Acoustic radiation force on an elastic sphere in a soft elastic medium

Yurii A. Ilinskii, Evgenia A. Zabolotskaya, Benjamin C. Treweek, and Mark F. Hamilton^{a)}
Applied Research Laboratories, The University of Texas at Austin, Austin, Texas 78713-8029, USA

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A theoretical framework in Lagrangian coordinates is developed for calculating the acoustic radiation force on an elastic sphere in a soft elastic medium. Advantages of using Lagrangian coordinates are that the surface of the sphere is fixed in the reference frame, and nonlinearity appears only in the stress tensor. The incident field is a time-harmonic compressional wave with arbitrary spatial structure, and there is no restriction on the size of the sphere. Bulk and shear viscosities are taken into account with complex wavenumbers. A solution is presented for the radiation force due to the scattered compressional wave. For an ideal liquid surrounding the sphere, there is no scattered shear wave contributing to the radiation force and the solution is complete. The theory reproduces established results obtained in Eulerian coordinates for an elastic sphere in a fluid.

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I. INTRODUCTION

Theory for the acoustic radiation force on a compressible sphere or gas bubble in a fluid is well established; see, for example, the review by Wang and Lee.¹ More recently, Sapozhnikov and Bailey² developed expressions for calculating the radiation force of an arbitrary acoustic beam on an elastic sphere in a fluid. Common to these and most other approaches is a theoretical framework formulated in Eulerian coordinates.

Acoustic radiation force on a scatterer in a soft elastic medium is also of interest, particularly in connection with biomedical applications of ultrasound in soft tissue. Displacement of a gas bubble and of a rigid sphere in a tissue-like medium has been calculated for a prescribed acoustic radiation force.³ It was assumed that the prescribed force is close to that which would exist if the sphere were surrounded by liquid. The theoretical model was used to determine the shear modulus of the surrounding elastic medium based on the transient response of the sphere to an impulsive force.^{4,5}

A theoretical framework for calculating the acoustic radiation force on an elastic sphere in a soft elastic medium is presented here. Soft is defined here as a medium characterized by a shear modulus that is several orders of magnitude smaller than the bulk modulus, as is the case for soft tissue. Shear and bulk viscosities are incorporated using complex elastic moduli. It is assumed that a time-harmonic compressional wave with arbitrary spatial structure is incident on an elastic sphere of arbitrary radius. The theory is developed in Lagrangian coordinates, which distinguishes it from previous analyses performed in Eulerian coordinates and primarily for fluids.

The two principal contributions in this paper are (1) a theoretical framework for calculating the acoustic radiation force on an elastic sphere using Lagrangian coordinates, and

(2) a method for calculating the radiation force on an elastic sphere surrounded by a soft elastic medium.

The first contribution, a theoretical framework based on Lagrangian coordinates, applies to scatterers surrounded by fluid as well as by a soft elastic medium. Advantages of the Lagrangian formalism are that the surface of the sphere is fixed in the reference frame, the integration is performed on the surface of the sphere, and nonlinearity appears only in the stress tensor. For example, on the moving surface of a compressible sphere surrounded by a fluid, the boundary conditions are simple and exact when expressed in Lagrangian coordinates, whereas in Eulerian coordinates, an approximation is required to render the surface stationary,⁶ and a kinematic nonlinearity appears in the momentum equation. Lagrangian coordinates are also more natural for describing nonlinear elasticity.

Much of the paper is devoted to demonstrating that the Lagrangian framework reproduces known results for the radiation force on elastic spheres in fluids obtained using Eulerian coordinates. These include the results derived by Sapozhnikov and Bailey² for a sound beam incident on a sphere of arbitrary size, and the results obtained by Gor'kov⁷ for both travelling and standing waves incident on a small particle. Results derived in the present work have been used by the authors to obtain analytical approximations for the force on an elastic sphere in the paraxial region of a non-axisymmetric incident beam.⁸ They have also been evaluated numerically to calculate both axial and transverse forces on an elastic sphere located off axis in a focused sound beam.⁹

The second contribution, a method for including the effect of the scattered shear wave when calculating the radiation force on a compressible sphere in a soft elastic medium, is described in Sec. VII. A suitable analytical expression for this contribution has not yet been obtained, although formally the method for doing so follows closely the procedure in Sec. IV for calculating the effect of the scattered compressional wave. While a paper by Cantrell¹⁰ provides a

^{a)}Electronic mail: hamilton@mail.utexas.edu

calculation of acoustic radiation stress in an elastic medium, it pertains only to the intrinsic time-averaged stress in a travelling plane wave, not to the radiation force associated with scattering from a sphere as considered in the present work.

II. BASIC RELATIONS

In the derivation of the radiation force presented here, viscosity, heat conduction, and specific properties of biological media such as relaxation and hysteresis are ignored. Effects of viscosity are taken into account *ad hoc* through the introduction of complex elastic moduli in Sec. V.

The equations for an elastic medium may be written in two forms. The first form corresponds to Cauchy's equations in Eulerian coordinates $\tilde{x}_n$:¹¹

$$\rho \frac{Dv_n}{Dt} = \frac{\partial \tilde{\sigma}_{nm}}{\partial \tilde{x}_m}, \quad \frac{\partial \rho}{\partial t} = -\frac{\partial(\rho v_m)}{\partial \tilde{x}_m}, \quad (1)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_m \frac{\partial}{\partial \tilde{x}_m} \quad (2)$$

is the total, or material, time derivative, tildes denote Eulerian coordinates, $\tilde{\sigma}_{nm}$ is a symmetric stress tensor, v_n is a particle velocity component, and ρ is the density of the medium. For an ideal fluid, the stress tensor is $\tilde{\sigma}_{nm} = -P\delta_{nm}$, where P is the pressure and δ_{nm} is the Kronecker delta.

The alternative form is expressed in Lagrangian coordinates x_n :¹¹

$$\rho_0 \ddot{u}_n = \frac{\partial \sigma_{nm}}{\partial x_m}, \quad (3)$$

where ρ_0 is the initial density, $u_n = \tilde{x}_n - x_n$ are the particle displacement components, and σ_{nm} is the first Piola-Kirchhoff stress tensor, also called the Piola-Kirchhoff pseudostress tensor. The tensor σ_{nm} is not symmetric. The quantity $\sigma_{nm}n_m$ is the n th component of the force that acts on a unit area having unit normal with component n_m in the reference (Lagrangian) space.

