Application of a forward model of axisymmetric shear wave propagation in viscoelastic media to shear wave elastography

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A simple but general solution of Navier's equation for axisymmetric shear wave propagation in a homogeneous isotropic viscoelastic medium is presented. It is well-suited for use as a forward model for some acoustic radiation force impulse based shear wave elastography applications because it does not require precise knowledge of the strength of the source, nor its spatial or temporal distribution. Instead, it depends on two assumptions: (1) the source distribution is axisymmetric and confined to a small region near the axis of symmetry, and (2) the propagation medium is isotropic and homogeneous. The model accounts for the vector polarization of shear waves and exactly represents geometric spreading of the shear wavefield, whether spherical, cylindrical, or neither. It makes no assumption about the frequency dependence of material parameters, i.e., it is material-model independent. Validation using measured shear wavefields excited by acoustic radiation force in a homogeneous gelatin sample show that the model accounts for well over 90% of the measured wavefield "energy." An optimal fit of the model to simulated shear wavefields with noise in a homogeneous viscoelastic medium enables estimation of both the shear storage modulus and shear wave attenuation to within 1%. © 2018 Acoustical Society of America.

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I. INTRODUCTION

Most shear wave elastography (SWE) techniques consist of three steps: (i) Excite a shear wave disturbance; (ii) measure the resulting response; (iii) fit the response to a model of shear wave propagation. Models used in acoustic radiation force impulse-based (ARFI-) SWE include one-dimensional time-of-flight models, and solutions based on spherical, cylindrical, or plane wave propagation.^{1–3} The fidelity of the model in representing the underlying measured phenomenon affects the accuracy of the parameter estimates obtained. A recent analytical study⁴ demonstrated that large biases can be introduced by incorrect modeling assumptions in the analysis, with the size of the bias depending on the frequency, the geometry of the excitation source, and the specific material model chosen. Indeed, material parameter estimates obtained from different ARFI-SWE implementations exhibit statistically significant variation in both clinical applications⁵ and phantom studies.^{6,7} This confounds direct comparisons between techniques and systems, raises questions about the accuracy of the methodology, and hinders clinical adoption of these techniques.⁵

Some of the variability in ARFI-SWE derived parameter estimates may be reduced by using forward models that more faithfully represent the vector nature of shear wave propagation and the viscoelasticity of the medium. There are two challenges encountered in developing a forward model of ARFI-excited

shear wave propagation in practical applications. One is that shear wave propagation in soft tissues is often dispersive⁸ and the second is that the excitation, i.e., the source of the shear waves, is not well-characterized and can vary from application to application, being dependent on system settings and acoustic properties of the soft tissue/medium under investigation.

In this paper, we present a simple but general solution of Navier's equation for axisymmetric shear wave propagation in a homogeneous isotropic viscoelastic medium. It is well-suited for use as a forward model for some ARFI-SWE applications because it overcomes the two challenges mentioned above. We account for dispersion by solving the problem in the frequency domain. That is, we separate the shear wave pulse into its temporal frequency components and analyze individual frequency components separately. Second, our model does not require precise knowledge of the spatial or temporal distribution of the source, nor its strength, so long as the source is axisymmetric.

Several past studies reported techniques to estimate phase velocity in dispersive media,^{2,9,10} and a few^{3,11–13} also estimated the shear loss modulus (or shear attenuation). The forward models used in these studies assume a direction of propagation (usually lateral) and a geometric spreading (usually cylindrical). These assumptions are often only approximately correct and thus can cause significant biases in the estimates obtained.^{4,13,14} Moreover, because these assumptions are justified in only a small subregion of the measured shear wavefield, they limit material parameter estimation to a small subset of the measured data and make these techniques sensitive to measurement noise. The proposed model accounts for the vector polarization of shear waves and is therefore

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applicable to the entire measured shear wavefield, thereby allowing the use of all available data. It also exactly models the geometric spreading of the shear wavefield, whether spherical, cylindrical, or neither. Furthermore, it makes no assumption about the frequency dependence of material parameters, i.e., it is material-model independent.

A second key feature of the model presented is that it requires neither precise knowledge of the spatial or temporal distribution of the source, nor its strength. Instead, it relies on the assumptions that the source distribution is axisymmetric and confined to a small region near the axis of symmetry, and that the medium through which the wavefield propagates is homogeneous. This allows us to infer, from the measured wavefield, a virtual source field that is consistent with the measurement. Our forward solution is thus a “hybrid” model of shear wave propagation, being a solution of the governing viscoelastic vector Navier equation that is informed by measurement data. We present two approaches to incorporate measured data. One of these uses the wavefield in an entire region of interest (ROI) to reduce effects of noise in the model predictions. When the propagation characteristics of the medium are unknown *a priori*, our forward model can be fit to a measurement and thus the medium parameters can be inferred.

The model was verified using simulated data in two contexts: First, when the medium parameters are known *a priori*, and second when they are unknown but determined as part of the data fitting process, i.e., an optimal model-based inversion technique. For experimental validation we used shear wavefields measured in a gel sample. The predictions of the model were compared to the measured wavefields and showed that the model can accurately represent all spatial features of the measurement. This qualitative impression was quantified through error metrics that showed that more than 90% of the measured wavefield “energy” was accounted for in our models.

The paper is organized as follows. In Sec. II we first describe the ARFI-SWE application that motivated the development of the model. Then we formulate the problem and solve for the forward model of viscoelastic shear wave propagation. In Sec. III, the model is both verified and validated using simulations and measurements in soft gels. The paper ends with a discussion (Sec. IV) and a summary (Sec. V).

