let us define a function $J_{ij}(t)$ as follows.

$$J_{ij}(t) := \begin{cases} 0 & \text{if } v_{ij}(t) < \Theta_j \text{ and } n_j \text{ is not} \\ & \text{firing at time } t; \\ v_{ij}(t) & \text{if } v_{ij}(t) \geq \Theta_j \text{ or } n_j \text{ is} \\ & \text{firing at time } t. \end{cases} \quad (8.19)$$

Then

$$J_{ij}(t)a_j^\dagger = \begin{cases} 0 & \text{if } v_{ij}(t) < \Theta_j \text{ and } n_j \text{ is not} \\ & \text{firing at time } t; \\ v_{ij}(t)a_j^\dagger & \text{if } v_{ij}(t) \geq \Theta_j \text{ or } n_j \text{ is} \\ & \text{firing at time } t. \end{cases} \quad (8.20)$$

where it is worth noting that the 0 in the first case of (8.20) refers to the zero operator.

Recall that $v_{ij}(t)$ is the value in n_j 's input node at time t after n_i has fired into n_j . If $v_{ij}(t) < \Theta_j$ and n_j is not firing at t the state of occupancy of n_j is not changed at t . If $v_{ij}(t) \geq \Theta_j$ and n_j is not firing at t , then the firing of n_j is initiated and the value $v_{ij}(t)$ is moved into the output node, forming the initial value of the outgoing action potential. Regardless of the value of $v_{ij}(t)$, if n_j is firing at t , then this value in the input node is ignored: that is, it has no impact on the action potential. For if n_j is firing, regardless of the input node value, it cannot be prepared to fire again since

$$J_{ij}(t)a_j^\dagger |\dots, \mathbf{1}_j, \dots\rangle = J_{ij}(t)(a_j^\dagger)^2 |\dots\rangle = 0. \quad (8.21)$$

These behaviors are reflected in (8.20). The first case reflects the impossibility of initiating a firing of n_j (by the application of $a_j^\dagger$) if its input value following n_i 's firing is below threshold. The second case reflects the firing of n_j at time t , initiated by n_i 's firing, if $v_{ij}(t) \geq \Theta_j$ and n_j is not already firing. The value $v_{ij}(t)$ is then pulled through to the output

node and forms the initial value of the outgoing action potential. The second case also takes care of the situation in which n_j is firing at t , namely, a second firing is aborted as in (8.21).

With this in mind, we now take note of the basic *hopping* (or *switching*) operators, namely the operators $a_j^\dagger a_i$ for which:

$$a_j^\dagger a_i |\mathbf{0}\rangle = 0 \quad (8.22)$$

$$a_j^\dagger a_i |\mathbf{1}_k\rangle = \begin{cases} 0 & \text{if } i \neq k \\ |\mathbf{1}_j\rangle & \text{if } i = k. \end{cases} \quad (8.23)$$

The operator $a_j^\dagger a_i$ may be interpreted as realizing the operation of turning n_j ON while turning n_i OFF and is not applicable to singly occupied states other than that in which n_i is ON, namely $|\mathbf{1}_i\rangle$: this non-applicability is the interpretation of zero, the quantum *falsum*, appearing as the result of the application. Then we have, in the only case which does not give zero, namely when n_i is firing so is in the state $|\mathbf{1}_i\rangle$ (at time 0):

$$J_{ij}(t) a_j^\dagger a_i |\mathbf{1}_i\rangle \quad (8.24)$$

$$= J_{ij}(t) |\mathbf{1}_j\rangle \quad (8.25)$$

$$= \begin{cases} 0 & \text{if } n_j \text{ is not firing at } t; \\ v_{ij}(t) |\mathbf{1}_j\rangle & \text{if } n_j \text{ is firing } t. \end{cases} \quad (8.26)$$

Thus $J_{ij}(t) a_j^\dagger a_i$ is the operator yielding the net effect on the network a (short) time t after n_i fires. Then $\sum_{i,j} J_{ij}(t) a_j^\dagger a_i$ is the operator representing the net effect after t of all possible firings at time 0. Since the basic hops (or switchings) cannot be reduced further it would appear that the operator $\sum_{i,j} J_{ij}(t) a_j^\dagger a_i$ should represent also a small deformation induced on the states of $E(\mathfrak{H})$. This would imply that the operator acts as a derivation for the algebra structure of $E(\mathfrak{H})$. The proof that it does so follows by applying it to a basic state written in the form $e_{i_1} \wedge \dots \wedge e_{i_m}$. Namely

$$\begin{aligned} \sum_{i,j} J_{ij}(t) a_j^\dagger a_i (e_{i_1} \wedge \dots \wedge e_{i_m}) \\ = \sum_j \left(\sum_i J_{ij}(t) a_j^\dagger a_i (e_{i_1} \wedge \dots \wedge e_{i_m}) \right) \end{aligned} \quad (8.27)$$

$$= \sum_j (J_{i_1 j}(t) a_j^\dagger a_{i_1} (e_{i_1} \wedge \dots \wedge e_{i_m}) + J_{i_2 j}(t) a_j^\dagger a_{i_2} (e_{i_1} \wedge \dots \wedge e_{i_m}) + \dots) \quad (8.28)$$

$$= \sum_j (J_{i_1 j}(t) (e_j \wedge e_{i_2} \wedge \dots \wedge e_{i_m}) + J_{i_2 j}(t) (e_{i_1} \wedge e_j \wedge e_{i_3} \wedge \dots \wedge e_{i_m}) + \dots) \quad (8.29)$$

$$= \left(\sum_j J_{i_1 j}(t) e_j \right) \wedge e_{i_2} \wedge \dots \wedge e_{i_m} + e_{i_1} \wedge \left(\sum_j J_{i_2 j}(t) e_j \right) \wedge e_{i_3} \wedge \dots \wedge e_{i_m} + \dots \quad (8.30)$$

Thus, the operator $\sum_{i,j} J_{ij}(t) a_j^\dagger a_i$ acts as a deformation of the state $e_{i_1} \wedge \dots \wedge e_{i_m}$ since if we put

$$\delta_t := \sum_{i,j} J_{ij}(t) a_j^\dagger a_i \quad (8.31)$$

we find that (8.30) yields

$$\delta_t(e_{i_1} \wedge \dots \wedge e_{i_m}) = \delta_t e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_m} + e_{i_1} \wedge \delta_t e_{i_2} \wedge e_{i_3} \wedge \dots \wedge e_{i_m} + \dots \quad (8.32)$$

so that δ_t is a derivation. Consequently, for any occupied state ξ of the type given above, we should have, for small t

$$\xi(t) \approx \xi(0) + t\delta_t\xi(0) \quad (8.33)$$

$$= (I + t \sum_{i,j} J_{ij}(t) a_j^\dagger a_i) \xi(0). \quad (8.34)$$

Thus, the operator on the right-hand side of the last equation implements the change over (short) intervals of time of the state of occupancy, or *firing pattern*, of the network. Consequently, we identify δ_t with the p-Hamiltonian or infinitesimal generator of the time development operator, namely

$$H(t) := \sum_{i,j} J_{ij}(t) a_j^\dagger a_i, \quad (8.35)$$

with the corresponding global time development operator given by

$$T(t) := \exp(t \sum_{i,j} J_{ij}(t) a_j^\dagger a_i). \quad (8.36)$$

We note that $H(t)$ is time dependent and does not give rise to an orthogonal time evolution operator unless $J_{ij}(t) = -J_{ji}(t)$, a completely unreasonable restriction upon real neuronal networks, and one we shall not impose. This allows for the presence of *autapse*-like terms in which $i = j$, corresponding to the cases in which neurons may be connected to themselves, a frequent occurrence in biological neural networks. In the complex case of actual physics, this expression describes the kinetic term for a class of many-fermion systems going under the rubric *quasispin* [10].

With H as in (8.35), the evolution operator is

$$\exp(tH) = I + tH + \frac{t^2}{2!} H^2 + \dots \quad (8.37)$$

Its effect on a state of single occupancy, $|\mathbf{1}_{i_0}\rangle$ say—to produce the state evolving from the firing of the single b-neuron n_{i_0} —is given for small t by the superposition

$$\exp(tH)|\mathbf{1}_{i_0}\rangle = |\mathbf{1}_{i_0}\rangle + t \sum_{i,j} J_{ij}(t) a_j^\dagger a_i |\mathbf{1}_{i_0}\rangle + \frac{t^2}{2!} \sum_{i,j,k,l} J_{ij}(t) J_{kl}(t) a_j^\dagger a_i a_l^\dagger a_k |\mathbf{1}_{i_0}\rangle + \dots \quad (8.38)$$

Note that since t is small, contributions to the superpositions associated with higher powers of t are progressively less likely. To interpret this, let us unfold the term in $t^2/2!$, namely:

$$\sum_{i,j,k,l} J_{ij}(t) J_{kl}(t) a_j^\dagger a_i^\dagger a_l^\dagger a_k |1_{i_0}\rangle = \sum_{i,j,k,l} J_{ij}(t) J_{kl}(t) a_j^\dagger a_i^\dagger \delta_{ki_0} |0\rangle \quad (8.39)$$

$$= \sum_{i,j,l} J_{ij}(t) J_{i_0l}(t) a_j^\dagger a_i |1_l\rangle \quad (8.40)$$

$$= \sum_{i,j,l} J_{ij}(t) J_{i_0l}(t) a_j^\dagger \delta_{il} |0_l\rangle \quad (8.41)$$

$$= \sum_{j,l} J_{lj}(t) J_{i_0l}(t) |1_j\rangle \quad (8.42)$$

$$= \sum_{j,l} J_{i_0l}(t) J_{lj}(t) |1_j\rangle \quad (8.43)$$

$$= \sum_j (J(t)^2)_{i_0j} |1_j\rangle \quad (8.44)$$

where $(J(t)^2)_{i_0j}$ represents the appropriate entry in the matrix $J(t)^2$. This term represents the superposition of the states of single occupancy which have been reached by firings of n_{i_0} along 2-step axonal paths. The next term is similar, involving three-step paths, and so on. Thus the formalism picks out sequential firings along paths of an increasing number of steps leading from n_{i_0} consequent upon its firing: these contribute to the superposition with probability decreasing as path length increases. Thus short paths are favored and this implies a certain nearest neighbor behavior. We note here that this behavior, where nearest neighbor effects dominate, is built into the Ising-like models. (A comparison between our model p-Hamiltonian and those of models from actual physics, including the Ising-like ones, will be made elsewhere.)