Equations (1) and (3) are equivalent because the stress and pseudostress tensors are related as follows:

$$\rho \sigma_{kl} = \rho_0 \tilde{\sigma}_{km} \frac{\partial x_l}{\partial \tilde{x}_m}, \quad \rho_0 \tilde{\sigma}_{km} = \rho \sigma_{kl} \frac{\partial \tilde{x}_m}{\partial x_l}. \quad (4)$$

For calculation of the acoustic radiation force acting on the surface of a sphere embedded in an elastic medium, it is more convenient to use Eq. (3) employing the Piola-Kirchhoff pseudostress tensor because the position of the spherical surface is fixed in the reference frame. Additionally, nonlinearity enters Eq. (3) only via σ_{nm} . While nonlinearity enters Eq. (1) via $\tilde{\sigma}_{nm}$ as well, the Reynolds stress tensor $\rho v_n v_m$ obtained after combining Eqs. (1) contributes a second source of nonlinearity.

In the present work, the problem is formulated in Lagrangian coordinates. For brevity, since only the pseudostress tensor appears in the analysis, the pseudostress tensor

shall henceforth be referred to simply as the stress tensor. Following Landau and Lifshitz¹² we write

$$\sigma_{nm} = \frac{\partial \mathcal{E}}{\partial (\partial u_n / \partial x_m)}, \quad (5)$$

where the strain energy density $\mathcal{E}$ is expressed through cubic order in the particle displacement as follows:

$$\begin{aligned} \mathcal{E} = & \frac{\mu}{4} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right)^2 + \frac{1}{2} \left(K - \frac{2\mu}{3} \right) \left(\frac{\partial u_l}{\partial x_l} \right)^2 \\ & + \left(\mu + \frac{A}{4} \right) \frac{\partial u_i}{\partial x_k} \frac{\partial u_l}{\partial x_i} \frac{\partial u_l}{\partial x_k} + \frac{1}{2} \left(K + B - \frac{2\mu}{3} \right) \frac{\partial u_l}{\partial x_l} \left(\frac{\partial u_i}{\partial x_k} \right)^2 \\ & + \frac{A}{12} \frac{\partial u_i}{\partial x_k} \frac{\partial u_k}{\partial x_l} \frac{\partial u_l}{\partial x_i} + \frac{B}{2} \frac{\partial u_i}{\partial x_k} \frac{\partial u_k}{\partial x_i} \frac{\partial u_l}{\partial x_l} + \frac{C}{3} \left(\frac{\partial u_l}{\partial x_l} \right)^3. \end{aligned} \quad (6)$$

Here, K and μ are the bulk and shear moduli, respectively, and A , B , and C are nonlinear elastic constants, also called third-order elastic constants. Next, the stress tensor is separated into two parts, one containing only terms that are linear in the particle displacement ($\sigma_{nm}^{(1)}$), and the other containing only terms that are quadratic ($\sigma_{nm}^{(2)}$):

$$\sigma_{nm} = \sigma_{nm}^{(1)} + \sigma_{nm}^{(2)}, \quad (7)$$

where

$$\sigma_{nm}^{(1)} = \mu \left(\frac{\partial u_n}{\partial x_m} + \frac{\partial u_m}{\partial x_n} \right) + \left(K - \frac{2\mu}{3} \right) \left(\frac{\partial u_l}{\partial x_l} \right) \delta_{nm} \quad (8)$$

and

$$\begin{aligned} \sigma_{nm}^{(2)} = & \left(\mu + \frac{A}{4} \right) \left(\frac{\partial u_l}{\partial x_n} \frac{\partial u_l}{\partial x_m} + \frac{\partial u_m}{\partial x_l} \frac{\partial u_n}{\partial x_l} + \frac{\partial u_l}{\partial x_m} \frac{\partial u_n}{\partial x_l} \right) \\ & + \frac{1}{2} \left(K + B - \frac{2\mu}{3} \right) \left[\left(\frac{\partial u_i}{\partial x_k} \right)^2 \delta_{nm} + 2 \frac{\partial u_l}{\partial x_l} \frac{\partial u_n}{\partial x_m} \right] \\ & + \frac{A}{4} \frac{\partial u_m}{\partial x_l} \frac{\partial u_l}{\partial x_n} + B \left(\frac{\partial u_l}{\partial x_l} \frac{\partial u_m}{\partial x_n} + \frac{1}{2} \frac{\partial u_i}{\partial x_k} \frac{\partial u_k}{\partial x_i} \delta_{nm} \right) \\ & + C \left(\frac{\partial u_l}{\partial x_l} \right)^2 \delta_{nm}. \end{aligned} \quad (9)$$

The particle displacement is separated accordingly,

$$u_n = u_n^{(1)} + u_n^{(2)}, \quad (10)$$

with the first- and second-order displacement fields $u_n^{(1)}$ and $u_n^{(2)}$, respectively, obtained by the method of successive approximations.

From Eq. (3), the first-order displacement field satisfies

$$\rho_0 \ddot{u}_n^{(1)} = \frac{\partial \sigma_{nm}^{(1)}}{\partial x_m}, \quad (11)$$

where here $\sigma_{nm}^{(1)}$ stands for Eq. (8) evaluated with u_n replaced by $u_n^{(1)}$. Solutions of this equation are the displacement fields for compressional and shear waves in the linear approximation. These waves propagate with speeds $c_l = [(K + \frac{4}{3}\mu)/\rho]^{1/2}$

and $c_t = (\mu/\rho)^{1/2}$, respectively, where henceforth the subscript 0 on the density is suppressed for notational consistency with the elastic constants.

If the first-order displacement field is time harmonic, then through quadratic nonlinearity the second-order field $u_n^{(2)}$ accounts for both static displacement due to the radiation force and second-harmonic generation. The latter is of no interest here and it is eliminated from consideration upon taking the time average. Solutions for the static displacement of both a spherical particle and a gas bubble in an elastic medium were derived in earlier work³ under the assumption that the acoustic radiation force producing the static displacement is known *a priori*. The present work is devoted to calculation of this radiation force, which is associated with the second-order stress tensor $\sigma_{nm}^{(2)}$. This stress is calculated at second order by evaluating Eq. (9) with u_n replaced by $u_n^{(1)}$, where $u_n^{(1)}$ is known from the solution of Eq. (11).

III. SIMPLIFICATION FOR SOFT ELASTIC MEDIA

Substantial simplification of Eq. (9) follows from recognizing that in media such as soft tissue, the ratio μ/K is typically of order 10^{-4} to 10^{-6} , with the value of the bulk modulus K for soft tissue typically close to that for water. It may also be concluded,¹³ based in part on measurements reported by Catheline *et al.*,¹⁴ that the quantities A and $(K+B)$ are both $O(\mu)$, which permits the first three terms in Eq. (9) to be ignored in comparison with the fourth and fifth terms.