II. FORWARD MODEL OF SHEAR WAVE PROPAGATION

A. Motivation

We consider the physical situation corresponding to a typical ARFI-SWE implementation using a diagnostic ultrasound transducer. An ARFI-push is created by focusing P-wave acoustic energy from the transducer elements at a desired location in the transducer’s field of view (FOV). The focus is usually chosen along the central axis of the transducer at a desired depth. The resulting shear wave is imaged at a high frame rate using plane-wave imaging. The acquired frames are used to compute a time-history of the displacement field. The fact that high frequency P-waves (sometimes called longitudinal or dilatational waves) are used to measure the evolving low frequency wave implies that the

measured particle velocity is due only to shear wavefields. That is, the P-wavefield excited by the ARFI-push simply moves too quickly to be captured by later P-wave imaging pulses. Therefore, we assume we are given measurements of the shear wave displacement field in a limited ROI. We also assume that the axis of focus is known to high precision and that the resulting radiation force excitation is symmetric about this axis. Our goal is to create a mathematical model of the shear wavefield that is consistent with both the measured data and the governing Navier equation.

B. Problem formulation

We consider an infinite viscoelastic medium excited by an axisymmetric source confined to a cylindrical region ($r < r_0$) around the axis of symmetry; see Fig. 1. Lacking any further information about the precise nature of the source, we restrict our attention to the source-free region $\Omega_O(r > r_0)$. We recall that Navier’s equation¹⁵ for time-harmonic, steady-state motion in a source-free homogeneous isotropic elastic medium may be written as

$$(\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{U}) - \mu\nabla \times (\nabla \times \mathbf{U}) + \rho\omega^2\mathbf{U} = 0. \quad (1)$$

Here, the displacement field $\mathbf{u}(\mathbf{x}, t)$ has been represented in the form

$$\mathbf{u}(\mathbf{x}, t) = \text{Re}\{\mathbf{U}(\mathbf{x})e^{-i\omega t}\}. \quad (2)$$

Here λ and μ are the Lamé parameters, ρ is the mass density, and ω is the angular frequency of the time-harmonic motion. In a viscoelastic medium, complex Lamé parameters can be used to represent the relation between the oscillating stresses and strains.¹⁶ For example, the complex shear modulus may be written as

$$\mu = \mu' - i\mu'', \quad (3)$$

where μ' is the *shear storage modulus* and μ'' is the *shear loss modulus*. We note that the negative sign in Eq. (3) is due to the $e^{-i\omega t}$ time dependence chosen in Eq. (2) and for $\mu'' \geq 0$. This ensures that a propagating shear wave decays

FIG. 1. (Color online) Axisymmetric source in an unbounded homogeneous medium. Shown is a ROI Ω that comprises of a region, Ω_S , near the axis of symmetry that fully contains the excitation source and a source-free region, Ω_O .

in amplitude due to the viscous losses in the medium rather than growing in amplitude. Choosing the opposite time dependence in Eq. (2) leads to the opposite sign in Eq. (3). If δ is the phase lag between the shear stress and shear strain, then the ratio

$$\tan(\delta) = \mu''/\mu', \quad (4)$$

is a measure of the fractional energy dissipation in the medium per oscillation cycle.¹⁶

The Helmholtz decomposition for a vector field is

$$\mathbf{U} = \nabla\phi + \nabla \times \mathbf{\Psi}, \quad (5)$$

which is supplemented by a gauge condition $\nabla \cdot \mathbf{\Psi} = 0$. If $\mathbf{U}$ satisfies Navier's equation for a homogeneous solid, the potentials ϕ and $\mathbf{\Psi}$ independently represent compressional and shear waves, respectively. This is because of the vector identities: $\nabla \times (\nabla\phi) = 0$, implying no shear, and $\nabla \cdot (\nabla \times \mathbf{\Psi}) = 0$, implying no volume change. Thus, they independently satisfy the scalar and vector Helmholtz equations¹⁵

$$\nabla^2\phi + k_p^2\phi = 0, \quad (6)$$

$$\nabla^2\mathbf{\Psi} + k_s^2\mathbf{\Psi} = 0. \quad (7)$$

Here k_p and k_s are the wavenumbers associated with wave speeds c_p and c_s for the dilatational or P-waves and the solenoidal or S-waves, respectively. They are related by

$$k_p = \frac{\omega}{c_p} \quad \text{and} \quad k_s = \frac{\omega}{c_s},$$

$$c_p = \sqrt{\frac{\lambda + 2\mu}{\rho}} \quad \text{and} \quad c_s = \sqrt{\frac{\mu}{\rho}}. \quad (8)$$

Equation (8) shows that the wavenumbers and wave speeds in a viscoelastic medium with complex elastic moduli are also complex, with real and imaginary components.

In what follows, we assume that the source excites axisymmetric wavefields for which

$$\phi = \phi(r, z), \quad (9)$$

$$\mathbf{\Psi} = \psi(r, z)\hat{e}_\theta. \quad (10)$$

Here $\hat{e}_\theta$ is a unit vector in the azimuthal direction. In ARFI-SWE only the shear disturbance contained in $\mathbf{\Psi}$ is measured. Dilatational P-waves, contained in ϕ , travel much faster than the shear waves and leave the imaging FOV before the plane-wave imaging of the shear disturbance has started. As a consequence, particle velocities associated with the P-wave are absent from the measurements. Therefore in our forward model we will assume $\nabla\phi = 0$. Moreover, only the axial, or depth, component of particle velocity is measured. Equation (7) in cylindrical coordinates (r, θ, z) and with $\mathbf{\Psi}$ as defined in Eq. (10) reduces to

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{r^2}\psi + k_s^2\psi = 0. \quad (11)$$

Substituting Eq. (10) into Eq. (5), along with $\nabla\phi = 0$, gives the radial and axial components of the displacement field as

$$U_r(r, z) = -\frac{\partial\psi}{\partial z}, \quad U_z(r, z) = \frac{\partial\psi}{\partial r} + \frac{1}{r}\psi. \quad (12)$$

Our goal is to solve Eq. (11) in the source-free region subject to boundary conditions

$$\frac{\partial\psi}{\partial r} + \frac{1}{r_0}\psi(r_0, z) = U_z^M(r_0, z) \quad \text{at} \quad r = r_0, \quad (13)$$

$$\lim_{r \rightarrow \infty} \left[r \left(\frac{\partial\psi}{\partial r} - ik_s\psi \right) \right] = 0 \quad \text{as} \quad r \rightarrow \infty. \quad (14)$$

Equation (13) enforces a match to $U_z^M(r_0, z)$ obtained from the measured wavefield. Equation (14) is a statement of the Sommerfeld radiation condition,¹⁷ which enforces the condition that there are no sources or reflections at infinity.