The p-Hamiltonian $H(t)$ in (8.35) has been derived on the assumption that external influences acting on the network at hand have been excluded. Such influences can have the effect of turning groups of neurons on or off or leaving them untouched, so we may assume them to be of the form

$$\sum_i (h_i(t) a_i^\dagger + l_i(t) a_i) + E(t) I \quad (8.45)$$

where $h_i(t)$, (respectively, $l_i(t)$, $E(t)$) subsumes the effects of external influences to switch b-neurons ON (respectively OFF and leave alone except for a scaling). We postpone a discussion of these cases until a sequel to this paper.

The derivational nature of operators of the form (8.35) has the consequence that the corresponding time evolution operator (8.36) is an *automorphism* of the exterior algebra $E(\mathfrak{H})$ ([20], chapter 1, section 2). This property will be of significance later.

We note also that each hopping operator summand in $H(t)$ maps each subspace $\bigwedge^n \mathfrak{H}$ into itself, so that both $H(t)$ and $T(t)$ assume block diagonal form, and so basic firing patterns represented by superpositions of the basic states of the form $|1_{i_1}, \dots, 1_{i_m}\rangle = e_{i_1} \wedge \dots \wedge e_{i_m}$, are mapped to states of the same grade (or occupation number): occupation number is preserved in time. This is because $H(t)$ by itself represents the network's infinitesimal time evolution in isolation. In order for occupation number to vary in time, terms of the form (8.45), representing external fields or stimuli, must be added to $H(t)$.

Classical theories involving spaces of states can deal only with states of single occupancy, in our terminology. Namely, the elements in the first grade summand $\mathfrak{H} \subset E(\mathfrak{H})$.

The states of higher occupancy, namely those in $\bigwedge^n \mathfrak{H}$ for $n > 1$, will be unavailable to such treatments. Classical treatments, moreover, cannot make operational or semantic distinctions between pure states and superpositional ones, even in principle. The decoherence or otherwise of superpositional states could have no significance in such theories. In Section 9.4, we shall argue that even in the simplest non-trivial case of three-neuron motif classes, both higher occupancy states and their superpositional nature have a discernible bearing upon their observed behavior.

8.2 The $H(t)$ eigenstates

In actual quantum mechanics the eigenvectors of a time independent Hamiltonian play a fundamental rôle. The normalized ones represent the stationary states (of observed energy). Only their unitary phases change in time. In our case, the eigenstates do not have complex phases, and the Hamiltonian analogue is time dependent. However, eigenvectors of our p-Hamiltonians still represent stationary *states* since the rays they determine remain fixed locally in time (that is, over short intervals) though their “phases” are real scale factors that vary exponentially in time. (Otherwise expressed, the firing pattern represented by such states, that is, the set of b-neurons that are firing, remains fixed for the possibly short time involved.) Namely, if

$$H(t)\xi = \lambda(t)\xi \quad (8.46)$$

then for very small t

$$\xi(t) = \exp(tH(t))\xi \quad (8.47)$$

$$= \exp(t\lambda(t))\xi \quad (8.48)$$

and more generally

$$\xi(t) = \exp\left(\int_0^t \lambda(s) ds\right)\xi \quad (8.49)$$

for larger t , as in (8.4). Thus, over short time intervals, the *states* (i.e., the rays) determined by such eigenvectors remain stable, though while in them *each* b-neuron’s output value scales by the *same* factor $\exp(\int_0^t \lambda(s) ds)$. This can be seen by considering the single occupancy, or first grade, component of the eigenstate ξ , whose coefficients represent the values in the output nodes of those b-neurons which are firing if the system were to enter that state at time 0. (As noted the space of these states is mapped into itself by the p-Hamiltonian.) Equation (8.49) asserts that over the next short time interval the nodal values scale by the exponential factor. Thus, for this possibly very short time interval after entering the eigenstate, the network’s firing b-neurons are synchronized in the sense that this scaling is the same for all of them, and this globalized behavior is *independent of local synaptic activity*. These eigenstates thus represent a sort of non-local coherence in the firing pattern of the network, though this stability may be fleeting.

To arrive at a provisional and tentative interpretation of the exponent we note from (8.31) and (8.35) that for such an eigenvector $\xi = \xi(0)$:

$$\xi(t) - \xi(0) \approx t\delta_t\xi(0) \quad (8.50)$$

$$= t\lambda(t)\xi(0) \quad (8.51)$$

so that $\lambda(t)$ reflects the change in the firing pattern over the time t . That is, it is interpretable as a measure of the overall instantaneous average firing rate. Consequently, $\int_0^t \lambda(s) ds$ is interpretable in some sense as a measure of the *firing activity* during the period t .

For all networks having p-Hamiltonians of the form $H(t)$ in (8.35) the two states of extreme occupancy, namely the vacuum state $|\mathbf{0}\rangle$ in which no b-neurons are firing, and the state of full occupancy $|\mathbf{1}_1, \dots, \mathbf{1}_N\rangle = e_1 \wedge \dots \wedge e_N$ in which all b-neurons are firing, have the following properties:

$$H(t)|\mathbf{0}\rangle = 0, \quad (8.52)$$

$$H(t)|\mathbf{1}_1, \dots, \mathbf{1}_N\rangle = \text{tr}J(t)|\mathbf{1}_1, \dots, \mathbf{1}_N\rangle, \quad (8.53)$$

where $\text{tr}J(t)$ denotes the trace of the matrix $J(t) := (J_{ij}(t))$. The first of these is obvious while the second follows at once from (8.31) and (8.32). The first of these equations asserts that a network in which no b-neurons are firing will remain in that state if no external influences are applied, as expected. The second asserts that if all b-neurons are firing then the short-term future behavior in the absence of external influences is entirely determined by the presence and behavior of autapses. If there are none, or if they are such that $\text{tr}J(t) = 0$, then the network remains in its state of full firing for the short term. If $\text{tr}J(t) < 0$, signifying overall autapse driven inhibition, then the values in all the output nodes will start to decay exponentially with the same exponent, namely $\int_0^t \text{tr}J(t) dt$, while if $\text{tr}J(t) > 0$, signifying overall autapse driven excitation, then the values will increase exponentially, with firing activity increasing synchronously and independently of synaptic activity, leading perhaps to seizure or some other catastrophic failure of the network. Note that it would take only a single excitatory autapse to precipitate such a short-term explosion. In reality, the complexity of the behavior of $J(t)$ would yield mediating behaviors. Thus, for instance, we might expect to find increased autapse involvement in epileptic tissue *in vitro*, with increased non-synaptic synchronized firing activity as seizure is approached. There seems to be some support for both of these conclusions: cf. [21, 22].

In general, the eigenstates with zero eigenvalues are stable and synchronized but apparently only in death, since there is no firing activity once these states are encountered, at least for the short time during which our considerations leading to the p-Hamiltonian above may be expected to obtain. Note that though OFF, a b-neuron may have non-zero values in its input node.

Finally, we note a property of these eigenstates that will be germane to their possible cognitive significance. Namely, as representatives of firing patterns, they are *local attractors*. Thus suppose ξ_λ denotes such an eigenstate at some time $t = 0$, (with eigenvalue $\lambda(t)$) and that $\xi_\lambda + \delta\xi_\lambda$ denotes a neighboring state. Here we assume that the variation δ is of parameters other than time. Then for small t ,

$$(\xi_\lambda + \delta\xi_\lambda)(t) = T(t)\xi_\lambda + T(t)\delta\xi_\lambda \quad (8.54)$$

$$= T(t)\xi_\lambda + \delta T(t)\xi_\lambda \quad (8.55)$$

$$= \exp\left(\int_0^t \lambda(s)ds\right)\xi_\lambda + \delta \exp\left(\int_0^t \lambda(s)ds\right)\xi_\lambda \quad (8.56)$$

$$= \exp\left(\int_0^t \lambda(s)ds\right)\xi_\lambda + \delta \left(\int_0^t \lambda(s)ds\right) \exp\left(\int_0^t \lambda(s)ds\right)\xi_\lambda \quad (8.57)$$

$$= \left(1 + \delta\left(\int_0^t \lambda(s)ds\right)\right) \exp\left(\int_0^t \lambda(s)ds\right)\xi_\lambda. \quad (8.58)$$

Now whatever parameters are involved in the variation of the integral in the first factor it will be a scalar quantity and therefore the right-hand side of the last equation represents the

same state, i.e., firing pattern, that ξ_λ itself represents. Thus a state near ξ_λ develops in a short time into ξ_λ itself. It is in this sense that an eigenstate is a local attractor.

