Time-domain analysis of power law attenuation in space-fractional wave equations

Xiaofeng Zhao and Robert J. McGough^{a)}

Department of Electrical and Computer Engineering, Michigan State University, East Lansing, Michigan 48864, USA

(Received 26 October 2017; revised 5 July 2018; accepted 6 July 2018; published online 30 July 2018)

Ultrasound attenuation in soft tissue follows a power law as a function of the ultrasound frequency, and in medical ultrasound, power law attenuation is often described by fractional calculus models that contain one or more time- or space-fractional derivatives. For certain time-fractional models, exact and approximate time-domain Green's functions are known, but similar expressions are not available for the space-fractional models that describe power law attenuation. To address this deficiency, a numerical approach for calculating time-domain Green's functions for the Chen–Holm space-fractional wave equation and Treeby–Cox space-fractional wave equation is introduced, where challenges associated with the numerical evaluation of a highly oscillatory improper integral are addressed with the Filon integration formula combined with the Pantis method. Numerical results are computed for both of these space-fractional wave equations at different distances in breast and liver with power law exponents of 1.5 and 1.139, respectively. The results show that these two space-fractional wave equations are causal and that away from the origin, the time-domain Green's function for the Treeby–Cox space-fractional wave equation is very similar to the time-domain Green's function for the time-fractional power law wave equation.

© 2018 Acoustical Society of America. <https://doi.org/10.1121/1.5047670>

[JFL]

Pages: 467–477

I. INTRODUCTION

The attenuation of ultrasound in soft tissue is described by a power law of the form $\alpha(\omega) = \alpha_0 \omega^y$, where this relation is a “power law” because the attenuation is proportional to ω raised to the y power. In the expression for $\alpha(\omega)$, α_0 is the attenuation constant in Np/cm/MHz^y or dB/cm/MHz^y, y is the power law exponent, and ω is the angular frequency in radians/s. Power law attenuation of compressional waves in medical ultrasound is often modeled with noninteger order derivatives with respect to the time and/or space variables, i.e., with fractional calculus. For time- and space- fractional models of power law attenuation, the close connection between power law attenuation and fractional calculus is evident in the expression for the derivative operation $(j\omega)^y$ in the frequency domain, which contains the power law term ω^y .

Time-fractional and space-fractional wave equations that describe power law attenuation are often characterized through the attenuation and phase velocity, which are obtained from the imaginary and real parts of the wavenumber k , respectively. Other methods for analyzing time-fractional wave equations include the Kramers–Kronig relations,¹ the Paley–Wiener theorem,² and time-causal theories.³ Time-fractional wave equations are also evaluated through time-domain analysis of the material impulse response function (MIRF)⁴ and time-domain Green's functions.^{5,6} Much of this analysis concentrates on the causality; however, the distinction between these is very subtle because the attenuation is very similar in each of the fractional

calculus models for medical ultrasound. When present, the noncausal components of power law attenuation models are only evident in time-domain Green's functions evaluated very close to the origin.⁵ For example, the noncausal components of the time-domain Green's function for the power law wave equation are observed in the results of numerical calculations within 10 nm of the source for breast and within 10 μ m (where μ m represents micrometers or 10^{-6} m) for liver. Elsewhere, examples of noncausal behavior in the time-domain are very difficult to identify because beyond a very short distance from the origin, the time-domain Green's function in noncausal fractional calculus models is very small for all times $t < 0$ compared to the value at the peak.⁵

Determining whether a space-fractional model is causal or noncausal requires a different approach because the dispersion relations for space-fractional equations are transcendental equations. This is further confounded by the various claims that appear in the literature, where one paper indicates that the Chen–Holm space-fractional wave equation is causal,⁷ two papers claim that the Chen–Holm space-fractional wave equation is noncausal,^{8,9} and one other paper suggests that the Chen–Holm and Treeby–Cox space-fractional wave equations are both noncausal.^{10,11} If these space-fractional wave equations are noncausal, then some evidence of noncausal behavior is expected very close to the origin, as demonstrated in Ref. 5 for two noncausal time-fractional wave equations. Similarly, if these space-fractional wave equations are causal, then the time-domain Green's functions are expected to consistently equal zero at all times $t < 0$, including all locations very close to the origin. Since these results have not yet been demonstrated for either of

^{a)}Electronic mail: mcgough@egr.msu.edu

these space-fractional wave equations, additional evaluations are needed.

To definitively establish whether the Chen–Holm and Treeby–Cox wave equations are causal or noncausal and also to evaluate the similarities and differences in the time-domain Green’s functions for both of these space-fractional wave equations, some new analytical expressions and numerical results are introduced. After the fractional partial differential equations, the dispersion relations, and other important expressions are provided for the Chen–Holm and the Treeby–Cox space-fractional wave equations, a numerical calculation approach that combines Filon’s integration formula with the Pantis method to evaluate a highly oscillatory improper integral is established for time-domain Green’s function calculations. The power law wave equation, which is a time-fractional wave equation, is also included as a reference. Analytical expressions showing that the Chen–Holm and the Treeby–Cox space-fractional wave equations are both causal are also introduced. The results show that there is a small but noticeable difference between numerically calculated values for the attenuation and dispersion and the values predicted by previously published models and that the discrepancy is eliminated when additional terms are included in the approximations for the attenuation and dispersion. Numerical evaluations of the time-domain Green’s functions also reinforce that both of these space-fractional wave equations are causal. Although both of these space-fractional wave equations demonstrate similar attenuation as a function of frequency, the phase velocities for the Chen–Holm and Treeby–Cox wave equations are quite different. The results also show that the time-domain Green’s functions for the Treeby–Cox wave equation and the power law wave equation yield approximately the same result a short distance from the origin. When convolved with a short input pulse, the time-domain Green’s functions for the Treeby–Cox wave equation and the power law wave equation are also nearly identical, but the results obtained with the time-domain Green’s function for the Chen–Holm wave equation are also different. Except for some very limited circumstances, the space-fractional Treeby–Cox wave equation and the time-fractional power law wave equation yield similar results, even though the former is causal and the latter is noncausal.

II. TIME-FRACTIONAL AND SPACE-FRACTIONAL WAVE EQUATIONS

A. Power law attenuation and dispersion

The attenuation $\alpha(\omega)$ of ultrasound in soft tissue as a function of frequency follows the power law

$$\alpha(\omega) = \alpha_0 |\omega|^y, \quad (1)$$

with attenuation constant α_0 , power law exponent y , and angular frequency ω . The phase velocity or dispersion $c(\omega)$ satisfies³

$$\frac{1}{c(\omega)} = \frac{1}{c_0} + \alpha_0 \tan\left(\frac{\pi y}{2}\right) |\omega|^{y-1}, \quad (2)$$

where c_0 is the sound speed at $\omega = 0$ for $1 < y \leq 2$ and c_0 is the sound speed at $\omega = \infty$ for $0 \leq y < 1$. The parameter c_0 only appears in Eq. (2), but α_0 and y appear in both Eqs. (1) and (2).