C. Solution

We transform the partial differential Eq. (11) into an ordinary differential equation by using a Fourier transform in the z -direction, i.e., we let

$$\psi(r, z) = \int_{-\infty}^{+\infty} \tilde{\psi}(r, k_z) e^{ik_z z} dk_z. \quad (15)$$

Substituting Eq. (15) in Eq. (11) gives, for each wavenumber k_z in the z -direction,

$$\frac{d^2\tilde{\psi}}{dr^2} + \frac{1}{r}\frac{d\tilde{\psi}}{dr} - \frac{1}{r^2}\tilde{\psi} + k_r^2\tilde{\psi} = 0. \quad (16)$$

Here we introduced $k_r(w, k_z)$ defined so that for k_s and k_z real

$$k_r = \sqrt{k_s^2 - k_z^2} = \begin{cases} +\sqrt{\frac{\omega^2}{c_s^2} - k_z^2} & k_z^2 \leq \frac{\omega^2}{c_s^2} \\ +i\sqrt{k_z^2 - \frac{\omega^2}{c_s^2}} & k_z^2 > \frac{\omega^2}{c_s^2} \end{cases} \quad (17)$$

The right most part in Eq. (17) is shown for illustrating the difference between propagating and evanescent waves. The wave components for which k_r is predominantly imaginary, i.e., for $k_z^2 > \omega^2/c_s^2$, are evanescent. They decay quickly away from the source and are only present in the near-field, i.e., they are evanescent. Note that c_s and k_s in a viscoelastic medium are, in general, complex. For complex k_s , we use the analytic continuation of the definition above. Hence k_r is also, in general, complex.

A general solution satisfying Eq. (16) can be written using Hankel functions of the first kind¹⁸ as

$$\tilde{\psi}(r, k_z) = A(k_z)H_1^{(1)}(k_r r) + B(k_z)H_1^{(2)}(k_r r). \quad (18)$$

The outgoing boundary condition (14) requires $B(k_z) = 0$. The coefficients $A(k_z)$ are determined by the boundary condition at $r = r_0$, i.e., Eq. (13), as follows. First we use Eq. (18) in Eq. (15) to obtain

$$\psi(r, z) = \int_{-\infty}^{+\infty} A(k_z) H_1^{(1)}(k_r r) e^{ik_z z} dk_z. \quad (19)$$

Substituting Eq. (19) into Eq. (12) yields the shear wave displacement field in the region $r \geq r_0$ in terms of the coefficients $A(k_z)$ as

$$U_r(r, z) = - \int_{-\infty}^{+\infty} ik_z A(k_z) H_1^{(1)}(k_r r) e^{ik_z z} dk_z, \quad (20a)$$

$$U_z(r, z) = \int_{-\infty}^{+\infty} k_r A(k_z) H_0^{(1)}(k_r r) e^{ik_z z} dk_z. \quad (20b)$$

The boundary condition in Eq. (13) requires

$$\int_{-\infty}^{+\infty} A(k_z) k_r H_0^{(1)}(k_r r_0) e^{ik_z z} dk_z = U_z^M(r_0, z). \quad (21)$$

Taking the inverse Fourier transform of Eq. (21) yields

$$A(k_z) = \frac{1}{k_r H_0^{(1)}(k_r r_0)} \tilde{U}_z^M(r_0, k_z), \quad (22)$$

where

$$\tilde{U}_z^M(r_0, k_z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} U_z^M(r_0, z) e^{-ik_z z} dz. \quad (23)$$

Equation (20) with coefficients $A(k_z)$ from Eq. (22) represents the general axisymmetric shear wavefield in the region $r \geq r_0$ that is consistent with both the governing Navier Eq. (11) and the boundary conditions (13) and (14).

D. Proof of equivalence or uniqueness

We now show that the shear wavefield computed in Eq. (20) can exactly represent any shear wavefield in an unbounded homogeneous viscoelastic medium excited by an axisymmetric source distribution that is completely contained within the region $r \leq r_0$. This is because the solution that satisfies both the governing Eq. (11) and the boundary conditions (13) and (14) in the source-free region Ω_0 is unique. To show this, we first consider a reference shear wavefield, represented by the vector potential Ψ_{ref} , excited by an axisymmetric source $\mathbf{S}$ in domain Ω . As shown in Eqs. (1)–(7), Ψ_{ref} satisfies the vector Helmholtz equation

$$\nabla^2 \Psi_{\text{ref}} + k_s^2 \Psi_{\text{ref}} = \mathbf{S} \quad \text{in } \Omega. \quad (24)$$

Ψ_{ref} is itself axisymmetric, so that $\Psi_{\text{ref}} = \psi_{\text{ref}}(r, z) \hat{e}_\theta$, and it is subject to the radiation boundary condition

$$\lim_{r \rightarrow \infty} \left[r \left(\frac{\partial \psi_{\text{ref}}}{\partial r} - ik_s \psi_{\text{ref}} \right) \right] = 0. \quad (25)$$

The problem of finding Ψ_{ref} that satisfies Eqs. (24) and (25) we will call Problem 1.