Further simplification follows from recognizing that for the problem under consideration, which is scattering of sound from a sphere in free space, potential forces produced by the stress tensor do not contribute to the net radiation force acting on the sphere. These potential forces, which may be expressed as the gradient of a scalar function, are volume forces that are offset by equal and opposite reaction forces associated with static deformation of the medium in response to the potential forces. In general, the force density (per unit volume) at second order appearing on the right-hand side of Eq. (3) is expressed as

$$f_n^{(2)} = \frac{\partial \sigma_{nm}^{(2)}}{\partial x_m}. \quad (12)$$

Quantities of the form $Q\delta_{nm}$ that appear in $\sigma_{nm}^{(2)}$, where Q is a scalar function, act as pressures and produce only potential forces. If P is the corresponding pressure then $Q = -P$ and from Eq. (12) the potential force $f_n^{(2)} = -\partial P/\partial x_n$ (or in vector form $\mathbf{f}^{(2)} = -\nabla P$) is the only force produced by the quantity $Q\delta_{nm}$. Returning to the remaining fourth and fifth terms in Eq. (9), we thus ignore the quantities containing δ_{nm} , leaving only the stress tensor

$$\sigma_{nm}^B = B \frac{\partial u_l}{\partial x_l} \frac{\partial u_m}{\partial x_n} \quad (13)$$

that contributes to the net acoustic radiation force acting on the sphere. Equation (13) produces both potential and non-

potential forces. Removal of the source of the remaining potential forces is addressed in Sec. IV.

The superscript B indicates that only the elastic constant B participates in the stress tensor of interest for sufficiently small values of the quantity μ/K . Since $K+B=O(\mu)$, it is consistent to replace B by $-K$ in Eq. (13), which is useful because very few measurements of the third-order elastic constants A , B , and C are available, particularly for tissue. We therefore write

$$\sigma_{nm}^B = -K \frac{\partial u_l}{\partial x_l} \frac{\partial u_m}{\partial x_n}, \quad (14)$$

retaining the notation B to identify the origin of this stress.

From Eq. (14) it is apparent that for soft elastic media, as for ideal fluids, the nonlinear constitutive behavior of the medium surrounding the sphere, which is characterized by the third-order elastic constants A , B , and C , does not affect the radiation force on the sphere. If there are constraints on the medium, for example when radiation forces are produced by plane waves confined within a tube, it may be necessary to retain quantities of the form $Q\delta_{nm}$ in the fourth and fifth terms in Eq. (9), in which case nonlinearity of the medium contributes to the radiation force through the elastic constant C .

It is assumed that a time-harmonic compressional wave is incident on the elastic sphere. Scattering from the sphere produces both shear and compressional waves in the first-order displacement field. The emphasis of the present work is on particle displacements associated with the scattered compressional wave when evaluating Eq. (14). Contributions from the scattered shear wave are discussed in Sec. VII.

IV. REMOVAL OF THE POTENTIAL FORCES

If only compressional waves in the medium surrounding the sphere are considered when evaluating Eq. (14), then the first-order particle displacement can be expressed as $u_n^{(1)} = \partial \varphi / \partial x_n$, where φ is the displacement potential for the irrotational field. The potential φ is assumed to be time-harmonic with angular frequency ω such that

$$\nabla^2 \varphi + k^2 \varphi = 0, \quad (15)$$

where $k = \omega/c_l$ is the wavenumber for compressional waves. One thus obtains

$$\frac{\partial u_l}{\partial x_l} = \frac{\partial^2 \varphi}{\partial x_l^2} = \nabla^2 \varphi = -k^2 \varphi, \quad (16)$$

and from Eqs. (12) and (14) it follows that the force density exerted on the medium by the compressional waves is

$$f_n^B = Kk^2 \frac{\partial}{\partial x_m} \left(\varphi \frac{\partial}{\partial x_n} \frac{\partial \varphi}{\partial x_m} \right). \quad (17)$$

Differentiation and rearrangement yields an expression in the form of a potential force,

$$f_n^B = \frac{1}{2} K k^2 \frac{\partial}{\partial x_n} \left[\left(\frac{\partial \varphi}{\partial x_m} \right)^2 - k^2 \varphi^2 \right], \quad (18)$$

where $(\partial \varphi / \partial x_m)^2 = \nabla \varphi \cdot \nabla \varphi$ is a scalar invariant.

By Newton's third law, the response of the medium to the time average of the applied potential force in Eq. (18) is a static deformation which produces a stress field that exactly offsets the applied force:

$$\sigma_{nm}^P = -P \delta_{nm}, \quad (19)$$

where

$$P = \frac{1}{2} K k^2 \left[\left(\frac{\partial \varphi}{\partial x_m} \right)^2 - k^2 \varphi^2 \right]. \quad (20)$$

The reaction force $\partial \sigma_{nm}^P / \partial x_m$ exerted by the medium is therefore the negative of Eq. (18). Consequently, σ_{nm}^P must be added to σ_{nm}^B when calculating the net radiation force on the sphere.

The net acoustic radiation force acting on a sphere of radius R is thus

$$F_n = \int_S \langle \sigma_{nm}^B + \sigma_{nm}^P \rangle \frac{x_m}{R} dS, \quad (21)$$

where the integral is performed over the surface of the sphere, x_m/R is the unit normal on the spherical surface, and the angular brackets indicate time average. The integrals over the stress tensors may be considered separately, with

$$F_n = F_n^B + F_n^P \quad (22)$$

where

$$\begin{aligned} F_n^B &= -K \int_S \left\langle \frac{\partial u_l}{\partial x_l} \frac{\partial u_m}{\partial x_n} \right\rangle \frac{x_m}{R} dS \\ &= K k^2 \int_S \left\langle \varphi \frac{\partial^2 \varphi}{\partial x_n \partial x_m} \right\rangle \frac{x_m}{R} dS \\ &= \frac{K k^2}{R} \int_S \left\langle \varphi \frac{\partial}{\partial x_n} \left(r \frac{\partial \varphi}{\partial r} - \varphi \right) \right\rangle dS \end{aligned} \quad (23)$$

and

$$\begin{aligned} F_n^P &= - \int_S \langle P \rangle \delta_{nm} \frac{x_m}{R} dS \\ &= - \frac{K k^2}{2} \int_S \langle (\nabla \varphi)^2 - k^2 \varphi^2 \rangle \frac{x_n}{R} dS. \end{aligned} \quad (24)$$

Combining Eqs. (22)–(24) and choosing $x_n = z$ (without lack of generality, as explained in different contexts below) yields for the total radiation force $F_z = F_z^B + F_z^P$ in the z direction

$$\begin{aligned} F_z &= \frac{K k^2}{R} \int_S \left\langle \varphi \frac{\partial}{\partial z} \left(r \frac{\partial \varphi}{\partial r} - \varphi \right) \right. \\ &\quad \left. - \frac{1}{2} [(\nabla \varphi)^2 - k^2 \varphi^2] R \cos \theta \right\rangle dS, \end{aligned} \quad (25)$$

where θ is the polar angle, with respect to the z axis, corresponding to the location of the surface element on the sphere. There is no restriction on Eq. (25) in terms of sphere radius.