B. The Chen–Holm space-fractional wave equation

The space-fractional wave equation developed by Chen and Holm⁷ is given by

$$\nabla^2 g - \frac{1}{c_0^2} \frac{\partial^2 g}{\partial t^2} - \tau \frac{\partial}{\partial t} (-\nabla^2)^{y/2} g = 0, \quad (3)$$

where $\tau = 2\alpha_0 c_0^{y-1}$ is the fractional relaxation time,⁷ and t is the time in seconds. The first two terms in Eq. (3) are recognized as the lossless wave equation, and the third term, which contains a fractional Laplacian, produces power law attenuation. There is no known exact analytical closed-form time-domain Green’s function for the Chen–Holm wave equation, so numerical evaluations are required.

To obtain an expression for time-domain Green’s function calculations, the transfer function $G(k, s)$ is derived by applying the negative of an impulsive forcing function to the right hand side of Eq. (3). This is followed by a Laplace transform in time, which is defined as $\mathcal{L}[h(t)] = \int_0^\infty h(t) dt$, and a Fourier transform in spherically symmetric three dimensional (3D) space with $r = \sqrt{x^2 + y^2 + z^2}$, which is given by $\mathcal{F}[f(r)] = (4\pi/k) \int_0^\infty f(r) \sin(kr) r dr$ (where this is the same expression given on p. 342 of Bracewell¹² with the spatial frequency variable s replaced by $k = 2\pi s$). Applying the differentiation properties of the Laplace (without initial conditions) and Fourier transforms, namely, $\mathcal{L}[(d/dt)h(t)] = sH(s)$, $\mathcal{L}[(d^2/dt^2)h(t)] = s^2H(s)$, and $\mathcal{F}[\nabla^2 f(r)] = -k^2 F(k)$, yields

$$-k^2 G(k, s) - \frac{s^2}{c_0^2} G(k, s) - s\tau k^y G(k, s) = -1. \quad (4)$$

Solving for $G(k, s)$, multiplying the numerator and denominator by c_0^2 , and completing the square in the denominator produces the expression

$$G(k, s) = \frac{c_0^2}{(s + \tau k^y c_0^2/2)^2 + k^2 c_0^2 - \tau^2 k^{2y} c_0^4/4}. \quad (5)$$

The inverse Laplace transform of Eq. (5) is

$$\hat{g}(k, t) = c_0 e^{-\tau k^y c_0^2 t/2} \frac{\sin\left(k c_0 t \sqrt{1 - \tau^2 k^{2y-2} c_0^2/4}\right)}{k \sqrt{1 - \tau^2 k^{2y-2} c_0^2/4}} u(t) \quad (6)$$

for $k < (2/(\tau c_0))^{1/(y-1)}$, and

$$\hat{g}(k, t) = c_0 e^{-\tau k^y c_0^2 t/2} \frac{\sinh\left(k c_0 t \sqrt{\tau^2 k^{2y-2} c_0^2/4 - 1}\right)}{k \sqrt{\tau^2 k^{2y-2} c_0^2/4 - 1}} u(t) \quad (7)$$

for $k > (2/(\tau c_0))^{1/(y-1)}$. At $k = (2/(\tau c_0))^{1/(y-1)}$, $\hat{g}(k, t) = \tau c_0^2 e^{-\tau k^y c_0^2 t/2} u(t)$, so $\hat{g}(k, t)$ is continuous with respect to k

and t . In all three of these expressions for $\hat{g}(k, t)$, $u(t)$ is the unit step function, which guarantees that the time-domain response is equal to 0 for all times $t < 0$ and that $\hat{g}(k, t)$ is causal. The 3D time-domain Green's function in free space is then obtained from Ref. 12 (again with the spatial frequency variable s replaced by $k = 2\pi s$):

$$g(r, t) = \frac{4\pi}{(2\pi)^3 r} \int_0^\infty \hat{g}(k, t) \sin(kr) k dk. \quad (8)$$

The integral in Eq. (8) also contains the unit step function $u(t)$ through the expressions for $\hat{g}(k, t)$, so the time-domain Green's function $g(r, t)$ for the Chen–Holm space-fractional wave equation is causal. Although Eqs. (6) and (7) are analytical expressions, the time-domain Green's function $g(r, t)$ in Eq. (8) is evaluated numerically.

Approximate expressions for the phase velocity and attenuation are derived from the dispersion relation

$$k^2 = \frac{\omega^2}{c_0^2} + j\omega\tau k^y, \quad (9)$$

with $\tau = 2\alpha_0 c_0^{y-1}$. After taking the square root of both sides of Eq. (9), factoring out ω/c_0 on the right hand side, and keeping the first three terms in the binomial approximation, the wavenumber is rewritten as $k \approx (\omega/c_0)(1 + j\alpha_0 c_0^{y+1} \omega^{-1} k^y + \alpha_0^2 c_0^{2y+2} \omega^{-2} k^{2y}/2)$. On the right hand side, k^y and k^{2y} are replaced by the approximations $(\omega/c_0)^y (1 + jy\alpha_0 c_0 \omega^{y-1})$ and $(\omega/c_0)^{2y}$, respectively, and all third order and higher terms with respect to α_0 are discarded. The approximate phase velocity is then obtained from the real part of the wavenumber divided by ω ,

$$\frac{1}{c(\omega)} \approx \frac{1}{c_0} - \left(y - \frac{1}{2}\right) \alpha_0^2 c_0 \omega^{2y-2}. \quad (10)$$

The approximate attenuation $\alpha(\omega) = \alpha_0 \omega^y$ is obtained from the imaginary part of the resulting approximation for k , which is the same expression given in Eq. (1). In Treeby and Cox,¹⁰ the attenuation for the Chen–Holm wave equation is also $\alpha(\omega) = \alpha_0 \omega^y$, but the approximate phase velocity for the Chen–Holm wave equation is instead expressed as $c(\omega) \approx c_0$.

C. The Treeby–Cox space-fractional wave equation

Treeby and Cox¹⁰ recognized that the Chen–Holm space-fractional wave equation correctly models the power law attenuation of Eq. (1) but that the dispersion for the Chen–Holm wave equation differs from the desired expression given in Eq. (2). To address this problem, Treeby and Cox inserted a second space-fractional term that accounts for the dispersion, and the resulting expression is given by¹⁰

$$\nabla^2 g - \frac{1}{c_0^2} \frac{\partial^2 g}{\partial t^2} + \left\{ -\tau \frac{\partial}{\partial t} (-\nabla^2)^{y/2} + \eta (-\nabla^2)^{(y+1)/2} \right\} g = 0, \quad (11)$$

where $\tau = 2\alpha_0 c_0^{y-1}$ and $\eta = 2\alpha_0 c_0^y \tan(\pi y/2)$. In Eq. (11), the third and fourth terms account for the effects of attenuation

and dispersion, respectively. Similar to the Chen–Holm wave equation, there is no known exact analytical closed-form time-domain Green's function for the Treeby–Cox wave equation, so numerical evaluations are required. As before, $G(k, s)$ is derived by applying the negative of an impulsive forcing function to the right hand side of Eq. (11), followed by a Laplace transform in time and a Fourier transform in space. Solving for $G(k, s)$, multiplying and dividing by c_0^2 , and completing the square in the denominator gives

$$G(k, s) = \frac{c_0^2}{(s + \tau k^y c_0^2/2)^2 + k^2 c_0^2 - \tau^2 k^{2y} c_0^4/4 - \eta c_0^2 k^{y+1}}. \quad (12)$$