We now consider a second problem, that we will call Problem 2. In Problem 2, attention is confined to the region $r > r_0$. We consider a shear wavefield represented by $\Psi_0 = \psi_0(r, z) \hat{e}_\theta$ defined in the source-free region $\Omega_0 (r > r_0)$ which satisfies

$$\nabla^2 \Psi_0 + k_s^2 \Psi_0 = 0 \quad \text{in } \Omega_0, \quad (26)$$

$$\Psi_0 = \Psi_{\text{ref}} \quad \text{at } r = r_0, \quad (27)$$

$$\lim_{r \rightarrow \infty} \left[r \left(\frac{\partial \psi_0}{\partial r} - ik_s \psi_0 \right) \right] = 0. \quad (28)$$

Now we shall show that Ψ_0 , the solution to Problem 2, is equal to Ψ_{ref} , the solution to Problem 1, where both are defined. We note that both Problem 1 defined by Eqs. (24) and (25), and Problem 2 defined by Eqs. (26)–(28), have unique solutions.¹⁷ Next we observe that since $\mathbf{S} = 0$ in Ω_0 , by Eq. (24) Ψ_{ref} satisfies Eq. (26). Furthermore, Ψ_{ref} also satisfies boundary conditions (27) and (28). Therefore, Ψ_{ref} is a solution to Problem 2. Since the unique solution to Problem 2 is Ψ_0 , we conclude that

$$\Psi_0 = \Psi_{\text{ref}} \quad \text{in } \Omega_0. \quad (29)$$

This shows that no matter what axisymmetric source distribution $\mathbf{S}$ gives rise to Ψ_{ref} , outside the source region, $\Psi_{\text{ref}} = \Psi_0$. Conversely, since Ψ_0 satisfies Eqs. (26)–(28) and since Eqs. (26)–(28) has a unique solution, then Ψ_0 can represent any axisymmetric shear wavefield in Ω_0 .

E. Evaluating source terms using data in Ω_0 : Least squares fit

When using Eq. (20b) to fit measured data, the presence of noise in the measurement at $r = r_0$ can cause errors in the source coefficients $A(k_z)$, and thereby lead to errors in the forward propagation model. The computation of the source coefficients can be made more robust by replacing the boundary condition (13) which was enforced at just one radial location $r = r_0$, to that of fitting Eq. (20b) to the entire measured wavefield outside of the source region. Effectively this is like attempting to enforce the boundary condition in Eq. (13) at all ($r \geq r_0$). This leads to an overdetermined system which can be solved as a linear least squares problem with the error functional

$$\mathcal{E}(A) = \int_{r \geq r_0} |\tilde{U}_z^M(r, k_z) - k_r H_0^{(1)}(k_r r) A(k_z)|^2 dr. \quad (30)$$

A robust estimate for the coefficients $A(k_z)$ can be obtained by minimizing $\mathcal{E}(A, k_z)$ considering one k_z at a time, or

$$A_{\text{opt}}(k_z) = \arg \min_A [\mathcal{E}(A)]. \quad (31)$$

The improved model-fitting of noisy wavefields using these optimal estimates of the source coefficients is demonstrated in Sec. III B, Fig. 2.

III. VERIFICATION AND VALIDATION

In this section, we consider three example applications of the proposed model (20b). We note that the coefficients $A(k_z)$ in the model are evaluated from measured wavefields. The data used to determine $A(k_z)$ may be corrupted by error (measurement error, modeling error, or both). In the first example in Sec. III B, therefore, we assess the

impact on our model of noise in the measurements used to evaluate the $A(k_z)$ coefficients. In the second example in Sec. III C, we consider the case in which the propagation speeds and loss within the medium are unknown *a priori*. We show that by fitting model (20) to the given data, the propagation speed and the shear loss modulus in the medium can be inferred. In both of these first examples, we use simulated data from a reference solution (in Sec. III A) that is, by construction, consistent with the original three-dimensional (3D) linear viscoelastic wave model from which Eq. (20b) was derived. In the third example in Sec. III D, therefore, we validate the model for use in ARFI-SWE applications. We do so by comparing Eq. (20) to ARFI-excited wavefields measured in gelatin. We assess the fit by computing the fraction of energy in the measured wavefield that is captured in the model.

A. Simulated data—Reference solution

Consider a concentrated time-harmonic force, $P e^{-i\omega t} \hat{e}_z$, acting at the origin in an unbounded isotropic viscoelastic medium with known material parameters. Since the source of the motion is time-harmonic, the displacement field can be represented in the form $\mathbf{u}(\mathbf{x}, t) = \text{Re}\{\mathbf{U}(\mathbf{x}) e^{-i\omega t}\}$. Then, similar to Eq. (1), $\mathbf{U}(\mathbf{x})$ is a solution of the time-harmonic Navier equation but now with a source term

$$(\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{U}) - \mu\nabla \times (\nabla \times \mathbf{U}) + \rho\omega^2 \mathbf{U} = -P\delta(\mathbf{x})\hat{e}_z, \quad (32)$$

subject to the Sommerfeld radiation condition given in Eq. (14). An analytical solution to this problem in a purely elastic medium is available, see Eq. (8.134a) in Ref. 19 or Eqs. (20) and (21) in Ref. 20. This solution can be adapted to a viscoelastic medium by the use of complex Lamé parameters, as mentioned earlier, and used for the verification of our proposed model. Using the solution, the axial component of the displacement field (i.e., the displacement field in the direction of the applied force) in the $r - z$ plane can be written as

$$U_z(r, z) = U_z^P + U_z^S, \quad (33)$$

$$U_z^P(r, z) = \frac{1}{4\pi\rho\omega^2} \frac{e^{ik_p R}}{R^3} \left[k_p^2 z^2 R^2 + \left(3 \frac{z^2}{R^2} - 1 \right) (ik_p R - 1) \right], \quad (34)$$

$$U_z^S(r, z) = \frac{1}{4\pi\rho\omega^2} \frac{e^{ik_s R}}{R^3} \left[k_s^2 R^2 \left(1 - \frac{z^2}{R^2} \right) - \left(3 \frac{z^2}{R^2} - 1 \right) (ik_s R - 1) \right]. \quad (35)$$

U_z^P is the particle displacement due to the dilatational P-wave with wavenumber k_p ; U_z^S is the particle displacement due to the solenoidal S-wave with wavenumber k_s ; and $R = \sqrt{r^2 + z^2}$ is the distance from the excitation point, which is chosen to be the origin here. As mentioned in the problem formulation in Sec. II B, P-waves are not present in the measured ARFI-SWE deformation field, and so in this study we will use only the shear wavefield solution in Eq. (35). We

note however, that the modeling approach used in Sec. II C can be modified to include the presence of P-waves in the solution if needed.