V. SPHERICAL HARMONIC EXPANSION

It remains to solve for the first-order incident and scattered fields in the linear approximation in order to calculate the integral over the surface of the sphere. Thus with

$$\varphi = \frac{1}{2} (\psi e^{-i\omega t} + \psi^* e^{i\omega t}) = \frac{1}{2} \psi e^{-i\omega t} + \text{c.c.} \quad (26)$$

for the compressional waves, Eqs. (23) and (24) become

$$F_z^B = \frac{K k^2}{4R} \int_S \left[\psi^* \frac{\partial}{\partial z} \left(r \frac{\partial \psi}{\partial r} - \psi \right) \right] dS + \text{c.c.}, \quad (27)$$

$$F_z^P = - \frac{K k^2}{4} \int_S \left(\frac{\partial \psi^*}{\partial x_m} \frac{\partial \psi}{\partial x_m} - k^2 \psi^* \psi \right) \cos \theta dS, \quad (28)$$

for the forces in the z direction.

It is assumed that a compressional wave is incident on the sphere. Both the sphere and the medium surrounding it are elastic. The displacement field $\mathbf{u}$ in both the sphere and surrounding medium is separated into its longitudinal (compressional, or irrotational) and transverse (shear, or solenoidal) components $\mathbf{u}_l$ and $\mathbf{u}_t$, respectively, as follows:

$$\mathbf{u} = \mathbf{u}_l + \mathbf{u}_t, \quad (29)$$

where¹⁵

$$\mathbf{u}_l = \frac{1}{2} \nabla \psi(r, \theta, \phi) e^{-i\omega t} + \text{c.c.}, \quad (30)$$

$$\mathbf{u}_t = \frac{1}{2} \nabla \times \nabla \times [\mathbf{r} \Pi(r, \theta, \phi)] e^{-i\omega t} + \text{c.c.} \quad (31)$$

Note that ϕ is the azimuthal angle in the spherical coordinate system, not to be confused with the displacement potential φ , where $\mathbf{u}_l = \nabla \varphi$.

The incident compressional wave field in the medium surrounding the sphere is expanded in spherical harmonics as

$$\psi_{\text{in}}(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^n a_n^m j_n(kr) P_n^m(\cos \theta) e^{im\phi}, \quad (32)$$

where j_n are spherical Bessel functions, P_n^m are associated Legendre polynomials, and $k = \omega/c_l$. Any compressional wave field can be described with this expansion. If the incident field is expressed in terms of acoustic pressure $\frac{1}{2} p_{\text{in}} e^{-i\omega t} + \text{c.c.}$ then $p_{\text{in}} = \rho \omega^2 \psi_{\text{in}}$. Often the field is assumed to be a travelling plane wave. For example, if $\psi_{\text{in}} = \psi_0 e^{i(kz - \omega t)}$ then $a_n = i^n (2n+1) \psi_0$, in which case the coefficients a_n^m do not depend on m because the field is symmetric about the z axis, independent of ϕ , and therefore $m = 0$.

The scattered potential fields ψ_{sc} and Π_{sc} in the medium surrounding the sphere are expressed as

$$\psi_{\text{sc}}(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^n A_n a_n^m h_n(kr) P_n^m(\cos \theta) e^{im\phi}, \quad (33)$$

$$\Pi_{\text{sc}}(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^n B_n a_n^m h_n(\kappa r) P_n^m(\cos \theta) e^{im\phi}, \quad (34)$$

where $h_n (\equiv h_n^{(1)})$ are spherical Hankel functions and $\kappa = \omega/c_t$. The dimensionless amplitudes A_n and B_n are scattering coefficients for the compressional and shear waves, respectively. On account of the spherical symmetry of the scatterer, A_n and B_n are independent of m . This property of the scattering coefficients is a consequence of Schur's Lemma.¹⁶

Equations (32)–(34) for the potential fields in the medium surrounding the sphere must be supplemented by the fields inside the sphere, designated by subscript s :

$$\psi_s(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^n C_n a_n^m j_n(k_s r) P_n^m(\cos \theta) e^{im\phi}, \quad (35)$$

$$\Pi_s(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^n D_n a_n^m j_n(\kappa_s r) P_n^m(\cos \theta) e^{im\phi}, \quad (36)$$

where $k_s = \omega/c_l^s$ and $\kappa_s = \omega/c_t^s$ are the wavenumbers, c_l^s and c_t^s the wave speeds. The four unknown coefficients A_n , B_n , C_n , and D_n are found by satisfying the boundary conditions for continuity of displacement and stress on the surface of the sphere using Eqs. (32)–(36); see, e.g., Ying and Truell.¹⁵ The resulting linear system of four equations in the four unknowns is presented in Appendix A.

Only the scattering coefficients A_n are needed in the present section because only the contribution from the incident and scattered compressional waves is taken into account in the radiation force. While radiation force due to momentum transfer associated with scattering of shear waves following mode conversion is not taken into account here (see Sec. VII), the shear modulus of the medium surrounding the sphere alters the radiation force by virtue of its effect on the coefficients A_n in the linear solution.

Substitution of $\psi = \psi_{\text{in}} + \psi_{\text{sc}}$ into Eqs. (27) and (28) for F_z^B and F_z^P and summing the results yields, making use of recurrence relations for the Bessel and Legendre functions,

$$F_z = i\pi K k^2 \sum_{n=0}^{\infty} \sum_{m=-n}^n \frac{(n+m+1)(n+m)!}{(2n+1)(2n+3)(n-m)!} \times (a_n^m)^* a_{n+1}^m (A_n^* + A_{n+1} + 2A_n^* A_{n+1}) + \text{c.c.} \quad (37)$$

Whereas F_z^B and F_z^P individually depend on sphere radius R , their sum $F_z = F_z^B + F_z^P$ resulting in Eq. (37) is independent of R . Both F_z^B and F_z^P , and therefore F_z , are valid for all values of kR and κR , i.e., for spheres of arbitrary radius R . Equation (37) describes the total radiation force acting in the z direction on an elastic sphere in an ideal fluid, and it is equivalent to Eq. (48) of Sapozhnikov and Bailey.²