9 Applications

In this section we offer some applications of the occupation number-*cum*-firing pattern formalism. Namely, a discussion of *context*, an abstract model of context storage and retrieval, and a discussion of (and some conjectures about) the nature of logic processing by schizophrenic patients. Finally, we give an analysis of the spectra of the 13 three-neuron motif classes as classified by Sporns, and argue for the structural relevance of their superpositional eigenstates.

9.1 Context

The notion of *context* is discussed in [2], chapter 7. The version of quantum theory used there is similar to ours, being adopted from the standard physical model, and stays mainly in the “single space” modality. To make the connection we return to our earlier treatment of **OL** in [8] (see also [23, 24]). Namely, we consider the logic of a single space, corresponding to the ortholattice of *propositions* in a Kripke model for **OL**, i.e., a proximity space, exemplified in this case by the ortholattice of closed subspaces of a Hilbert space. In this view, the Kripkean propositions are regarded as representing *concepts*, or *perceptions*, and it is illuminating to contemplate their structure in the language of proximity spaces, divorced from the distraction of the linear structure, which is in a sense incidental. Namely, a *concept* associates states according to the transitive closure of the proximity, or *similarity*, relation. For example, in a putative space of *tastes* the concept of *confection* might collect states associated with sweetness and other attributes. Then, if a taste state x is such that every taste state similar to x is also similar to one recognized as confectionary, x should itself be recognized as confectionary. That is to say, with C representing the set of states representing the concept *confection*, we have:

if x is such that for each $y \approx x$ there exists a $z \in C$ such that $y \approx z$, then $x \in C$.

In a proximity space this is equivalent to $C = C^{\perp\perp}$ or C being a proposition. In the proximity space based on a Hilbert space, this is equivalent to C being a closed subspace. Thus, by enforcing transitive closure via similarity in this way, states are corralled into the “conceptual space” *confection*. (See [17, 18].)

Now, vectors in a Hilbert space may be resolved along many choices of basis elements. In [2], each basis choice for a given *concept*, i.e., closed subspace and therefore a Hilbert space in its own right, is identified as a *context* for that concept. A change of context for a concept is then tantamount to an orthogonal transformation of the representing space or subspace. For a space $\mathfrak{H}_A$, let us represent a given context by the basis $\{\xi_1, \dots, \xi_n\}$. Then we may write $\mathfrak{H}_A = \mathbb{R}\xi_1 \oplus \dots \oplus \mathbb{R}\xi_n$ so that $E(\mathfrak{H}_A) = E(\mathbb{R}\xi_1) \otimes \dots \otimes E(\mathbb{R}\xi_n)$, by the basic isomorphism (9.35). In general, tensor decompositions of a state space of the form $E(\mathfrak{H}_A)$ into spaces of a similar form give rise to decompositions of the space $\mathfrak{H}_A$ and, conversely, tensor products of this form correspond to contexts associated with the direct sums of the underlying spaces. A *context* is thus identified in general as a resolution of a “conceptual” space (of states of a network) into a direct sum of subconcepts/subspaces. A different resolving set of subconcepts/subspaces affords a different context in which to view or apprehend the same space of concepts.

We note that the presence of tensor products entails much possible entanglement among states sharing a context, a point emphasized in [2].

9.2 Context formation and retrieval

We shall exploit some of the algebraic properties of these Fermi–Dirac spaces, or exterior algebras, to give an account of certain neural-like processes, such as context formation and retrieval. We shall present this first in schematic form which clearly reveals the mechanisms involved, and then narrow the focus somewhat to draw certain conclusions about how the actual process might work in vivo.

Before proceeding let us note the following. For a vector space A equipped with a map $m : A \otimes A \rightarrow A$ (such as any exterior algebra, with algebra product), we obtain a maps $m \otimes 1 : A \otimes A \otimes A \rightarrow A \otimes A$, $m \otimes 1 \otimes 1 : A \otimes A \otimes A \otimes A \rightarrow A \otimes A \otimes A$, and so on. By composing such maps we obtain a map

$$\bigotimes_{i=1}^n A \rightarrow A \quad (9.1)$$

for any n . For two vector spaces A, B equipped with maps $c_A : A \rightarrow \mathbb{R}$, and $c_B : B \rightarrow \mathbb{R}$, such as the counits of exterior (Hopf) algebras (Appendix A.2), we have the maps

$$c_A \otimes 1_B : A \otimes B \rightarrow \mathbb{R} \otimes B \cong B \text{ and } 1_A \otimes c_B : A \otimes B \rightarrow A \otimes \mathbb{R} \cong A. \quad (9.2)$$

Our basic assumption is that the firing patterns of our networks correspond to putative “cognitive states,” representing for instance memory fragments or sense impressions. Consider a finite sequence of maps of firing patterns $f_i : E(\mathfrak{H}_{A_i}) \rightarrow E(\mathfrak{H}_M)$, with $i = 1, \dots, n$. The intuition here is of incoming firing patterns of networks N_{A_i} , representing memory fragments, or sense impressions, mutually associated in some way, possibly by time or space or emotional state, being consigned to, or encoded into, a “memory module”, or *Gestalt*, namely the space of firing patterns of some network N_M . Owing to the complexity of the maps that will arise, we shall drop their inclusion as labels except where this is necessary for explanatory purposes: the interest of this exercise lies in the structural aspects of the resulting patterns of arrows and spaces. In addition, we shall simplify the notation by writing $E_i := E(\mathfrak{H}_{A_i})$ and $E_M := E(\mathfrak{H}_M)$. Thus we have a sequence of maps of firing patterns of the form $E_i \rightarrow E_M$, $i = 1, \dots, n$. We shall assume in addition that these maps remain fixed once established, for the time intervals of interest. From this we obtain the maps

$$\bigotimes_{i=1}^n E_i \rightarrow \bigotimes_{i=1}^n E_M \quad (9.3)$$

and

$$\bigotimes_{i=1}^n E_M \rightarrow E_M \quad (9.4)$$

from (9.1), which may be composed to obtain the map

$$\bigotimes_{i=1}^n E_i \rightarrow E_M. \quad (9.5)$$

That is to say, the sequence of firing patterns in the domain are eventually “stored in” the space of firing patterns on the right, which functions like a long-term memory module (LTM). This is *context formation* as discussed in Section 9.1. Now let us *access* or retrieve one of these stored spaces of firing patterns, E_{i_0} say. That is to say, it is fed out to an output network with space of firing patterns $E_{D_{i_0}}$, a short-term memory module (STM) perhaps,

by some external prompt, such as the Proustian taste of a madeleine. This is expressed by a map

$$E_{i_0} \rightarrow E_{D_{i_0}}. \quad (9.6)$$

Tensoring this with the map in (9.5) we obtain

$$\bigotimes_{i=1}^n E_i \otimes E_{i_0} = E_1 \otimes \cdots \otimes E_{i_0} \otimes E_{i_0} \otimes \cdots \otimes E_n \rightarrow E_M \otimes E_{D_{i_0}}. \quad (9.7)$$

Now, by composing this map with the map

$$1 \otimes \cdots \otimes 1 \otimes \psi_{i_0} \otimes 1 \otimes \cdots \otimes 1 : E_1 \otimes \cdots \otimes E_n \rightarrow E_1 \otimes \cdots \otimes E_{i_0} \otimes E_{i_0} \otimes \cdots \otimes E_n \quad (9.8)$$

where $\psi_{i_0} : E_{i_0} \rightarrow E_{i_0} \otimes E_{i_0}$ is the appropriate coproduct, we obtain a map

$$\bigotimes_{i=1}^n E_i \rightarrow E_M \otimes E_{D_{i_0}}. \quad (9.9)$$

This in turn yields the two maps

$$\bigotimes_{i=1}^n E_i \rightarrow E_M \quad (9.10)$$

and

$$\bigotimes_{i=1}^n E_i \rightarrow E_{D_{i_0}} \quad (9.11)$$

from (9.2) applied to the range of the map (9.9).

To summarize, we have encoded into the formalism:

- the sequential long-term storage of a set of network inputs into a single memory module: (9.5);
- the retrieval later of one of these stored inputs by feeding it out to an STM with space of firing patterns $E_{D_{i_0}}$: (9.6).