B. Fitting simulated data: Known material parameters

The shear wavefield in Eq. (35) is shown graphically in Fig. 2(a). The wavefield was evaluated over a 129×101 pixel rectangular grid covering a region of size 10×8 shear wavelengths. The voxel size, expressed in cylindrical coordinates (r, θ, z) , was thus parameterized by the shear wavelength, λ , as $\Delta r = 10\lambda/128 \approx 0.08\lambda$ and $\Delta z = 8\lambda/100 = 0.08\lambda$. It should be noted that while image resolution is usually expressed in terms of the wavelength of the imaging P-waves in ultrasound imaging, the ROI in this simulation was parameterized using the shear wavelength. Since this is an axisymmetric wavefield, the out of plane voxel size, $r d\theta$, is arbitrary. In an elastic medium with shear modulus $\mu = 4 \text{ kPa}$ and density $\rho = 1000 \text{ kg/m}^3$, the shear wavelength at 500 Hz is $\lambda = 4 \text{ mm}$. The simulated wavefield then has an equivalent ROI size (*width* $\times$ *depth*) of $40 \times 32 \text{ mm}$. This was roughly similar to our ARFI-SWE measurements in soft gel shown later in Figs. 8 and 9 which had an ROI size of $38 \times 30 \text{ mm}$.

We use this wavefield as the reference to demonstrate the axisymmetric shear wave propagation model developed in Sec. II C. To do this, we use the wavefield solution evaluated at $r_0 = \lambda$ and use that as the boundary condition in Eq. (13) to compute the propagated wavefield $U_z(r, z)$ using Eq. (20b). The source in this reference solution is a point source at the origin, and so any finite r_0 will completely enclose this “source region”; we arbitrarily chose it to be one shear wavelength. This region is demarcated in Fig. 2(a) by two vertical lines on either side of the axis of symmetry. This propagated wavefield U_z is shown in Fig. 2(b) and its deviation from the reference wavefield U_z^S is shown as a percentage error in Fig. 2(c).

We note that the solution in Eq. (20) is comprised of a sum of Hankel functions, which individually represent cylindrical or conical waves. Nevertheless, Figs. 2(b) and 2(c) demonstrate that the combination can model even spherical wave propagation from a point source, as in this case. The modeling error shown in Fig. 2(c) is caused by shear waves emanating from mirror image sources above and below the actual source. These are an artifact of the finite range Fourier transform along z , Eq. (15). This error can be reduced by increasing the z -range of the measurement, or by zero-padding, thus moving the image sources farther from the ROI. In the example shown here, with a z -range of size 8λ , the propagated solution is no more than 1% different from the reference wavefield, measured as a percentage of the maximum displacement in the source-free region.

The effect of noise in the coefficients can affect the accuracy of the model, but apparently not too adversely. To demonstrate this we added noise to the wavefield in Fig. 2(a) as shown in Fig. 2(d). The added noise was random, independent, and identically distributed (IID) from pixel to pixel. The forward model was used as before to obtain the denoised wavefield shown in Fig. 2(e). Further error

FIG. 2. (Color online) Application of the axisymmetric propagation solution [Eq. (20b)] to a wavefield in a medium with known material parameters. (a) The real part of the simulated wavefield given by Eq. (35) with $\tan(\delta) = 0.1$. (b) The propagated wavefield based on Eq. (20b). The coefficients $A(k_z)$ are computed using Eq. (22) with $r_0 = \lambda$. (c) The error [(a),(b)] in the region $r > r_0$ expressed as a percentage of maximum displacement. The pattern seen is caused by shear waves emanating from the mirror-image sources above and below the actual excitation source—an artifact of the finite range Fourier transform along z used to compute the coefficients $A(k_z)$. In this example, the propagated solution is no more than 1% different from the reference. (d)–(f) are the corresponding plots for the data with added artificial noise with a SNR = 4, demonstrating an application of denoising. (f) shows the increased deviation from the reference solution due to the noise. This deviation can be reduced by using not just one r_0 but all available data in the region $r > r_0$, as explained in Sec. III B. (g) and (h) show the improvement in performance when all the data are used. As shown in the table, the mean deviation is reduced by a factor of 2 approximately.

reduction can be achieved by using more data for the estimation of the coefficients, $A(k_z)$ in Eq. (31), and thereby gaining the benefits of averaging. Figures 2(g) and 2(h) show that using all the available data in region $r \geq r_0$ reduces the mean deviation in the wavefield by a factor of 2. Data from Fig. 2, at the location $r = r_0$, are analyzed in greater detail in Fig. 3.

C. Fitting simulated data: Unknown material parameters

The general solution in Eq. (20b) can also be applied to situations where the propagation parameters of the medium are not known *a priori*, provided they are homogeneous. We can use the model to estimate the unknown parameters using an iterative model-based inversion technique. In particular, we can estimate the complex shear wavenumber, k_s . From k_s and knowledge of both frequency, ω , and mass density, ρ , the complex shear modulus can be computed using Eq. (8).

The approach we take is to find the complex wavenumber k_s that achieves the best match between the given wavefield, $U_z^M(r, z)$ and the forward propagated model, U_z , obtained using Eq. (20b). To do so, we minimize the normalized cost function

$$\mathcal{C}(k_s) = \frac{\int_{\Omega_0} |U_z^M(r, z) - U_z(r, z; k_s)|^2 dr dz}{\int_{\Omega_0} |U_z^M(r, z)|^2 dr dz}. \quad (36)$$