After the inverse Laplace transform of Eq. (12) is evaluated, the result is

$$\hat{g}(k, t) = c_0 e^{-\tau k^y c_0^2 t/2} \times \frac{\sin\left(k c_0 t \sqrt{1 - \tau^2 k^{2y-2} c_0^2/4 - \eta k^{y-1}}\right)}{k \sqrt{1 - \tau^2 k^{2y-2} c_0^2/4 - \eta k^{y-1}}} u(t) \quad (13)$$

for $k < \kappa$, and

$$\hat{g}(k, t) = c_0 e^{-\tau k^y c_0^2 t/2} \times \frac{\sinh\left(k c_0 t \sqrt{\tau^2 k^{2y-2} c_0^2/4 + \eta k^{y-1} - 1}\right)}{k \sqrt{\tau^2 k^{2y-2} c_0^2/4 + \eta k^{y-1} - 1}} u(t) \quad (14)$$

for $k > \kappa$, where $\kappa = [(\eta + \sqrt{\eta^2 + \tau^2 c_0^2})/2]^{-1/(y-1)}$. At $k = \kappa$, $\hat{g}(k, t) = t c_0^2 e^{-[(\eta + \sqrt{\eta^2 + \tau^2 c_0^2})/2]^{-y/(y-1)} \tau c_0^2 t/2} u(t)$, so the expression for $\hat{g}(k, t)$ is continuous with respect to k and t . The unit step function $u(t)$ in all three of these expressions for $\hat{g}(k, t)$ indicates that $\hat{g}(k, t)$ is causal. The 3D time-domain Green's function for the Treeby–Cox space-fractional wave equation in free space is calculated with Eq. (8) combined with the above expressions for $\hat{g}(k, t)$. Due to the presence of $u(t)$ in each expression for $\hat{g}(k, t)$, the time-domain Green's function $g(r, t)$ for the Treeby–Cox space-fractional wave equation is also causal.

The phase velocity and attenuation for the Treeby–Cox space-fractional wave equation are derived from the dispersion relation

$$k^2 = \frac{\omega^2}{c_0^2} + j2\alpha_0 c_0^{y-1} \omega k^y + 2\alpha_0 c_0^y \tan(\pi y/2) k^{y+1}, \quad (15)$$

starting with the assumption that the attenuation and dispersion are both small relative to ω/c_0 . The binomial approximation is then applied to Eq. (15), and the first three terms are retained, yielding

$$k \approx \frac{\omega}{c_0} \left[1 + \alpha_0 c_0^{y+1} \omega^{-1} (j k^y + \tan(\pi y/2) c_0 \omega^{-1} k^{y+1}) - \frac{1}{2} \alpha_0^2 c_0^{2y+2} \omega^{-2} (j k^y + \tan(\pi y/2) c_0 \omega^{-1} k^{y+1})^2 \right]. \quad (16)$$

On the right hand side of Eq. (16), k^y and k^{y+1} are replaced with the approximations $(\omega/c_0)^y[1 + y\alpha_0 c_0 \omega^{y-1}(j + \tan(\pi y/2))]$ and $(\omega/c_0)^{y+1}[1 + (y+1)\alpha_0 c_0 \omega^{y-1}(j + \tan(\pi y/2))]$, respectively, and all third order and higher terms with respect to α_0 are discarded. The approximate phase velocity is obtained from the real part of the wavenumber divided by ω ,

$$\frac{1}{c(\omega)} \approx \frac{1}{c_0} + \tan(\pi y/2) \alpha_0 \omega^{y-1} + \left[-y + \frac{1}{2} + \left(y + \frac{1}{2} \right) \tan^2(\pi y/2) \right] \alpha_0^2 c_0 \omega^{2y-2}, \quad (17)$$

and the approximate attenuation is the imaginary part of the wavenumber,

$$\alpha(\omega) \approx \alpha_0 \omega^y + 2y \tan(\pi y/2) \alpha_0^2 c_0 \omega^{2y-1}. \quad (18)$$

Treeby and Cox¹⁰ instead express the approximate phase velocity and attenuation for the Treeby–Cox wave equation as $1/c(\omega) \approx 1/c_0 + \tan(\pi y/2) \alpha_0 \omega^{y-1}$ and $\alpha(\omega) \approx \alpha_0 \omega^y$, respectively.

D. The power law wave equation

Equations (1) and (2) are exactly satisfied by the power law wave equation,⁶ which is the time-fractional wave equation given by

$$\nabla^2 g - \frac{1}{c_0^2} \frac{\partial^2 g}{\partial t^2} - \frac{2\alpha_0}{c_0 \cos(\pi y/2)} \frac{\partial^{y+1} g}{\partial t^{y+1}} - \frac{\alpha_0^2}{\cos^2(\pi y/2)} \frac{\partial^{2y} g}{\partial t^{2y}} = 0. \quad (19)$$

The power law wave equation is time-fractional because the time derivative in the third term is not an integer-order derivative for all noninteger values of y between 0 and 2. The 3D frequency-domain Green's function for the power law wave equation in free space is⁶

$$G(r, \omega) = \frac{e^{-j\omega r/c_0} e^{-\alpha_0(j\omega)^y r / \cos(\pi y/2)}}{4\pi r}, \quad (20)$$

and the 3D time-domain Green's function for the power law wave equation in free space is⁶

$$g(r, t) = \frac{1}{4\pi r} \frac{1}{(\alpha_0 r)^{1/y}} \tilde{f}_y \left[\frac{t - r/c_0}{(\alpha_0 r)^{1/y}} \right], \quad (21)$$

where $\tilde{f}_y[\cdot]$ represents the probability density function (pdf) for a maximally skewed stable distribution with stability parameter y . For numerical calculations with Eq. (21), the stable pdf indicated is readily computed in Matlab with the STABLE toolbox.¹⁹

E. Definition of causality

Causality, as defined for signal analysis,¹³ requires that the impulse response $h(t)$ is equal to zero at all times prior to

the application of an input at $t=0$. In other words, a causal system requires that the impulse response $h(t)=0$ at all times $t < 0$. In the context of acoustic wave propagation in lossy soft tissue, causality is described in terms of the time response of the time-domain Green's function⁵ (or equivalently, the material impulse response function⁴), where causality requires that the time-domain Green's function $g(r, t) = 0$ for all times $t < 0$ for an input applied at time $t = 0$.