If the incident field is symmetric about the z axis, then $m=0$ and Eqs. (32), (33), and (37) assume the reduced forms

$$\psi_{\text{in}}(r, \theta) + \psi_{\text{sc}}(r, \theta) = \sum_{n=0}^{\infty} a_n [j_n(kr) + A_n h_n(kr)] P_n(\cos \theta) \quad (38)$$

and

$$F_z = i\pi K k^2 \sum_{n=0}^{\infty} \frac{(n+1)}{(2n+1)(2n+3)} \times a_n^* a_{n+1} (A_n^* + A_{n+1} + 2A_n^* A_{n+1}) + \text{c.c.}, \quad (39)$$

where $P_n(\cos \theta)$ are Legendre polynomials. Equation (39) is equivalent to Eq. (16) of Sapozhnikov and Bailey.²

The above results were developed for lossless elastic media. Losses are taken into account by introducing the complex elastic moduli $\tilde{K} = K - i\omega\zeta$ and $\tilde{\mu} = \mu - i\omega\eta$, where ζ is the bulk viscosity and η the shear viscosity. The real wavenumbers above (and in Appendix A), defined by $k = \omega[\rho/(K + \frac{4}{3}\mu)]^{1/2}$ and $\kappa = \omega(\rho/\mu)^{1/2}$, are thus replaced by

$$\tilde{k} = \omega \sqrt{\frac{\rho}{K + 4\mu/3 - i\omega(\zeta + 4\eta/3)}}, \quad \tilde{\kappa} = \omega \sqrt{\frac{\rho}{\mu - i\omega\eta}}. \quad (40)$$

These expressions are used for the sphere (then with subscript or superscript s on the relevant quantities) as well as the surrounding medium.

Finally, when calculating the remaining force components F_x and F_y , it proves convenient to express the potential fields as expansions in terms of normalized spherical harmonics, e.g.,

$$\psi_{\text{in}}(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^n \tilde{a}_n^m j_n(kr) Y_{nm}(\theta, \phi) \quad (41)$$

in place of Eq. (32), where

$$Y_{nm}(\theta, \phi) = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} P_n^m(\cos \theta) e^{im\phi}. \quad (42)$$

The advantage is associated with the fact that the spherical harmonics are orthonormal, thus enabling the use of established relations for rotating coordinate systems in order to calculate the radiation force in directions other than along the z axis. The coefficients a_n^m in Eq. (32) are related to the $\tilde{a}_n^m$ in Eq. (41) by

$$a_n^m = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} \tilde{a}_n^m. \quad (43)$$

Expressed in terms of $\tilde{a}_n^m$, Eq. (37) becomes

$$F_z = \frac{i}{4} K k^2 \sum_{n=0}^{\infty} \sum_{m=-n}^n \sqrt{\frac{(n+m+1)(n-m+1)}{(2n+1)(2n+3)}} \times (\tilde{a}_n^m)^* \tilde{a}_{n+1}^m (A_n^* + A_{n+1} + 2A_n^* A_{n+1}) + \text{c.c.}, \quad (44)$$

which for an axisymmetric field reduces to

$$F_z = \frac{i}{4} K k^2 \sum_{n=0}^{\infty} \frac{(n+1)}{\sqrt{(2n+1)(2n+3)}} \times \tilde{a}_n^* \tilde{a}_{n+1} (A_n^* + A_{n+1} + 2A_n^* A_{n+1}) + \text{c.c.} \quad (45)$$

The x and y components of the radiation force are obtained by rotation of the coordinate system and use of the Wigner D-matrix to transform the coefficients a_n^m . In addition to Eq. (44) one obtains

$$F_x = \frac{i}{4} K k^2 \sum_{n=0}^{\infty} \sum_{m=-n}^n \sqrt{\frac{(n+m+1)(n-m+1)}{(2n+1)(2n+3)}} \times [\tilde{a}_n^m]_x [\tilde{a}_{n+1}^m]_x (A_n^* + A_{n+1} + 2A_n^* A_{n+1}) + \text{c.c.} \quad (46)$$

and

$$F_y = \frac{i}{4} K k^2 \sum_{n=0}^{\infty} \sum_{m=-n}^n \sqrt{\frac{(n+m+1)(n-m+1)}{(2n+1)(2n+3)}} \times [\tilde{a}_n^m]_y [\tilde{a}_{n+1}^m]_y (A_n^* + A_{n+1} + 2A_n^* A_{n+1}) + \text{c.c.}, \quad (47)$$

where the coefficients $[\tilde{a}_n^m]_x$ and $[\tilde{a}_n^m]_y$ are developed in Appendix B.

VI. RADIATION FORCE ON A SMALL SPHERE

While Eqs. (37) and (39) are valid for spheres of arbitrary size, we consider here the case of a small sphere, referred to here as a particle because in the absence of shear forces, it can be shown that the familiar results obtained by Gor'kov are recovered.

In the case of a sphere with radius much less than all wavelengths, it is sufficient to assume an axisymmetric field incident on the sphere ($m=0$). Only the leading terms $n=0$ and $n=1$ in Eq. (37) are significant, which yields

$$F_z = i\pi K k^2 \left[\frac{1}{3} (A_0^* + A_1 + 2A_0^* A_1) a_0^* a_1 + \frac{2}{15} A_1^* a_1^* a_2 \right] + \text{c.c.}, \quad (48)$$

where the superscript $m=0$ on a_n^m is suppressed. The coefficients A_0 and A_1 are associated with monopole and dipole scattering, respectively.

We begin with evaluation of A_0 . Since $B_0 = D_0 = 0$ in Eqs. (A1)–(A4) because the coefficients for the shear waves do not participate for $n=0$,¹⁵ one is left with the two Eqs. (A1) and (A3) for the unknowns A_0 and C_0 . Solving for A_0 and using small-argument expansions of the Bessel functions ($kR \ll 1$, $k_s R \ll 1$) yields

$$A_0 = \frac{i}{3} (kR)^3 \left(\frac{K}{K_s} - 1 \right) \left[\left(1 + \frac{4\mu}{3K_s} \right) - \frac{\omega^2}{\omega_R^2} - \frac{i}{3} (kR)^3 \left(\frac{K}{K_s} - 1 \right) \right]^{-1}, \quad (49)$$

where

$$\omega_R = \sqrt{\frac{3K_s}{\rho R^2}} \quad (50)$$

is a reference natural frequency of the sphere in the absence of shear forces. There is no dependence on μ_s for a small particle. The term $4\mu/3K_s$ is the correction to the natural frequency that accounts for the shear modulus of the surrounding medium.¹⁷ For a gas bubble one has $K_s = \gamma P_0$, where γ is the ratio of specific heats (or, more generally, the polytropic index) and P_0 is ambient pressure, in which case ω_R is the familiar Minnaert frequency for a bubble in liquid. The term $(i/3)(kR)^3(K/K_s - 1)$ accounts for radiation damping of the oscillations. Damping due to viscosity in the medium is included by introducing complex elastic moduli as discussed in connection with Eqs. (40).