We note that Eq. (36) is closely related to the error functional [Eq. (30)] which was defined for each individual z -wavenumber, k_z . However, it should be noted that Eq. (30) is used to determine the optimal coefficients, $A(k_z)$, for a given k_s —this procedure being a part of the forward model. In the inversion, the optimal k_s is determined by minimizing Eq. (36) iteratively in three steps: (i) Choose radius r_0 and an initial guess for $\text{Re}(k_s)$, $\text{Im}(k_s)$. (ii) For the current guess, k_s , evaluate A by minimizing Eq. (30) with respect to A . (iii) Iteratively update k_s to minimize Eq. (36). The minimization was performed using the routine *fminsearch* in the *Optimization Toolbox* in MATLAB (Mathworks Inc., Natick, MA). The result is a set of two real positive numbers $[\text{Re}(k_s), \text{Im}(k_s)]$, that together form the complex wavenumber k_s , which when used with the model [Eq. (20b)] best fits the given wavefield, i.e.,

$$k_s^{\text{opt}} = \arg \min_{k_s} [\mathcal{C}(k_s)]. \quad (37)$$

FIG. 3. (Color online) (a) shows a line plot of the shear wave displacement, U_z , at $r = r_0$, from the point source solution in Fig. 2(a), the noisy data in Figs. 2(d) and 2(e), and the model fit in Fig. 2(g) using the data in the region $r \geq r_0$. (b) The z -spectrum of the corresponding displacements at $r = r_0$ are shown. The point source solution has a finite and symmetric spectrum centered at the origin (or $k_z = 0$). The noisy data consists of a broadband noise spectrum superimposed on the exact solution, as expected. The model fit using all data in the region $r \geq r_0$ rejects much of the noise spectrum which is random in nature and does not conform to Navier's equation. As seen in the error plots in Figs. 2(f) and 2(h) the model fit using all the data (h) is thus much better at noise rejection than the fit using a single radial location (f).

Results from this procedure are shown in Figs. 4 and 5. Both use the same reference wavefield, but Fig. 4 is a simulation of a low-noise scenario with a signal-to-noise ratio (SNR) = 40 while Fig. 5 is a simulation of a high-noise scenario with a SNR = 4. In these simulations, the shear wavefield was evaluated over a 129×191 pixel rectangular grid covering a region of size $(10 \times 8)\lambda$, where λ is the shear wavelength. The voxel size, expressed in cylindrical coordinates (r, θ, z) , is $\Delta r = 10\lambda/128 \approx 0.08\lambda$ and $\Delta z = 8\lambda/191 \approx 0.04\lambda$. The greater resolution in the depth direction was chosen to mimic the typical situation in ultrasound imaging. We again chose $r_0 = \lambda$, i.e., one shear wavelength away from the

point excitation source. This choice obviously satisfies the condition for completely enclosing the source excitation, since the source is just a point in this case. The initial guess for $\text{Re}(k_s)$ and $\text{Im}(k_s)$ were chosen to be 50% of their target values, but it was observed that the optimization is not sensitive to the initial guess at least in an interval of $\pm 50\%$ of the target values.

1. Effect of noise on material parameter estimates

Measurement noise is a significant limiting factor in ultrasound-based elastography especially at greater imaging

FIG. 4. (Color online) Inferring *a priori* unknown material parameters using the axisymmetric propagation solution [Eq. (20b)]. (a) The real part of a simulated wavefield given by Eq. (35), $\tan(\delta) = 0.1$ and added artificial noise with a SNR = 40. (b) The propagated wavefield based on Eq. (20b). The coefficients $A(k_z)$ are computed using the least squares problem defined in Eq. (31) with $r_0 = \lambda$. (c) The error [(a)–(b)] in the region $r > r_0$ expressed as a percentage of the maximum displacement, which shows that the error is almost entirely the noise that was added artificially. Plot (d) shows the phase and plot (e) shows the amplitude of the wavefield along $z = 0$, both as functions of radius. We note that the effect of noise is more evident in the amplitude plot (e) than in the phase plot (d). Plot (f) shows the real part of the wavefields along $z = 0$. Plots (d)–(f) show good agreement between the given data and its optimal fit.

FIG. 5. (Color online) Results similar to results in Fig. 4 but with a SNR = 4 corresponding to a high noise scenario. Plots (d)–(f) show that the phase, amplitude, and the real part of the given wavefield are all noisy, but the optimization based inverse solution nevertheless fits the given wavefield well, plots (a) and (b), and the error in plot (c) is again primarily composed of the added noise. Despite the high noise level ($\approx 25\%$), the parameters $\text{Re}(k_s)$ and $\text{Im}(k_s)$ were estimated to within 0.1% of their target values.

depths. In homogeneous regions of tissue, the proposed forward model allows the use of the entire measured wavefield for material property estimation and thereby makes the proposed inversion technique remarkably resilient to the presence of measurement noise. The corrupting influence of noise on estimates of the shear modulus obtained from our inversion technique was studied using a Monte-Carlo simulation with $N = 100$. Normally distributed random noise with a root-mean-square (RMS) noise level ranging from 0% to 30% was added to the simulated shear wavefield from Eq. (35) and also used for the results in Figs. 4 and 5. For each noise level, random noise was added to the wavefield 100 times (noise was random, IID from pixel to pixel) and estimates of the storage and loss shear moduli were obtained each time. The mean, median, and the standard deviation of the resulting modulus estimates yielded an estimate of the accuracy and precision of the proposed method in estimating the medium parameters in the presence of noise.

We compared the performance of our model-based inversion technique to our implementation of a shear wave spectroscopy (SWS) technique described in Ref. 2. In our implementation, seven rows of pixels (three rows above and three below the row containing the point source location) were used, which together span a region that is one-quarter of a shear wavelength thick in the depth direction. As in Ref. 2, the asymptotic behavior of the Bessel function in the far field was assumed. The phase and amplitude were fit over 80% of the lateral width available after excluding one shear wavelength near the source at one end and one shear wavelength at the far boundary of the wavefield ($\approx 10\%$ each). This corresponded to fitting about three shear wavelengths worth of data in the lateral direction. The estimation of the loss modulus was done by fitting the amplitude decay after

applying a geometrical spreading correction factor. Though this was not proposed in Ref. 2, it can be considered an extension of SWS that also uses the amplitude information present in the wavefield.

We used a spherical ($1/R$) spreading model for the SWS results shown in Fig. 6 which is exactly correct for the case of geometrical spreading from a point source used in this simulation. An advantage of the SWS technique is that it runs a lot faster than our iterative inversion technique (on the order of seconds vs minutes), and so the Monte Carlo simulation for the SWS results were done with a higher $N = 1000$. SWS yields reasonably accurate and precise results at low noise levels ($\leq 5\%$, or SNR = 20, for this noise model). At higher noise levels, however, SWS results deviate significantly from the correct values. This is because SWS does not account for the vector nature of shear wave propagation and therefore can only use a small region of the wavefield data. This makes SWS more sensitive to measurement noise than our proposed inversion technique. The high fidelity of our model to the underlying physics and the use of the entire wavefield led to errors of less than $\pm 1\%$ for this noise model, up to a noise level of 25% (SNR = 4) as shown in Fig. 6.