III. METHODS

A. Dispersion relations

The attenuation and phase velocity of the power law wave equation are calculated with Eqs. (1) and (2), respectively. For the Chen–Holm and the Treeby–Cox space-fractional wave equations, there is no exact analytical solution for the complex wavenumber $k(\omega)$, so this quantity is calculated using the “fsolve” routine in Matlab. The Matlab “fsolve” routine applies the Levenberg–Marquardt algorithm to numerically determine the complex values of $k(\omega)$ that satisfy the dispersion relations in Eqs. (9) and (15) using the initial value $k(\omega) = \omega/c_0$ at each value of ω . Once the numerical result is obtained, the attenuation $\alpha(\omega)$ is equal to the imaginary part of $k(\omega)$, and the reciprocal of the phase velocity $1/c(\omega)$ is equal to the real part of $k(\omega)$ divided by ω .

B. The Pantis method

Numerical calculations of the time-domain Green's function for the Chen–Holm and Treeby–Cox wave equations are challenging because Eq. (8) numerically evaluates a highly oscillatory improper integral. According to Davis and Rabinowitz (Ref. 14, p. 119), “numerical integration of oscillatory integrands is beset with difficulties,” which suggests that a numerical approach designed specifically for oscillatory improper integrals is required. The Pantis method^{15,16} provides an ideal solution to this problem, where the improper integral in Eq. (8) is rewritten as the sum of a proper integral and an improper integral according to

$$g(r, t) = \frac{4\pi}{(2\pi)^3 r} \int_0^\infty k \hat{g}(k, t) \sin(kr) dk = \frac{4\pi}{(2\pi)^3 r} [I_1(r, t) + I_2(r, t)], \quad (22)$$

with

$$I_1(r, t) = \int_0^{m\pi/r} k \hat{g}(k, t) \sin(kr) dk, \quad (23)$$

and

$$I_2(r, t) = \int_{m\pi/r}^\infty k \hat{g}(k, t) \sin(kr) dk. \quad (24)$$

The integral $I_1(r, t)$ is evaluated with Filon's method,^{14,17,18} which approximates $k \hat{g}(k, t)$ with a piecewise second order polynomial. Filon's method is implemented through a Matlab routine from Ref. 18, where the input parameters

include the Matlab function handle, the lower and upper limits of integration, and the number of points at which the integrand is evaluated. Each numerical evaluation of $I_1(r, t)$ with Filon's method is calculated with 70 000 abscissas, which is sufficient for convergence at all temporal and spatial points evaluated here for both breast and liver. After integrating by parts and recognizing that $[k\hat{g}(k, t)]|_{k=\infty} = 0$, the integral $I_2(r, t)$ over the upper subinterval is rewritten as

$$I_2(r, t) = k\hat{g}(k, t) \frac{\cos(kr)}{r} \Big|_{k=m\pi/r} - \frac{\partial}{\partial k} [k\hat{g}(k, t)] \frac{\sin(kr)}{r^2} \Big|_{k=m\pi/r} - \int_{m\pi/r}^{\infty} \frac{\partial^2}{\partial k^2} [k\hat{g}(k, t)] \frac{\sin(kr)}{r^2} dk. \quad (25)$$

For these calculations, m is an odd integer, and only the first term in Eq. (25) is retained. Thus, $I_2(r, t) \approx -m\pi\hat{g}(m\pi/r, t)/r^2$, which is an effective approximation for these calculations when m is sufficiently large and when the contribution from $I_2(r, t)$ is relatively small. Also, when m is an odd integer, the second term in Eq. (25) disappears, and when m is odd, $I_1(r, t)$ is consistently positive for the expressions considered here. An odd value of m is advantageous for these calculations because an even value of m can yield a negative value for $I_1(r, t)$, which is undesirable because $g(r, t) \geq 0$ for all values of r and t . The value $m = 1601$ yields effective results for all (r, t) pairs evaluated here. Also, to avoid problems with numerical overflow when the argument of the $\sinh(\cdot)$ function grows large, $\exp(-a)\sinh(b)$ is instead calculated using $(e^{b-a} - e^{-b-a})/2$.

C. Time windows for computed time-domain Green's functions

The start and end times for all time-domain Green's function calculations are adjusted manually for each location such that each waveform fills a significant portion of the time window. For both the Chen–Holm wave equation and the Treeby–Cox wave equation, $g(r, t) = 0$ at $t = 0$ because $\hat{g}(k, t) = 0$ for all values of k when $t = 0$, which is the analytical result obtained through direct substitution of $t = 0$ into Eqs. (6) and (7) and Eqs. (13) and (14). The unit step function $u(t)$ guarantees that $\hat{g}(k, t) = 0$ for all $t < 0$, so $g(r, t) = 0$ for all $t < 0$ in the Chen–Holm and Treeby–Cox wave equations. For

the power law wave equation, the time-domain Green's function is calculated in Matlab with the STABLE toolbox,¹⁹ which numerically evaluates the expression for the stable pdf in Eq. (21). The time window defined for time domain Green's function calculations very close to the source includes negative time values to capture any noncausal components of the response in these locations. In all plots, the arrival time r/c_0 in a lossless medium is also indicated to provide a time reference.

IV. RESULTS

A. Phase velocity and attenuation in breast and liver

Figure 1 shows the exact and approximate frequency-dependent phase velocity and attenuation for the Chen–Holm space-fractional wave equation using values for human breast with $y = 1.5$, $\alpha_0 = 0.086$ Np/cm/MHz^{1.5}, and $c_0 = 1450$ m/s and using values for human liver with $y = 1.139$, $\alpha_0 = 0.0459$ Np/cm/MHz^{1.139}, and $c_0 = 1540$ m/s. The frequency range is 0.1–40 MHz, and the step size on the horizontal axis is $\Delta f = 0.3$ MHz for each of the phase velocity and attenuation plots. For the Chen–Holm wave equation, the result obtained with the Matlab “fsolve” routine applied to Eq. (9) is indicated by dashed lines with star markers (*), and the approximate solution from Eq. (10) is indicated by dot-dashed lines with solid dot markers (●). The zero-order approximation to the phase velocity $c(\omega) \approx c_0$ is indicated by a solid line with circle markers (○) in Figs. 1(a) and 1(c). Figures 1(a) and 1(c) show that the approximation to $c(\omega)$ introduced in Eq. (10) demonstrates excellent agreement with the numerical solution of Eq. (9). Figures 1(a) and 1(c) also indicate that the phase velocity of the Chen–Holm wave equation is not quite equal to c_0 because $c(\omega)$ increases slightly as the frequency increases. In addition, Fig. 1(b) shows that the attenuation predicted by the Chen–Holm wave equation follows the power law indicated by Eq. (1) very closely in breast.

Figure 2 shows the exact and approximate values for the phase velocity and attenuation of the Treeby–Cox space-fractional wave equation and the exact phase velocity and attenuation of the power law wave equation using the same parameters evaluated in Fig. 1. For the Treeby–Cox wave equation, the numerical results obtained with the Matlab “fsolve” routine applied to Eq. (15) are indicated by dashed lines with star markers (*), and the approximate solution from Eq. (17) is indicated by dot-dashed lines with solid dot markers (●). For the power law wave equation, which

FIG. 1. Phase velocity and attenuation in breast and liver obtained from the dispersion relation in Eq. (9) for the Chen–Holm wave equation (*) and two different approximations (● and ○) to the dispersion relation for the Chen–Holm wave equation.