To assess the effect of finite shear modulus on radiation force associated with monopole scattering, we consider the case of a gas bubble ($K/K_s \gg 1$, $\rho/\rho_s \gg 1$) driven below resonance ($\omega^2/\omega_R^2 \ll 1$) and ignoring radiation damping. Equation (49) then reduces to

$$A_0 \simeq \frac{i}{3} (kR)^3 \frac{K}{\gamma P_0} \left(1 + \frac{4}{3} \frac{\mu}{\gamma P_0} \right)^{-1}. \quad (51)$$

Reported measurements of shear moduli for soft tissue have values ranging from on the order of 1 kPa at the low end (breast, liver) up to 70 kPa at the high end (cancerous tissue).¹⁸ Taking $\mu = 70$ kPa, $P_0 = 100$ kPa, and $\gamma = 1.4$ yields $\mu/\gamma P_0 = 0.5$, for which A_0 , and therefore the contribution to F_z due to monopole scattering, is 40% less in magnitude than the value predicted for a gas bubble in liquid. Since $\rho_s \ll \rho$, the magnitude of A_1/A_0 is of order $\gamma P_0/K \sim 10^{-4}$, so the contribution from the dipole term is negligible, and F_z is approximately proportional to A_0 .

To proceed further analytically, in order to obtain a useful approximation for A_1 , it is helpful to ignore shear forces in both the particle and the medium ($\mu = \mu_s = 0$), in which case it is also helpful to introduce the notation f_1 and f_2 employed by Gor'kov.⁷ Equation (49) then reduces to

$$A_0 = -\frac{i}{3} (kR)^3 f_1 \left[1 - \frac{\omega^2}{\omega_R^2} + \frac{i}{3} (kR)^3 f_1 \right]^{-1}, \quad (52)$$

where

$$f_1 = \frac{K_s - K}{K_s}. \quad (53)$$

In the absence of shear forces, one has $B_n = D_n = 0$ in Eqs. (A1)–(A4), and then Eqs. (A1) and (A3) for A_1 and C_1 , with small-argument approximations for the Bessel functions, yield

$$A_1 = \frac{i}{6} (kR)^3 f_2 \left[1 - \frac{i}{6} (kR)^3 f_2 \right]^{-1}, \quad (54)$$

where

$$f_2 = \frac{2(\rho_s - \rho)}{2\rho_s + \rho}. \quad (55)$$

It is useful for comparison with Gor'kov's results to consider the expansions of Eqs. (52) and (54) for small kR below resonance ($\omega^2/\omega_R^2 \ll 1$):

$$A_0 = -i\frac{1}{3}(kR)^3 f_1 - \frac{1}{9}(kR)^6 f_1^2, \quad (56)$$

$$A_1 = i\frac{1}{6}(kR)^3 f_2 - \frac{1}{36}(kR)^6 f_2^2. \quad (57)$$

Equations (56) and (57) are consistent with the optical theorem, a consequence of which for $|\text{Re}(A_n)| \ll |\text{Im}(A_n)|$ is that $\text{Re}(A_n) \simeq -[\text{Im}(A_n)]^2$.

Equation (48) is now evaluated for plane progressive and standing incident waves. For the progressive wave $\psi_{\text{in}} = \psi_0 e^{ikz}$, the coefficients in Eq. (32) are $a_n = i^n (2n+1)\psi_0$ with the superscript $m=0$ again suppressed, for which Eq. (48) becomes

$$F_z = -2\pi K k^2 |\psi_0|^2 \text{Re}(A_0^* + 3A_1 + 2A_0^* A_1). \quad (58)$$

For $\omega^2/\omega_R^2 \ll 1$, substitution of Eqs. (56) and (57) yields at leading order in kR

$$F_z = \frac{2}{9} \pi K k^2 (kR)^6 |\psi_0|^2 \left(f_1^2 + \frac{3}{4} f_2^2 + f_1 f_2 \right), \quad (59)$$

which agrees with Eq. (10) of Gor'kov.⁷

The coefficients a_n for standing waves are constructed from the coefficients for progressive waves. With the introduction of z_0 as a fixed coordinate used to translate the incident field relative to the origin, the coefficients for $\psi_{\text{in}} = \psi_0 e^{ik(z+z_0)}$ are $a_n = i^n (2n+1)\psi_0 e^{ikz_0} = (2n+1)\psi_0 e^{i(kz_0 + n\pi/2)}$. The coefficients for the standing wave

$$\psi_{\text{in}} = \psi_0 \cos k(z+z_0) \quad (60)$$

can now be obtained via superposition by writing $\psi_0 \cos k(z+z_0) = \frac{1}{2} \psi_0 e^{ik(z+z_0)} + \text{c.c.}$ such that $a_n = \frac{1}{2} (2n+1)\psi_0 e^{i(kz_0 + n\pi/2)} + \text{c.c.}$, or with ψ_0 assumed real

$$a_n = (2n+1)\psi_0 \cos(kz_0 + n\pi/2), \quad (61)$$

and thus

$$\begin{aligned} a_0 &= \psi_0 \cos kz_0, & a_1 &= -3\psi_0 \sin kz_0, \\ a_2 &= -5\psi_0 \cos kz_0. \end{aligned} \quad (62)$$

Substitution of the relations in Eqs. (62) into (48) yields

$$F_z = \pi K k^2 \psi_0^2 \text{Im}(A_0^* + 3A_1 + 2A_0^* A_1) \sin 2kz_0. \quad (63)$$

For $\omega^2/\omega_R^2 \ll 1$, substitution of Eqs. (56) and (57) yields at leading order in kR

$$F_z = \pi K k^2 (kR)^3 \psi_0^2 \left(\frac{1}{3} f_1 + \frac{1}{2} f_2 \right) \sin 2kz_0, \quad (64)$$

which agrees with Eq. (13) of Gor'kov⁷ for a standing wave described by Eq. (60) and a particle located at $z=0$.

See Marston^{19,20} for various higher-order (in kR) improvements of Gor'kov's results.