The formalism then delivered the maps (9.10) and (9.11) at a later time, which have the following interpretation. Retrieving one space of firing patterns from storage has the effect of a little later retrieving all of them willy-nilly: the STM $E_{D_{i_0}}$ now appears as the target of the map. Moreover, the circumstances of the states returned to the memory module might now be affected by interference and entanglement effects produced by the presence of $E_{D_{i_0}}$. A little later the formalism produces the maps (9.10) and (9.11). These show that the memory fragments have been restored to the LTM by being copied back into it, and possibly changed by that process, while being available to the STM. Let us interpret these two maps in more detail. Note first that firing patterns are mapped by counits onto the appropriate *vacua*. In fact, such projections onto the vacuum state correspond to a quantum-like collapse, which in actual physics is precipitated by some external “measurement” event. This is to say that all the b-neurons that were ON before such a collapse, are now OFF. And that can only happen if they have all *fired*. Thus (9.10) describes the situation in which the STM space $E_{D_{i_0}}$ has fired its ON b-neurons—a single memorandum or memory fragment has been recalled—while the LTM space E_M of patterns is maintained or reinforced, while (9.11) describes the situation in which the LTM space E_M has fired all its ON b-neurons—the contextually associated memories have been recalled—while the STM space $E_{D_{i_0}}$ maintains its firing patterns. (We have ignored issues of timing here. They will be considered in detail elsewhere.)

The LTM is now ready to repeat this process if another memory fragment is accessed. It would seem that the order of access now must count, since the different processes may alter circumstances. This simulates *contextual* memory retrieval [25]. Note that each time a type is retrieved, the whole collection of them is re-established in the LTM so retrieval has a reinforcing effect. This mimics exactly the operation of memory retrieval in actual brains. Evoking one memory evokes associated memories. Moreover, the stored memories may be affected by this act of retrieval, though this retrieval reinforces their continued presence.

(Informally, and abusing the notation slightly, the inputs to the memory “module” E_M are the E_i s, one at a time, as in reality. For instance, if $M_{\text{childhood}}$ is the LTM associated with the “childhood” *Gestalt*, then E_{tea} could label the network states enabled by the first taste of Aunt Léonie’s tea. Its affect is then “stored” in $M_{\text{childhood}}$ via $E_{\text{tea}} \rightarrow E_{M_{\text{childhood}}}$, and so on. In reality, there may be thousands of these. Then, one of these experiences (an element of $E_{\text{madeleine}}$, say) is recalled (into the STM $E_{D_{\text{madeleine}}}$) perhaps by tasting one again. This is expressed by the map $E_{\text{madeleine}} \rightarrow E_{D_{\text{madeleine}}}$. Then the formalism yields the map (9.9): $E_{\text{Aunt Léonie}} \otimes \cdots \otimes E_{\text{tea}} \otimes \cdots \otimes E_{\text{madeleine}} \rightarrow E_{M_{\text{childhood}}} \otimes E_{D_{\text{madeleine}}}$. This says: *all* the associated memories are recalled into $E_{D_{\text{madeleine}}}$, and when put back into the memory module for “childhood” they may be changed by the presence of the $\otimes E_{D_{\text{madeleine}}}$ on the right-hand side via possible entanglement. Note that $E_{\text{madeleine}}$, etc., may also have been stored elsewhere, such as in the LTM for *confection*, $E_{M_{\text{confection}}}$, and therefore memories of other kinds of confection etc. may be brought forth, resulting in an experience of great complexity.)

To see how such a scheme may work in actual brains, we shall adopt possible candidates for the so-called LTM and STMs. Namely, given the input networks as above with spaces of firing patterns $E_i = E(\mathfrak{H}_{A_i})$ we note that there are canonical quantum-like measurement operations upon eigenspaces of the corresponding p-Hamiltonians. Thus, let us choose an eigenvector ξ_{λ_i} , say, in E_i for each i . Then the ray $\mathbb{R}\xi_{\lambda_i}$ is a subspace of E_i , and each vector $\xi \in E_i$ can be written uniquely as $\xi = \alpha\xi_{\lambda_i} + \eta$ for some real α and vector $\eta \in E_i$. Thus we have a projection $p_i : E_i \rightarrow E_i$ in which each vector ξ is mapped to the corresponding $\alpha\xi_{\lambda_i}$. Note that this projection, as a map of firing patterns, persists in time since

$$T(t)p_i(\xi) = \alpha T(t)\xi_{\lambda_i} \quad (9.12)$$

$$= \alpha \exp\left(\int_0^t \lambda_i(s) ds\right) \xi_{\lambda_i} \quad (9.13)$$

$$= \exp\left(\int_0^t \lambda_i(s) ds\right) \alpha \xi_{\lambda_i} \quad (9.14)$$

$$= \exp\left(\int_0^t \lambda_i(s) ds\right) p_i(\xi) \quad (9.15)$$

which defines the same state/firing pattern as $p_i(\xi)$ for any ξ . Moreover, since these eigenstates are local attractors (Section 8.2), disturbances in the form of small deformations will be corrected in short order. Thus, if such a firing pattern represents a *memorandum*, it will be *maintained* relative to small disturbances, or errors of recall. Such a collapse, precipitated by some external prompt or measurement-like act such as the taste of a madeleine, is a model of a single item retrieval or *focus of attention* [26], since the firing pattern itself is the experience of recall. Note also that it is retrieval according to *content addressability*.

We shall try to implement the above scheme by positing an LTM-like networked brain region into whose space of firing patterns, E_M , the whole context $\bigotimes_{i=1}^n E_i$ may be embedded as a subspace. For instance, some part, or all, of the medial temporal lobe may play

this rôle. Then the map (9.3) can be taken to be this embedding. Now the networks N_{A_i} themselves play the rôles of STMs and the retrieval maps (9.4) are the projections described above. Forgetfulness, and therefore the ST in STM, would then be modeled by the dual possibilities of distraction via quantum-like entanglement and decay via quantum-like decoherence of the eigenstate involved (i.e., the ray that is the range in (9.11)). At the same time, the entire context is written back to the LTM (9.10): its presence there has been reinforced by the application of the coalgebra map $\psi_{i_0} : E_{i_0} \rightarrow E_{i_0} \otimes E_{i_0}$, which may be interpreted as implementing long-term storage in a manner explained for instance in [15]: hence the LT in LTM. (We shall elaborate on this model elsewhere.) Although this treatment is rather crude, it does implement a unitary model since the representations of retrieval from both the LTM and STMs will be identical, being via collapse onto eigenstates. See [26] for a comprehensive account of these matters.

Here we note that the pattern of transformations of states depicted above (9.3) to (9.11) may be viewed as a form of quantum-like computation, a notion that may be formalized into a fragment of a known computational scheme whose computational capacity may be analyzed. This program will be carried out elsewhere.

9.3 Schizophrenia and the von Domarus Principle

Schizophrenic patients seem to be blind to contexts in a very general sense. Cf. [27] for a discussion. In our picture context forming is enabled by the process of tensoring different spaces together as in the last section. This tensoring might or might not be implemented at the network level by the tight interneural binding of actual networks. We shall assume that in normal circumstances it is so implemented. In this case, schizophrenia may be conjectured to be a disease of disconnection [28], and one moreover implicating the interneural substrate itself in certain brain regions such as the prefrontal cortex. Similar conjectures might be proposed with regard to memory disrupting conditions such as dementias of Alzheimer's type, affecting other brain regions, such as the hippocampus and/or the amygdala. There seems to be some experimental support for both of these conjectures: cf. [29, 30].

There are other conclusions to be arrived at. If tensoring is thwarted at the network level, we cannot, at this level, implement the isomorphism of (9.35) which entails "many spaces" on one side but only one space on the other. That is to say, the Fermi–Dirac space of firing patterns cannot be implemented explicitly as the state space of an actual network. Consequently, thinkers with this deficit are condemned to use the single space version of the exterior algebra in those places where it is used, instead of the tensor product representation. Since this version of the exterior algebra seems to account for von Domarus errors, as discussed in [27], we may conclude that schizophrenic patients would be more prone to errors of von Domarus type. This seems to be borne out in practice and indeed this is the circumstance that led von Domarus to his conclusion in the first place. Here we shall suggest a more precise argument in favor of this conclusion.

The von Domarus Principle asserts that if two entities share an attribute, then they are identical [31, 32]. von Domarus reported various uses of this principle by schizophrenic patients, for example the identification of a shoe and a pen because of their common attribute of "leaving a mark." Many other cases of even more bizarre identifications are on record going back to Bleuler: a man identifies himself with the country of Switzerland because he hears that Switzerland abhors war and so does he. von Domarus hypothesized that this propensity was the characterizing feature of schizophrenic reasoning, though later experiments showed it to be ubiquitous in human thinking [33, 34]. In that case, why are schizophrenic patients more prone to its use?