FIG. 2. Phase velocity and attenuation in breast and liver obtained from the dispersion relation in Eq. (15) for the Treeby–Cox wave equation (*), the approximations to the dispersion relation for the Treeby–Cox wave equation given in Eqs. (17) and (18) (•), and the attenuation and phase velocity for the power law wave equation given in Eqs. (1) and (2) (○), respectively.

exactly satisfies Eqs. (1) and (2), the results are indicated by solid lines with circle markers (○). Figures 2(a)–2(d) show that the approximations introduced in Eqs. (17) and (18) closely match the numerical values obtained from Eq. (15). Figure 2 also shows that the phase velocity and attenuation curves are similar for the power law wave equation and the Treeby–Cox wave equation; however, there is a small but noticeable difference between these curves that is associated with the higher order terms on the right hand side of Eqs. (17) and (18). In addition, Fig. 2(b) indicates that the difference between the attenuation for the Treeby–Cox wave equation and the power law wave equation increases slightly as the frequency grows larger.

B. Time-domain Green's functions calculated for breast

To demonstrate some of the similarities and differences between these three fractional wave equations evaluated at

various distances, Fig. 3 shows the time-domain Green's functions multiplied by $4\pi r$ for the power law wave equation (solid line), the Chen–Holm wave equation (dashed line), and the Treeby–Cox wave equation (dot-dashed line) calculated for human breast with the parameters given in Sec. IV A. The results are computed at $r = 1 \text{ nm}$, $r = 10 \text{ nm}$, $r = 100 \text{ nm}$, $r = 1 \mu\text{m}$, $r = 100 \mu\text{m}$, $r = 1 \text{ mm}$, $r = 1 \text{ cm}$, and $r = 10 \text{ cm}$. The units defined for the horizontal axis in Fig. 3 are picoseconds, nanoseconds, or microseconds. Figures 3(a) and 3(b) show that the Green's function for the power law wave equation is noncausal at $r = 1 \text{ nm}$ and $r = 10 \text{ nm}$ since the Green's function for the power law wave equation is clearly nonzero before time $t = 0$ in these two subfigures. However, the Green's functions for the Chen–Holm wave equation and the Treeby–Cox wave equation are both causal as these two Green's functions are always equal to zero for all times $t \leq 0$. The amplitudes of the Green's functions for these three wave equations are similar in each individual subfigure, and the amplitudes change by a significant amount

FIG. 3. Time-domain Green's functions scaled by $4\pi r$ calculated for breast with $y = 1.5$, $\alpha_0 = 0.086 \text{ Np/cm/MHz}^{1.5}$, and $c_0 = 1450 \text{ m/s}$ at (a) $r = 1 \text{ nm}$, (b) $r = 10 \text{ nm}$, (c) $r = 100 \text{ nm}$, (d) $r = 1 \mu\text{m}$, (e) $r = 100 \mu\text{m}$, (f) $r = 1 \text{ mm}$, (g) $r = 1 \text{ cm}$, and (h) $r = 10 \text{ cm}$ with the power law (solid line), Chen–Holm (dashed line), and Treeby–Cox (dot-dashed line) wave equations. No noncausal behavior is observed at any distance in the time-domain Green's functions for the Chen–Holm and Treeby–Cox wave equations evaluated for breast, and beyond about $r = 100 \mu\text{m}$, the time-domain Green's functions for the power law wave equation and the Treeby–Cox wave equation are nearly indistinguishable.

in each successive subfigure. For example, the peaks in Fig. 3(a) are all approximately $4 \times 10^{11} \text{ s}^{-1}$, and the peaks in Fig. 3(b) are all around $9 \times 10^{10} \text{ s}^{-1}$. Three distinct peaks are observed in Figs. 3(a) and 3(b), and then the peaks of the time-domain Green's functions for the power law wave equation and the Treeby–Cox wave equation start to move much closer together in Figs. 3(c) and 3(d) while the peak for the Chen–Holm wave equation remains distinct from the peaks for the other two fractional wave equations in each subfigure. In Figs. 3(e)–3(h), the waveforms for the Chen–Holm wave equation are consistently quite different from those obtained with the power law and Treeby–Cox wave equations. This is expected because the phase velocity for the Chen–Holm wave equation is approximately equal to c_0 for all frequency components; however, the phase velocity demonstrates much greater variation as a function of frequency in the power law and Treeby–Cox wave equations. Furthermore, in Figs. 3(e)–3(h), the time-domain Green's functions for the power law wave equation and the Treeby–Cox wave equation are nearly indistinguishable, which suggests that these two time-domain Green's functions are converging to the same result.

C. Time-domain Green's functions calculated for liver

Figure 4 shows the time-domain Green's functions multiplied by $4\pi r$ for the power law wave equation (solid line), the Chen–Holm wave equation (dashed line), and the Treeby–Cox wave equation (dot-dashed line) calculated for human liver with the parameters given in Sec. IV A. In Fig. 4, the three time-domain Green's functions are evaluated at $r = 100 \text{ zm}$, $r = 1 \text{ am}$, $r = 10 \text{ am}$, $r = 100 \text{ am}$, $r = 100 \mu\text{m}$, $r = 1 \text{ mm}$, $r = 1 \text{ cm}$, and $r = 10 \text{ cm}$. The units defined for the horizontal axis in Fig. 4 are yoctoseconds (ys or 10^{-24} seconds),

zeptoseconds (zs or 10^{-21} seconds), nanoseconds (ns), or microseconds (μs). In Fig. 4, the first row of subfigures shows that at relatively short distances, the time-domain Green's functions for the power law wave equation, the Chen–Holm wave equation, and the Treeby–Cox wave equation are again all noticeably different. Figure 4(a) shows that the time-domain Green's function for the power law wave equation is noncausal at $r = 100 \text{ zm}$ since this time-domain Green's function is clearly nonzero before time $t = 0$. However, the time-domain Green's functions for the Chen–Holm wave equation and the Treeby–Cox wave equation are both causal as these are consistently equal to zero for all times $t \leq 0$. The amplitudes of the Green's functions for these three wave equations are again similar in each individual subfigure, and the amplitudes again decrease by a significant amount in each successive subfigure. For instance, the peaks in Fig. 4(a) are all approximately $2 \times 10^{22} \text{ s}^{-1}$, and the peaks in Fig. 4(b) are all around $3 \times 10^{21} \text{ s}^{-1}$. Three distinct peaks are observed in Figs. 4(a)–4(d), while the peaks for the power law wave equation and the Treeby–Cox wave equation are very close together in Figs. 4(e)–4(h). In each subfigure, the location of the peak for the Chen–Holm wave equation is distinct from the peak locations for the other two wave equations. Furthermore, in Figs. 4(e)–4(h), the time-domain Green's functions for the power law wave equation and the Treeby–Cox wave equation are nearly indistinguishable, which suggests that these two time-domain Green's functions are converging to the same result.