2. Effect of source size on material parameter estimates

The proposed model [Eq. (20b)] accounts for geometric spreading exactly regardless of source size and shape, provided only that it is axisymmetric. This is demonstrated by comparing the performance of our inversion technique with that of our implementation of SWS (Ref. 2) based on both a spherical and cylindrical spreading assumption. A shear

FIG. 6. (Color online) Accuracy of the shear storage and loss moduli estimates using the proposed model in Eq. (20b) are remarkably resilient to measurement noise. Median error of less than 1% is observed in parameter estimates obtained in a Monte-Carlo simulation with a sample size of $N = 100$ with normally distributed random noise added to a simulated wavefield. An implementation of SWS evaluated using a similar simulation ($N = 1000$) produces greater than 100% error when the RMS noise level approaches 25% (SNR = 4). The error bars shown are one standard deviation estimated from the sample size N . The error bars for SWS exceed the y-axis limits for noise levels above 20% and are therefore suppressed. It is notable that our implementation of SWS biased the shear storage modulus high but shear loss modulus low in the presence of high noise.

wavefield produced by a finite line source, of linear dimension L and oriented along the axis of symmetry, was simulated over the same ROI used in Figs. 4 and 5. The wavefield produced by the finite line source is simulated as a superposition of individual point source solutions, given by Eq. (35), spaced apart by the z -resolution, $\Delta z = 0.04\lambda$. The resulting wavefield was used to estimate the shear storage and loss moduli using both the procedures described earlier, see Fig. 7. SWS using both geometric spreading assumptions, spherical or cylindrical, estimated the shear storage modulus reasonably accurately over the range of source sizes from 0 to 4λ , where λ is the shear wavelength. The choice of spreading assumption, however, had a big impact on the estimate of the shear loss modulus. At small source sizes, the spherical assumption performed well, while at large source sizes, the cylindrical assumption hit closer to the target, as expected. The proposed model which accounts for geometrical spreading exactly, estimated both storage and loss moduli accurately over the entire range of source sizes. The small deviation from near-zero error seen at source sizes L exceeding 2λ was likely an artifact caused by the finite size of the FOV chosen for the simulation, which was 8λ deep.

D. Fitting measurement data: Unknown material parameters

We next consider the application of our model to ARFI-SWE measurement data. The data were collected on a Verasonics research ultrasound system (Verasonics, Inc., Model V1, Kirkland, WA) at the Brigham and Women's Hospital in Boston, MA. The implementation is based on code from Ref. 21. We used an ATL L7-4 transducer with 128 elements and an aperture of 38 mm operated at its center frequency of 5 MHz. A shear disturbance was excited using three ARFI pushes along the center axis of the transducer aperture in a tissue-mimicking elastography phantom made of homogeneous gelatin prepared using the method described in Ref. 22. The central 118 out of the total 128 elements in the transducer aperture were used for the ARFI push, i.e., 5 elements at each end of the transducer aperture were silent. A Hamming apodization (symmetric with $N = 118$) was applied to the central 118 elements during the push. The three successive pushes were applied at depths of 21.6, 20.2, and 18.8 mm, respectively. All three pushes were $100 \mu\text{s}$ long with a wait time of $100 \mu\text{s}$ between them. Three

FIG. 7. (Color online) Accuracy of the shear storage and loss moduli estimates using the proposed model-based inversion technique are not affected by the size of the source region. These estimates were obtained from simulated wavefields with a finite length source, with a linear axial dimension L in the range of $[0-4\lambda]$, where λ is the shear wavelength. SWS based on a spherical or cylindrical spreading assumption produces large errors in the loss modulus estimate when the assumption chosen is inappropriate for the source size, while the proposed model [Eq. (20b)] accounts for geometrical spreading nearly exactly over the range of source sizes.

FIG. 8. (Color online) Estimation of unknown material parameters in a tissue-mimicking gel using the axisymmetric propagation solution [Eq. (20b)]. (a) The real part of a measured wavefield at a temporal frequency of 300 Hz. (b) The propagated wavefield based on Eq. (20b). The coefficients $A(k_z)$ are computed using the least squares problem defined in Eq. (31) with $r_0 = 3.1$ mm. (c) The error, i.e., (a)–(b), in the region $r > r_0$ expressed as a percentage of the maximum displacement. The propagated wavefield captures 96% of the energy in the measured wavefield. Plot (d) shows the phase and plot (e) shows the amplitude of the wavefield along $z = 0$, both as functions of radius. We note again that the effect of noise is more evident in the amplitude plot (e) than in the phase plot (d). Plot (f) shows the real part of the wavefields along $z = 0$. Plots (d)–(f) show good agreement between the given data and its optimal fit, at a shear modulus estimate of $\mu = 3.3 - i0.57$ kPa.

pushes were used mainly for the purpose of generating an adequate signal. The distance between the three push locations was small compared to the shear wavelength. The resulting shear wave was tracked for 10 ms starting 1 ms after the last push pulse, with an imaging pulse repetition

frequency of 10 kHz. During the 1 ms before the first frame is measured, the P-waves had traveled about 1.5 m, or well beyond our 40 mm ROI. An algorithm based on two-dimensional (2D)-autocorrelation²³ was used to compute the shear deformation field.

FIG. 9. (Color online) Results similar to results in Fig. 8 but at a temporal frequency of 600 Hz. Note that the wavenumber in plots (a) and (b) is visibly higher than the wavenumber at 300 Hz as expected, and the amplitude in plot (e) exhibits considerable noise near the far edges of the imaged wavefield. Nevertheless, the propagated wavefield (b) captures 93% of the energy in the measured wavefield (a), at a shear modulus estimate of $\mu = 3.5 - i0.81$ kPa.