Let us suppose that, as argued above, schizophrenic patients are condemned to use the “single space” logic **OL** associated with the ortholattice of closed subspaces of a Hilbert space as in the discussion in the last section. Then the concept or *Gestalt* corresponding, for instance, to the entity known as a *cannon ball* would be a certain subspace $\mathfrak{H}_{\text{cannonball}}$ containing a basis of subconcepts representing the essential attributes of this notion, such as $\{\xi_{\text{heavy}}, \xi_{\text{metallic}}, \xi_{\text{round}}, \dots, \xi_n\}$. Similarly, the entity known as a *pingpong ball* has a conceptual subspace $\mathfrak{H}_{\text{pingpongball}}$ with basis $\{\xi_{\text{light}}, \xi_{\text{round}}, \xi_{\text{hollow}}, \dots, \xi_m\}$. The whole space $\mathfrak{H}_{\text{cannonball}}$ is represented by the single vector $\xi_{\text{cannonball}} := \xi_{\text{heavy}} \wedge \xi_{\text{metallic}} \wedge \xi_{\text{round}} \wedge \dots \wedge \xi_n$, as in the Plücker embedding (Appendix Section A.2) which in a sense encapsulates or lumps together all the essential attributes of a cannon ball as it is conceived by the user of a putative such system. It does so, moreover, while suppressing the context forming aspect of the basis, which is denied to this thinker. Similarly $\xi_{\text{pingpongball}} := \xi_{\text{light}} \wedge \xi_{\text{round}} \wedge \xi_{\text{hollow}} \wedge \dots \wedge \xi_m$ encapsulates all the attributes of the user’s conception of a pingpong ball. Assuming the user can make such an appraisal, he or she would then find that $\xi_{\text{cannonball}} \wedge \xi_{\text{pingpongball}} = 0$ because of the common presence of the single attribute *roundness*. Now, a user of such a system, having only the notion of a single space, may not be aware of, or miss, the fact that the vectors $\xi_{\text{cannonball}}$ and $\xi_{\text{pingpongball}}$ are not in the original ambient space but belong to different spaces. If confused with ordinary states or vectors in some ambient space the condition $\xi_{\text{cannonball}} \wedge \xi_{\text{pingpongball}} = 0$ would signal (erroneously in this case) the colinearity of $\xi_{\text{pingpongball}}$ and $\xi_{\text{cannonball}}$, or equivalently that $\mathfrak{H}_{\text{pingpongball}} = \mathfrak{H}_{\text{cannonball}}$, by the injectivity of the Plücker embedding (Appendix Section A.2). Thus a cannon ball is identical to a pingpong ball as far as this user is concerned. This is exactly the von Domarus Principle in action: a common attribute collapses all the others. (This argument presupposes that humans, among other animals, are equipped with the circuitry to make these appraisals on the fly, implementing the exterior product by means of the creation/annihilation operators perhaps, and that the vanishing of $\xi_{\text{cannonball}} \wedge \xi_{\text{pingpongball}}$, etc., is a type of built-in stopping rule. That is to say, a rule stopping further processing while initiating avoidance behavior. Such a system would be simultaneously more prone to von Domarus errors and selectively advantageous under conditions of threat, where rapidity of the decision process is of the essence.)

Neurotypical subjects may presumably also use such mechanisms but are not condemned to their sole use. They may have other mechanisms to recognize entities. For example, as in Section 9.2, suppose we have the following decomposition

$$\mathfrak{H}_{\text{cannonball}} \cong \mathbb{R}\xi_{\text{heavy}} \oplus \mathbb{R}\xi_{\text{metallic}} \oplus \mathbb{R}\xi_{\text{round}} \oplus \dots \oplus \mathbb{R}\xi_n \quad (9.16)$$

which entails

$$E(\mathfrak{H}_{\text{cannonball}}) \cong E(\mathbb{R}\xi_{\text{heavy}}) \otimes E(\mathbb{R}\xi_{\text{metallic}}) \otimes E(\mathbb{R}\xi_{\text{round}}) \otimes \dots \otimes E(\mathbb{R}\xi_n). \quad (9.17)$$

Similarly, we obtain the decomposition

$$E(\mathfrak{H}_{\text{pingpongball}}) \cong E(\mathbb{R}\xi_{\text{light}}) \otimes E(\mathbb{R}\xi_{\text{round}}) \otimes E(\mathbb{R}\xi_{\text{hollow}}) \otimes \dots \otimes E(\mathbb{R}\xi_m). \quad (9.18)$$

Pattern matching procedures would now not necessarily risk confusing a cannon ball with a pingpong ball despite the appearance of $E(\mathbb{R}\xi_{\text{round}})$ in both contexts. Thus von Domarus errors are avoided at the possible risk of time being wasted in carrying out such a pattern matching procedure. If the former single space logic were wired into an organism’s central nervous system, the selective advantage would be clear. Although von Domarus errors are avoided in the many space formulation—which signifies unrestricted connectivity—there is now also the risk of context distraction, brought about by possible entanglement among

components of the context. This kind of context distraction in neurotypical subjects seems to be borne out in experimental studies [27].

9.4 The three b-neuron motif classes and their spectra

As a test of these hypothetical constructs we shall, in this section, analyze the spectra—i.e., the set of p-Hamiltonian eigenstates—for the much-studied 13 three-neuron *motif classes* [7].

An M *motif* is a directed network of M nodes containing at least $M - 1$ links or edges [7]. For each M , such a network is called a *motif class*. For $M = 3$, there are 13 motif classes that have been classified and their frequencies have been studied in cat and macaque cortices [7]. The number of motifs in a class increases rapidly with M . For instance, for $M = 4$ there are 199 classes and for $M = 7$ there are 880,471,142 classes. In this section, we shall consider the eigenstates for the p-Hamiltonians of the $M = 3$ cases and employ the classification of them by Sporns [7]. In particular, we shall refer to the data encapsulated in Fig. 6.4, p. 109 of [7]. For simplicity, we shall consider these classes in isolation from the networks they are presumed to be embedded within. (The treatment of many-network dynamics will be considered elsewhere.)

In the Sporns classification, the first class, denoted $M1$ is depicted in Fig. 2.

Its p-Hamiltonian is

$$H_1 := J_{12}a_2^\dagger a_1 + J_{32}a_2^\dagger a_3, \quad (9.19)$$

where for convenience we have dropped the references to t . In light of (8.52), we may ignore the scalar values representing the vacuum state and in light of (8.53) we may ignore the state of full occupation in all of the three node motif classes since there are no autapse-like terms of the form J_{ii} in their p-Hamiltonians. Therefore, we may write the relevant eigenvectors to be sought in the general form

$$\xi = c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_1 \wedge e_2 + c_5 e_2 \wedge e_3 + c_6 e_1 \wedge e_3. \quad (9.20)$$

A simple calculation yields

$$H_1 \xi = (J_{12}c_1 + J_{32}c_3)e_2 + J_{32}c_6 e_1 \wedge e_2 + J_{12}c_6 e_2 \wedge e_3. \quad (9.21)$$

As noted in the last section the 6×6 matrix determined by the last equation acting on the vector of c s is of the block diagonal form $J^{(1)} \oplus J^{(2)}$, where $J^{(1)}$ is the 3×3 matrix involving the singly occupied states and $J^{(2)}$ is the 3×3 matrix involving the doubly occupied states. Since for matrices A and B , $\det(A \oplus B) = \det A \cdot \det B$ we may treat the eigenvalue problems separately for each occupancy class. We shall label the singly occupied cases by O1 and the doubly occupied cases by O2. This applies to all 13 three-node motif classes.

From (9.21) we have

$$J^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ J_{12} & 0 & J_{32} \\ 0 & 0 & 0 \end{pmatrix} \quad (9.22)$$

Fig. 2 M1

Thus the eigenvalues are given by the roots of the characteristic polynomial $\det(J^{(1)} - \lambda I) = -\lambda^3$ so that in this case there are no non-zero O1 eigenvalues.

Here

$$J^{(2)} = \begin{pmatrix} 0 & 0 & J_{32} \\ 0 & 0 & J_{12} \\ 0 & 0 & 0 \end{pmatrix} \quad (9.23)$$

and again $\det(J^{(2)} - \lambda I) = -\lambda^3$ so that there are also no non-zero O2 eigenvalues.

Continuing along the Sporns list, the first case in which there are non-zero eigenvalues occurs at the motif class *M4*, depicted in Fig. 3.

In this case

$$H_4 = J_{12}a_2^\dagger a_1 + J_{21}a_1^\dagger a_2 + J_{32}a_2^\dagger a_3. \quad (9.24)$$

Here we have

$$J^{(1)} = \begin{pmatrix} 0 & J_{21} & 0 \\ J_{12} & 0 & J_{32} \\ 0 & 0 & 0 \end{pmatrix} \quad (9.25)$$

and $\det(J^{(1)} - \lambda I) = \lambda(J_{12}J_{21} - \lambda^2)$ so the non-zero O1 eigenvalues are

$$\lambda = \pm(J_{12}J_{21})^{\frac{1}{2}} \quad (9.26)$$

where we must have $J_{12}J_{21} > 0$. The O1 eigenstates are superpositions of $|1_1\rangle$ and $|1_2\rangle$.

For O2 we have

$$J^{(2)} = \begin{pmatrix} 0 & 0 & J_{32} \\ 0 & 0 & J_{21} \\ 0 & J_{12} & 0 \end{pmatrix} \quad (9.27)$$

and the characteristic polynomial in this case is $-\lambda(J_{12}J_{21} - \lambda^2)$ so the eigenvalues are the same as in the O1 case with the O2 eigenstates being superpositions of all possible double occupancies, namely $|1_1, 1_2\rangle$, $|1_2, 1_3\rangle$ and $|1_1, 1_3\rangle$.

M5 has no non-zero eigenstates.

We will consider next three cases of some interest, namely *M6*, *M8* and *M9*.

M6 is depicted in Fig. 4.