D. Amplitude and full width at half maximum (FWHM) values in breast and liver

Figure 5 shows the amplitudes and full width at half maximum (FWHM) values for the time-domain Green's

FIG. 4. Time-domain Green's functions scaled by $4\pi r$ calculated for liver with $y = 1.139$, $\alpha_0 = 0.0459 \text{ Np/cm/MHz}^{1.139}$, and $c_0 = 1540 \text{ m/s}$ at (a) $r = 100 \text{ zm}$, (b) $r = 1 \text{ am}$, (c) $r = 10 \text{ am}$, (d) $r = 100 \text{ am}$, (e) $r = 100 \mu\text{m}$, (f) $r = 1 \text{ mm}$, (g) $r = 1 \text{ cm}$, and (h) $r = 10 \text{ cm}$ with the power law (solid line), Chen–Holm (dashed line), and Treeby–Cox (dot-dashed line) wave equations. No noncausal behavior is observed at any distance in the time-domain Green's functions for the Chen–Holm and Treeby–Cox wave equations evaluated for liver, and beyond about $r = 100 \mu\text{m}$, the time-domain Green's functions for the power law wave equation and the Treeby–Cox wave equation are nearly indistinguishable.

FIG. 5. The (a), (c) amplitudes and (b), (d) FWHM values of the time-domain Green's functions calculated for the power law, Chen-Holm, and Treeby-Cox wave equations in (a), (b) breast and (c), (d) liver. The amplitudes of all three time-domain Green's functions decrease as the distance increases while the FWHM values of all three time-domain Green's functions increase as the distance increases. The amplitudes of the time-domain Green's functions for all three of these fractional wave equations are very similar at each distance, and the FWHM values are all approximately the same at longer distances, although there is a small difference in the FWHM values at shorter distances that diminishes with increasing distance.

functions multiplied by $4\pi r$ in breast and liver for the power law wave equation (solid line), the Chen-Holm wave equation (dashed line), and the Treeby-Cox wave equation (dot-dashed line), where these four subfigures summarize the effects of attenuation and dispersion in the time-domain. The initial distances are $r = 100$ nm in breast and $r = 100$ μm in liver, and each plot ends at $r = 10$ cm. Figures 5(a) and 5(c) indicate that the amplitudes of the time-domain Green's functions for these three fractional wave equations decrease with frequency as the distance increases due to the effect of attenuation. Also, the amplitudes of the time-domain Green's functions for these three wave equations are all very similar at all distances in both breast and liver. The slopes of the amplitudes in Fig. 5(a) and 5(c) are -0.6677 and -0.8790 , respectively, where the absolute values of these two quantities are approximately equal to the reciprocals of the power law exponents in breast and liver, respectively. Figures 5(b) and 5(d) show that the FWHM values of the time-domain Green's functions for these three fractional wave equations increase with distance due to the effect of dispersion. In Figs. 5(b) and 5(d), the value of the FWHM of the time-domain Green's functions for these three fractional wave equations differs slightly close to the source, but at longer distances, the FWHM values of the time-domain Green's functions for all three wave equations yield nearly the same result even though the shape of the time-domain Green's function for the Chen-Holm wave equation is distinct from the shapes of the time-domain Green's functions

for the power law and Treeby-Cox wave equations. In Figs. 5(b) and 5(d), the slopes of the FWHM values are 0.6711 and 0.8825 , which are approximately equal to the reciprocals of the power law exponents in breast and liver, respectively.

E. Time-domain Green's functions convolved with three cycle Hanning-weighted pulses

Figure 6 shows the waveforms obtained when a three cycle Hanning-weighted pulse^{20,21} is convolved with the time-domain Green's functions for the power law wave equation (solid line), the Chen-Holm wave equation (dashed line), and the Treeby-Cox wave equation (dot-dashed line) multiplied by $4\pi r$ evaluated at $r = 1$ cm in breast with the parameters from Sec. IV A. This figure shows how power law attenuation and phase velocity influence the shape of a short pulse. Input pulses with two different center frequencies, $f_0 = 7.5$ MHz and $f_0 = 29$ MHz, are evaluated to highlight the differences in the resulting waveforms. Figure 6 indicates that the convolution results for the power law wave equation and the Treeby-Cox wave equation are nearly indistinguishable for pulses with $f_0 = 7.5$ MHz and $f_0 = 29$ MHz while the convolution results for the Chen-Holm wave equation evaluated at these two frequencies contain a significant time delay because of the substantial difference in the phase velocity.

Figure 7 shows the results obtained when a three cycle Hanning-weighted pulse is convolved with time-domain Green's functions for the power law wave equation (solid

FIG. 6. Simulated three-cycle Hanning-weighted pulses with center frequencies (a) $f_0 = 7.5$ MHz and (b) $f_0 = 29$ MHz convolved with time-domain Green's functions for the power law, Chen-Holm, and Treeby-Cox wave equations multiplied by $4\pi r$ evaluated in breast at $r = 1$ cm.

FIG. 7. Simulated three-cycle Hanning-weighted pulses with center frequencies (a) $f_0 = 7.5$ MHz and (b) $f_0 = 29$ MHz convolved with time-domain Green's functions for the power law, Chen–Holm, and Treeby–Cox wave equations multiplied by $4\pi r$ evaluated in liver at $r = 1$ cm.

line), the Chen–Holm wave equation (dashed line), and the Treeby–Cox wave equation (dot-dashed line) multiplied by $4\pi r$ evaluated at $r = 1$ cm in liver with the parameters from Sec. IV A. The center frequencies for these two pulses are equal to $f_0 = 7.5$ MHz and $f_0 = 29$ MHz. The convolution results for the power law wave equation and the Treeby–Cox wave equation once again track very closely for both pulses in Fig. 7 while the convolution result for the Chen–Holm wave equation contains a noticeable time delay.

V. DISCUSSION

A. The causality of ultrasound attenuation models for soft tissue

Ascertaining whether a particular ultrasound attenuation model is causal or noncausal is sometimes challenging, especially considering that several contradictory statements appear in the ultrasound literature for equivalent models. For example, Ref. 22 claims that the frequency response for the power law exponent $y=1$ given by $H(\omega) = e^{-\beta x|\omega|/(2\pi)} e^{-j\omega\tau x} e^{j\omega\beta x \ln(\omega)/\pi^2}$ is causal, whereas Ref. 6 indicates that this expression is noncausal because the inverse Fourier transform of $H(\omega)$ is nonzero for all values of t . Similarly, Ref. 23 suggests that the combination of $\alpha(\omega) = \alpha_0 \omega^{1+\varepsilon}$ and $1/c(\omega) - 1/c(\omega_0) = \alpha_0 \tan[(1+\varepsilon)\pi/2](\omega^\varepsilon - \omega_0^\varepsilon)$ is causal for $0 < \varepsilon \leq 1$ where ε is related to the power law exponent y via $y = 1 + \varepsilon$, whereas Ref. 5 shows that the time-domain Green's functions that satisfy Eqs. (1) and (2) describe a nonzero time response for times $t < 0$ when $y = 1.139$ and when $y = 1.5$.