VII. CONTRIBUTION FROM SCATTERED SHEAR WAVE

In Secs. IV and V, only the potential (longitudinal) component of the scattered wave field was taken into account when calculating the radiation force. If an elastic solid or a viscous liquid surrounds the sphere, then a solenoidal (transverse) component is generated in the scattered field that contributes to the radiation force even if the incident field is purely potential.

The calculation begins as before with the force separated as in Eqs. (21) and (22) when only compressional waves are involved. The latter equation is expressed here as

$$G_n = G_n^B + G_n^P, \quad (65)$$

where the notation G_n used here rather than F_n as in Eq. (22) is to make clear the distinction between the two contributions to the force on the sphere. Instead of the final form of Eq. (23), the following expression for G_n^B must be used:

$$G_n^B = \frac{Kk^2}{R} \int_S \left\langle \varphi \frac{\partial u_m}{\partial x_n} \right\rangle x_m dS. \quad (66)$$

To calculate the force due to the scattered compressional wave, $\partial u_m / \partial x_n$ is replaced by its potential part and Eq. (23) is recovered. For the force due to the scattered shear wave, the potential φ in Eq. (66) is still evaluated for the scattered compressional wave, but $\partial u_m / \partial x_n$ is evaluated only for the scattered shear wave. Use of Eqs. (32) and (33) to evaluate φ and Eq. (34) to evaluate $\partial u_m / \partial x_n$ yields

$$\begin{aligned} G_z^B &= \pi K k^2 \kappa R \sum_{n=1}^{\infty} \frac{h_n(\kappa R)}{2n+1} [j_n^*(\kappa R) + A_n^* h_n^*(\kappa R)] \\ &\times \left[\sum_{m=-n}^n (a_n^m)^* \frac{n(n+2)(n+m+1)!}{(2n+3)(n-m)!} a_{n+1}^m B_{n+1} \right. \\ &- \left. \sum_{m=-n+1}^{n-1} (a_n^m)^* \frac{(n-1)(n+1)(n+m)!}{(2n-1)(n-m-1)!} a_{n-1}^m B_{n-1} \right] \\ &+ \text{c.c.} \end{aligned} \quad (67)$$

for the component of G_n^B in the z direction, which for an axisymmetric incident wave field reduces to

$$\begin{aligned} G_z^B &= \pi K k^2 \kappa R \sum_{n=1}^{\infty} \frac{h_n(\kappa R)}{2n+1} [j_n^*(\kappa R) + A_n^* h_n^*(\kappa R)] \\ &\times a_n^* \left[\frac{n(n+1)(n+2)}{2n+3} a_{n+1} B_{n+1} \right. \\ &- \left. \frac{(n-1)n(n+1)}{2n-1} a_{n-1} B_{n-1} \right] + \text{c.c.} \end{aligned} \quad (68)$$

While Eqs. (67) and (68) resemble Eqs. (37) and (39), respectively, the former are incomplete because the

contribution from G_z^P in Eq. (65) must be included to obtain the actual force $G_z = G_z^B + G_z^P$ due to the scattered shear wave.

Calculation of G_n^P is nontrivial. First, Eqs. (12) and (14) are used to obtain the force density g_n^B due to the scattered shear wave, which is analogous to f_n^B and is written as

$$g_n^B = Kk^2 \frac{\partial}{\partial x_m} \left(\phi \frac{\partial u_m}{\partial x_n} \right). \quad (69)$$

Next, Helmholtz decomposition of g_n^B is performed in order to express the vector as $\mathbf{g}^B = \nabla Q + \nabla \times \mathbf{S}$, such that the potential component may be identified and the integral $G_n^P = -\int_S \langle Q \rangle (x_n/R) dS$ may be formed. An analytical Helmholtz decomposition of g_n^B is currently unavailable. Suitable analytical and numerical approximations of G_n^P are under investigation and will be reported elsewhere.

Finally, Eq. (65) is the force on the sphere due to shear stresses in the medium surrounding the sphere only at the instant when the radiation force is initially applied. Displacement of the sphere due to continuous action of a radiation force causes elastic deformation of the surrounding medium, and at subsequent instants in time the resulting reaction force acting on the sphere must also be taken into account. After sufficient time has elapsed in the presence of a constant radiation force, the sphere comes to rest when the reaction force due to deformation of the medium increases to where it offsets the radiation force, and the net force on the sphere is ultimately zero.

VIII. CONCLUSION

A theoretical framework in Lagrangian coordinates is developed for calculating the acoustic radiation force on an elastic sphere in a soft elastic medium. Bulk and shear viscosities are taken into account through complex wavenumbers. Solutions are presented for the contributions to the radiation force from the scattered compressional wave. For an elastic sphere in an ideal fluid the solutions are complete, and they reproduce previous results obtained in Eulerian coordinates. Ongoing work to obtain solutions for the radiation force due to the scattered shear wave and due to deformation of the surrounding medium will be reported elsewhere.

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APPENDIX A: SCATTERING COEFFICIENTS

The equations for the four unknown coefficients A_n , B_n , C_n , and D_n appearing in Eqs. (33)–(36) are (see, e.g., Ying and Truell¹⁵)

$$\begin{aligned} & A_n(kR)h_{n+1}(kR) - B_n n(\kappa R)h_{n+1}(\kappa R) \\ & - C_n(k_s R)j_{n+1}(k_s R) + D_n n(\kappa_s R)j_{n+1}(\kappa_s R) \\ & = -(kR)j_{n+1}(kR), \end{aligned} \quad (A1)$$

$$\begin{aligned} & A_n h_n(kR) + B_n[(n+1)h_n(\kappa R) - (\kappa R)h_{n+1}(\kappa R)] \\ & - C_n j_n(k_s R) - D_n[(n+1)j_n(\kappa_s R) - (\kappa_s R)j_{n+1}(\kappa_s R)] \\ & = -j_n(kR), \end{aligned} \quad (A2)$$

$$\begin{aligned} & A_n \left[2(n+2)(kR)h_{n+1}(kR) - (\kappa R)^2 h_n(kR) \right] \\ & - B_n n \left[2(n+2)(\kappa R)h_{n+1}(\kappa R) - (\kappa R)^2 h_n(\kappa R) \right] \\ & - \frac{\rho_s \kappa^2}{\rho \kappa_s^2} \left\{ C_n \left[2(n+2)(k_s R)j_{n+1}(k_s R) - (\kappa_s R)^2 j_n(k_s R) \right] \right. \\ & \left. - D_n n \left[2(n+2)(\kappa_s R)j_{n+1}(\kappa_s R) - (\kappa_s R)^2 j_n(\kappa_s R) \right] \right\} \\ & = - \left[2(n+2)(kR)j_{n+1}(kR) - (\kappa R)^2 j_n(kR) \right], \end{aligned} \quad (A3)$$