Here the eigenvalues both for the O1 and O2 states are the same as in *M4*, but the O2 eigenstates are superpositions of $|1_2, 1_3\rangle$ and $|1_1, 1_3\rangle$.

M8 is depicted in Fig. 5.

For O1 we have non-zero eigenvalues $\lambda = \pm(J_{21}J_{12})^{\frac{1}{2}}$, where we must have $J_{21}J_{12} > 0$, and the eigenstates are superpositions of $|1_1\rangle$ and $|1_2\rangle$. For O2, we have the same non-zero eigenvalues with eigenstates being superpositions of all possible O2 states.

We pause at this point to take stock. The eigenstates we have found are, at least in the short term, stable, with synchronized exponential scaling, and alive, as opposed to the states with zero eigenvalues, which may be stable but are essentially dead (though quantum-like effects, such as tunneling, may vitiate the permanence of this state). We shall assume that these live states are the usable ones, either bestowing selective advantages because of their

Fig. 3 *M4*

Fig. 4 M6

functional attributes, whatever those are, or surviving in the volatile in vivo environment per organism simply because they demonstrate a certain robustness.

Starting with the base case *M4* we note that there is a certain probability, at each moment, that the condition required for the formation of an eigenstate of the *live* type, namely $J_{12}J_{21} > 0$, holds. In that case, live states are possible and the independent ones, per occupation number, are four in number, namely the two O1 states, being superpositions of $|1_1\rangle$ and $|1_2\rangle$, and the two O2 states, being superpositions of all possible double occupancies. Let us dub this collection of data the *spectral profile* of the network involved. Thus *M4*'s spectral profile might be written

M4	O1	O2
Eigenvalues:	$\pm(J_{12}J_{21})^{\frac{1}{2}}$	$\pm(J_{12}J_{21})^{\frac{1}{2}}$
Constraints:	$J_{12}J_{21} > 0$	$J_{12}J_{21} > 0$
Probabilities:	0.34	0.34
Basis elements used:	$\{ 1_1\rangle, 1_2\rangle\}$ Whole basis $\{ 1_i, 1_j\rangle, i < j$	
Total Eigenstate Count: 4		

where the probabilities are that of the constraints being met instantaneously under the hypothesis of complete Poissonian randomness (cf. Appendix B), and where by “basis elements used” we mean the set of basis vectors contributing as summands to the eigenvectors found. The “eigenstate count” is the total number of independent eigenvectors per occupancy class. That is, the number of independent eigenvectors in each occupancy class considered separately and added together.

Now compare these results with those obtained for *M8*. The spectral profile of *M8* is exactly the same as that of *M4*. The difference in *M8* resides in the topology of the diagram, in which there is an edge not involved in the formation of the *M4* O1 eigenstates, namely the $n_3 \rightarrow n_1$ edge. This allows interference with the live O1 eigenstates, coming from the extraneous edge, whose output/input activity could destroy the superpositions that constitute the live O1 eigenstates. Consequently, everything else being equal for these two networks, we would expect the frequency of *M8*'s appearance to be suppressed relative to that of *M4*'s, and that does seem to be the case. (Thus the size of a set of basis elements used relative to the size of the whole network might be taken as a crude estimate of the fragility of the corresponding live eigenstate: the smaller this ratio, the more fragile the state, since it is possibly subject to more interference from outside itself.)

Fig. 5 M8

Fig. 6 M7

Next we compare $M4$ with $M6$. The latter's spectral profile is as follows.

M6	O1	O2
Eigenvalues:	$\pm(J_{12}J_{21})^{\frac{1}{2}}$	$\pm(J_{12}J_{21})^{\frac{1}{2}}$
Constraints:	$J_{12}J_{21} > 0$	$J_{12}J_{21} > 0$
Probabilities:	0.34	0.34
Basis elements used:	Whole basis $\{ 1_i >\}$ $\{ 1_1, 1_2 >, 1_2, 1_3 >\}$	
Total Eigenstate Count: 4		

The $M6$ profile is the same as that of $M4$ except that the set of O1 basis elements used is bigger while the set O2 basis elements used is smaller. Consequently the difference between these two cases is difficult to call but is likely to depend on other factors such as the specific topologies involved. Thus, if the O2 states are more strongly determining, then their relative fragility in the $M6$ case may cause the frequency of the appearance of $M6$ relative to $M4$ to be suppressed, as seems to be the case. This topic will be considered elsewhere.

We shall continue with details of two more of these motif classes (Figs. 6 and 7).

M7	O1	O2
Eigenvalues:	$(J_{31}J_{12}J_{23})^{\frac{1}{3}}$	$(-J_{31}J_{12}J_{23})^{\frac{1}{3}}$
Constraints:	$J_{31}J_{12}J_{23} \neq 0$	$J_{31}J_{12}J_{23} \neq 0$
Probabilities:	1	1
Basis elements used:	Whole basis $\{ 1_i \rangle$ Whole basis $\{ 1_i, 1_j \rangle, i < j$	
Total Eigenstate Count:2		

This case would appear to be the least frequent of the live cases we have encountered so far, since it has the smallest eigenstate count. So we would expect this motif class to be suppressed relative to the others we have considered so far. This seems to be the case.

M9	O1	O2
Eigenvalues:	$\pm(J_{23}J_{32} + J_{21}J_{12})^{\frac{1}{2}}$	$\pm(J_{23}J_{32} + J_{21}J_{12})^{\frac{1}{2}}$
Constraints:	$J_{23}J_{32} + J_{21}J_{12} > 0$	$J_{23}J_{32} + J_{21}J_{12} > 0$
Probabilities:	0.5	0.5
Basis elements used:	Whole basis $\{ 1_i >\}$ Whole basis $\{ 1_i, 1_j >, i < j$	
Total Eigenstate Count:4		

Fig. 7 M9

In this case (known as the *dual dyad*) the constraint is the same for O1 and O2 and the set of basis elements used is as large as it can be. (For the probability estimate we use the last paragraph of Appendix B with $\mathbf{v} = (J_{23}, J_{21})$ and $\mathbf{w} = (J_{32}, J_{12})$). The dual dyad and *M4* have the most eigenstates among the motif classes so far considered, but the probabilities of their occurrence are greater in the case of the dual dyad. Moreover, the dual dyad's O1 eigenstates are more robust against external interference than those of *M4*, using the largest possible set of basis elements. It would seem therefore that *M9* should be more frequently found than any of the others so far considered and this seems to have been confirmed across many studies.

The motif classes *M10* through *M13* present problems since from the *M10* case onwards we are faced with cubic equations for the eigenvalues which preclude simple closed form expressions. However, from *M11* up there are single live eigenstates in each motif and occupancy class, since cubic equations with real coefficients have at least one real root, which is not zero in any of these cases. In the case of *M10*, the O1 eigenproblem is cubic, and the single real eigenvalue does not constrain the *J*s at all. However, the O2 case is quadratic, and the constraint is the same as that of *M4* and *M6*. For *M11* and up, all the eigenproblems are cubic and none of the resulting eigenvalues are constraining. We surmise that these latter motifs are less fragile than *M10* and although their live eigenstates are fewer in number they are not constraining and robust, and may appear more frequently, though not as frequently as *M9* which has the most eigenstates. This seems to be confirmed by experiment, at least in the last two cases *M12* and *M13*.

9.4.1 Eigenstates that mix occupation number

The block structure of the p-Hamiltonian in the case of isolated networks such as those considered above justifies the separation of eigenstates by occupation number. However, it is significant that in some cases eigenstates are possible that superpose states of different occupation number, namely those eigenstates having the same eigenvalue. This sort of degeneracy in the case of actual physics has been found in certain exotic states of matter, called topological, in which there are low energy degenerate states. These states are associated with topological features and are “topologically” protected against the agencies of decoherence of superpositions to which normal matter is prone. For this reason, these materials are of central interest in the hopeful discipline of quantum computing. They include quantum Hall liquids, and more general *quantum spin liquids*, such as the mineral smithite (also known as herbertsmithite) [35, 36].

In our simple examples this kind of degeneracy occurs in the cases *M4*, *M6*, *M8* and *M9*. In these cases, there are eigenvalues common to eigenstates of both occupation classes, and so there exist eigenstates that are superpositions of states of both classes. If this degeneracy bestowed extra “topological” protection upon these states then we may expect to find a boost in the frequency of their appearance above the common run of motif classes. A glance at the data (Fig. 6.4, p. 109 of [7]) seems to confirm this in all the cases listed above with the exception of *M8*. Perhaps such protection is vitiated in this case by the intrusion of the $n_1 \rightarrow n_3$ edge as discussed above. The possible connection between our results and the theory known as *topological quantum computing* will be taken up elsewhere. It is to be noted that:

- the superpositional live eigenstates seem to play a discernible rôle in the histories of even such simple networks as these motif classes;

- the *interpretation* of these vectors as superpositional states is beyond the remit of any purely classical network model.