Similar issues arise in the ultrasound literature for space-fractional models of ultrasound attenuation. Reference 7 asserts that the Chen–Holm space-fractional wave equation is causal because of similarities with the parabolic anomalous diffusion equation, whereas Ref. 9 suggests that the Chen–Holm wave equation is noncausal because the phase velocity for this space-fractional wave equation is inconsistent with the desired expression in Eq. (2). In addition, Ref. 11 claims that space-fractional wave equations are noncausal because of the nonlocality of fractional Laplacians.

With regards to the causality of the Chen–Holm and Treeby–Cox wave equations, the presence of the unit step (or Heaviside) function $u(t)$ in each time-domain Green's function definitively establishes the causality of these space-

fractional wave equations in terms of the definition provided in Sec. II E. Interestingly, this means that two different causal models with fractional Laplacians describe the power law attenuation observed in soft tissue. This is possible because the attenuations of these two space-fractional models differ slightly [see Eqs. (1) and (18)], and the corresponding phase velocities also differ slightly [see Eqs. (2), (10), and (17)].

B. Noncausal versus nonlocal

Reference 11 treats noncausality and nonlocality as equivalent notions, although noncausality and nonlocality are two completely different concepts. In a causal system, the effect never precedes the cause,²⁴ whereas nonlocal behavior describes memory effects.²⁵ Causality, as defined in Sec. II E, strictly describes the time response of a system, whereas locality is equally applicable to the temporal and spatial responses. A system that is nonlocal in time depends on the past states,²⁶ whereas a system that is nonlocal in space describes long-range spatial interactions.²⁷ Time-fractional derivatives in lossy wave equations describe temporal memory,²⁸ and space-fractional derivatives (including fractional Laplacians) are nonlocal operators that are averaged over space.²⁵ Both time-fractional and space-fractional models are equally applicable to lossy wave propagation in medical ultrasound, and time- and space-fractional models are also closely connected for the fractional advection-dispersion equation.²⁹

Nevertheless, Ref. 11 raises some interesting issues. In particular, the equation of state introduced in Ref. 10, which relates the density to the pressure, contains a nonlocal fractional Laplacian. In this expression, a local change in density causes a nonlocal change in pressure, and a local change in pressure causes a nonlocal change in density. Although neither is an example of noncausal behavior according to the definition given in Sec. II E, these effectively show examples of long-range interactions predicted by the space-fractional equation of state in Ref. 10.

C. Numerical evaluations of the spherically symmetric inverse 3D Fourier transform

Neither the Matlab “quadgk” routine, which is based on adaptive Gauss–Kronrod quadrature, nor the Matlab “integral” routine, which is based on global adaptive

quadrature, is able to consistently generate the correct result when applied to Eq. (8). Although Filon's formula provides an effective method for evaluating definite integrals with oscillatory integrands, Filon's formula alone is often insufficient when applied to the improper integral in Eq. (8) because the upper limit of integration is infinite and because the envelope of the integrand decays slowly. This problem is solved when Filon's formula is combined with the Pantis method,^{15,16} which consistently converges to the correct result for all of the parameter combinations evaluated here. To establish numerical convergence at a given distance, the result is compared to the value obtained with twice as many Filon abscissas and a value for m that is twice as large. For these evaluations, if the Euclidean norm of the difference between the two results is within 1×10^{-3} , then the results are sufficiently close and convergence is achieved.

D. The effect of attenuation and phase velocity on calculations of time-domain Green's functions

Figures 3 and 4 show that the amplitudes are similar for all three Green's functions in each subfigure. The amplitudes for all three of these Green's functions are proportional to the scale factor $(\alpha_0 r)^{-1/y}$, which was also demonstrated for the three time-fractional wave equations evaluated in Zhao and McGough.⁵ The peak values are all about the same for each subplot in Figs. 3 and 4 and in Figs. 5(a) and 5(c), which confirms that the attenuation is approximately the same in all three of these fractional wave equations. Figures 3 and 4 also indicate that the time-domain Green's functions for the Treeby-Cox wave equation and the power law wave equation converge to the same result at longer distances. The temporal spreading is also about the same for all three of these fractional wave equations, as shown in Figs. 5(b) and 5(d), where the amount of temporal spreading is determined by the factor $(\alpha_0 r)^{1/y}$ that appears in the argument of the stable distribution in Eq. (21). Since the attenuations of all three fractional wave equations are about the same, this suggests that the attenuation $\alpha(\omega)$ is primarily responsible for both the decay in the peak values and the amount of temporal waveform spreading in these three fractional wave equations, whereas the phase velocity $c(\omega)$, particularly the first nonconstant term in the series approximation for $c(\omega)$, is responsible for the "skew" or (a)symmetry in each of the time-domain Green's functions shown in Figs. 3 and 4.

E. Intended applications

The approach introduced in Sec. III B evaluates time-domain Green's functions of space-fractional wave equations at a single point in space quickly and efficiently, which is advantageous for several different types of calculations and comparisons. For example, this approach facilitates comparisons between time-domain Green's functions for space-fractional and time-fractional wave equations, as demonstrated in Figs. 3 and 4. Comparisons with other time-fractional wave equations such as the Szabo wave equation^{3,31} and the Caputo wave equation³² are also enabled by these calculations, as are validations of calculations with 3D FFTs (Ref. 30) and calculations in k-Wave.³³ Furthermore, comparisons between

diffraction patterns for different transducer and material parameters is now possible by superposing contributions from time-domain Green's functions for both space-fractional and time-fractional wave equations, which we ultimately expect to integrate into FOCUS, the "Fast Object-oriented C++ Ultrasound Simulator."³⁴ We also expect to find these calculations useful for modeling power law attenuation in shear waves.⁸

VI. CONCLUSION

Phase velocities and attenuation values were evaluated over a range of ultrasound frequencies, and time-domain Green's functions were calculated for the Chen-Holm, Treeby-Cox, and power law wave equations in breast and liver. In addition, the amplitudes and FWHM values of the time-domain Green's functions for these three fractional wave equations were calculated, and three-cycle Hanning-weighted pulses at two different center frequencies were convolved with the time-domain Green's functions for three fractional wave equations. An additional term was derived for the power series that describes $c(\omega)$ for the Chen-Holm wave equation, and additional terms were derived for the power series that describe $c(\omega)$ and $\alpha(\omega)$ for the Treeby-Cox wave equation, where these new power series match the results obtained by numerically evaluating the dispersion relation more closely than previous approximations. Causality was also demonstrated analytically in the time domain for both the Chen-Holm and Treeby-Cox wave equations. The Pantis method was introduced as an effective approach for evaluating the highly oscillatory improper integrals that arise in numerical calculations of the time-domain Green's functions for the Chen-Holm and Treeby-Cox space-fractional wave equations. The time-domain Green's functions for all three time-domain Green's functions show a similar amount of temporal spreading and amplitude reduction, but the time-domain Green's functions for the Treeby-Cox and power law wave equations are skewed whereas the time-domain Green's function for the Chen-Holm wave equation is symmetric. Thus, the Chen-Holm and Treeby-Cox space-fractional wave equations both demonstrate approximately the same amount of attenuation and waveform spreading, but the phase velocities, time-domain Green's functions, and convolution results obtained with these time-domain Green's functions all differ significantly. Also, although the attenuation and phase velocity for the Treeby-Cox and power law wave equation differ slightly, the time-domain Green's functions and the convolution results obtained with the Green's functions for these two fractional wave equations converge to the same result.