$$\begin{aligned} & A_n [(n-1)h_n(kR) - (kR)h_{n+1}(kR)] \\ & + B_n \left[(n^2 - 1)h_n(\kappa R) - \frac{1}{2}(\kappa R)^2 h_n(\kappa R) \right. \\ & \left. + (\kappa R)h_{n+1}(\kappa R) \right] - \frac{\rho_s \kappa^2}{\rho \kappa_s^2} \left\{ C_n [(n-1)j_n(k_s R) \right. \\ & \left. - (k_s R)j_{n+1}(k_s R)] + D_n \left[(n^2 - 1)j_n(\kappa_s R) \right. \right. \\ & \left. \left. - \frac{1}{2}(\kappa_s R)^2 j_n(\kappa_s R) + (\kappa_s R)j_{n+1}(\kappa_s R) \right] \right\} \\ & = -[(n-1)j_n(kR) - (kR)j_{n+1}(kR)]. \end{aligned} \quad (A4)$$

The subscript s distinguishes wavenumbers and density in the sphere from the corresponding quantities for the medium outside the sphere. Equations (A1)–(A4) are combined to solve for the scattering coefficients A_n required to calculate the radiation force on the sphere due to scattering of compressional waves.

Losses in the medium and the sphere are taken into account according to Eqs. (40).

APPENDIX B: WIGNER D-MATRIX TRANSFORMATION

Spherical harmonics $Y_{nm'}(\theta', \phi')$ in a rotated coordinate system (x', y', z') , which is obtained from an original coordinate system (x, y, z) by rotating with Euler angles (α, β, γ) , are linear combinations of spherical harmonics $Y_{nm}(\theta, \phi)$ in the original coordinate system:²¹

$$Y_{nm'}(\theta', \phi') = \sum_{m=-n}^n D_{nm'}^n(\alpha, \beta, \gamma) Y_{nm}(\theta, \phi), \quad (B1)$$

where $D_{nm'}^n(\alpha, \beta, \gamma)$ is the Wigner D-matrix and is written as

$$D_{nm'}^n(\alpha, \beta, \gamma) = e^{-im'\alpha} d_{nm'}^n(\beta) e^{-im'\gamma}. \quad (B2)$$

The Wigner (small) d-matrix $d_{nm'}^n(\beta)$ depends on which convention is used for Euler angle rotations and axes. The z - y - z convention (i.e., rotation by α about the z axis, then rotation by β about the new y axis, then rotation by γ about the final z' axis) is often used because the elements of $d_{nm'}^n(\beta)$ are real in this case. One such expression is from Eq. (5) in Sec. 4.3 of Varshalovich *et al.*,²¹

$$d_{nm'}^n(\beta) = (-1)^{m-m'} \sqrt{(n+m)!(n-m)!(n+m')!(n-m')!} \sum_s (-1)^s \frac{[\cos(\beta/2)]^{2n-2s-m+m'} [\sin(\beta/2)]^{2s+m-m'}}{s!(n-m-s)!(n+m'-s)!(m-m'+s)!}, \quad (\text{B3})$$

with summation limits defined such that the factorials are non-negative.

When incident coefficients $\tilde{a}_n^m$ are included, as in Eq. (41), it follows that

$$\begin{aligned} & \sum_{m'=-n}^n \tilde{a}_n^{m'} Y_{nm'}(\theta', \phi') \\ &= \sum_{m'=-n}^n \tilde{a}_n^{m'} \sum_{m=-n}^n D_{nm'}^n(\alpha, \beta, \gamma) Y_{nm}(\theta, \phi) \\ &= \sum_{m=-n}^n Y_{nm}(\theta, \phi) \sum_{m'=-n}^n \tilde{a}_n^{m'} D_{nm'}^n(\alpha, \beta, \gamma) \\ &= \sum_{m=-n}^n \tilde{a}_n^m Y_{nm}(\theta, \phi), \end{aligned} \quad (\text{B4})$$

where

$$\tilde{a}_n^m = \sum_{m'=-n}^n \tilde{a}_n^{m'} D_{nm'}^n(\alpha, \beta, \gamma). \quad (\text{B5})$$

Equivalently, $\tilde{a}_n^{m'}$ can be written as

$$\begin{aligned} \tilde{a}_n^{m'} &= \sum_{m=-n}^n \tilde{a}_n^m [D^{-1}(\alpha, \beta, \gamma)]_{m'm}^n \\ &= \sum_{m=-n}^n \tilde{a}_n^m D_{m'm}^n(-\gamma, -\beta, -\alpha). \end{aligned} \quad (\text{B6})$$

In the z - y - z convention, the new z' axis can be oriented with the original x axis via Euler angles $(0, \pi/2, \pi/2)$ or with the original y axis via Euler angles $(-\pi/2, -\pi/2, 0)$. (These choices align x' and y' with original coordinate axes as well, but there are other valid choices to correctly align z' .) Then, from expansion coefficients $\tilde{a}_n^m (\equiv [\tilde{a}_n^m]_z)$ about the z axis, one can obtain expansion coefficients about the x and y axes

$$[\tilde{a}_n^{m'}]_x = \sum_{m=-n}^n [\tilde{a}_n^m]_z D_{m'm}^n(-\pi/2, -\pi/2, 0), \quad (\text{B7})$$

and

$$[\tilde{a}_n^{m'}]_y = \sum_{m=-n}^n [\tilde{a}_n^m]_z D_{m'm}^n(0, \pi/2, \pi/2), \quad (\text{B8})$$

respectively. The relations for F_x and F_y have the same form as Eq. (44) but with altered coefficients:

$$\begin{aligned} F_x &= \frac{i}{4} K k^2 \sum_{n=0}^{\infty} \sum_{m=-n}^n \sqrt{\frac{(n+m+1)(n-m+1)}{(2n+1)(2n+3)}} \\ &\quad \times [\tilde{a}_n^m]_x^* [\tilde{a}_{n+1}^m]_x (A_n^* + A_{n+1} + 2A_n^* A_{n+1}) + \text{c.c.}, \end{aligned} \quad (\text{B9})$$

and

$$\begin{aligned} F_y &= \frac{i}{4} K k^2 \sum_{n=0}^{\infty} \sum_{m=-n}^n \sqrt{\frac{(n+m+1)(n-m+1)}{(2n+1)(2n+3)}} \\ &\quad \times [\tilde{a}_n^m]_y^* [\tilde{a}_{n+1}^m]_y (A_n^* + A_{n+1} + 2A_n^* A_{n+1}) + \text{c.c.} \end{aligned} \quad (\text{B10})$$

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