10 Discussion

We have derived quantum-like virtual dynamical properties, prefiguring a proper computational treatment, of neural networks from general principles, without relying on any detailed assumptions concerning the highly complex biophysics and biochemistry that must underlie neuronal network behavior. Such phenomena as quantum-like superposition and entanglement emerge as attributes of the structure, which, since they involve only virtual quanta, are not subject to the usual objection that physical decoherence would be too rapid to have any effect in a hot brain. We have shown that the formalism leads immediately to a class of quasispin-like models for such networks and may be used to deduce the flow of resource distribution in the processes of context formation and retrieval in such systems, thus providing computational resources for future use. The formalism also identifies mechanisms to explain aspects of schizophrenia, dementias and epileptic-like seizures. An analysis of the 13 three-neuron motif classes leads to the conclusion that the existence of stable states interpretable as quantum-like superpositions has an observable bearing on their behavior and integrity, even in such simple examples as those considered here. In a classical treatment, the structural attributes of such states, *qua* superpositions, would be entirely invisible.

Our considerations started, in [8], with the simple logic of confusable states, from which we derived a virtual quantum-like dynamics for certain networks, independently of further physical input other than abstract specifications of neuronal and network configurations. Consequently, the question of the ultimate underlying provenance of such confusability remains in abeyance. On the other hand, other compelling mechanisms to explain some of the quantum-like aspects of mental function have been put forward recently [37, 38]. The question of how these approaches may be related will be taken up elsewhere.

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Research Data No data was used or generated during the course of this study.

Appendix A: Some multilinear algebra

References for this appendix include [39–43].

A.1 Tensor products

We take vector spaces over a field (or modules over a ring) k which we may take to be $\mathbb{R}$, the field we will be using here. For n of them, $V_1, V_2, \dots, V_n$ and another one W (they do not need to be finite dimensional here) one may have *multilinear* functions $f: V_1 \times V_2 \times \dots \times V_n \rightarrow W$ meaning that f is linear in each variable separately. Linear means it preserves vector addition and scalar multiplication. One might envisage a complicated theory of such multilinear functions generalizing the theory of linear functions of a single variable. However such a theory is not necessary, since for any such collection of vector spaces $V_1, V_2, \dots, V_n$ there exists a single vector space $T(V_1, V_2, \dots, V_n)$ satisfying the following Universal Mapping Property (UMP):

There exists a multilinear map $\iota: V_1 \times V_2 \times \dots \times V_n \rightarrow T(V_1, V_2, \dots, V_n)$ such that for any multilinear map $f: V_1 \times V_2 \times \dots \times V_n \rightarrow W$ there exists a unique linear map $\tilde{f}: T(V_1, V_2, \dots, V_n) \rightarrow W$ such that $\tilde{f} \circ \iota = f$.

It is easy to prove that such a $T(V_1, V_2, \dots, V_n)$ must be unique up to isomorphism of vector spaces. It is called the *tensor product* of the vector spaces involved and written $V_1 \otimes V_2 \otimes \dots \otimes V_n = \bigotimes_{i=1}^n V_i$. Usually the base field or ring is appended to the tensor sign as in $\otimes_k$ since often algebraists have many fields and/or rings to deal with simultaneously. If the field is not in doubt, as here, it is omitted. In this way, multilinear maps are turned into linear ones and there is no need for a separate theory. One may think of the tensor product as the vector space generated by basis vectors of the form $a_1 \otimes a_2 \otimes \dots \otimes a_n$, $a_i \in V_i$ subject to linear additivity in each variable and the scalar multiplication property $\lambda a_1 \otimes a_2 \otimes \dots \otimes a_n = a_1 \otimes \lambda a_2 \otimes \dots \otimes a_n = \dots = a_1 \otimes a_2 \otimes \dots \otimes \lambda a_n$, $\lambda \in k$. If V_i has dimension d_i then $\dim(\bigotimes_{i=1}^n V_i) = d_1 d_2 \dots d_n$.

Properties of tensor products may be derived entirely through the use of the UMP stated above. For instance, suppose linear maps $f_i: V_i \rightarrow W_i$, $i = 1, \dots, n$ are given. Then $f_1 \times \dots \times f_n: V_1 \times \dots \times V_n \rightarrow W_1 \times \dots \times W_n$ is multilinear so that its composition with the linear ι -map of $W_1 \times \dots \times W_n$ into $W_1 \otimes \dots \otimes W_n$ is also multilinear so that it may be lifted to produce a linear map $V_1 \otimes \dots \otimes V_n \rightarrow W_1 \otimes \dots \otimes W_n$ as in the diagram above. This map is denoted $\bigotimes_{i=1}^n f_i = f_1 \otimes \dots \otimes f_n$.

A.2 Exterior algebra

With notation as in the last section, let $V^p := \overbrace{V \times \dots \times V}^p$ for $p \geq 2$. Then, a multilinear function $f: V^p \rightarrow W$ is said to be *alternating* if $f(v_1, v_2, \dots, v_i, v_i, \dots, v_p) = 0$. It follows from multilinearity that interchanging any pair of adjacent variables changes the sign of the value of f and from this that the same holds for the interchange of any pair of variables. Then it follows that the repetition of any pair of variables causes the value of f to vanish. There is a UMP for alternating maps similar to the one for general multilinear maps. The unique vector space that plays the rôle of the tensor product $\bigotimes^p V$ in this case is written $\bigwedge^p V$, and called the *exterior* product. It is generated by elements of the form $v_1 \wedge v_2 \wedge \dots \wedge v_p$, $v_i \in V$. This element is multilinear in its arguments and alternating in the sense described above for f . Thus for instance $v \wedge v = 0$ and if $v \wedge w = 0$ then v and w generate the same one-dimensional subspace, i.e., are *colinear*. (For, if v and w were not linearly dependent they could be included in a basis for V in which case $v \wedge w$ would be a basis element of $\bigwedge^2 V$ which could not be the zero vector.) The map corresponding to ι in the last section sends $(v_1, \dots, v_p)$ to $v_1 \wedge v_2 \wedge \dots \wedge v_p$. It is not hard to show that, if the dimension of V is n , then $\dim \bigwedge^p V = \binom{n}{p}$. Note that $\dim \bigwedge^n V = 1$ and that $\bigwedge^p V = \{0\}$

if $p > n$. A useful intuition is that the exterior product $v_1 \wedge v_2 \wedge \dots \wedge v_p$ is a vector representing the volume of the polytope bounded by the vectors $v_1, \dots, v_p$ normal to the surface of this polytope.

These exterior products may be assembled into a unital associative algebra (i.e., an associative algebra containing a unit) having a certain universal property with respect to linear maps $f : V \rightarrow A$ into such an algebra A having the property that $f(v)^2 = 0$ for all $v \in V$. Namely, there exists an associative unital algebra $E(V)$ for any vector space V , and a linear map $\kappa : V \rightarrow E(V)$ having the property mentioned, such that if $f : V \rightarrow A$ is any linear map into any associative unital algebra A having that property, then there exists a unique algebra morphism $\tilde{f} : E(V) \rightarrow A$ such that $\tilde{f} \circ \kappa = f$. $E(V)$ is then necessarily unique up to algebra isomorphism. One may take this algebra to be the *exterior algebra* of V defined by

$$E(V) := \bigoplus_{k \geq 0} \bigwedge^k V. \quad (9.28)$$

In case V is finite dimensional, of dimension n , say, this direct sum terminates to give

$$E(V) = \bigwedge^0 V \oplus \bigwedge^1 V \oplus \bigwedge^2 V \oplus \dots \oplus \bigwedge^n V \quad (9.29)$$

$$= \mathbb{R} \oplus V \oplus \bigwedge^2 V \oplus \dots \oplus \bigwedge^n V. \quad (9.30)$$

Here we take $\bigwedge^0 V = \mathbb{R}$ and $\bigwedge^1 V = V$. Note that $\dim E(V) = 2^n$. The algebra multiplication is given by wedging together two elements of the summands—called *homogenous elements*—in the order given, and extending by linearity to the whole space, with elements in $\bigwedge^0 V = \mathbb{R}$ just acting as scalars in the usual way. The map $\kappa : V \rightarrow E(V)$ is given by $\kappa(v) = v$ considered to lie in the summand $\bigwedge^1 V = V$. This algebra has many interesting properties and symmetries which were understood by H. Grassmann in the middle of the 19th century but whose published treatment of it was famously misunderstood by his contemporaries, probably because of limitations in the notations of the time. We shall rehearse a few of these properties here. First we note that for two finite dimensional vector spaces V and W the map

$$\bigwedge^m V \otimes \bigwedge^n W \longrightarrow \bigwedge^{m+n} (V \oplus W) \quad (9.31)$$

given by $(v_1 \wedge \dots \wedge v_m) \otimes (w_1 \wedge \dots \wedge w_n) \mapsto v_1 \wedge \dots \wedge v_m \wedge w_1 \wedge \dots \wedge w_n$ induces an isomorphism of vector spaces

$$\bigoplus_{k=0}^p \left(\bigwedge^k V \otimes \bigwedge^{p-k} W \right) \cong \bigwedge^p (V \oplus W) \quad (9.32)$$

from which we obtain an isomorphism of vector spaces

$$E(V \oplus W) \cong E(V) \otimes E(W) \quad (9.33)$$

which is *not* an isomorphism of algebras when the ordinary tensor product algebra multiplication is used on the right-hand side of (9.33). There is, however, an algebra product structure on the right-hand side that does render that isomorphism above an isomorphism of algebras: it is called the *graded product* and it is described as follows. For homogeneous

elements $a, c \in E(V)$, $b, d \in E(W)$, the *graded product* on the algebra $E(V) \otimes E(W)$ is determined by the definition:

$$(a \otimes b)(c \otimes d) := (-1)^{\deg(b)\deg(c)}(ac \otimes bd), \quad (9.34)$$

where the degree $\deg(f)$ of a homogeneous element f is the power of the exterior product to which it belongs, also called the *grade* of f .