ACKNOWLEDGMENTS

This work was supported in part by NIH Grant Nos. EB012079 and EB023051. The authors would like to thank John Nolan for providing the STABLE toolbox.

¹K. R. Waters, M. S. Hughes, J. Mobley, G. H. Brandenburger, and J. G. Miller, "On the applicability of Kramers-Kronig relations for ultrasonic attenuation obeying a frequency power law," *J. Acoust. Soc. Am.* **108**(2), 556–563 (2000).

- ²P. He, "Simulation of ultrasound pulse propagation in lossy media obeying a frequency power law," *IEEE Trans. Ultrason. Ferroelectr. Freq. Control* **45**(1), 114–125 (1998).
- ³T. L. Szabo, "Causal theories and data for acoustic attenuation obeying a frequency power-law," *J. Acoust. Soc. Am.* **97**(1), 14–24 (1995).
- ⁴T. L. Szabo, "The material impulse response for broadband pulses in lossy media," in *2003 IEEE Symposium on Ultrasonics*, Vol. 1, pp. 748–751.
- ⁵X. Zhao and R. J. McGough, "Time-domain comparisons of power law attenuation in causal and noncausal time-fractional wave equations," *J. Acoust. Soc. Am.* **139**(5), 3021–3031 (2016).
- ⁶J. F. Kelly, R. J. McGough, and M. M. Meerschaert, "Analytical time-domain Green's functions for power-law media," *J. Acoust. Soc. Am.* **124**(5), 2861–2872 (2008).
- ⁷W. Chen and S. Holm, "Fractional Laplacian time-space models for linear and nonlinear lossy media exhibiting arbitrary frequency power-law dependency," *J. Acoust. Soc. Am.* **115**(4), 1424–1430 (2004).
- ⁸S. Holm and R. Sinkus, "A unifying fractional wave equation for compressional and shear waves," *J. Acoust. Soc. Am.* **127**(1), 542–548 (2010).
- ⁹S. Holm and S. P. Näsholm, "A causal and fractional all-frequency wave equation for lossy media," *J. Acoust. Soc. Am.* **130**(4), 2195–2202 (2011).
- ¹⁰B. E. Treeby and B. T. Cox, "Modeling power law absorption and dispersion for acoustic propagation using the fractional Laplacian," *J. Acoust. Soc. Am.* **127**(5), 2741–2748 (2010).
- ¹¹R. Kowar, "Causality analysis of the fractional state and wave equations based on fractional Laplacians," *Wave Motion* **58**, 244–258 (2015).
- ¹²R. N. Bracewell, *The Fourier Transform and Its Applications*, 3rd ed. (McGraw-Hill, New York, 2000), p. 342.
- ¹³A. Papoulis, *Signal Analysis* (McGraw-Hill, New York, 1977), p. 14.
- ¹⁴P. J. Davis and P. Rabinowitz, *Methods of Numerical Integration*. (Academic, New York, 1975), p. 119.
- ¹⁵G. Pantis, "The evaluation of integrals with oscillatory integrands," *J. Comput. Phys.* **17**(2), 229–233 (1975).
- ¹⁶P. K. Kythe and M. R. Schäferkötter, *Handbook of Computational Methods for Integration* (CRC, Boca Raton, FL, 2004), pp. 209–211.
- ¹⁷L. N. G. Filon III, "On a quadrature formula for trigonometric integrals," *Proc. R. Soc. Edinburgh* **49**, 38–47 (1930).
- ¹⁸G. R. Lindfield and J. E. T. Penny, *Numerical methods using Matlab*, 3rd ed. (Academic, Waltham, MA, 2012), pp. 211–214.
- ¹⁹J. P. Nolan, "Numerical calculation of stable densities and distribution functions," *Commun. Stat. Stochastic Models* **13**(4), 759–774 (1997).
- ²⁰D. Chen, J. F. Kelly, and R. J. McGough, "A fast nearfield method for calculations of time-harmonic and transient pressures produced by triangular pistons," *J. Acoust. Soc. Am.* **120**(5), 2450–2459 (2006).
- ²¹J. F. Kelly and R. J. McGough, "A time-space decomposition method for calculating the nearfield pressure generated by a pulsed circular piston," *IEEE Trans. Ultrason. Ferroelectr. Freq. Control* **53**(6), 1150–1159 (2006).
- ²²K. V. Gurumurthy and R. M. Arthur, "A dispersive model for the propagation of ultrasound in soft-tissue," *Ultrason. Imag.* **4**(4), 355–377 (1982).
- ²³J. Mobley, K. R. Waters, and J. G. Miller, "Finite-bandwidth effects on the causal prediction of ultrasonic attenuation of the power-law form," *J. Acoust. Soc. Am.* **114**(5), 2782–2790 (2003).
- ²⁴T. L. Szabo, *Diagnostic Ultrasound Imaging: Inside Out* (Academic, Waltham, MA, 2014), p. 89.
- ²⁵R. Herrmann, *Fractional Calculus: An Introduction for Physicists* (World Scientific, Singapore, 2011), p. 88.
- ²⁶S. Das, *Functional Fractional Calculus* (Springer, Berlin, 2011), p. 38.
- ²⁷V. E. Tarasov, *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media* (Springer, New York, 2010), p. 222.
- ²⁸S. Holm, "Four ways to justify temporal memory operators in the lossy wave equation," in *2015 IEEE International Ultrasonics Symposium (IUS)*, pp. 1–4.
- ²⁹J. F. Kelly and M. M. Meerschaert, "Space-time duality for the fractional advection-dispersion equation," *Water Resour. Res.* **53**(4), 3464–3475, <https://doi.org/10.1002/2016WR019668> (2017).
- ³⁰B. E. Treeby and B. T. Cox, "A k-space Green's function solution for acoustic initial value problems in homogeneous media with power law absorption," *J. Acoust. Soc. Am.* **129**(6), 3652–3660 (2011).
- ³¹T. L. Szabo, "Time-domain wave-equations for lossy media obeying a frequency power-law," *J. Acoust. Soc. Am.* **96**(1), 491–500 (1994).
- ³²M. Caputo, "Linear models of dissipation whose Q is almost frequency independent-II," *Geophys. J. R. Astron. Soc.* **13**, 529–539 (1967).
- ³³B. E. Treeby and B. T. Cox, "k-Wave: MATLAB toolbox for the simulation and reconstruction of photoacoustic wave fields," *J. Biomed. Opt.* **15**(2), 021314 (2010).
- ³⁴FOCUS, "The 'Fast Object-oriented C++ Ultrasound Simulator,'" <https://www.egr.msu.edu/~fultras-web> (Last viewed July 02, 2018).