The shear wavefield was obtained by Fourier transforming the particle velocity data at each pixel along the time-dimension. This leads to a complex amplitude and phase at each pixel as a function of the frequency. The measurement provided shear wavefields in the frequency range 100–1200 Hz, with a peak in the energy content of the shear waves near 500 Hz. Results are shown in Figs. 8 and 9, for two different representative temporal frequencies, 300 and 600 Hz, respectively.

We estimated the complex wavenumber k_s of the gelatin medium by following the procedure described in Sec. III C, with a choice of $r_0 = 3.1$ mm. The choice of r_0 is based on balancing two considerations: source size and signal strength in the fitted wavefields. In an idealized scenario, with a finite source and infinite sensitivity, the parameters in the model-fit are insensitive to the choice of r_0 , so long as the source is completely enclosed within $r < r_0$. In the measured wavefields, however, a practical approximation was necessary: signal strength in the model-fit can be maximized by choosing r_0 as small as possible, but the model of a “source-free” wavefield only fits if r_0 lies outside of the source region. The first priority, therefore, is to ensure that r_0 is large enough so that a significant part of the source does not lie outside $r < r_0$; this was best accomplished by choosing r_0 roughly on the order of one shear wavelength in all studied cases. The chosen source region is indicated in Figs. 8(b)–8(f) and Figs. 9(b)–9(f) by the gray region around the axis of symmetry. A 30 mm deep axial range (from 5 to 35 mm) was used for wavefield fitting as shown.

The propagated wavefields computed using our model, i.e., Eq. (20b), capture 96% and 93% of the energy present in the wavefields at 300 and 600 Hz, respectively, thus indicating a good match between the model and measurement at both temporal frequencies. Shear storage and loss moduli estimates of 3.3 and 0.57 kPa, respectively, at 300 Hz and 3.5 kPa and 0.81 kPa, respectively, at 600 Hz yielded the best match between the proposed model and the measured wavefields. Note that both the storage and loss modulus estimates at 600 Hz differ from, and are higher than, the ones used at 300 Hz, indicating the presence of dispersion in the medium.

IV. DISCUSSION

We have presented a forward model (20) of shear wave propagation excited by an axisymmetric source in an infinite homogeneous isotropic viscoelastic medium. The model is based on an exact solution of Navier’s equation. The main features of the forward model include:

- (1) The model does not depend on detailed knowledge of the source distribution, except its axis of symmetry, and fully accounts for the resulting 3D axisymmetric propagation. It thereby accounts exactly for geometrical spreading that can be spherical, cylindrical, or neither.
- (2) It accounts for vector polarization effects and it can model shear waves propagating in multiple directions.
- (3) It does not rely on a far-field assumption, i.e., it makes no assumption about the direction of propagation, nor of the geometric spreading, and includes the effect of evanescent waves.

- (4) It assumes shear wave propagation in an unbounded, homogeneous, isotropic, linear viscoelastic solid. It is material-model independent, i.e., it does not assume any particular frequency-dependence of the material parameters.

Though successful in the applications above, the proposed model has some limitations. The model assumes axisymmetry and so it will not apply to wavefields that are strongly non-axisymmetric; the latter were studied experimentally in Ref. 14 and analytically in Ref. 4. A second limitation is the assumption of homogeneity and isotropy which precludes its use in the imaging of anisotropic tissues such as muscle fibres, or in inhomogeneous tissues with tumours and/or non-uniform sclerosis. This assumption is reasonable, however, for the characterization of diffuse diseases in relatively homogeneous tissues, such as liver fibrosis and steatosis. A related limitation is that the model does not account for reflections and refraction at tissue boundaries. This can lead to errors when reflecting boundaries are located near the ROI and the medium’s shear-wave attenuation is small. The finite-range Fourier transform in the axial direction makes the model less accurate when the source is close to the top or bottom boundary of the ROI. The effects of reflections can be mitigated by truncating the time-domain data at the instant when the primary wavefront reaches the boundary of the ROI.

The proposed model is well-suited for axisymmetric propagation from a confined source. Two examples where such vibration sources are found in elastography practice are ARFI-SWE and electrode vibration elastography²⁴ where a vibrating needle is the source of shear wave excitation. Since the proposed model does not require detailed knowledge of the source distribution, it is hoped to prove suitable for clinical applications, where patient to patient variability can lead to alterations of the shape of the ARFI or electrode push region. The ability to resolve 3D propagation eliminates artifacts associated with 2D model assumptions like plane-stress or plane-strain, and an assumed direction of propagation, all of which produce biases in the modulus value estimated. Our model also accounts for out of plane geometrical spreading which is often inaccurately approximated based on a spherical, cylindrical, or plane-wave assumption. Our examples show that this has a big impact on the accuracy of the shear loss modulus estimate.

V. SUMMARY

We present a simple but general solution of Navier’s equation for axisymmetric shear wave propagation in a homogeneous isotropic viscoelastic medium. The solution is a hybrid model that satisfies Navier’s equation and is consistent with a measured shear wavefield. It is well-suited for use as a forward model for some ARFI-based SWE applications. We have applied the model to both simulated and measured data, and demonstrated its applicability in the presence of noise, for a range of source sizes, and in the presence of uncertainty in the medium parameters. By fitting the model to the measured wavefield, *a priori* unknown medium properties can be inferred. The model accounts for 3D effects

while making minimal assumptions about the spatial distribution and amplitude of the source. The model is agnostic to the direction of wave propagation and exactly accounts for geometrical spreading. The inversion scheme based on this model is robust to noise and accurately estimates both the shear storage and loss moduli of the homogeneous medium. Validation using ARFI-excited wavefields in a tissue-mimicking gelatin sample show that the model accounts for well over 90% of the measured wavefield energy. It makes no assumption about the frequency dependence of material parameters, and thus when used in SWE with wide-band ARFI-excitation, it holds the promise of accurately characterizing dispersion in homogeneous soft tissues. To the extent that use of this model may improve ARFI-SWE estimated tissue parameters, it is hoped to ultimately aid in the diagnosis and treatment monitoring of diffuse diseases such as liver fibrosis¹⁰ and steatosis,²⁵ and in the identification of acute rejection in liver transplants.³

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