Let us consider the case when V is one-dimensional, with basis element e , say. Then clearly $E(V) = E(\mathbb{R}e) = \mathbb{R} \oplus V \cong \mathbb{R} \oplus \mathbb{R}e$ in our earlier notation, where, as an element in the algebra of $E(\mathbb{R}e)$, $e^2 = e \wedge e = 0$. It is immediate that $E(\mathbb{R}e)$ is commutative as an algebra. Now, for any finite dimensional vector space V with basis $\{e_1, \dots, e_n\}$ we have

$$E(V) \cong E(\mathbb{R}e_1 \oplus \dots \oplus \mathbb{R}e_n) \cong E(\mathbb{R}e_1) \otimes \dots \otimes E(\mathbb{R}e_n) \quad (9.35)$$

as vector spaces. As noted above, this is not an isomorphism of algebras with the ordinary tensor product multiplication on the right-hand side, since this would be commutative as all of the $E(\mathbb{R}e_i)$ are, while the left hand side is not. However, as mentioned, the right-hand side with graded product is isomorphic with the exterior product on the left-hand side. Thus, the exterior algebra may be described in terms of graded tensor products of algebras isomorphic with $E(\mathbb{R}e)$ (cf. [42]). The reader may note the similarity of such tensor products to the notion of qubit registers in the parlance of quantum computation.

If $f : V \rightarrow W$ is a linear map of vector spaces, there is a unique map of algebras $E(f) : E(V) \rightarrow E(W)$ that extends $f : V \rightarrow W$. This may be proved using the UMP for exterior algebra. It is easy to see that it is given by the linear extension of the assignments $E(f)(1) = 1$, $E(f)(v_1 \wedge \dots \wedge v_k) = f(v_1) \wedge \dots \wedge f(v_k)$.

The exterior algebra of a finite dimensional k vector space V has additional elements of structure which are not usually invoked in the physical context of the Fermi–Dirac space of a many fermion system, but will be of interest to us here. First consider the vector space maps given by $V \rightarrow V \oplus V$, $v \mapsto (v, v)$ and $V \rightarrow \{0\}$, $v \mapsto 0$. As in Appendix A.2 these respectively induce algebra maps

- $\psi_V : E(V) \rightarrow E(V \oplus V) \cong E(V) \otimes E(V)$ called the *coproduct* which is an algebra map given on elements $v \in V \subset E(V)$ by

$$\psi_V(v) = 1 \otimes v + v \otimes 1. \quad (9.36)$$

This coproduct makes $E(V)$ a *coalgebra*;

- $c_V : E(V) \rightarrow E(\{0\}) = k$, called the *counit*, given by the projection of $E(V)$ onto its first component.

Together with the algebra structure these maps give $E(V)$ the structure of a *Hopf algebra* in which the product, coproduct, unit, and counit intertwine in certain ways that need not concern us here.

A.3 The Plücker embedding

For a (real) vector space V , let $\mathbf{Gr}(p, V)$ denote the family of subspaces of V of dimension p , the notation $\mathbf{Gr}$ being in honor of Grassmann. The special case of $p = 1$ is called the *projective space* of V , and is denoted by $\mathbf{P}(V)$. Exterior products may be used to obtain an explicit representation of $\mathbf{Gr}(p, V)$, namely the map

$$\psi : \mathbf{Gr}(p, V) \longrightarrow \mathbf{P}\left(\bigwedge^p V\right) \quad (9.37)$$

given, for a p -dimensional subspace $W \subseteq V$, with basis $\{w_1, \dots, w_p\}$, by

$$\psi(W) := \mathbb{R}w_1 \wedge \dots \wedge w_p \quad (9.38)$$

is well defined. For, another basis of W is related to this basis by a matrix with a non-vanishing determinant and the corresponding exterior product is the previous one, namely $w_1 \wedge \dots \wedge w_p$, multiplied by this determinant and so specifies the same element in $\mathbf{P}(\bigwedge^p V)$. Moreover, it is not hard to see that $w \in W$ if and only if $w \wedge \psi(W) = 0$, showing that ψ is one-to-one, or *injective*. It is called the *Plücker embedding*.

Intuitively this last result can be interpreted as follows: if the volume of the $(p + 1)$ -dimensional polytope obtained by adding w as another side to the p -dimensional polytope with sides $w_1, \dots, w_p$ is zero, then w must lie in the polytope, and conversely. This is easily seen when $p = 2$: if adding a third vector to the two-dimensional polytope with sides w_1, w_2 produces a (three-dimensional) polytope of zero volume, then w must lie in the plane determined by w_1, w_2 , and conversely.

Appendix B: Some probabilities

The statistics of neuronal firing is a very large field but the assumption that it is a Poisson process seems to be reasonable as a general approximation. In that case, the probability of a firing at any particular time t is zero. Thus, from (8.19)

$$\begin{aligned} \text{prob}(J_{ij}(t) = 0) \\ = \text{prob}((v_{ij}(t) < \Theta_j) \text{ and } (n_j \text{ is not firing at time } t)) \end{aligned} \quad (9.39)$$

$$= \text{prob}(v_{ij}(t) < \Theta_j) \text{ prob}(n_j \text{ is not firing at time } t) \quad (9.40)$$

since these events are independent,

$$= \text{prob}(v_{ij}(t) < \Theta_j). \quad (9.41)$$

To estimate this probability we note that the full range of membrane potential values available to a typical neuron goes from -70 mV to 40 mV, approximately, with an approximate value of the threshold value being -50 mV. We therefore estimate the last named probability by taking the ratio of possible range of values of $v_{ij}(t) < \Theta_j$ to the range of all possible values for $v_{ij}(t)$ to obtain

$$\text{prob}(J_{ij}(t) = 0) = \frac{20 \text{ mV}}{110 \text{ mV}} = 0.18. \quad (9.42)$$

Similarly, we find from (8.19), that

$$(J_{ij}(t) > 0) \text{ iff } ((v_{ij}(t) > 0) \text{ and } ((v_{ij}(t) \geq \Theta_j) \text{ or } (n_j \text{ is firing at } t))).$$

or

$$(J_{ij}(t) > 0) \text{ iff } ((v_{ij}(t) > 0) \text{ or } (n_j \text{ is firing at } t)) \text{ since } \Theta_j = -50 \text{ mV.}$$

Consequently,

$$\text{prob}(J_{ij}(t) > 0) = \text{prob}(v_{ij}(t) > 0) \quad (9.43)$$

$$= \frac{40 \text{ mV}}{110 \text{ mV}} \quad (9.44)$$

$$= 0.36. \quad (9.45)$$

It follows that

$$\text{prob}(J_{ij}(t) < 0) = \frac{50 \text{ mV}}{110 \text{ mV}} = 0.45. \quad (9.46)$$

Also,

$$\begin{aligned} \text{prob}(J_{ij}(t)J_{kl}(t) > 0) \\ = \text{prob}((J_{ij}(t) > 0 \text{ and } J_{kl}(t) > 0) \text{ or} \\ (J_{ij}(t) < 0 \text{ and } J_{kl}(t) < 0)) \end{aligned} \quad (9.47)$$

$$\begin{aligned} = \text{prob}(J_{ij}(t) > 0 \text{ and } J_{kl}(t) > 0) + \\ + \text{prob}(J_{ij}(t) < 0 \text{ and } J_{kl}(t) < 0) \end{aligned} \quad (9.48)$$

$$\begin{aligned} = \text{prob}(J_{ij}(t) > 0) \text{prob}(J_{kl}(t) > 0) + \\ + \text{prob}(J_{ij}(t) < 0) \text{prob}((J_{kl}(t) < 0)) \end{aligned} \quad (9.49)$$

$$= (0.36)^2 + (0.45)^2 \quad (9.50)$$

$$= 0.34 \quad (9.51)$$

while

$$\text{prob}(J_{ij}(t)J_{kl}(t)J_{mn}(t) \neq 0) = 1. \quad (9.52)$$

Now consider two randomly chosen two-dimensional real vectors $\mathbf{v}$ and $\mathbf{w}$. Since $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos \theta_{\mathbf{v}, \mathbf{w}}$, where $\theta_{\mathbf{v}, \mathbf{w}}$ denotes the angle subtended by the vectors $\mathbf{v}$ and $\mathbf{w}$, we have

$$\text{prob}(\mathbf{v} \cdot \mathbf{w} > 0) = \text{prob}(\cos \theta_{\mathbf{v}, \mathbf{w}} > 0) \quad (9.53)$$

$$= \text{prob}\left(\theta_{\mathbf{v}, \mathbf{w}} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right) \quad (9.54)$$

$$= \frac{\pi}{2\pi} \quad (9.55)$$

$$= 0.5. \quad (9.56)$$

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